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Searching Market Equilibria under Uncertain Utilities

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Abstract

Our basic model is a noncooperative multi-player game in which the governments of neighboring countries trade emission reductions. We prove the existence of a market equilibrium (combining properties of Pareto and Nash equilibria) and study algorithms of searching a market equilibrium. The algorithms are interpreted as repeated auctions in which the auctioneer has no information on countries' costs and benefits and every government has no information on the costs and benefits of other countries. In each round of the auction, the auctioneer offers individual prices for emission reductions and observes countries' best replies. We consider several auctioneer's policies and provide conditions that guarantee approaching a market equilibrium. From a game-theoretical point of view, the repeated auction describes a process of learning in a noncooperative repeated game with incomplete information.

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Introduction

The problem we shall be concerned with has a reasonable background in economic theory. At the general level it is the problem of finding an exchange equilibrium between agents which produce and consume a public good, where each agent's valuation and production costs are unknown to other agents, where each agent can only contribute by participation in the production of the public good (contributing by paying money is not allowed) and where a central authority, that can impose a solution, is lacking. As an example one could mention the early history of the Netherlands where inhabitants in an area threatened by floods had to join hands in building and maintaining the dykes. An example from more recent history are multilateral negotiations on reciprocal reduction of arms. In this paper we shall focus on international environmental cooperation. Many international environmental conventions have the form of agreements between governments to reduce emissions of transboundary pollutants reciprocally. As recent examples one can mention the Second Sulphur Protocol (1994) and the Framework Convention on Climate Change (1992). The commitments can vary much: in the Second Sulphur Protocol from reduction of emissions by 87 percent, to 4 percent in 2010 relative to 1980. This observation raises the question whether it is possible to design procedures to improve the process of negotiations to obtain commitment that are preferred by all parties in the convention.

In economic theory this problem has been rather neglected. Economists have mainly concentrated on the question whether there will be sufficient incentives to participate in a convention. Such issues of formation and stability of the coalition of states are discussed, for example in [Barrett 1990, 1994]. Closer related to our research question is the work of [Maeler 1989], [Tulkens 1991], [Chandler, Tulkens 1990]. They have proposed algorithms to find an equilibrium solution that specifies the emission reduction commitments of participants. The drawback of their approach is that money transfers between parties are involved in searching the cooperative solution. These publications lack in realism, since multilateral agreements where some countries pay the other ones for cleaning up are quite exceptional. In this paper we concentrate on the case of reciprocal transboundary pollution where countries "pay" each other by reducing their emissions on a "quid pro quo"

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base. The case of reciprocal emission reduction as trade, which is the dominant actual practice, has been researched by [Hoel 1991], [Nentjes 1993, 1994a, 1994b] and [Pethig 1982], but only for the simple two country case and they give only static cooperative solutions. The question how these can be found is not discussed. The n-country case has been investigated by [Nentjes 1990], but analytically it lacks in rigour.

In the game theory the analogous ideas can be found in the works [Ehtamo, Hamalainen 1993], [Ehtamo, Hamalainen, Verkama 1994] in which the problem of computing Pareto optimal solutions with distributed algorithms was analyzed.

In this paper we combine the mathematical model of noncooperative games with the economic model of "trading" emission reductions at the auction. Specific for our approach is that we interpret international environmental negotiations as a kind of multilateral trade between governments in which the "goods" traded consist of emission reduction by each party. The basic idea is that country i is willing to increase its emission control effort if it gets in return a sufficient reduction of transboundary emissions "imported" from other countries, $i = 1, ..., n$. In the process each country tries to maximize a national utility function in which the costs of reducing national emissions are balanced against the national benefits of a lower pollution load arising from reducing emission in all countries that participate in the international convention.

The first analytical problem to be solved is whether there exists an equilibrium solution to such a multilateral exchange of emission reductions. We shall call it a market equilibrium. A next question is, if such an equilibrium exists, under which conditions the equilibrium solution is optimal in the Pareto sense. These questions will be discussed in sections 2 and 3. The definition of a market equilibrium has the decomposition property: each equation depends on only one utility function. In this sense the definition of a market equilibrium looks in form like the definition of a Nash equilibrium. In section 1 we will establish links, show differences between market and non-cooperative Nash equilibria and prove that a market equilibrium dominates a non-cooperative Nash equilibrium. After solving these problems we shall focus on the dynamic question of how to discover the equilibrium. In sections 4, 5 and 6 we shall design algorithms that approach an equilibrium in a stepwise way. The algorithms are formulated in terms of an auction. An auctioneer proposes prices, for each country a specific price, or exchange rate, which defines how much reduction in national pollution load the country receives in return for a unit of reduction in its emissions. The countries-participants are replying iteratively by mentioning emission reductions they are willing to make, given the prices (exchange rates) proposed by the auctioneer. In the process the auctioneer has information about the emission transport coefficients between the participants. He uses it to translate the proposed emission reductions of the countries in the reduction of the pollution load per country. The auctioneer does not have exact information about the utility functions of the participants. He may have only rough estimates of their rate of growth and convexity. The auctioneer compares emission/pollution load reductions offered by the participants with those demanded by the participants. In case of a gap between "supply" and "demand" he proposes new prices. From their side, the participants reply by the emission reductions based only on their own utility functions. In section 5 we prove a convergence result for a particular auction algorithm. In section 6 we discuss several modifications of searching algorithms.

We conclude the Introduction with a brief game-theoretical characterization of the proposed approach. We view trading on emission reductions as a noncooperative game between the governments (see, e.g., [Germeyer 1971, 1976], [Basar, Olsder 1982], [Vorobyev 1985], [Ehtamo, Hamalainen 1993]). A market equilibrium closely related to the classical Pareto and Nash equilibria is treated as one of the acceptable situations in the game.

After stating the existence results we pass to the question: How can the participants choose an acceptable situation? Unfortunately, game theory does not give general answers to this question. We follow the approach of the theory of repeated games which assumes that the players learn in an infinite sequence of game rounds (see, e.g., [Brown 1951], [Robinson 1951], [Axelrod 1984] [Smale 1980], [Fudenberg, Krebs 1993], [Nowak, Sigmund 1992], [Kaniovski, Young 1995], [Kryazhimskii, Tarasyev 1998]). The proposed auction determines a learning process for the governments. A strong uncertainty in information (a government has no information on the utility functions of other countries) is partially compensated by the auctioneer who regulates individual decisions indirectly and can, to a certain extend, be associated with the market player described in [Zangwill, Garcia 1981].

1 Market equilibrium, Nash equilibrium and Pareto maximum

We deal with a model of trading emission reductions (see [Hoel 1991], [Nentjes 1990, 1994], [Pethig 1982]). The model involves n countries and an auctioneer. Each country i controls its emission reduction value, $x_i \geq 0$. Country i is interested in the maximization of its utility function, w_i given by

$$w_i(x) = -C_i(x_i) + B_i\left(\sum_{j=1}^{n} a_{ji}x_j\right). \tag{1.1}$$

Here $x = (x_1, \dots, x_n)$ is the full emission reduction vector, $C_i(x_i)$ is the cost paid by country i for the emission reduction x_i, $B_i(\sum_{j=1}^{n} a_{ji}x_j)$ is the ecological benefit gained by country i thanks to the reduction of the total pollution load to its territory, $\sum_{j=1}^{n} a_{ji}x_j$, and a_{ji}, a transport coefficient, is a proportion of the emission of country j which is transported to country i. It is assumed that $a_{ji} > 0$ and $\sum_{i=1}^{n} a_{ji} \leq 1$. Every cost function C_i is convex and monotone increasing. Every benefit function B_i is strictly concave, monotone increasing and has a finite saturation level $\bar{y}_i$, that is, remains constant on the interval $[\bar{y}_i, \infty)$. Finally, C_i and B_i are assumed to be twice differentiable, which implies

$$C_i'(x_i) > 0, \quad C_i''(x_i) \geq 0 \quad (x_i \geq 0), \tag{1.2}$$

$$B_i'(y_i) > 0, \quad B_i''(y_i) < 0 \quad (0 \leq y_i < \bar{y}_i), \quad B_i'(y_i) = 0, \quad (y_i \geq \bar{y}_i). \tag{1.3}$$

We view the process of choosing an emission reduction vector x as an n-person noncooperative game between the countries (see, e.g., [Basar, Olsder 1982], [Vorobyev 1985], [Barrett 1990, 1994], [Ehtamo, Hamalainen 1993]). The admissible strategies of country i are emission reductions $x_i \geq 0$ and its payoff function is w_i. We assume that in trading emission reductions (in international negotiations), or, equivalently, searching a solution of the game, a delegate of country i is fully informed about the transport matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ & & \dots & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

and the utility function of the government it represents (w_i), and has practically no information on the utility functions of other countries. The countries enter the game with no emission reductions; the initial emission reduction vector, x^0, is, therefore, zero. We shall call an emission reduction vector $x = (x_1, \dots, x_n)$ *positive* if $x_1, \dots, x_n$ are positive.

