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Contents

Abstract

We develop a procedure and the requisite theory for incorporating preference information in a novel way in the efficiency analysis of Decision Making Units. The efficiency of Decision Making Units is defined in the spirit of Data Envelopment Analysis (DEA), complemented with Decision Maker's preference information concerning the desirable structure of inputs and outputs. Our procedure begins by aiding the Decision Maker in searching for the most preferred combination of inputs and outputs of Decision Making Units (for short, Most Preferred Solution) which are efficient in DEA. Then, assuming that the Decision Maker's Most Preferred Solution maximizes his/her underlying (unknown) value function at the moment when the search is terminated, we approximate the indifference contour of the value function at this point with its possible tangent hyperplanes. Value Efficiency scores are then calculated for each Decision Making Unit comparing the inefficient units to units having the same value as the Most Preferred Solution. The resulting Value Efficiency scores are optimistic approximations of the true scores. The procedure and the resulting efficiency scores are immediately applicable to solving practical problems.

Keywords: Efficiency Analysis, Data Envelopment Analysis, Multiple Criteria Decision Making, Value Function

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Value Efficiency Analysis for Incorporating Preference Information in Data Envelopment Analysis[1]

Merja Halme, Tarja Joro, Pekka Korhonen
Seppo Salo and Jyrki Wallenius

1. Introduction

Data Envelopment Analysis (DEA), originally proposed by Charnes, Cooper and Rhodes [1978 and 1979], has become one of the most widely used methods in management science. DEA measures the relative efficiency of comparable entities called Decision Making Units (DMUs) essentially performing the same task using similar multiple inputs to produce similar multiple outputs. The purpose of DEA is to empirically estimate the so-called efficient frontier based on the set of available DMUs. A DMU is efficient if there is no other unit – existing or virtual – that can either produce more outputs by consuming the same amount or less of inputs or produce the same amount or more of outputs by consuming less or the same amount of inputs as the DMU under consideration. The former approach is referred to as the output oriented and the latter as the input oriented DEA. DEA provides the user with information about the efficient and inefficient units, as well as the efficiency scores and reference sets for inefficient units. The results of the DEA analysis, especially the efficiency scores, are used in practical applications as performance indicators of DMUs.

When Decision Making Units are evaluated in practice, there is always a reason for this. It might be the allocation of existing or additional resources to units, need to make the operations more profitable by improving the performance of inefficient units, or the desire to reward the most efficient units. The results of the analysis provide a basis for such decisions. Generally there exists a Decision Maker (DM) who has preferences over outputs and inputs. The underlying assumption of the original DEA, however, is that no output or input is more important than another. In such a situation, a DMU which, for example, is a superior producer of a marginally important output is diagnosed as efficient even if it performs poorly with respect to all other outputs. Hence, in the original DEA the efficiency scores are not necessarily good performance indicators. We use Figure 1.1 to clarify our point. The example consists of five DMUs, each producing two outputs and all consuming the same amount of one input. We can see that DMUs 1,

[1] An abbreviated version is forthcoming in Management Science

2 and 3 are efficient and 4 and 5 inefficient. Thus DMUs 1, 2 and 3 all receive an efficiency score of 1. Let us assume that for some reason the DM considers output 1 to be much more important than output 2. In this case DMU_1 would be far more preferred to DMU_3. The DM might even prefer DMU_5 to DMU_3, even though the former is inefficient.

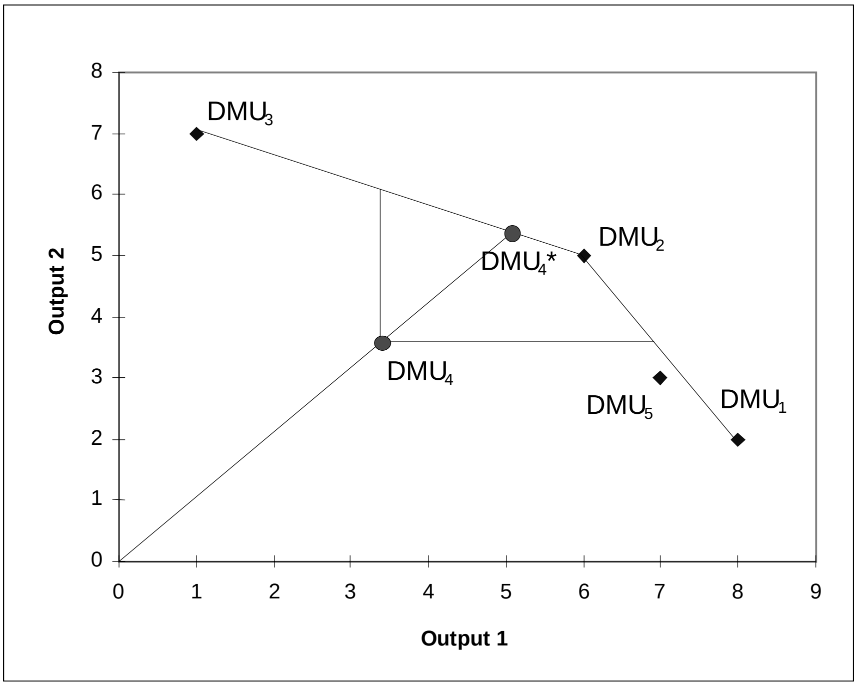

Figure 1.1 Classical DEA

Although DEA calculations were originally value-free, several attempts have been made to incorporate preference information in DEA, resulting in different models. Such models can be divided into two categories: (1) models that use preference information to set targets for inefficient DMUs, and (2) models that use preference information to produce more meaningful efficiency scores. We briefly discuss both types of models.

Golany [1988] and Thanassoulis and Dyson [1992], among others, have developed target setting models. Golany introduced a model that allows the DM to select the preferred set of output levels given the input levels of a DMU. In standard DEA, the target for DMU_4 in Figure 1.1 would be DMU_4*, but in Golany's model the DM is able to choose any point from the efficient frontier restricted by the dotted lines. Thanassoulis and Dyson have introduced models which can be used to estimate alternative input/output target levels to render relatively inefficient DMUs efficient. The models can incorporate preferences over potential improvements to individual input/output levels, and thus the DM may select the preferred target in the input/output space. In the DEA language, such models are not radial. Such models do not, however, aim at producing efficiency scores, and thus perform a task different from the one we have set for us in this paper.

In the category of efficiency score models, the traditional way to incorporate preference information in DEA is to restrict the flexibility of weights. Several weight flexibility restriction schemes have been proposed by Charnes, Cooper, Wei and Huang [1989, 1990], Dyson and Thanassoulis [1988], Thompson, Langemeier, Lee, Lee and Thrall [1990], Thompson, Singleton, Thrall and Smith [1986], and Wong and Beasley [1990], among others. Generally speaking, weight restrictions result in the reduction of the number of efficient DMUs. Zhu [1996] introduced a model that calculates efficiency scores incorporating the DM's preference information. His model can also be used for target setting; it offers to the DM targets that do not dominate the DMU under consideration. In Zhu's model the preferences are elicited as weights, reflecting the relative degree of desirability of the adjustments in the current input or output levels. However, it is far from trivial to select the bounds for the weights or, more generally, elicit the DM's preference structure. Golany and Roll [1994] introduced a model that does not use weights to elicit preference information; instead they incorporated preference information in the form of hypothetical DMUs in an otherwise standard DEA model.

In the Multiple Criteria Decision Making (MCDM) literature we can find numerous arguments against using importance weights as a means to elicit and represent DM's preference information (for example, Steuer [1986, pp. 193-200], Korhonen and Wallenius [1989], and Wierzbicki [1986]). It seems particularly difficult to understand that the intuitively appealing notion "the greater the importance -- the larger the weight" does not always work. When the weights have a straightforward interpretation, such as prices in economics, their definition and use is also straightforward. We suggest that the DM's preferences are incorporated in efficiency analysis by explicitly locating his/her most preferred input-output vector on the efficient frontier. We call this vector the DM's *Most Preferred Solution* (MPS). It is a vector on the efficient frontier which he/she prefers to any other vector at the moment of the final choice. In this paper we use an interactive Multiple Objective Linear Programming (MOLP) search procedure to locate the MPS, but any approach (using weights or any other preference information) resulting in the MPS is applicable for carrying out the proposed analysis (e.g., Steuer [1986]). We may also use a "goal focusing" approach for this purpose (see, e.g., Charnes, Cooper, Rousseau, Schrinnar, Terleckyj, and Levy [1980]). Conceptually, an MPS can be defined as the point at which the DM's value function assumes its maximum when the search terminates. Using the knowledge of the MPS, the DM's (unknown) value function is approximated using so-called tangent cones at the MPS. The efficiency of each DMU is then determined with respect to this tangent cone. As a result we obtain scores that we call V*alue Efficiency* scores, because the efficiency of each DMU is determined by means of an approximation of the indifference surface of an implicitly known value function at the MPS.

The rest of this paper is organized as follows. Section 2 sets the stage by discussing preliminary considerations. In Section 3 we develop the procedure and the requisite theory for incorporating preference information in DEA. Section 4 illustrates the procedure with a numerical example, and Section 5 discusses the use of Value Efficiency Analysis with real data. Section 6 concludes the paper.

2. DEA and Multiple Objective Linear Programming

Assume we have n DMUs each consuming m inputs and producing p outputs. Let $\mathbf{X} \in \Re_{+}^{m \times n}$ and $\mathbf{Y} \in \Re_{+}^{p \times n}$ be the matrices, consisting of nonnegative elements, containing the observed input and output measures for the DMUs. We further assume that there are no duplicated units in the data set. We denote by $\boldsymbol{x}_j$ (the jth column of $\mathbf{X}$) the vector of inputs consumed by DMUj, and by x_{ij} the quantity of input i consumed by DMUj. A similar notation is used for outputs. Furthermore, we denote $\boldsymbol{1} = [1, ..., 1]^T$.

The traditional CCR-models, as introduced by Charnes et al. [1978] are fractional linear programs which can easily be formulated and solved as linear programs. To condense the text, in the sequel we consider solely output oriented DEA models. The discussion, with appropriate modifications, holds for input oriented models as well. Later Banker, Charnes and Cooper [1984] developed the so-called BCC models with variable returns to scale. The CCR and BCC models are the basic model types in DEA. The output oriented CCR- and BCC-models are given in (2.1a), (2.1b), (2.2a), and (2.2b). Note that, following Charnes and Cooper, the original primal formulation is called the dual and vice versa.

Output-Oriented CCR Primal (CCR_P - O)	**Output-Oriented CCR Dual (CCR_D - O)**
max $Z_O = \theta + \varepsilon(\boldsymbol{1}^T\boldsymbol{s}^+ + \boldsymbol{1}^T\boldsymbol{s}^-)$ [2] s.t. (2.1a) $\mathbf{Y}\lambda - \theta\boldsymbol{y}_0 - \boldsymbol{s}^+ = \boldsymbol{0}$ $\mathbf{X}\lambda + \boldsymbol{s}^- = \boldsymbol{x}_0$ $\lambda, \boldsymbol{s}^-, \boldsymbol{s}^+ \geq \boldsymbol{0}$ $\varepsilon > 0$ ("Non-Archimedean") [3]	*min* $W_O = \nu^T\boldsymbol{x}_0$ s.t. (2.1b) $\mu^T\boldsymbol{y}_0 = 1$ $-\mu^T\mathbf{Y} + \nu^T\mathbf{X} \geq \boldsymbol{0}^T$ $\mu, \nu \geq \varepsilon\boldsymbol{1}$ $\varepsilon > 0$
Output-Oriented BCC Primal (BCC_P - O)	**Output-Oriented BCC Dual (BCC_D - O)**
max $Z_O = \theta + \varepsilon(\boldsymbol{1}^T\boldsymbol{s}^+ + \boldsymbol{1}^T\boldsymbol{s}^-)$ s.t. (2.2a) $\mathbf{Y}\lambda - \theta\boldsymbol{y}_0 - \boldsymbol{s}^+ = \boldsymbol{0}$ $\mathbf{X}\lambda + \boldsymbol{s}^- = \boldsymbol{x}_0$ $\boldsymbol{1}^T\lambda = 1$ $\lambda, \boldsymbol{s}^-, \boldsymbol{s}^+ \geq \boldsymbol{0}$ $\varepsilon > 0$	*min* $W_O = \nu^T\boldsymbol{x}_0 + u$ s.t. (2.2b) $\mu^T\boldsymbol{y}_0 = 1$ $-\mu^T\mathbf{Y} + \nu^T\mathbf{X} + u\boldsymbol{1}^T \geq \boldsymbol{0}^T$ $\mu, \nu \geq \varepsilon\boldsymbol{1}$ $\varepsilon > 0$

[2] For clarity, throughout the paper we assume that the units of all slacks are the same. See Thrall (1996) for a discussion.

[3] For more details, see Arnold, Bardhan, Cooper, and Gallegos [1997].

To unify the presentation we formulate a general model (for short, GEN) which includes CCR- and BCC-models as special cases. Matrix $\mathbf{A} \in \Re^{k \times n}$ and vector $\boldsymbol{b} \in \Re^k$ are used to specify the set of feasible λ-variables.

Output-Oriented GEN Primal (GEN$_P$ - O)	Output-Oriented GEN Dual (GEN$_D$ - O)
$\max Z_O = \theta + \varepsilon(\mathbf{1}^T\boldsymbol{s}^+ + \mathbf{1}^T\boldsymbol{s}^-)$	$\min W_O = \nu^T\boldsymbol{x}_0 + \boldsymbol{u}^T\boldsymbol{b}$
s.t. (2.3a)	s.t. (2.3b)
$\mathbf{Y}\lambda - \theta\boldsymbol{y}_0 - \boldsymbol{s}^+ = \mathbf{0}$	$\mu^T\boldsymbol{y}_0 = 1$
$\mathbf{X}\lambda + \boldsymbol{s}^- = \boldsymbol{x}_0$	$-\mu^T\mathbf{Y} + \nu^T\mathbf{X} + \boldsymbol{u}^T\mathbf{A} \geq \mathbf{0}^T$
$\mathbf{A}\lambda \leq \boldsymbol{b}$	$\mu, \nu \geq \varepsilon\mathbf{1}$
$\lambda, \boldsymbol{s}^-, \boldsymbol{s}^+ \geq \mathbf{0}$	$\boldsymbol{u} \geq \mathbf{0}$
$\varepsilon > 0$	$\varepsilon > 0$

A DMU is efficient iff Z* = 1 and all slack variables $\boldsymbol{s}^-$, $\boldsymbol{s}^+$ equal zero; otherwise it is inefficient (Charnes et al. 1994).

When it is not necessary to emphasize the different roles of inputs and outputs, we denote $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{x} \end{bmatrix}$ and $\mathbf{U} = \begin{bmatrix} \mathbf{Y} \\ -\mathbf{X} \end{bmatrix}$, and modify the problem (2.3) accordingly. Because the results concerning $\boldsymbol{u}$ and $\mathbf{U}$ are valid for $\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{x} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{Y} \\ \mathbf{X} \end{bmatrix}$ as well, for simplicity, we often refer to $\boldsymbol{u}$ and $\mathbf{U}$, although we are factually interested in results concerning $\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{x} \end{bmatrix}$ and $\begin{bmatrix} \mathbf{Y} \\ \mathbf{X} \end{bmatrix}$. Define the sets $\Lambda = \{ \lambda \mid \lambda \in \Re^n_+ \text{ and } \mathbf{A}\lambda \leq \boldsymbol{b}\}$ and $T = \{ \boldsymbol{u} \mid \boldsymbol{u} = \mathbf{U}\lambda, \lambda \in \Lambda\}$. We assume that $\mathbf{e}_i \in \Lambda$, $i = 1,\ldots, n$, where $\mathbf{e}_i$ is the i^{th} unit vector in $\Re^n$. All efficient DMUs lie on the efficient frontier, which is defined as a subset of points of set T satisfying the efficiency condition above. The definition of efficiency and the corresponding definition for weak efficiency, can be given in the following equivalent form:

Definition 1. A point $\mathbf{U}\lambda^* = \boldsymbol{u}^*$ is *efficient* iff there does not exist another $\boldsymbol{u} \in T$ such that $\boldsymbol{u} \geq \boldsymbol{u}^*$, and $\boldsymbol{u} \neq \boldsymbol{u}^*$.

Definition 2. A point $\boldsymbol{u}^* \in T$ is *weakly efficient* iff there does not exist another $\boldsymbol{u} \in T$ such that $\boldsymbol{u} > \boldsymbol{u}^*$.

In DEA, the efficiency of a DMU is traditionally determined either by maximizing outputs subject to given input levels or minimizing inputs subject to given output levels. A model considering both input minimization and output maximization was introduced as early as 1985 (Charnes, Cooper, Golany, Seiford, and Stutz [1985]). Other models considering simultaneous input minimization and output maximization exist (see, for example, Warwick DEA-User Manual, Thanassoulis and Dyson [1992] and Joro, Korhonen, and Wallenius [1995]).

The efficiency of DMUs can also be determined using the following Multiple Objective Linear Programming (MOLP) model:

$$\max \quad \mathbf{U}\lambda = \begin{bmatrix} \mathbf{Y} \\ -\mathbf{X} \end{bmatrix} \lambda$$

$$\text{s.t.} \qquad \lambda \in \Lambda = \{ \lambda \mid \lambda \in \Re^n_+ \text{ and, } \mathbf{A}\lambda \le \boldsymbol{b} \}. \tag{2.4}$$

Model (2.4) -- like any multiple criteria model -- has no unique solution. Its solutions are defined analogously to the efficient solutions in DEA. Specifically, in the MOLP-literature (see, e.g., Steuer [1986]), the concept of efficiency is used to refer to the solutions in the decision variable space (set Λ) and the concept of dominance is used to refer to the efficient solutions in the criterion space (set T). Weakly efficient solutions of problem (2.4) are defined according to Definition 2.