We consider a market equilibrium as a desired solution of the game. A positive emission reduction vector $x^M = (x_1^M, ..., x_n^M)$ will be called a *market equilibrium* if for each country i, the function $w_i(\lambda x^M)$ $(\lambda > 0)$, is maximized at $\lambda = 1$,

$$x_i^M = \text{argmax}\{w_i(\lambda x^M) : \lambda > 0\};$$

equivalently,

$$\frac{dw_i(\lambda x^M)}{d\lambda} \mid_{\lambda=1} = 0 \quad (i = 1, \ldots, n). \tag{1.4}$$

Analogous definitions were given in [Ehtamo, Hamalainen 1993] and [Ehtamo, Hamalainen, Verkama 1994].

The relations (1.4) show that x^M solves the equations

$$\langle \nabla\ w_i(x), x \rangle = 0 \quad (i = 1, ..., n); \tag{1.5}$$

here $\langle \cdot, \cdot \rangle$ stands for the scalar product in the n-dimensional Euclidean space.

Taking into account the form of w_i, (1.1), one easily specifies (1.5) into

$$-x_i C_i'(x_i) + (\sum_{j=1}^{n} a_{ji} x_j) B_i'(\sum_{j=1}^{n} a_{ji} x_j) = 0 \quad (i = 1, \ldots, n). \tag{1.6}$$

The equations (1.6) define a set of n offer curves that specify how much emission reduction x_i country i is willing to supply in response to the deposition reduction $\sum_{j=1}^{n} a_{ji} x_j$ it receives thanks to emission control of all countries. The ratio

$$p_i = p_i(x) = \frac{\sum_{j=1}^{n} a_{ji} x_j}{x_i} \tag{1.7}$$

represents the *rate of exchange* (at the emission reduction vector x). It shows to how many units of the deposition reduction country i is willing to change a unit of its emission reduction. Using the rates of exchange, we represent (1.6) as

$$-C_i'(x_i) + p_i B_i'(p_i x_i) = 0 \tag{1.8}$$

and thus arrive at the next characterization of the market equilibria: A positive emission reduction vector x is a market equilibrium if and only if it solves the system of algebraic equations (1.8) where p_i is given by (1.7).

Let us recall the notion of a Nash equilibrium. An emission reduction vector $x^N = (x_1^N, ..., x_n^N)$ is a *Nash equilibrium* if

$$\max_{x_i \geq 0} w_i(x_1^N, ..., x_i, ..., x_n^N) = w_i(x_1^N, ..., x_i^N, ..., x_n^N) \quad (i = 1, ..., n). \tag{1.9}$$

Since the functions w_i are strictly concave, the relations (1.9) are equivalent to the requirement that all partial derivatives $\partial w_i(x^N)/\partial x_i$ vanish:

$$\frac{\partial w_i}{\partial x_i}(x^N) = 0 \quad (i = 1, \ldots, n);$$

more specifically, x^N is a solution to

$$-C_i'(x_i) + a_{ii} B_i'(\sum_{j=1}^{n} a_{ji} x_j) = 0 \quad (i = 1, \ldots, n). \tag{1.10}$$

Using the rates of exchange p_i (see (1.7)), we represent (1.10) in the form

$$-C_i'(x_i) + a_{ii}B_i'(p_i x_i) = 0 \quad (i = 1, \ldots, n) \tag{1.11}$$

which looks very much like the market equilibrium equations (1.8). A single difference between the Nash equilibrium equations (1.11) and the market equilibrium equations (1.8) is that in the former the "self-transport" coefficients, a_{ii}, stand on the place of the rates of exchange, p_i. More specifically, the equations are identical in the sense that both state that the national government of each country i reduces its emission up to the level x_i where its *marginal control cost*, $C_i'(x_i)$, equals its national *marginal benefit*. The equations differ in understandings of the marginal benefits. At a Nash equilibrium, x^N, country i treats its marginal benefit as the derivative of its benefit function $x \mapsto B_i(\sum_{j=1}^n a_{ji}x_j)$ in the direction parallel to the axis x_i, and at a market equilibrium, x^M, as the derivative in the direction of x^M. In simpler words, to identify a Nash equilibrium emission reduction, x_i^N, country i looks at the impact of its own emission reduction on its marginal benefit, whereas in identifying a market equilibrium emission reduction, x_i^M, it takes into account the impact on its marginal benefit of the deposition reductions, which all countries exchange, with the rate $p_i = p_i(x^M)$, to its own abatement.

In what follows, we assume that a starting null-point, x^0, is a Nash equilibrium x^N. Thus $x_i^N = 0$ $(i = 1, \ldots, n)$. Every market equilibrium, x^M, which is by definition positive, dominates the Nash equilibrium in all coordinates, $x_i^M > 0$ $(i = 1, \ldots, n)$ which says that for each country a market equilibrium emission reduction exceeds the initial Nash equilibrium one. In other words, for each country a market equilibrium is ecologically "cleaner" than the initial Nash equilibrium.

Let us compare the market equilibricity with the Pareto optimality (the analysis will be continued in section 3). An emission reduction vector $x^P = (x_1^P, ..., x_n^P)$ is called a *Pareto maximum* if for every emission reduction vector $x \neq x^P$ there is country j for which $w_j(x) < w_j(x^P)$. By the Germeyer's theorem (see [Germeyer 1971, 1976]) the set of all Pareto maxima coincides with the set of all solutions of the parametric family of the maximization problems

$$\text{maximize } w(x, \gamma), \quad x_i \geq 0 \tag{1.12}$$

where

$$w(x, \gamma) = w(x_1, ..., x_n, \gamma_1, ..., \gamma_n) = \sum_{k=1}^n \gamma_k w_k(x)$$

$$\gamma_k \geq 0 \quad (k = 1, \ldots, n), \quad \sum_{k=1}^n \gamma_k = 1 \tag{1.13}$$

(the theorem is applicable since the utility functions w_i are strictly concave). Due to the strict concavity of w_i a maximizer in (1.12) is characterized by

$$\sum_{k=1}^n \gamma_k \frac{\partial w_k}{\partial x_i}(x) = 0 \quad (i = 1, \ldots, n). \tag{1.14}$$

Thus, all Pareto maxima are characterized as the solutions of (1.14) with arbitrary γ_k satisfying (1.13). Note that (1.14) expresses the fact that the rows of of the Jacobi matrix

$$DW(x) = \begin{pmatrix} \partial w_1(x)/\partial x_1 & \partial w_1(x)/\partial x_2 & \ldots & \partial w_1(x)/\partial x_n \\ \partial w_2(x)/\partial x_1 & \partial w_2(x)/\partial x_2 & \ldots & \partial w_2(x)/\partial x_n \\ & & \ldots & \\ \partial w_n(x)/\partial x_1 & \partial w_n(x)/\partial x_2 & \ldots & \partial w_n(x)/\partial x_n \end{pmatrix} \tag{1.15}$$

are linearly dependent with nonnegative coefficients γ_k.

Now let us recall that a market equilibrium x solves the equations (1.5). The latter shows that the columns of the Jacobi matrix $DW(x)$ are linearly dependent with coefficients $x_i > 0$ $(i = 1, \dots, n)$. Hence, in the market equilibrium x the rows of $DW(x)$ are also linearly dependent with some coefficients γ_k^* not all of which vanish. If all γ_k^* are nonnegative then, with no loss of generality, $\sum_{k=1}^n \gamma_k^* = 1$; hence, the market equilibrium x is a Pareto maximum.