One possible, currently popular way to perform the search for solutions on the efficient frontier of a MOLP-problem is to use *the achievement (scalarizing) function* (ASF) suggested by Wierzbicki [1980]. To characterize the efficient set of problem (2.5), we may use the following formulation:

$$\min s(\boldsymbol{g}, \boldsymbol{u}, \boldsymbol{w}, \delta) = \min \{\max_{i \in P} [(g_i - u_i) / w_i] + \delta \sum_{i \in P} (g_i - u_i)\}$$

$$\text{s.t.} \qquad \boldsymbol{u} \in T, \tag{2.5}$$

where s is the ASF, $\boldsymbol{w} = \begin{bmatrix} \boldsymbol{w}^y \\ \boldsymbol{w}^x \end{bmatrix} > \boldsymbol{0}$, $\boldsymbol{w} \in \Re^{m+p}$ is a vector of weights, $\delta > 0$ is "Non-Archimedean" and $P = \{1, 2, ..., m+p\}$. Vector $\boldsymbol{g} = \begin{bmatrix} \boldsymbol{g}^y \\ -\boldsymbol{g}^x \end{bmatrix} \in \Re^{m+p}$ is a given point, the components of which are called *aspiration levels*. Using (2.5), we may project any given (feasible or infeasible) point $\boldsymbol{g} \in \Re^{m+p}$ onto the set of efficient solutions of (2.4). Varying the vector of aspiration levels, all efficient solutions of (2.4) can be generated (Wierzbicki [1986]). In Joro et al. [1995], we have shown that, using the above formulation, the projection problem can be presented as in (2.6a) and (2.6b).

Reference Point Model Primal (REF_p)	Reference Point Model Dual (REF_D)
max $\sigma + \varepsilon(\mathbf{1}^T\mathbf{s}^+ + \mathbf{1}^T\mathbf{s}^-)$ s.t. (2.6a) $\mathbf{Y}\lambda - \sigma\mathbf{w}^y - \mathbf{s}^+ = \mathbf{g}^y$ $\mathbf{X}\lambda + \sigma\mathbf{w}^x + \mathbf{s}^- = \mathbf{g}^x$ $\mathbf{A}\lambda \le \mathbf{b}$ $\lambda, \mathbf{s}^-, \mathbf{s}^+ \ge \mathbf{0}$ $\varepsilon > 0$	*min* $\nu^T\mathbf{g}^x - \mu\mathbf{g}^y + \eta^T\mathbf{b}$ s.t. (2.6b) $-\mu^T\mathbf{Y} + \nu^T\mathbf{X} + \eta^T\mathbf{A} \ge \mathbf{0}$ $\mu^T\mathbf{w}^y + \nu^T\mathbf{w}^x = 1$ $\mu, \nu \ge \varepsilon\mathbf{1}$ $\eta \ge \mathbf{0}$ $\varepsilon > 0$

Vector $\mathbf{g}^x$ consists of aspiration levels for inputs and $\mathbf{g}^y$ of aspiration levels for outputs. Vectors $\mathbf{w}^x > \mathbf{0}$ and $\mathbf{w}^y > \mathbf{0}$ are the weighting vectors for inputs and outputs, respectively. If a particular unit's efficiency has to be checked, vector **g** is replaced by its input/output vector. We refer to model (2.6) as the reference point model à la Wierzbicki [1980].

The reference point model is a generalization of the traditional DEA input and output oriented CCR and BCC models. Vectors $\mathbf{w}^x > \mathbf{0}$ and $\mathbf{w}^y > \mathbf{0}$ give freedom to project the inefficient input/output vector to any point on the efficient frontier dominating the input/output vector under consideration. Obviously, the radial projection ($\mathbf{w}^x = \mathbf{x}_0$ and $\mathbf{w}^y = \mathbf{y}_0$) can be performed as a special case. The first non-radial projection was proposed by Banker and Morey [1986] as early as in the eighties. When the weight vector in (2.6a) coincides with the input/output vector of the unit under diagnosis, the model is radial with respect to $\mathbf{y}$ and $\mathbf{x}$. The optimal solution provides us with a lower bound estimate of the percentage the inputs have to be decreased and outputs increased in order to make the unit efficient.

3. Value Efficiency Analysis

3.1. An Introduction

Our purpose is to assist the DM to evaluate the *value* of each vector $\mathbf{u} = \begin{bmatrix} \mathbf{y} \\ -\mathbf{x} \end{bmatrix} \in T$ to him/her. Actually, our approach makes it possible to evaluate the value of any $\mathbf{u} \in \Re^{m+p}$. The evaluation could be done easily, if we explicitly knew the DM's value function. However, generally in practice it is not realistic to assume that the value function is known or that it could reliably be estimated. That is why we use a different approach to incorporating a DM's preferences in the efficiency analysis. Our approach is based on the idea of locating the DM's MPS. The only assumption that we make about the DM's value function is that it is pseudoconcave at the moment when the search for the MPS is terminated. We first characterize the set of the tangent hyperplanes of the contours of all possible pseudoconcave value functions. We then use this information to evaluate the value of each DMU to the DM in the spirit of DEA.

The MPS is a solution which is preferred by the DM to any other solution. Assuming a rational DM who prefers more of any output and less of any input, it is obvious that the MPS is efficient. Unfortunately defining the MPS in this way provides no practical tool for efficiency analysis. It is not realistic to assume that the DM is generally able to compare all possible solutions to the final solution at the end of the search. In practice, the MPS is a solution at which the search process ends. It is difficult to know how good it is. In this paper, we assume that the MPS is the solution at which the DM's value function $v(\boldsymbol{u})$: $\Re^{m+p} \rightarrow \Re^{1}$ obtains its maximum over T. Note that we do not need to make any assumptions whatsoever concerning the value function during the search process. We only need the assumptions at the moment of termination in order to be able to say "something" about the quality of the final solution. The weaker these assumptions are, the better. We assume that the choice of the MPS was based on the DM's value function $v(\boldsymbol{u})$, $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{x} \end{bmatrix} \in \Re^{m+p}$, which is strictly increasing (i.e. strictly increasing in $\boldsymbol{y}$ and strictly decreasing in $\boldsymbol{x}$) and with a (local) maximal value $v(\boldsymbol{u}^*)$ over T, $\boldsymbol{u}^* = \begin{bmatrix} \boldsymbol{y}^* \\ -\boldsymbol{x}^* \end{bmatrix} \in \Re^{m+p}$. Furthermore, we assume that v is pseudoconcave, because then its local optimum over a convex set is also global (Bazaraa and Shetty [1979], p. 510) and the optimality conditions can easily be verified. This assumption guarantees that the DM has found his/her most preferred solution in the original meaning of the word.

Next we define the concept of V*alue Efficiency*.

Definition 3. Assuming that $\boldsymbol{u}^* \in$ T is the DM's most preferred solution, point $\boldsymbol{u} \in \Re^{m+p}$ is V*alue Efficient* iff $v(\boldsymbol{u}) \geq v(\boldsymbol{u}^*)$.

If the point $\boldsymbol{u} \in \Re^{m+p}$ is not V*alue Efficient*, it is called V*alue Inefficient*.

Definition 4. The *weighted true Value Efficiency score* for point $\boldsymbol{u}^0$ is defined as follows:

$$E_t^w(\boldsymbol{u}^0) = \gamma^t,$$

where γ^t is the optimal value of the objective function of the following problem:

$$\begin{aligned} &\sup \quad \gamma \\ &\text{s.t.} \\ &\qquad \boldsymbol{u} - \gamma \boldsymbol{w} \geq \boldsymbol{u}^0 \\ &\qquad \boldsymbol{u} \in \mathrm{V} = \{\boldsymbol{u} \mid v(\boldsymbol{u}) \leq v(\boldsymbol{u}^*)\} \\ &\qquad \boldsymbol{w} > \boldsymbol{0}. \end{aligned} \tag{3.1}$$

Note that we have to use "sup" in the above formulation, because we did not assume the continuity of function v, and thus set V is not necessarily closed.

Remark. It is evident for an increasing value function γ that $\gamma > 0$ iff the point $\boldsymbol{u}^0$ is Value Inefficient. When $\gamma = 0$, the point is Value Efficient; and if $\gamma < 0$, it is Value "Superefficient", i.e. $v(\boldsymbol{u}^0) > v(\boldsymbol{u}^*)$.

Lemma 1. Let $v(\boldsymbol{u})$ be strictly increasing. Then, for any finite $\boldsymbol{u}^*$, $\boldsymbol{u}^0$, and $\boldsymbol{w} > \boldsymbol{0}$, problem (3.1) has a finite solution γ^s corresponding to a finite input/output point $\boldsymbol{u}^s = \boldsymbol{u}^0 + \gamma^s \boldsymbol{w}$.

Proof. Assume first that $v(\boldsymbol{u}^0) < v(\boldsymbol{u}^*)$. (This assumption includes the cases: $\boldsymbol{u}^0 \in$ T and $v(\boldsymbol{u}^0) \neq v(\boldsymbol{u}^*)$). Because $\boldsymbol{w} > \boldsymbol{0}$, $\exists\ \gamma^1$ such that $\boldsymbol{u}^0 + \gamma^1\boldsymbol{w} > \boldsymbol{u}^*$. Because v is strictly increasing, $v(\boldsymbol{u}^0 + \gamma^1\boldsymbol{w}) > v(\boldsymbol{u}^*)$ and $v(\boldsymbol{u}^0 + \gamma\,\boldsymbol{w})$ is strictly increasing in γ. Hence it follows that $\exists \gamma^s < \gamma^1$, where

$$\gamma^s = \sup\{\gamma \mid v(\boldsymbol{u}^0 + \gamma\boldsymbol{w}) \leq v(\boldsymbol{u}^*)\}.$$

The proof for the case $v(\boldsymbol{u}^0) > v(\boldsymbol{u}^*)$ is analogous and the case $v(\boldsymbol{u}^0) = v(\boldsymbol{u}^*)$ trivially gives $\gamma=0$.

In (3.1), if we set $\boldsymbol{w} = \boldsymbol{u}^0$ the model becomes radial in $\boldsymbol{y}$ and radial in $\boldsymbol{x}$ and we may interpret γ^s as the percentage of improvement needed in both inputs and outputs to make $\boldsymbol{u}^0$ Value Efficient.

It is not realistic to assume that we know the DM's value function. Hence we only assume that we know its form at the time the search for the MPS is terminated. If the DM explored the neighborhood of the MPS in a systematic manner and was unable to find a more preferable input/output vector, we may conclude that the MPS has been found, provided that the value function assumption from above is valid (for a more detailed description of the neighborhood exploration, see Korhonen and Laakso [1986]).

To keep the presentation brief, we do not discuss in any detail how to carry out the search for the MPS. If suffices to say that we believe that an interactive system would be necessary for the DM to support him/her in the process. Numerous procedures and accompanying software systems are available for this purpose. The Pareto Race interface by Korhonen and Wallenius [1988], which is implemented in the VIG software, is a case in point. Its use is illustrated in Section 4.

The MPS lies on the indifference contour of the DM's value function possessing the highest possible value among all feasible input/output vectors in T. Accordingly the MPS has the highest possible V*alue Efficiency* for the DM. Value Inefficient DMUs should increase their performance to reach the contour on which the MPS lies in order to achieve the same Value Efficiency.

It is interesting to compare Value Efficiency to the concepts of classical efficiency analysis: technical and overall efficiency (Farrell, [1957]). (See, for example, Norman and Stoker [1991] for a discussion of classical efficiency analysis and DEA.) Figures 3.1a and 3.1b illustrate different situations. Again, we assume that the DMUs produce two outputs and all consume the same amount of one input.

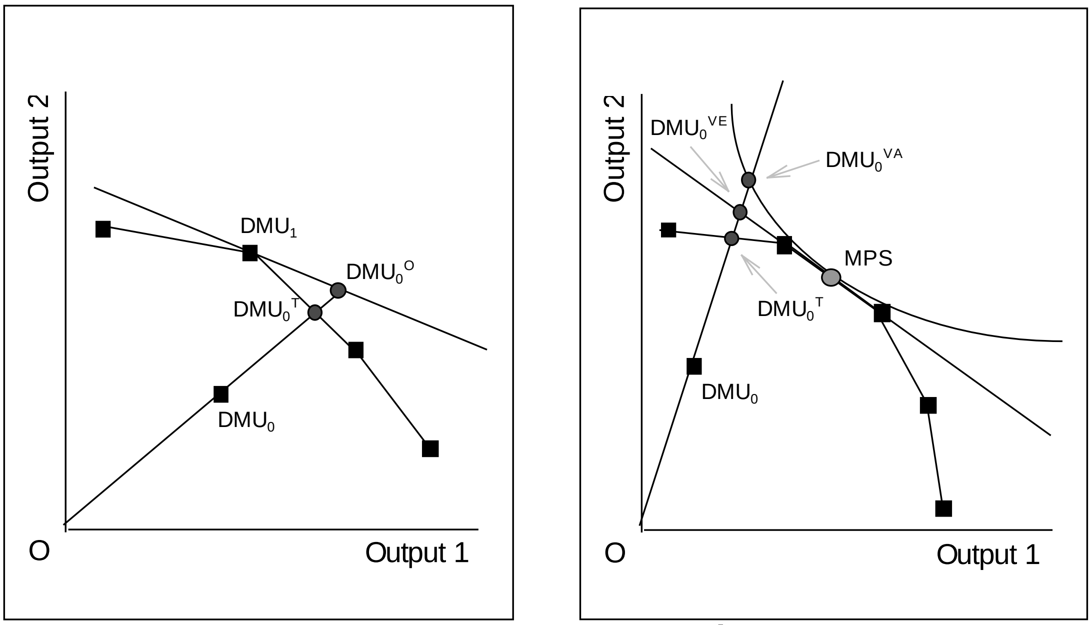

Figure 3.1a *Figure 3.1b*

Figure 3.1: Classical Efficiency Vs Value Efficiency

Figure 3.1a illustrates the concept of classical efficiency. The downward sloping line through DMU_0^O represents the revenue equation, thus containing information about the prices. Only DMU_1 is overall efficient. For DMU_0 the ratio O-DMU_0/O-DMU_0^T reflects technical efficiency, and ratio O-DMU_0/O-DMU_0^O overall efficiency.

Next, we seek to clarify the connection between classical overall efficiency and Value Efficiency. Classical overall efficiency is based on the idea of maximizing a known revenue (cost) function. In Value Efficiency analysis, this revenue function is replaced by a more general unknown pseudoconcave value function. Furthermore, we assume that the maximum of this function is known, but its precise form is unknown. Based on this information, in Value Efficiency analysis, we estimate "overall efficiency". More precisely, we postulate that the value function $v(\boldsymbol{u})$ is a pseudoconcave, strictly increasing function, which obtains its maximum over T at the MPS $\boldsymbol{u}^*$. The contours of a pseudoconcave function lie above their tangent hyperplanes. Hence we use the tangent hyperplane at the MPS as a linear approximation of $v(\boldsymbol{u})$. In Figure 3.1b, the ratio O-DMU_0/O-DMU_0^T reflects technical efficiency. The ratio DMU_0-DMU_0^{VA}/O-DMU_0 reflects (true) Value Efficiency that is not possible to determine. That is why we use the ratio DMU_0-DMU_0^{VE}/O-DMU_0 to approximate the Value Efficiency score. For an efficient unit, Value Efficiency score is zero. The requisite theory is developed in subsections 3.2 and 3.3. Our approximation of the Value Efficiency score is optimistic: it provides a lower bound for the actual Value Efficiency score.

3.2. Some Mathematical Considerations

In this subsection we present the requisite mathematical theory to formulate an operational model for computing Value Efficiency scores.

Definition 5. A nonempty set G_x defined in an n-dimensional Euclidean space $\Re^n$ is called a *(pointed) cone* with vertex $\boldsymbol{x}$, if $\boldsymbol{x} + \boldsymbol{y} \in G_x \Rightarrow \boldsymbol{x} + \lambda\boldsymbol{y} \in G_x$ for all $\lambda \geq 0$. The cone with the origin as vertex is denoted by G.

Note that vertex $\mathbf{x} \in G_x$. A singleton $\{\boldsymbol{x}\}$ is also a cone with vertex $\boldsymbol{x}$.

Definition 6. Let X be a nonempty polytope in $\Re^n$ and let $\boldsymbol{x} \in$ X. The pointed cone D($\boldsymbol{x}$) in $\Re^n$ is called the *cone of feasible directions* of X at $\boldsymbol{x}$, if

$$D(\boldsymbol{x}) = \{\boldsymbol{d} \mid \boldsymbol{x} + \lambda\boldsymbol{d} \in X \text{ for all } \lambda \in (0,\delta) \text{ for some } \delta > 0\}.$$

Each $\boldsymbol{d} \in D(\boldsymbol{x})$, $\boldsymbol{d} \neq \boldsymbol{0}$, is called a feasible direction. The cone $G_x = \{\boldsymbol{y} \mid \boldsymbol{y} = \boldsymbol{x} + \boldsymbol{d}, \boldsymbol{d} \in D(\boldsymbol{x})\}$ is called the *tangent cone* of X at $\boldsymbol{x}$ and the cone $W_x = \{\boldsymbol{s} \mid \boldsymbol{s} = \boldsymbol{y} + \boldsymbol{z}, \boldsymbol{y} \in G_x, \boldsymbol{z} \in \Re^n_-\}$ the *augmented tangent cone* of X at $\boldsymbol{x}$.

Note that both G_x and W_x are closed and convex. For any $\boldsymbol{s} \in W_x$ there is an $\boldsymbol{y} \in G_x$ such that $\boldsymbol{s} \leq \boldsymbol{y}$ and all points $\boldsymbol{z} \leq \boldsymbol{s}$ are in W_x.

We illustrate the tangent cone and the augmented tangent cone in Figure 3.2. The area defines the polytope X. The tangent cone G_x at $\boldsymbol{x}$ is spanned by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, and the augmented tangent cone W_x by vectors $\boldsymbol{a}$ and $\boldsymbol{c}$.