We formulate this observation as a lemma.

Lemma 1.1 *Let a market equilibrium x be such that the rows of the Jacobi matrix $DW(x)$ are linearly dependent with nonnegative coefficients γ_k^* $(k = 1, \dots, n)$ not all of which vanish. Then x is a Pareto maximum.*

In section 3 we shall give simple conditions that guarantee that the coefficients γ_k^* are nonnegative.

Example.

To take a closer look at market equilibria and its relationships to Nash equilibria and Pareto maxima, let us consider a simplified model. Assume that the countries' cost and benefit functions are given by

$$C_i(x_i) = c_i x_i$$

$$B_i(y_i) = \begin{cases} d_i y_i - \frac{b_i}{2} y_i^2 & 0 \le y \le d/b \\ d^2/2b & y > d/b \end{cases}.$$

Here d_i and b_i are positive constants. We also assume that the transport matrix A is nondegenerate. The equations (1.10) for a Nash equilibrium take the form

$$\sum_{j=1}^n a_{ji} x_j = \frac{a_{ii} d_i - c_i}{a_{ii} b_i} \quad (i = 1, \dots, n). \tag{1.16}$$

The system (1.16) determines the unique Nash equilibrium $x^N = 0$ if $a_{ii} d_i - c_i = 0$ $(i = 1, \dots, n)$.

The equations (1.14) for Pareto maxima are transformed into

$$-\gamma_i c_i + \sum_{k=1}^n \gamma_k d_k a_{ik} = \sum_{k=1}^n \gamma_k b_k a_{ik} y_k \quad (i = 1, \dots, n); \tag{1.17}$$

here vectors $y = y(x) = (y_1(x), ..., y_n(x))$ are determined by

$$y_k = y_k(x) = \sum_{j=1}^n a_{jk} x_j \quad (k = 1, \dots, n). \tag{1.18}$$

The system (1.17) can be rewritten in the matrix form

$$-C\gamma + AD\gamma = AB\Gamma y \tag{1.19}$$

or

$$-C\gamma + AD\gamma = ABY\gamma \tag{1.20}$$

where

$$B = \begin{pmatrix} b_1 & ... & 0 \\ & ... & \\ 0 & ... & b_n \end{pmatrix}, \quad C = \begin{pmatrix} c_1 & ... & 0 \\ & ... & \\ 0 & ... & c_n \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & ... & 0 \\ & ... & \\ 0 & ... & d_n \end{pmatrix},$$

$$Y = \begin{pmatrix} y_1 & ... & 0 \\ & ... & \\ 0 & ... & y_n \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \gamma_1 & ... & 0 \\ & ... & \\ 0 & ... & \gamma_n \end{pmatrix}, \quad \gamma = \begin{pmatrix} \gamma_1 \\ ... \\ \gamma_n \end{pmatrix}.$$

Resolving (1.19) with respect to y under the assumption that $\gamma_k > 0$ $(k = 1, ..., n)$, we arrive at a linear system of equations for x,

$$y = A^T x = (AB\Gamma)^{-1}(-C\gamma + AD\gamma).$$

Hence, arbitrary positive γ_k satisfying $\sum_{k=1}^{n} \gamma_k = 1$ determine a unique a Pareto maximum by the formula

$$x = (A^T)^{-1}(AB\Gamma)^{-1}(-C\gamma + AD\gamma).$$

In order to describe all Pareto points, we resolve the system (1.20) with respect to the vector of weight coefficients, γ. Nontrivial solutions of this system exist if its discriminant is zero:

$$\det(ABY + C - AD) = 0. \tag{1.21}$$

We treat (1.21) as an equation of power n with respect to y. The equation describes a surface in the n-dimensional space. The highest power in (1.21) has the term $\det(A) b_1 b_2 \dots b_n y_1 y_2 \dots y_n$ which shows that the curvature of the surface is determined by the sign of $\det(A)$. For example, in the two-dimensional space $(n = 2)$ the surface (1.21) is a hyperbola

$$\begin{aligned} &\det(A) b_1 b_2 y_1 y_2 + (-\det(A) b_1 d_2 + a_{11} b_1 c_2) y_1 + (-\det(A) b_2 d_1 + a_{22} b_2 c_1) y_2 + \\ &\det(A) d_1 d_2 + c_1 c_2 - a_{11} d_1 c_2 - a_{22} d_2 c_1 = 0. \end{aligned} \tag{1.22}$$

We conclude that all Pareto maxima x are such that $y = y(x)$ given by (1.18) lies on the surface (1.21).

The equations (1.6) for a market equilibrium have the form

$$c_i x_i = d_i y_i - b_i y_i^2 \quad (i = 1, \dots, n). \tag{1.23}$$

Each equation describes a parabolic surface. The market equilibria are represented by all positive vectors at which all surfaces intersect (all the surfaces obviously intersect at the origin). The market equilibria with the Pareto property lie in the intersection of all parabolic surfaces (1.23) and all surfaces (1.21), (1.18).

The equations (1.23) can be rewritten in the form

$$w_i(x) = \frac{b_i}{2} y_i^2 \quad (i = 1, \dots, n).$$

At a market equilibrium x, the right hand sides are positive, since b_i and a_{ji} are positive by assumption. Hence, $w_i(x) > w_i(0) = 0$ $(i = 1, \dots, n)$; in other words, in every market equilibrium each country has a higher utility than in the initial Nash equilibrium $x^N = 0$.

2 Existence of market equilibrium

In this section we provide conditions sufficient for the existence of a market equilibrium. First of all, we assume that the second derivatives of the cost and benefit functions are bounded,

$$\infty > \sigma_i \geq C_i''(x_i) \geq 0 \quad (x_i \geq 0, \quad i = 1, \dots, n), \tag{2.1}$$

$$\infty > b_i \geq -B_i''(y_i) > 0 \quad (y_i \geq 0, \quad i = 1, \dots, n). \tag{2.2}$$

Note that the functions $y_i B_i'(y_i)$ are bounded,

$$0 \leq y_i B_i'(y_i) \leq \xi_i^0 < \infty \quad (y_i \geq 0, \quad i = 1, \dots, n). \tag{2.3}$$

This follows from the assumption that $B_i(y_i)$ is constant on $[\bar{y}_i, \infty)$. We shall use the notation

$$d_i = B_i'(0). \tag{2.4}$$

We assume that for all $i = 1, \dots, n$

$$d_i > b_i \alpha_i, \tag{2.5}$$

$$d_i \beta_i > \sigma_i + b_i a_{ii}^2 \tag{2.6}$$

where

$$\alpha_i = \sum_{j \neq i} \frac{a_{ji} \xi_j^0}{a_{jj} d_j}, \tag{2.7}$$

$$\beta_i = \frac{a_{ii} d_i}{\xi_i^0} \sum_{j \neq i} a_{ji}; \tag{2.8}$$

here σ_i, b_i, ξ_i^0, and d_i come from (2.1), (2.2), (2.3) and (2.4).

The conditions (2.1), (2.2), (2.5), (2.6) guarantee the existence of a market equilibrium. Note that the conditions do not involve the marginal costs $C_i'(x_i)$ which constitute strictly private information. To verify (2.5), (2.6) we must know bounds for the second derivatives of the cost and benefit functions, the transport coefficients and the derivatives of the benefit functions at the origin (in fact, the latter derivatives, d_i, can be replaced by available lower bounds for d_i).