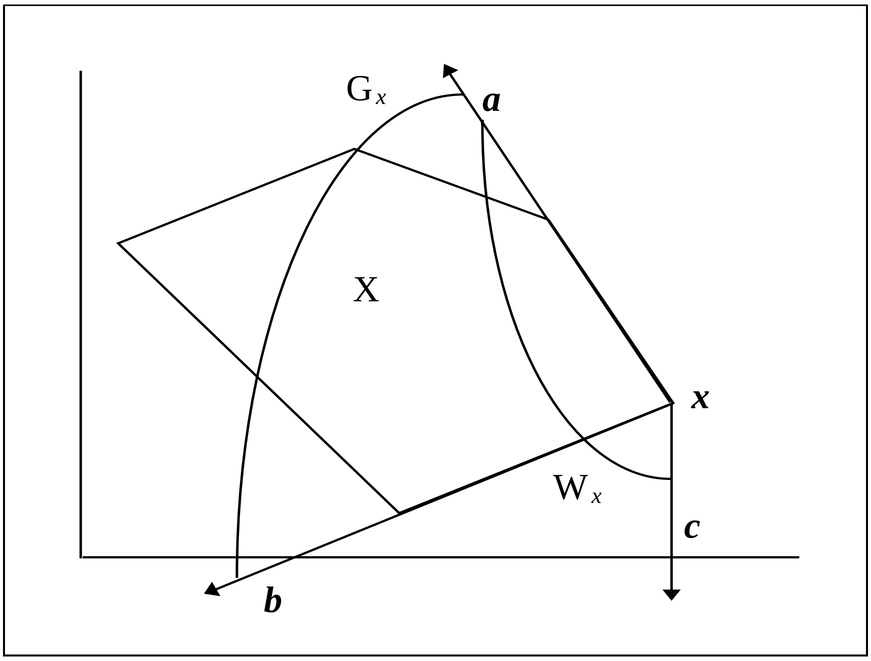

Figure 3.2: *An Example Illustrating the Tangent Cone and the Augmented Tangent Cone*

Lemma 2. Let $X = \{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ be a nonempty polytope, where $\mathbf{A} \in \Re^{k\times n}$, $\boldsymbol{b} \in \Re^k$, and $\boldsymbol{x}^0 \in$ X an arbitrary point. Then $G_{x^0} = X^0$, where $X^0 = \{ \boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, x_j \geq 0$ if $x_j^0 = 0$, and otherwise x_j is free, $j = 1, 2, \ldots, n \}$.

Proof. Clearly the tangent cone of an affine set $X_a = \{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}\}$ at $\boldsymbol{x}^0$ is X_a itself. Moreover, the tangent cone of the closed halfspace $H_j = \{\boldsymbol{x} \mid x_j \geq 0\}$ at $\boldsymbol{x}^0$ is $\Re^n$ if $x_j^0 > 0$ and H_j, if $x_j^0 = 0$, $j = 1, \ldots, n$. Because X is the intersection of X_a and the halfspaces H_j, $j = 1,\ldots,n$, the tangent cone of X at $\boldsymbol{x}^0$ is the intersection of their tangent cones, respectively, i.e. set X^0.

Lemma 3. Let $U = \{\boldsymbol{u} \in \Re^m \mid \boldsymbol{u} = B\boldsymbol{x}, \boldsymbol{x} \in X\}$, where $X = \{\boldsymbol{x} \mid \mathbf{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$, be a linear transformation of a nonempty polytope X and $\boldsymbol{u}^0 \in U$ an arbitrary point. Let $\boldsymbol{x}^0 \in X$ be any point such that $\boldsymbol{u}^0 = B\boldsymbol{x}^0$. Then the tangent cone of U at $\boldsymbol{u}^0$ is $G_{\boldsymbol{u}^0} = BG_{\boldsymbol{x}^0} = \{\boldsymbol{u} \mid \boldsymbol{u} = B\boldsymbol{x}, \boldsymbol{x} \in G_{\boldsymbol{x}^0}\}$.

Proof. Any $\boldsymbol{u} \in G_{\boldsymbol{u}^0}$, $\boldsymbol{u} \neq \boldsymbol{u}^0$, defines a feasible direction $\boldsymbol{u} - \boldsymbol{u}^0$ for U at $\boldsymbol{u}^0$, which must be generated by a feasible direction $\boldsymbol{x} - \boldsymbol{x}^0$ for X at $\boldsymbol{x}^0$. Thus $G_{\boldsymbol{u}^0} \subset BG_{\boldsymbol{x}^0}$. Any $\boldsymbol{x} \in G_{\boldsymbol{x}^0}$, $\boldsymbol{x} \neq \boldsymbol{x}^0$, defines a feasible direction $\boldsymbol{x} - \boldsymbol{x}^0$ for X at $\boldsymbol{x}^0$, which defines a feasible direction $\boldsymbol{u} - \boldsymbol{u}^0$ for U at $\boldsymbol{u}^0$. Thus $G_{\boldsymbol{u}^0} \supset BG_{\boldsymbol{x}^0}$.

Definition 7. A differentiable function $f{:}\Re^n \rightarrow \Re$ is *pseudoconcave* on a convex set S iff for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in S$ such that $\nabla f(\boldsymbol{x}_1)^T(\boldsymbol{x}_2 - \boldsymbol{x}_1) \leq 0 \Rightarrow f(\boldsymbol{x}_2) \leq f(\boldsymbol{x}_1)$.

Note that pseudoconcave functions are by definition differentiable and therefore continuous.

Let $X \subset \Re^n$ be a nonempty polytope and $\boldsymbol{x}^* \in X$. Define $\Xi(\boldsymbol{x}^*)$ as the set of increasing pseudoconcave functions $\xi{:}\Re^n \rightarrow \Re$ which obtain their maximum in X at $\boldsymbol{x}^* \in X$.

Lemma 4. Let $\boldsymbol{x}^* \in X$ and $\Xi(\boldsymbol{x}^*) \neq \emptyset$. Denote the *augmented tangent cone* of X at $\boldsymbol{x}^*$ by $W_{\boldsymbol{x}^*}$. Then $\boldsymbol{x} \in W_{\boldsymbol{x}^*}$ iff $\xi(\boldsymbol{x}) \leq \xi(\boldsymbol{x}^*)$ for all $\xi \in \Xi(\boldsymbol{x}^*)$.

Proof. Let $\boldsymbol{x} \in W_{\boldsymbol{x}^*}$. Then there is $\boldsymbol{y} \in G_{\boldsymbol{x}^*}$ such that $\boldsymbol{x} \leq \boldsymbol{y}$. ξ is increasing $\Rightarrow \xi(\boldsymbol{x}) \leq \xi(\boldsymbol{y})$. $\boldsymbol{y} - \boldsymbol{x}^* \in D(\boldsymbol{x}^*)$ and ξ obtains its maximum in X at $\boldsymbol{x}^* \Rightarrow \nabla\xi(\boldsymbol{x}^*)^T(\boldsymbol{y} - \boldsymbol{x}^*) \leq 0$. Because ξ is pseudoconcave, $\xi(\boldsymbol{y}) \leq \xi(\boldsymbol{x}^*) \Rightarrow \xi(\boldsymbol{x}) \leq \xi(\boldsymbol{x}^*)$ for all $\xi \in \Xi(\boldsymbol{x}^*)$. To prove the second part, let $\boldsymbol{x} \in \Re^n$ for which $\xi(\boldsymbol{x}) \leq \xi(\boldsymbol{x}^*)$ for all $\xi \in \Xi(\boldsymbol{x}^*)$. Assume $\boldsymbol{x} \notin W_{\boldsymbol{x}^*}$. Then, $\boldsymbol{x}$ can be strongly separated from $W_{\boldsymbol{x}^*}$, i.e. $\exists \boldsymbol{p} \in \Re^n$ such that $\boldsymbol{p}^T\boldsymbol{x} > \boldsymbol{p}^T\boldsymbol{y}$ for all $\boldsymbol{y} \in W_{\boldsymbol{x}^*}$, i.e. for any $\boldsymbol{y} = \boldsymbol{x}^* + \boldsymbol{d} + \boldsymbol{z}$, where $\boldsymbol{d} \in D(\boldsymbol{x}^*)$, $\boldsymbol{z} \leq \boldsymbol{0}$. Hence $\boldsymbol{p}^T\boldsymbol{d} \leq 0$ and $\boldsymbol{p}^T\boldsymbol{z} \leq 0$ ($\Rightarrow \boldsymbol{p} \geq \boldsymbol{0}$), because otherwise $\boldsymbol{p}^T\boldsymbol{d}$ or $\boldsymbol{p}^T\boldsymbol{z}$ could be positive and arbitrarily large. Therefore a pseudoconcave increasing function $\xi(\boldsymbol{x}) = \boldsymbol{p}^T\boldsymbol{x}$ obtains its maximum in X at $\boldsymbol{x}^*$, i.e. $\xi \in \Xi(\boldsymbol{x}^*)$, and $\xi(\boldsymbol{x}) > \xi(\boldsymbol{x}^*)$ contrary to the assumption that $\xi(\boldsymbol{x}) \leq \xi(\boldsymbol{x}^*)$.

3.3. Determination of Value Efficiency Scores

Now we are in a position to formulate and prove the requisite theorems for evaluating Value Efficiency. We make use of lemmas 2, 3 and 4 by substituting $\Re^{m+p}$ for $\Re^n$, set Λ for set X, set T for set U and set $\Xi(\boldsymbol{u}^*)$ for set $\Xi(\boldsymbol{x}^*)$, where $\Xi(\boldsymbol{u}^*)$ is the set of pseudoconcave increasing functions $v(\boldsymbol{u})$, which obtain their maximum in T at $\boldsymbol{u}^*$.

Lemma 4 is employed when approximating the set V = $\{\boldsymbol{u} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{x} \end{bmatrix} \mid v(\boldsymbol{u}) \leq v(\boldsymbol{u}^*)\}$ where v may be any function in $\Xi(\boldsymbol{u}^*)$. This means that when the projections of inefficient units are restricted to the indifference contours of this set, the resulting efficiency scores are always surely better than the true ones.

Theorem 1. $W_{\boldsymbol{u}^*}$ is the largest cone with the property $W_{\boldsymbol{u}^*} \subset V = \{\boldsymbol{u} \mid v(\boldsymbol{u}) \leq v(\boldsymbol{u}^*)$, for any $v \in \Xi(\boldsymbol{u}^*)\}$.

Proof. Evident from Lemma 4.

Thus V is approximated by the cone W, the tangent cone of T at $\boldsymbol{u}^*$ with all input/output points weakly dominated by T appended, which guarantees that the resulting scores are optimistic (not greater than the real ones). Without supplementary information this is the best approximation available for set V in the sense that it is the largest set contained in all the sets of input/output points which are not preferred by any pseudoconcave increasing value function $v(\boldsymbol{u}) \in \Xi(\boldsymbol{u}^*)$.

Theorem 2. Let $\boldsymbol{u}^* = \begin{bmatrix} \boldsymbol{y}^* \\ -\boldsymbol{x}^* \end{bmatrix} \in$ T be the DM's Most Preferred Solution. Then $\boldsymbol{u} \in \Re^{m+p}$, an arbitrary point in the input/output space, is Value Inefficient with respect to any strictly increasing pseudoconcave value function $v(\boldsymbol{u})$, $\boldsymbol{u} = \begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{x} \end{bmatrix}$ with a maximum at point $\boldsymbol{u}^*$, if the optimum value Z* of the following problem is strictly positive:

$$
\begin{aligned}
&\max \quad Z = \sigma + \varepsilon(\mathbf{1}^T\boldsymbol{s}^+ + \mathbf{1}^T\boldsymbol{s}^-)\\
&\text{s.t.}\\
&\quad \mathbf{Y}\lambda - \sigma\boldsymbol{w}^y - \boldsymbol{s}^+ = \boldsymbol{y},\\
&\quad \mathbf{X}\lambda + \sigma\boldsymbol{w}^x + \boldsymbol{s}^- = \boldsymbol{x},\\
&\quad \mathbf{A}\lambda + \mu = \boldsymbol{b},\\
(3.2)&\\
&\quad \boldsymbol{s}^-, \boldsymbol{s}^+ \geq \mathbf{0},\\
&\quad \varepsilon > 0, \quad \text{("Non-Archimedean")}\\
&\quad \lambda_j \geq 0, \text{ if } \lambda_j^* = 0,\ j = 1,2,\ldots,n\\
&\quad \mu_j \geq 0, \text{ if } \mu_j^* = 0,\ j = 1,2,\ldots,k
\end{aligned}
$$

where $\lambda^* \in \Lambda$, μ^* correspond to the Most Preferred Solution:

$$
\begin{aligned}
\boldsymbol{y}^* &= \mathbf{Y}\lambda^*\\
\boldsymbol{x}^* &= \mathbf{X}\lambda^*.
\end{aligned}
$$

Note: For easy reference to the traditional Output oriented DEA models we have given the output and input parts separately.

Proof. By lemmas 2 and 3 the tangent cone of T at $\boldsymbol{u}^*$ is the set $\{ \begin{bmatrix} \boldsymbol{v} \\ -\boldsymbol{z} \end{bmatrix} \mid \boldsymbol{v} = \mathbf{Y}\lambda, \boldsymbol{z} = \mathbf{X}\lambda, \lambda \in G_{\lambda*}\}$, where the tangent cone of Λ at λ^* is $G_{\lambda*} = \{\lambda \mid \mathbf{A}\lambda + \mu = \boldsymbol{b}, \lambda_j \geq 0$ if $\lambda_j^* = 0$, $j = 1, 2, \ldots, n$, $\mu_j \geq 0$ if $\mu_j^* = 0$, $j = 1, 2, \ldots, k\}$. The augmented tangent cone W_{u*} of T at $\boldsymbol{u}^*$ is the set $\{\begin{bmatrix} \boldsymbol{v} \\ -\boldsymbol{z} \end{bmatrix} \mid \boldsymbol{v} = \mathbf{Y}\lambda + \boldsymbol{d}^y, \boldsymbol{z} = \mathbf{X}\lambda + \boldsymbol{d}^x, \boldsymbol{d}^y \leq \boldsymbol{0}, \boldsymbol{d}^x \geq \boldsymbol{0}, \lambda \in G_{\lambda*}\}$. Therefore (3.2) has a solution with $\sigma \geq 0$ only if $\begin{bmatrix} \boldsymbol{y} \\ -\boldsymbol{x} \end{bmatrix} \in W_{u*}$. Now let Z*, λ^s, σ^s, μ^s be a solution of (3.2). With $\varepsilon > 0$, Z* > 0 only if either $\sigma^s > 0$ or $\sigma^s = 0$ and $(\boldsymbol{s}^-, \boldsymbol{s}^+) \neq (\boldsymbol{0}, \boldsymbol{0})$. In either case, $\begin{bmatrix} \boldsymbol{v}^s \\ -\boldsymbol{z}^s \end{bmatrix} \in W_{u*}$, $\boldsymbol{y}^s = \mathbf{Y}\lambda^s \geq \boldsymbol{y}$, $\boldsymbol{x}^s = \mathbf{X}\lambda^s \leq \boldsymbol{x}$ and $(\boldsymbol{y}, \boldsymbol{x}) \neq (\boldsymbol{y}^s, \boldsymbol{x}^s)$. Thus $v(\boldsymbol{y}, -\boldsymbol{x}) < v(\boldsymbol{y}^s, -\boldsymbol{x}^s) \leq v(\boldsymbol{y}^*, -\boldsymbol{x}^*)$ and by Theorem 1, $(\boldsymbol{y}, \boldsymbol{x})$ is Value Inefficient.

Definition 9. The *(weighted) Value Efficiency score* for point $\boldsymbol{u}^0$ is defined as:

$$E^w(\boldsymbol{y}^0, -\boldsymbol{x}^0) = \sigma^s,$$

where σ^s is the value of σ at the optimal solution of problem (3.2).

Note that $\sigma^s > 0$ means that the point $\boldsymbol{u}^0$ is V*alue Inefficient*. It is also V*alue Inefficient*, if $\sigma^s = 0$, and $\boldsymbol{1}^T(\boldsymbol{s}^+ + \boldsymbol{s}^-) > 0$; otherwise $\sigma^s = 0$ means that the point is diagnosed V*alue Efficient*. However, the point is not necessarily **truly** V*alue Efficient*. Formulation (3.2) only guarantees that we use the largest possible set guaranteed not to include Value Efficient points except $\boldsymbol{u}^*$ to diagnose Value Inefficiency, but it is not the set consisting of all value inefficient points. If $\sigma^s < 0$, the point is diagnosed V*alue "Superefficient"*. In that case it does not belong to the original set of given units.

Remark. It is important to note that the Most Preferred Solution $\boldsymbol{u}^* = \begin{bmatrix} \boldsymbol{y}^* \\ -\boldsymbol{x}^* \end{bmatrix}$ was assumed efficient in T. Then $D_{u*} = \{\boldsymbol{u} \mid \boldsymbol{u} = \boldsymbol{u}^* + \boldsymbol{w}, \boldsymbol{w} > \boldsymbol{0}\}$ is separated from T and also from W_{u*}. As discussed in the proof of Lemma 1, for any finite $\boldsymbol{w} > 0$ and $\boldsymbol{u}$ there is a finite σ so that $\boldsymbol{u} + \sigma\boldsymbol{w} > \boldsymbol{u}^* \Rightarrow \boldsymbol{u} + \sigma\boldsymbol{w} \in D_{u*} \Rightarrow \boldsymbol{u} + \sigma\boldsymbol{w} \notin W_{u*}$ and a finite σ so that $\boldsymbol{u} + \sigma\boldsymbol{w} < \boldsymbol{u}^* \Rightarrow \boldsymbol{u} + \sigma\boldsymbol{w} \in W_{u*}$. Therefore for an efficient MPS the solution of (3.2) is guaranteed to be bounded. If $\boldsymbol{u}^*$ is not efficient, e.g. an interior point of T, in which case $W_{u*} = \Re^{m+p}$ and (3.2) is guaranteed to be unbounded.