To give a strict formulation of the existence result we need several additional definitions. Let us set

$$x_i^0 = \frac{\xi_i^0}{a_{ii} d_i}, \tag{2.9}$$

and choose a positive ε smaller than the minimum of x_i^0, $(i = 1, \dots, n)$ and satisfying the inequalities

$$\varepsilon \leq \frac{-a_{ii} b_i \alpha_i + (a_{ii}^2 b_i^2 \alpha_i^2 + \alpha_i (d_i - b_i \alpha_i)(\sigma_i + b_i a_{ii}^2))^{1/2}}{\sigma_i + b_i a_{ii}^2}, \tag{2.10}$$

$$\varepsilon \leq \frac{1}{\beta_i} \left[\left(\frac{d_i \beta_i - \sigma_i}{b_i} \right)^{1/2} - a_{ii}. \right]. \tag{2.11}$$

Note that the assumptions (2.5) and (2.6) imply that the right hand sides in (2.10) and (2.10) are positive, hence, the desired positive ε exists. Introduce the parallelepiped

$$\Pi_\varepsilon = \{x \in R^n : \ \varepsilon \leq x_i \leq x_i^0, \ i = 1, ..., n\} \tag{2.12}$$

(which is nonempty due to the choice of ε).

Theorem 2.1 *Let (2.1), (2.2), (2.5), (2.6) hold. Then*

(i) *there exists a continuous operator z mapping Π_ε into itself, which associates to every $x \in \Pi_\varepsilon$ a solution $z(x)$ of the system (1.8), (1.7)),*

(ii) *there exists a market equilibrium belonging to Π_ε.*

Proof. Take arbitrary $x \in \Pi_\varepsilon$. Let $p_i = p_i(x)$ $(i = 1, \dots, n)$ be the exchange rates (1.7). Using (2.9), we estimate p_i as follows:

$$p_i \leq a_{ii} + \frac{\sum_{j \neq i} a_{ji} x_j^0}{\varepsilon} \leq a_{ii} + \frac{1}{\varepsilon} \sum_{j \neq i} \frac{a_{ji} \xi_j^0}{a_{jj} d_j} = a_{ii} + \frac{\alpha_i}{\varepsilon}, \tag{2.13}$$

$$p_i \geq a_{ii} + \frac{\varepsilon \sum_{j \neq i} a_{ji}}{x_i^0} \geq a_{ii} + \varepsilon \frac{a_{ii} d_i \sum_{j \neq i} a_{ji}}{\xi_i^0} = a_{ii} + \beta_i \varepsilon. \tag{2.14}$$

The vector x satisfies the market equilibrium equation (1.8) if and only if it solves the equivalent system of equations

$$r_i(z_i) = 0 \quad (i = 1, \ldots, n) \tag{2.15}$$

where

$$r_i(z_i) = C_i'(z_i) - p_i B_i'(p_i z_i).$$

The function r_i is strictly monotone increasing, since $r_i'(z_i) = C_i''(z_i) - p_i^2 B_i''(p_i z_i) > 0$. By assumption the point $x^N = 0$ satisfies the Nash equilibrium equation (1.10):

$$C_i'(0) = a_{ii} B_i'(0) = a_{ii} d_i. \tag{2.16}$$

Hence,

$$r_i(0) = C_i'(0) - p_i B_i'(0) = (a_{ii} - p_i) d_i < 0; \tag{2.17}$$

the inequality holds due to (2.14). For $\overline{y}_i$, the saturation point of the benefit function B_i', we have $B_i'(\overline{y}_i)$ which implies

$$r_i \left(\frac{\overline{y}_i}{p_i} \right) = C_i' \left(\frac{\overline{y}_i}{p_i} \right) - p_i B_i' (\overline{y}_i) = C_i' \left(\frac{\overline{y}_i}{p_i} \right) > 0.$$

Therefore, the interval $[0, \overline{y}_i / p_i]$ contains a single positive root, z_i^0, of the ith equation in (2.15).

Let us prove that $z_i^0 \leq x_i^0$. Assume the contrary. Then necessarily $r(z_i) < 0$ for all $z_i \in [0, x_i^0]$. Hence,

$$x_i^0 C_i'(x_i^0) < p_i x_i^0 B_i'(p_i x_i^0) \leq \xi_i^0$$

(see (2.3)).

Substituting (2.9), we get

$$x_i^0 C_i'(x_i^0) = \frac{\xi_i^0}{a_{ii} d_i} C_i' \left(\frac{\xi_i^0}{a_{ii} d_i} \right) < \xi_i^0.$$

Consequently,

$$C_i' \left(\frac{\xi_i^0}{a_{ii} d_i} \right) < a_{ii} d_i.$$

Since C_i' is increasing (see (1.2)), we get

$$C_i'(0) < a_{ii} d_i$$

which contradicts (2.16). A contradiction proves that $z_i^0 \leq x_i^0$.

Let us show that $z_i^0 \geq \varepsilon$. From (2.17) and $r_i(z_i^0) = 0$, due to the Lagrange mean value theorem, we deduce that

$$r_i(z_i^0) - r_i(0) = (p_i - a_{ii}) d_i = r_i'(\eta_i) z_i^0 = (C_i''(\eta_i) - p_i^2 B_i''(p_i \eta_i)) z_i^0$$

for some $\eta_i \in (0, z_i^0)$.

Referring to the estimates (2.1) and (2.2), we get

$$z_i^0 = \frac{d_i (p_i - a_{ii})}{C_i''(\eta_i) - p_i^2 B_i''(p_i \eta_i)} \geq \frac{d_i (p_i - a_{ii})}{\sigma_i + b_i p_i^2} = Q_i(p_i). \tag{2.18}$$

Now we use (2.13), (2.14) and continue as follows:

$$z_i^0 \geq \min\left\{Q_i(s_i):\ a_{ii}+\beta_i\varepsilon \leq s_i \leq a_{ii}+\frac{\alpha_i}{\varepsilon}\right\} = m_i(\varepsilon).$$

The derivative

$$Q_i'(s_i) = \frac{d_i(-b_i s_i^2 + 2a_{ii}b_i s_i + \sigma_i)}{b_i s_i^3}$$

vanishes on the interval $[a_{ii},\infty)$ at a single point $a_{ii}+(a_{ii}^2+\sigma_i/b_i)^{1/2}$. One can easily state that this point is a global maximum of $Q_i'(s_i)$ on $[a_{ii},\infty)$. Hence, the minima of $Q_i(s_i)$ on the interval $[a_{ii}+\beta_i\varepsilon, a_{ii}+\alpha_i/\varepsilon]$ are in the endpoints:

$$\begin{aligned} m_i(\varepsilon) &= \min\left\{Q_i(a_{ii}+\beta_i\varepsilon),\ Q_i(a_{ii}+\frac{\alpha_i}{\varepsilon})\right\} \\ &= \varepsilon d_i \min\left\{\frac{\alpha_i}{\sigma_i\varepsilon^2+b_i(a_{ii}\varepsilon+\alpha_i)^2}, \frac{\beta_i}{\sigma_i+b_i(a_{ii}+\beta_i\varepsilon)^2}\right\}. \end{aligned}$$

For stating the desired estimate $z_i^0 \geq \varepsilon$ it is sufficient to prove that $m_i(\varepsilon)\geq\varepsilon$ or, more specifically,

$$\frac{d_i\alpha_i\varepsilon}{\sigma_i\varepsilon^2+b_i(a_{ii}\varepsilon+\alpha_i)^2} \geq \varepsilon$$

and

$$\frac{d_i\beta_i\varepsilon}{\sigma_i+b_i(a_{ii}+\beta_i\varepsilon)^2} \geq \varepsilon.$$

A simpler form of these inequalities is (2.10), (2.11). By assumption the inequalities (2.10) and (2.10) hold true. Therefore, $m_i(\varepsilon)\geq\varepsilon$ which implies $z_i^0\geq\varepsilon$. Thus, a single root of the ith equation in (2.15), z_i^0, lies in the interval $[\varepsilon, x_i^0]$. Due to the arbitrariness of $x\in\Pi_\varepsilon$, we conclude that there is an operator z that associates to each $x\in\Pi_\varepsilon$ a solution $z^0 = z(x)$ of the system (2.15), (1.7)) which belongs to Π_ε.

Statement (i) is proved.

The operator z is obviously continuous and carries Π_ε into itself. By the lemma of Schauder this operator has a fixed point $x^M\in\Pi_\varepsilon$. Evidently, x^M solves the system (2.15), (1.7) which is equivalent to the market equilibrium system (1.8), (1.7). Since x^M is positive, it is a market equilibrium.

Statement (ii) is proved.

The proof is accomplished.

Example.

Let us give an illustration for the conditions (2.5), (2.6). Assume that the cost functions are linear and all benefit functions are identical and have the form

$$B_i(y_i) = \begin{cases} dy_i - \frac{b}{2}y_i^2, & 0\leq y_i\leq d/b \\ d^2/2b, & y_i\geq d/b \end{cases}.$$

Obviously $d_i = d$. The estimates (2.1), (2.2) and (2.3) hold with $\sigma_i = 0$, $b_i = b$ and $\xi_i^0 = d^2/4b$. The relation (2.5) turns into

$$\frac{d}{b} > \sum_{j\neq i}\frac{a_{ji}d^2}{4a_{jj}bd}$$

which is equivalent to

$$4 > \sum_{j\neq i}\frac{a_{ji}}{a_{jj}}. \qquad (2.19)$$

The relation (2.6) turns into

$$\frac{d}{b} > \frac{a_{ii}d^2}{4bd\sum_{j\neq i} a_{ji}}$$

which is equivalent to

$$\frac{\sum_{j\neq i} a_{ji}}{a_{ii}} > \frac{1}{4}. \tag{2.20}$$

By Theorem 2.1 the inequalities (2.19) and (2.20) guarantee the existence of a market equilibrium. Note that (2.19) and (2.20) involve only the transport coefficients. The inequality (2.19) requires that the pollution amount transported to country i from other countries must be not too high relative to their self-pollution loads. This requirement can intuitively be explained as follows. In the opposite situation when the external pollution load to some country i is very high, the exchange rate p_i (1.7) is very high, hence, the marginal benefit $p_i B_i'(p_i x_i)$ is very high relative to the marginal cost $C_i'(x_i)$; therefore, the market equilibrium equation (1.8) is never satisfied. The inequality (2.20) is complementary to (2.20). It says that the pollution amount transported to country i from other countries must be not too low relative to the self-pollution load of country i. The latter requirement well agrees with intuition. Indeed, in the opposite situation when the external pollution load to some country is very low, the country is not interested in the exchange of emission reductions, hence, a market equilibrium is never reached.

3 Pareto market equilibria

Let us draw our attention to market equilibria belonging to the set of Pareto maxima; we shall call them *Pareto market equilibria.* In this section we give an explicit characterization of some Pareto market equilibria. The argument is based on Lemma 1.1.

We assume the conditions (2.1), (2.2), (2.5), (2.6) which guarantee the existence of a market equilibrium in the parallelepiped Π_ε (2.12) due to Theorem 2.1. Like in the previous section, Π_ε is determined by the values x_i^0 (2.9) and a positive ε smaller than the minimum of x_i^0, $(i = 1, \ldots, n)$ and satisfying the inequalities (2.10) and (2.11).

For every $i = 1, \ldots, n$ we set

$$A_i = \sum_{j=1}^{n} a_{ji}.$$

Let Ω_i be the collection of all subsets K_i of the set $\{k = 1, \ldots, n : k \neq i\}$ such that $|K_i| = \left[\frac{n}{2}\right] - 1$. Here and below $|K_i|$ stands for the number of elements of the set K_i and $[n/2]$ denotes the integer part of $n/2$.

A condition sufficient for the Pareto optimality of a market equilibrium has the form:

$$C_i'(\varepsilon) - a_{ii}B_i'(A_i\varepsilon) \geq \max_{K_i\in\Omega_i} \sum_{k\in K_i} a_{ik}B_k'(A_k\varepsilon). \tag{3.1}$$

Proposition 3.1 *Let the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) be satisfied. Then every market equilibrium belonging to the parallelepiped* Π_ε *is a Pareto market equilibrium.*

Proof. In section 1 we noticed that for a market equilibrium x the rows of the Jacobi matrix $DW(x)$ (1.15) are linearly dependent with some coefficients γ_k^* $(k = 1, \ldots, n)$ not all of which vanish. We shall prove that for arbitrary market equilibrium $x \in \Pi_\varepsilon$ all γ_k^* are nonnegative (or, equivalently, have a common sign). This observation will complete the proof due to Lemma 1.1.

Using the form of the utility functions w_i (1.1), we specify the Jacobi matrix:

$$DW(x) = \begin{pmatrix} -C_1'(x_1) + a_{11}B_1'(y_1) & a_{21}B_1'(y_1) & \dots & a_{n1}B_1'(y_1) \\ a_{12}B_2'(y_2) & -C_2'(x_2) + a_{22}B_2'(y_2) & \dots & a_{n2}B_2'(x_2) \\ & & \dots & \\ a_{1n}B_n'(y_n) & a_{2n}B_n'(y_n) & \dots & -C_n'(x_n) + a_{nn}B_n'(y_n) \end{pmatrix}$$

here

$$y_k = \sum_{j=1}^{n} a_{jk}x_j \quad (k = 1, \dots, n).$$

We have

$$\sum_{j \neq i} \gamma_j^* a_{ij} B_j'(y_j) + \gamma_i^*(-C_i'(x_i) + a_{ii}B_i'(y_i)) = 0, \quad (i = 1, \dots, n). \tag{3.2}$$

Assume that among the coefficients γ_k^* there are both positive and negative ones. Introduce notations for the sets of positive and negative coefficients:

$$\Gamma^+ = \{k : \ \gamma_k^* > 0\}, \quad \Gamma^- = \{k : \ \gamma_k^* < 0\}.$$

We have $\Gamma^+ \neq \emptyset$ and $\Gamma^- \neq \emptyset$. Without loss of generality we can assume that the number of positive coefficients is greater than the number of negative ones, $|\Gamma^+| \geq |\Gamma^-|$, hence, $|\Gamma^-| \leq [\frac{n}{2}]$. Let us fix the index of a negative coefficient whose modulus is maximal:

$$l = \text{argmax}\{|\gamma_k^*| : k \in \Gamma^-\}.$$

Now select in (3.2) the linear combination corresponding to the column $i = l$:

$$\sum_{j \neq l} \gamma_j^* a_{lj} B_j'(y_j(x)) + \gamma_l^*(-C_l'(x_l) + a_{ll}B_l'(y_l(x))) = 0.$$

Since $a_{lj}B_j'(y_j(x)) > 0$, we have

$$\sum_{k \in \Gamma^- \setminus \{l\}} \gamma_k^* a_{lk} B_k'(y_k) + \gamma_l^*(-C_l'(x_l) + a_{ll}B_l'(y_l)) < 0.$$

Hence,

$$C_l'(x_l) - a_{ll}B_l'(y_l) < \sum_{k \in \Gamma^- \setminus \{l\}} \frac{|\gamma_k^*|}{|\gamma_l^*|} a_{ik} B_k'(y_l) \leq \sum_{k \in \gamma^- \setminus \{l\}} a_{ik} B_k'(y_l).$$

Taking into account the inequalities $\varepsilon \leq x_i$, $A_i\varepsilon \leq y_i$ and the fact that the functions C_i' are increasing and the functions B_i' are decreasing, we continue as follows:

$$C_l'(\varepsilon) - a_{ll}B_l'(A_i\varepsilon) < \sum_{k \in \Gamma^- \setminus \{l\}} a_{ik} B_k'(y_l). \tag{3.3}$$

The right hand side does not exceed the right hand side in (3.1) for $i = l$. Therefore, (3.3) contradicts the assumption (3.1). A contradiction shows that all γ_k^* have a common sign, which accomplishes the proof.

Remark 3.1 For $n = 2, 3$, we have $([n/2] - 1) = 0$ and the condition (3.1) is satisfied automatically.

4 Searching Pareto market equilibria: repeated auction

It is a common view that Pareto maxima are best in the cooperative game (in a Pareto maximum no change in the emission reductions leads all countries to better utilities). How can the governments arrive at a Pareto maximum? Let us imagine for a moment that there is an international agency able to collect full information on the national emission control cost functions, transport coefficients and national benefit functions. The agency could find all Pareto maxima and communicate them to the governments which could cooperatively select an appropriate one. Unfortunately, this simple way of resolving the game is not realistic because the full information is never available; the cost functions constitute a strictly private information for the national governments. We shall show that under appropriate conditions, there is a realistic negotiation process allowing the governments to find a Pareto market equilibrium. The process is interpreted as a repeated auction in which the governments update their decisions without exchanging information on the national costs and benefits.