4. An Illustrative Example

We illustrate our Value Efficiency model with a simple example. Throughout the example we use the generalization (2.6a) of the traditional BCC model (combined BCC model). In terms of formulas (2.6a) and (3.2) this means that the constraint $\boldsymbol{A}\lambda \leq \boldsymbol{b}$ becomes $\boldsymbol{1}^T\lambda = 1$; we also set $\boldsymbol{w}^x = \boldsymbol{g}^x = \boldsymbol{x}_0$ and $\boldsymbol{w}^y = \boldsymbol{g}^y = \boldsymbol{y}_0$. Note that in traditional DEA analysis efficient DMUs receive a score of 1. In combined models where both inputs and outputs are treated simultaneously, efficient DMUs receive a score of 0, and inefficient units a positive score. This is because the interpretation of the score in combined models -- when weights are set as above -- is the percentage by which the inefficient units should simultaneously increase their outputs and decrease their inputs to become efficient.

Assume that there are six DMUs, each requiring one input and producing one output. See Table 4.1 and Figures 4.2a, 4.2b and 4.2c.

TABLE 4.1: A Simple Example

	DMU_1	DMU_2	DMU_3	DMU_4	DMU_5	DMU_6
Output	1	4	7	9	12	8
Input	3	3	5	7	11	10

As discussed in the body of this paper, the DM's preferences are incorporated in the efficiency analysis via his/her Most Preferred Solution. Hence we must first identify the DM's Most Preferred Solution over the set consisting of all convex combinations of existing DMUs. We begin by formulating model (2.4) as a bi-criteria problem, where we wish to maximize the output and minimize the input:

$$\begin{aligned} &max \quad \lambda_1 + 4\lambda_2 + 7\lambda_3 + 9\lambda_4 + 12\lambda_5 + 8\lambda_6 \\ &min \quad 3\lambda_1 + 3\lambda_2 + 5\lambda_3 + 7\lambda_4 + 11\lambda_5 + 10\lambda_6 \\ &s.t. \quad \mathbf{1}^T\lambda = 1, \\ &\qquad \lambda \geq \mathbf{0}. \end{aligned}$$

Several Multiple Objective Linear Programming methods can be used to solve the above model. In fact, a two-criteria problem is so trivial that a good visual representation of the points in the input-output scape is sufficient for enabling the DM to locate the MPS. However, we illustrate a technique suitable for solving more general multiple criteria problems.

Reflecting our own bias, we have used the VIG software to perform the search for the Most Preferred Solution. VIG implements Pareto Race, a dynamic and visual "free-search" type of interactive procedure for Multiple Objective Linear Programming. It enables a DM to freely search any part of the efficient frontier by controlling the speed and direction of motion. The objective function values are represented in numeric form and as bar graphs on the computer screen. The theoretical foundations of Pareto Race are based on the reference direction approach developed by Korhonen and Laakso [1986]. In the reference direction approach, any direction specified by the DM is projected onto the efficient frontier. Pareto Race is the implementation of the dynamic version of the reference direction approach as proposed by Korhonen and Wallenius [1988]. In Pareto Race, a reference direction is determined by the system on the basis of preference information received from the DM. By pressing number keys corresponding to the ordinal numbers of the objectives, the DM expresses which objectives he/she would like to improve and how strongly. In this way he/she implicitly specifies a reference direction. Figure 4.1 shows the Pareto Race interface for the search, embedded in the VIG software (Korhonen and Wallenius 1988).

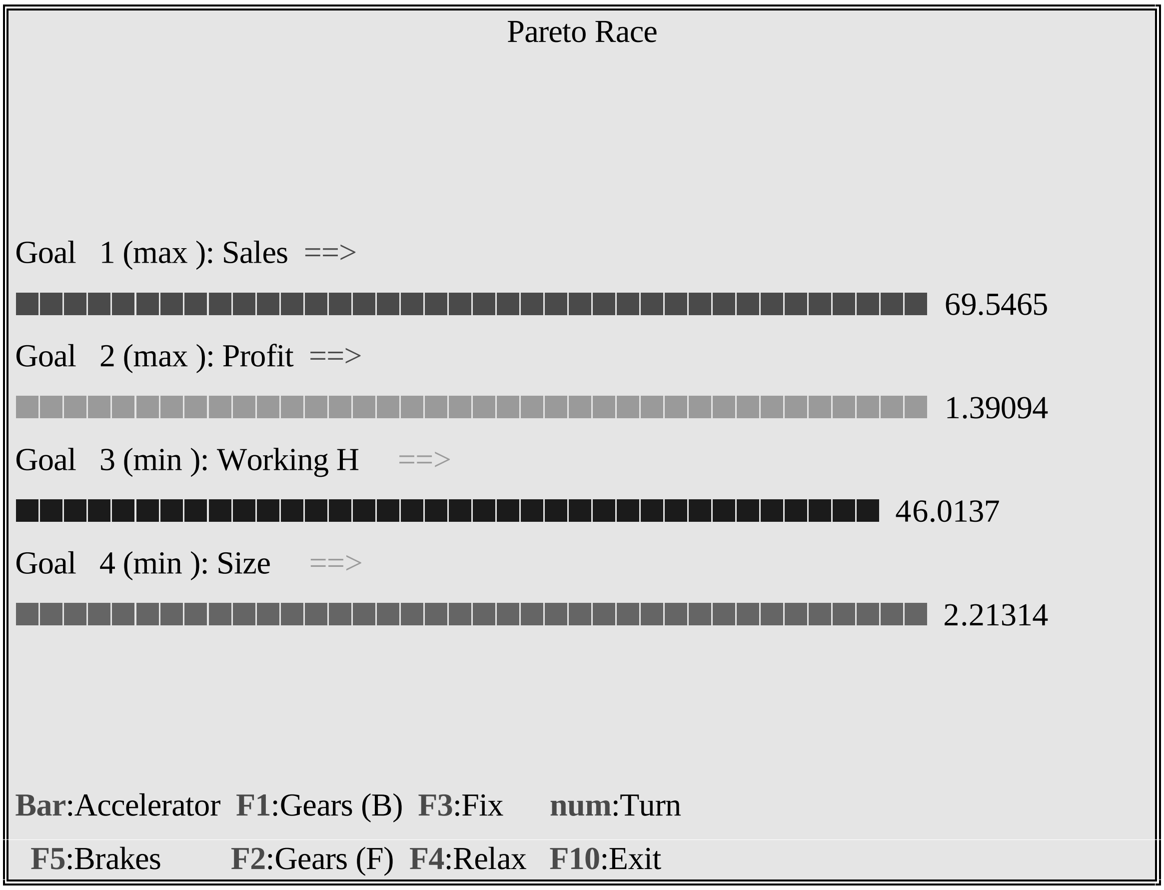

Figure 4.1: Pareto Race Screen

For illustrative purposes let us assume that the DM chooses the input/output vector (4, 5.5) shown on the screen in Figure 4.1 to be his/her Most Preferred Solution. This situation corresponds to Figure 4.2a. The Value Efficiency scores are calculated with formula (3.2). For example, for DMU_5 the formulation is:

$$\begin{aligned}
&\max\ \sigma + \varepsilon(s^+ + s^-)\\
&\text{s. t.}\quad \lambda_1 + 4\lambda_2 + 7\lambda_3 + 9\lambda_4 + 12\lambda_5 + 8\lambda_6 - 12\sigma - s^+ = 12,\\
&\qquad 3\lambda_1 + 3\lambda_2 + 5\lambda_3 + 7\lambda_4 + 11\lambda_5 + 10\lambda_6 + 11\sigma + s^- = 11,\\
&\qquad \mathbf{1}^T\lambda = 1,\\
&\qquad \lambda_1, \lambda_4, \lambda_5, \lambda_6 \geq 0,\\
&\qquad \lambda_2, \lambda_3 \text{ unrestricted.}
\end{aligned}$$

It is important to note that the nonnegativity constraints for λ_2 and λ_3 have been relaxed. This is because the chosen Most Preferred Solution can be presented as a convex combination of them, i.e. their values in the solution of model (2.4) are strictly positive. All other λs are equal to zero and they need the nonnegativity constraints in the above formulation. The corresponding tangent cone is illustrated in Figure 4.2a.

The optimal solution to the above problem produces $\sigma = 0.14$, hence DMU_5 is Value Inefficient. So are DMUs 1, 4, and 6. The MPS being (4, 5.5), only DMUs 2 and 3 are

Value Efficient. Note that all units that are inefficient in classical DEA analysis are also Value Inefficient, but efficient units can be either Value Inefficient or Value Efficient. See Table 4.2.

Let us also consider a situation where the MPS corresponds to an existing DMU. Assume the DM chooses his/her MPS to be (5, 7) (Figure 4.2b). Since the MPS coincides with an existing unit DMU_3, it can be represented by that unit solely. Hence in this case the only strictly positive λ-variable in model (2.4) is λ_3, and we relax its nonnegativity constraint in model (3.2). Now for DMU_5 the optimization produces $\sigma = 0.04$. It is still Value Inefficient, but receives a better score than with the previous MPS. The corresponding tangent cone is illustrated in Figure 4.2b. In this case DMUs 2, 3 and 4 are approximated to be Value Efficient and DMUs 1, 5, and 6 Value Inefficient. In Figure 4.2c DMU_2 is the MPS. That case illustrates the situation in which the DMU_1 is value inefficient, although $\sigma = 0$, since one slack is positive. See Table 4.2.