Generally, in the negotiations on emission control, each government demands a reduction in pollution load and in return offers an emission reduction. The government's supply of emission reduction depends on the current rate of exchange, p_i (1.7). In the repeated auction, all delegates of the national governments elect an auctioneer and accept the next "rule of the game". In every round of the auction, the auctioneer proposes a rate of exchange, or a price, p_i for each country i, and each national delegate responds by stating the reduction in emissions his government offers for this price. The (imaginary) emission reduction vectors at the background of the proposed rates of exchange are viewed as auctioneer's plans.

A round of the auction proceeds as follows. The auctioneer offers a *plan*, a positive emission reduction vector x, and computes the associated rates of exchange, or *prices*, $p_i = p_i(x)$ $(i = 1, \ldots, n)$, by the formula (1.7). Each country i finds its emission reduction $z_i(p_i)$, as its *best reply to the price* p_i, that is, a maximizer of its *priced utility*

$$w_i(z_i, p_i) = -C_i(z_i) + B_i(p_i z_i)$$

over all $z_i \geq 0$. The auctioneer analyses all best replies $z_i(p_i)$ $(i = 1, \ldots, n)$ and works out an updated plan for the next round. In updating the plans the auctioneer does not use any information on the national cost and benefit functions. The goal of the repeated auction is to guide the vector of best the replies, $(z_1(p_1), \ldots, z_n(p_n))$, to a Pareto market equilibrium in an infinite sequence of rounds.

Let us take a closer look at the countries' best replies. Obviously the best reply $z_i(p_i)$ is unique and coincides with a nonnegative root of the ith equation in (1.8). Since the price $p_i = p_i(x)$ (1.7) is determined by the plan x, we shall also call $z_i(p_i)$ the *best reply to the plan* x and denote it $z_i(x)$. We shall deal with the *best reply operator* z that associates to every plan x the vector of countries' best replies, $(z_1(x), \ldots, z_n(x))$.

Remark 4.1 Due to statement (i) of Theorem 2.1, the conditions (2.1), (2.2), (2.5), (2.6) yield that the best reply operator z maps the parallelepiped Π_ε into itself. We shall use this observation later.

Consider a fixed point of the best reply operator, $x^* = z(x^*)$. Obviously, x_i^* solves the equation (2.2) with $p_i = p_i(x^*)$ (see (1.7)). Hence, the fixed point x^* is a solution of the system of equations (1.8) which describes market equilibria. Therefore, if x^* is positive, it constitutes a market equilibrium. We arrive at the next observation.

Proposition 4.1 *Every positive fixed point of the best reply operator is a market equilibrium.*

Due to Proposition 3.1 this statement is specified as follows:

Proposition 4.2 *Let the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) be satisfied. Then every positive fixed point of the best reply operator belonging to the parallelepiped* Π_ε *is a Pareto market equilibrium.*

We conclude that, under the assumptions of Proposition 4.2, the repeated auction eventually finds a Pareto market equilibrium if the auctioneer's plans approach a positive fixed point of the best reply operator in the parallelepiped Π_ε.

Now let us consider in more detail how the repeated auction proceeds. In each round, the governments give out their best replies to auctioneer's plan. Therefore, the auction dynamics is determined by auctioneer's strategy for updating plans; we shall call it a search strategy. Formally, we understand a *search strategy* as an arbitrary function U that associates to every (natural) round number k, every positive vector x^k (the auctioneer's plan in round k) and every nonnegative vector z^k (the collection of countries' best replies to x^k) a positive vector $x^{k+1} = u(k, x^k, z^k)$ (the plan for round $k+1$).

Let us suppose that some search strategy U ($x^{k+1} = U(k, x^k, z^k)$) be chosen. Then the repeated auction proceeds as follows.

Round 0.
The auctioneer chooses some (positive) plan x^1 for round 1.

Round k, $(k \geq 1)$.
Step 1.
For the positive plan $x^k = (x_1^k, ..., x_n^k)$ worked out at the previous round, the auctioneer computes the prices

$$p_i^k = p_i(x^k) = \frac{\sum_{j=1}^n a_{ji} x_j^k}{x_i^k} = \frac{y_i(x^k)}{x_i^k} \quad (i = 1, \ldots, n)$$

and offers the price p_i^k to country i.
Step 2.
Each country i finds its best reply, $z_i^k = z_i^k(p^k) = z_i^k(x^k)$, to the price p_i^k; recall that z_i^k is a maximizer of the priced utility $w_i(z_i, p_i^k) = -C_i(z_i) + B_i(p_i^k z_i)$ over all $z_i \geq 0$.
Step 3.
Each country i communicates its best reply z_i^k to the auctioneer.
Step 4.
The auctioneer puts his latest plan x^k and the best reply vector $z^k = (z_1^k, ..., z_n^k)$ in the search strategy U and works out a plan x^{k+1} for the next round: $x^{k+1} = U(k, x^k, z^k)$.

We shall say that the search strategy $x^{k+1} = U(k, x^k, z^k)$ with the initial plan x^1 *finds* a Pareto market equilibrium x^M in the repeated auction if the best reply vectors z^k converge to x^M as the round numbers k go to infinity: $z^k \to x^M$ as $k \to \infty$. Note that in the assumptions of Proposition 4.2, x^M can be defined as a fixed point of the best reply operator in the parallelepiped Π_ε. In this case the best replies converge to x^M if and only if the auctioneer's plans converge to x^M: $x^k \to x^M$ as $k \to \infty$.

From the point of view of the theory of repeated games the proposed repeated auction determines a learning process for the governments; the auctioneer can, to a certain extend, be associated with the market player described in [Zangwill, Garcia 1981].

5 Finding Pareto market equilibria by following best replies

In this section we focus on the *following-best-replies* search strategy which prescribes the auctioneer to take the latest best reply for a new plan:

$$x^{k+1} = z(x^k). \tag{5.1}$$

Our goal is to give conditions that guarantee that the following-best-replies search strategy finds a Pareto market equilibrium in the repeated auction. Basing on Proposition 4.2, we look for a market equilibrium represented as a fixed point of the best reply operator in the parallelepiped Π_ε (2.12).

A key point in our analysis is an upper estimate for the norm of the partial derivatives of the best reply operator $z : x \mapsto z(x) = (z_1(x), \ldots z_n(x))$ in Π_ε. The differentiation of the market equilibrium equation (1.8), in which $z = z(x)$ and $p_i = p_i(x)$ is given by (1.7), yields

$$C_i''(z_i(x))\frac{\partial z_i(x)}{\partial x_j} - \frac{\partial p_i(x)}{\partial x_j}B_i'(p_i(x)z_i(x)) - p_iB_i''(p_i(x)z_i(x))(\frac{\partial p_i(x)}{\partial x_j}z_i(x) + p_i(x)\frac{\partial z_i(x)}{\partial x_j(x)}) = 0.$$

Resolving this equality with respect to $\partial z_i(x)/\partial x_j$, we obtain

$$\frac{\partial z_i(x)}{\partial x_j} = H_i(p_i(x), z_i(x))\frac{\partial p_i(x)}{\partial x_j} \tag{5.2}$$

where

$$H_i(p_i(x), z_i(x)) = \frac{B_i'(p_i(x)z_i(x)) + p_i(x)z_i(x)B_i''(p_i(x)z_i(x))}{C_i''(z_i(x)) - B_i''(p_i(x)z_i(x))p_i^2(x)}.$$

Now we estimate the numerator and denominator as follows:

$$B_i'(p_i(x)z_i(x)) + p_i(x)z_i(x)B_i''(p_i(x)z_i(x)) \le D_i,$$

$$C_i''(z_i) - B_i''(p_iz_i)p_i^2 \ge B_ip_i^2;$$

here

$$D_i = \max_{0\le y_i\le \overline{y}_i}(B_i'(y_i) + y_iB_i''(y_i)), \tag{5.3}$$

$$B_i = \min_{0\le y_i\le \overline{y}_i}(-B_i''(y_i)) > 0. \tag{5.4}$$

Hence,

$$H_i(p_i(x), z_i(x)) \le \frac{D_i}{B_ip_i^2(x)}. \tag{5.5}$$

Due to (1.7),

$$\frac{\partial p_i}{\partial x_j}\begin{cases} a_{ji}/x_i & \text{if } j \ne i \\ \left(-\sum_{k\ne i} a_{ji}x_k\right)/x_i^2 & \text{if } j = i \end{cases}.$$

For $x \in \Pi_\varepsilon$ this formula yields

$$\sum_{j=1}^n \left|\frac{\partial p_i(x)}{\partial x_j}\right| = \frac{p_i(x) - q_i}{x_i} \le \frac{p_i(x) - q_i}{\varepsilon} \tag{5.6}$$

where

$$q_i = a_{ii} - \sum_{j\ne i} a_{ji},$$

and

$$\frac{1}{\varepsilon}\sum_{j=1}^{n} x_j \left|\frac{\partial p_i(x)}{\partial x_j}\right| = \frac{2(p_i(x) - a_{ii})}{\varepsilon} \tag{5.7}$$

(note that (1.7) easily implies $p_i(x) \geq a_{ii}$; hence, the right hand sides in the obtained estimates are nonnegative).

Combining the equality (5.2), and the inequalities (5.5), (5.6) and (5.7), we get:

$$\sum_{j=1}^{n}\left|\frac{\partial z_i(x)}{\partial x_j}(x)\right| \leq \frac{D_i}{\varepsilon B_i}\min\{u_i(p_i(x)), v_i(p_i(x))\} \quad (x \in \Pi_\varepsilon); \tag{5.8}$$

here

$$u_i(p_i) = \frac{(p_i - q_i)}{p_i^2},$$

$$v_i(p_i) = \frac{2(p_i - a_{ii})}{p_i^2}.$$

Since $p_i(x) \geq a_{ii}$, we have

$$u_i(p(x)) \leq u_i^0, \quad v_i(p(x)) \leq v_i^0$$

where u_i^0 and v_i^0 are, respectively, the maximum values of $u_i(p_i)$ and $v_i(p_i)$ over all $p_{ii} \geq a_{ii}$.

The differentiation of $v_i(p_i)$ implies

$$v_i'(p_i) = \frac{2(2a_{ii} - p_i)}{p_i^3}.$$

A single maximum point is $p_i = 2a_{ii}$ and the maximum value is

$$v_i^0 = \frac{1}{2a_{ii}}. \tag{5.9}$$

Differentiate $u_i(p_i)$. We get

$$u_i'(p_i) = \frac{2q_i - p_i}{p_i^3}.$$

If $a_{ii} \geq 2q_i$, then $u_i'(p_i) \leq 0$ for all $p_i \geq a_{ii}$, the point $p_i = a_{ii}$ is the maximum of $u_i(p_i)$, and the maximum value is

$$u_i^0 = \frac{a_{ii} - q_i}{a_{ii}^2} \geq \frac{1}{2a_{ii}}. \tag{5.10}$$

If $a_{ii} < 2q_i$, then the maximum is attained at $p_i = 2q_i$, and

$$u_i^0 = \frac{1}{4q_i} < \frac{1}{2a_{ii}}. \tag{5.11}$$

Taking into account (5.9), (5.10), (5.11), we arrive at

$$\min\{u_i(p_i), v_i(p_i)\} \leq \min\{u_i^0, v_i^0\} = \frac{1}{2}\min\left\{\frac{1}{a_{ii}}, \frac{1}{2q_i}\right\} = \frac{1}{a_i}$$

where

$$a_i = 2\left(a_{ii} + \max\left\{0, a_{ii} - 2\sum_{j\neq i} a_{ji}\right\}\right). \tag{5.12}$$

Now we specify (5.8) as follows:

$$\max_{x\in P_\varepsilon}\sum_{j=1}^{n}\left|\frac{\partial z_i(x)}{\partial x_j}(x)\right| \leq \frac{D_i}{\varepsilon a_i B_i}. \tag{5.13}$$

This estimate yields the next result on finding a Pareto market equilibrium via the following-best-replies search strategy.

Proposition 5.1 *Let the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) be satisfied, and*

$$\frac{1}{\varepsilon} \max_{i=1,\ldots,n} \frac{D_i}{a_i B_i} \leq \rho < 1 \tag{5.14}$$

where D_i, B_i and a_i are given by (5.3), (5.4) and (5.12). Then the following-best-replies search strategy (5.1) with the initial plan $x^1 \in \Pi_\varepsilon$ finds a Pareto market equilibrium in the repeated auction.

Proof. By Remark 4.1 the best reply operator z maps the parallelepiped Π_ε into itself. It is obviously continuous. Hence, Π_ε contains a fixed point, x^M of z.

As noted in the previous section, x^M is a market equilibrium. By Proposition 3.1 x^M is a Pareto market equilibrium.

Consider the max-norm in the n-dimensional space: $\|x\| = \max_{i\in 1,\ldots,n} |x_i|$. Using (5.13), for be the best reply operator z and arbitrary $x, y \in \Pi_\varepsilon$ we obtain:

$$\begin{aligned}
\|z(x) - z(y)\| &= \max_i |z_i(x) - z_i(y)| = \max_{i=1,\ldots,n} \left| \sum_{j=1}^{n} \max_{0\leq\eta\leq 1} \frac{\partial z_i}{\partial x_j}(y + \eta(x-y))(x_j - y_j) \right| \\
&\leq \max_{i=1,\ldots,n} \left| \sum_{j=1}^{n} \max_{0\leq\eta\leq 1} \frac{\partial z_i}{\partial x_j}(y + \eta(x-y)) \right| \max_{k=1,\ldots,n} |x_k - y_k| \\
&\leq \max_{i=1,\ldots,n} \frac{1}{\varepsilon} \frac{D_i}{A_i B_i} \leq \max_{k=1,\ldots,n} |x_k - y_k| \leq \rho \|x - y\|
\end{aligned}$$

Since $\rho < 1$, the best reply operator z is a contraction operator on Π_ε. Therefore, x^M is a unique fixed point of z in Π_ε.

Let $\{x^k\}$ and $\{z^k\}$ be, respectively, the sequences of the plans and best replies in the repeated auction corresponding to the following-best-replies search strategy (5.1). Since the initial plan x^1 lies in Π_ε and z is a contraction operator on Π_ε, (5.1) shows that $\{x^k\}$ converges to x^M (see, e.g., [Vasin, Ageev 1995], Theorem 2.2). As soon as x^M is a fixed point of the operator z the best replies $z^k = z(x^k) = x^{k+1}$, also converge to x^M.

We have shown that the following-best-replies search strategy with the initial plan x^1 finds a Pareto market equilibrium in the repeated auction.

6 Other search strategies

In stating that the following-best-replies search strategy finds a Pareto market equilibrium (Proposition 5.1) we used the method of contraction operators well-known in the theory of approximation of fixed points. Other theoretical methods can also be utilized for the design of search strategies. We shall consider a few applications of the method of nonexpansion operators and its modifications. Our analysis will not be as detailed as in the previous section; we shall not provide any specific relations between the parameters which can be sufficient for the nonexpansion property of the best reply operator. The technique we shall refer to operates with the Euclidean norm, $\|x\|_e = (x_1^2 + \ldots + x_n^2)^{1/2}$ (in contrast to the proof of Proposition 5.1, where the max-norm, $\|x\| = \max_{i\in 1,\ldots,n} |x_i|$, was utilized).

We shall assume that the best reply operator z is a *nonexpansion operator* in the parallelepiped Π_ε, that is, the inequality $\|z(x) - z(y)\|_e \leq \|x - y\|_e$ holds for all $x, y \in \Pi_\varepsilon$. This assumption allows to employ the (more cautious than following-best-replies) *τ-following-best-replies* search strategy, which prescribes the auctioneer to move the latest plan x^k towards the latest best reply z^k and stop at a point where a chosen τ proportion of the distance between x^k and z^k is covered:

$$x^{k+1} = \tau x^k + (1-\tau)z^k, \quad 0 < \tau < 1. \tag{6.1}$$

In our situation, one of the known fixed point approximation results (see, e.g., [Ortega, Rheinboldt 1970]; [Vasin, Ageev 1995], Theorem 2.4) can be specified as follows:

If the best reply operator z maps Π_ε into itself (see Remark 4.1) and the initial plan x^1 lies in Π_ε, then the plans x^k and best replies z^k in the repeated auction corresponding to the τ-following-best-replies search strategy converge to a fixed point of z in Π_ε. This observation and a simple argument used in the proof of Proposition 5.1 leads to the next proposition.

Proposition 6.1 *Let the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) be satisfied, and the best reply operator z be a nonexpansion operator on Π_ε. Then the τ-following-best-replies search strategy (6.1) with the initial plan $x^1 \in \Pi_\varepsilon$ finds a Pareto market equilibrium in the repeated auction.*

A modification of the τ-following-best-replies search strategy allows to select a fixed point closest to a desired emission reduction vector $\overline{x}$. If the best reply operator z is a nonexpansion operator and maps Π_ε into itself, then the set S_ε of all fixed points of the operator z in Π_ε is nonempty, closed and convex (see [Vasin, Ageev 1995], Theorem 2.1), and, hence, for every $\overline{x} \in \Pi_\varepsilon$ there is a unique fixed point $\overline{x} \in S_\varepsilon$ closest to $\overline{x}$: $\|\overline{x}^M - \overline{x}\|_e = \min_{x \in S_\varepsilon} \|x - \overline{x}\|_e$. In the assumptions of Theorem 2.1, all fixed points from S_ε constitute Pareto market equilibria. In this case we shall call $\overline{x}^M$ the $\overline{x}$-*closest* Pareto market equilibrium (in Π_ε).

We define the $\overline{x}$- *extremal following-best-replies* search strategy by the formula

$$x^{k+1} = (1-\tau_k)\overline{x} + \tau_k z(x^k) \tag{6.2}$$

where $0 < \tau_k < 1$. This search strategy is able to find the $\overline{x}$- closest Pareto market equilibrium under an appropriate choice of the coefficients τ_k. A sequence (τ_k) will be called called *admissible* if $0 < \tau_k < 1$, $\tau_{k+1} > \tau_k$, $\lim_{k\to\infty} \tau_k = 1$, and there is a sequence of natural numbers $(n(k))$ such that $n(k+1) > n(k)$, and

$$\lim_{k\to\infty} \frac{(1-\tau_{(k+n(k))})}{(1-\tau_k)} = 1, \quad \lim_{k\to\infty} n(k)(1-\tau_k) = \infty.$$

An example of an admissible sequence is $\tau_k = 1 - 1/k^p$ where $p \in (0,1)$; one can set $n(k) = k^q$ with $q \in (p,1)$.

Referring to [Vasin, Ageev 1995], Theorem 2.8, we arrive at the next statement.

Proposition 6.2 *Let the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) be satisfied, the best reply operator z be a nonexpansion operator on Π_ε, $\overline{x}$ be an arbitrary vector from Π_ε, and (τ_k) be an admissible sequence. Then the $\overline{x}$-extremal following-best-replies search strategy (6.2) with the initial plan $x^1 \in \Pi_\varepsilon$ finds the $\overline{x}$- closest Pareto market equilibrium $\overline{x}^M$ in the repeated auction.*

The following-best-replies and τ-following-best-replies search strategies are particular realizations of a more general Mann's iteration scheme with averaging, which is defined by the formula

$$x^{k+1} = z(v^k), \quad v^k = \sum_{j=1}^{k} m_{kj} x_j. \tag{6.3}$$

The coefficients m_{kj} are chosen so as to meet the *normal Mann conditions*: $m_{kj} \geq 0$; $m_{kj} = 0$ for $j > k$; $\sum_{j=1}^{k} m_{kj} = 1$; $\lim_{k\to\infty} m_{kj} = 0$; $m_{k+1j} = (1 - m_{k+1k+1})m_{kj}$ for $j = 1, \ldots, k$; finally, either $m_{kk} = 1$ for all k, or $m_{kk} < 1$ for all k.

We shall treat (6.3) as a generalized *averaging search strategy.* In contrast to the search strategies considered earlier, the averaging search strategy has full memory. In round k of the auction, the auctioneer using the averaging search strategy composes the average plan v_k by giving different weights m_{kj} to all plans x_j ($j = 1, \ldots, k$) realized in the past, and offers the prices $p_i^k = p_i(v_k)$; the best reply vector $z_k = z(v_k)$ emerging due to these prices determines the new auctioneer's plan x_{k+1}.

If averaging takes into account only latest plan x^k, that is, $m_{kk} = 1$ and $m_{kj} = 0$ ($k \neq j$), the averaging search strategy (6.3) turns into the following-best-replies search strategy (5.1).

If we put

$$m_{k1} = (1-\tau)^{k-1}, \quad m_{kj} = \tau(1-\tau)^{k-j} \quad (j = 2, \ldots, k), \quad m_{kj} = 0 \quad (j > k),$$

the averaging search strategy can be represented in the form

$$v^{k+1} = (1-\tau)v^k + \tau z(v^k),$$

which imitates the τ-following-best-replies search strategy (6.1). The latter, as we see, takes into account the whole history of plans, $m_{kj} \neq 0$ ($j \leq k$); for large k the latest plan has the strongest impact: $m_{kk} > m_{kj}$.

In the averaging search strategy it is admissible to give equal preferences to all plans in the record: $m_{kj} = 1/k$ ($j \leq k$) (Chesaro weights), which implies that the average plans v_k evolve with harmonic coefficients:

$$v^{k+1} = \left(1 - \frac{1}{k}\right) v^k + \frac{1}{k} z(v^k).$$

The application of [Vasin, Ageev 1995], Theorem 2.28, yields the next result.

Proposition 6.3 *Let the conditions (2.1), (2.2), (2.5), (2.6) and (3.1) be satisfied, the best reply operator z be a nonexpansion operator on Π_ε, and m_{kj} ($k, j = 1, 2, \ldots$) satisfy the normal Mann condition. Then the averaging search strategy (6.3) with the initial plan $x^1 \in \Pi_\varepsilon$ finds a Pareto market equilibrium x^M in the repeated auction; moreover the average plans v_k defined in (6.3) converge to x^M.*

We considered several search strategies based on standard successive approximations methods. Nonstandard strategies that strongly take into account the specific structure of the best reply operator may essentially extend the variety of situations in which a Pareto market equilibrium is found.

As an example of a nonstandard search strategy, we mention the procedure suggested in [Ehtamo, Hamalainen, Verkama 1994]:

$$x^{k+1} = \frac{y^{k+1}}{\sum_{i=1}^{n} y_i^{k+1}},$$

$$y_i^{k+1} = x_i^k + \max_{m=1,\ldots,n} (z_i(x^k) - z_m(x^k)),$$

$$x^k \in S_{n-1} = \left\{ x \in R^n : \ x_i \geq 0, \ \sum_{i=1}^{n} x_i = 1 \right\}.$$

The rigorous specification of the conditions that ensure that this and other nonstandard search strategies find a Pareto market equilibrium is a challenging subject for future research.

Conclusion

In this paper we explicitly formulated conditions that guarantee the existence of a market equilibrium and a Pareto market equilibrium in a noncooperative multi-person game arising in trading emission reductions. We showed that the notion of the market Pareto equilibrium and successive fixed point approximation algorithms can be applied in a learning process associated with international negotiations on multilateral reduction of emissions. The negotiations can be viewed as a repeated auction in which emission reductions depend on the rates of exchange (prices) that are offered. Under appropriate conditions, we designed a type of auction that would enable the participants to find Pareto equilibrium emission reductions dominating the initial Nash equilibrium. The auction would function if the participants have only information on their own national cost and benefit functions and their individual rates of exchange. The auctioneer having neither information on costs, nor on benefits operates with the matrix of transport coefficients and emission reductions proposed by the delegates. Therefore, the auction is indeed a mechanism that uses the information actually dispersed among the participants and allows them to coordinate decisions on emission reduction in a Pareto optimal way.

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