TABLE 4.2: Results of Efficiency Analysis with the BCC Model and Three Cases of the Value Efficiency Model

								Slacks		**Efficiency Score**
	Ineff. Units	**DMU_1 λ_1**	**DMU_2 λ_2**	**DMU_3 λ_3**	**DMU_4 λ_4**	**DMU_5 λ_5**	**DMU_6 λ_6**	**s+**	**s-**	**σ**
BCC-Model (Combined Score)										
	DMU_1		1					3		0
	DMU_6				0.77	0.23				0.21
Value Efficiency Model with MPS_1										
	DMU_1		1.82	-0.82						0.55
	DMU_4		-0.82	1.82						0.05
	DMU_5		-2.23	3.23						0.14
	DMU_6		-1.09	2.09						0.28
Value Efficiency Model with MPS_2										
	DMU_1		1.82	-0.82						0.55
	DMU_5			-1.76	2.76					0.04
	DMU_6			-0.39	1.39					0.22
Value Efficiency Model with MPS_3										
	DMU_1		1					3		0
	DMU_4		-0.82	1.82						0.05
	DMU_5		-2.23	3.23						0.14
	DMU_6		-1.09	2.09						0.28
MPS_1, MPS_2, and MPS_3 refer to three different most preferred solutions in Figures 4.2a, b, and c. Note: Combined Score refers to the model that treats inputs and outputs simultaneously. DMUs having an efficiency score equal to 0 are efficient. DMUs having a positive efficiency score are inefficient.										

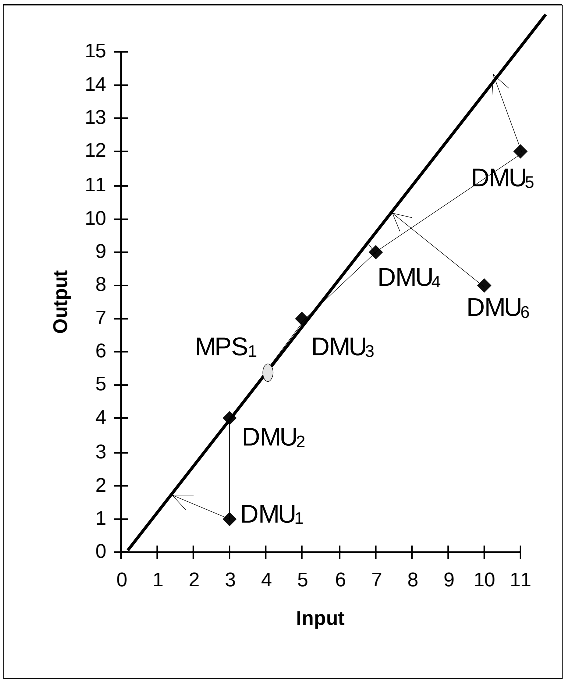

Figure 4.2a

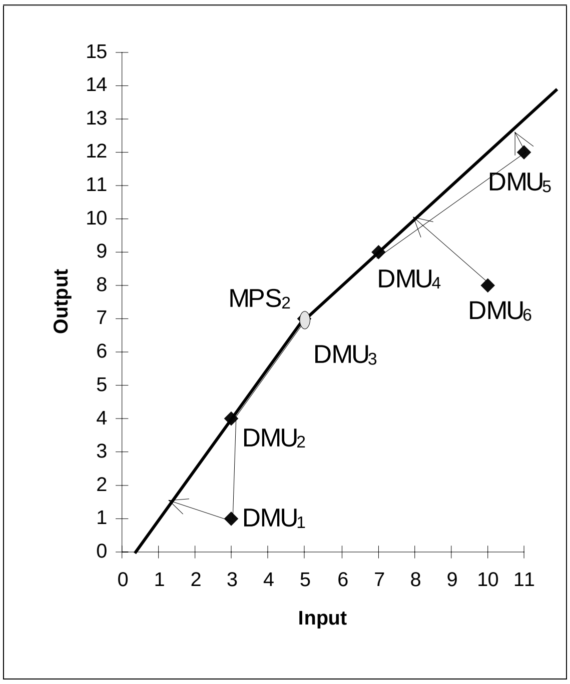

Figure 4.2b

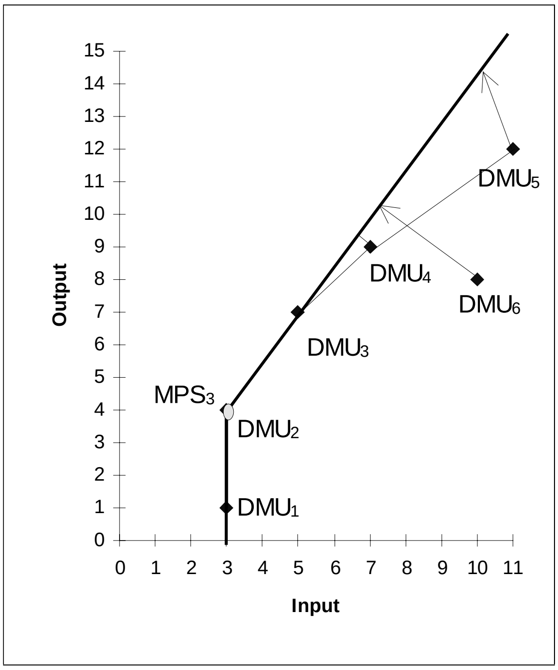

Figure 4.2c

Figures 4.2a, 4.2b and 4.2c: Three Cases Illustrating the Determining of Value Efficiency

5. Value Efficiency Analysis with Real Data

In this section we perform our Value Efficiency analysis with real-world data to demonstrate its use in a realistic setting. The data is from Charnes, Cooper and Li [1989], where they evaluated the efficiency in the economic performance of 28 key Chinese cities during 1983 and 1984. The cities played a critical role in the government's program of economic development. In total, 3 inputs and 3 outputs were used. We reproduce the data from Charnes et al. [1989] in Table 5.1.

TABLE 5.1. Outputs and Inputs of Key Chinese Cities: 1983 Statistics

DMUs	Outputs			Inputs		
(Cities)	GIOV	P&T	RS	LABOR	WF	INV
DMU_1	6785798	1594957	1088699	483.01	1397736	616961
DMU_2	2505984	545140	835745	371.95	355509	385453
DMU_3	2292025	406947	473600	268.23	685584	341941
DMU_4	1158016	135939	336165	202.02	452713	117424
DMU_5	1244124	204909	317709	197.93	471650	112634
DMU_6	1187130	190178	605037	178.96	423124	189743
DMU_7	658910	86514	239760	148.04	367012	97004
DMU_8	993238	1411954	353896	184.93	408311	111904
DMU_9	854188	135327	239360	123.33	245542	91861
DMU_{10}	606743	78357	208188	116.91	305316	91710
DMU_{11}	736545	114365	298112	129.62	295812	92409
DMU_{12}	454684	67154	233733	106.26	198703	53499
DMU_{13}	494196	78992	118553	89.70	210891	95642
DMU_{14}	842854	149186	243361	109.26	282209	84202
DMU_{15}	776285	116974	234875	85.50	184992	49357
DMU_{16}	490998	117854	118924	72.17	222327	73907
DMU_{17}	482448	67857	158250	76.18	161159	47977
DMU_{18}	515237	114883	101231	73.21	144163	43312
DMU_{19}	625514	173099	130423	86.72	190043	55326
DMU_{20}	382880	74126	123968	69.09	158436	66640
DMU_{21}	867467	65229	262876	77.69	135046	46198
DMU_{22}	830142	128279	242773	97.42	206926	66120
DMU_{23}	521684	37245	184055	54.96	79563	43192
DMU_{24}	869973	86859	194416	67.00	144092	43350
DMU_{25}	604715	55989	127586	46.30	100431	31428
DMU_{26}	601299	37088	224855	65.12	96873	28112
DMU_{27}	145792	11816	24442	20.09	50717	54650
DMU_{28}	319218	31726	169051	69.81	117790	30976

Outputs:	Inputs:
Gross Industrial Output Value (GIOV)	Labor (10,000 persons)
Profits and Taxes (P&T)	Working Funds (WF)
Retail Sales (RS)	Investments (INV)
Note: GIOV, P&T, RS are measured in 10,000 rmb, the Chinese currency	Note: WF and INV are measured in 10,000 rmb, the Chinese currency

In Table 5.2 we have produced output-oriented BCC and CCR efficiency scores, and the corresponding Value Efficiency scores. We have *assumed* DMU_{27} to be the Most Preferred Solution (MPS). For locating the MPS, a Pareto Race model with 6 columns and 28 rows (plus one row for the BCC constraint) is formulated. The search for the MPS is quite straightforward. Since DMU_{27} is BCC efficient, it is a valid MPS for BCC Value Efficiency analysis. When computing the BCC Value Efficiency scores, the lambda variable corresponding to DMU_{27} has been defined free.

TABLE 5.2. Results of the Analysis

OUTPUT ORIENTED MODEL				
	BCC MODEL		CCR MODEL	
DMU	DEA Efficiency Score	Value Efficiency Score	DEA Efficiency Score	Value Efficiency Score
1	1.00	1.00	1.00	1.00
2	1.00	0.87	1.00	0.69
3	0.70	0.58	0.66	0.63
4	0.85	0.44	0.52	0.45
5	0.86	0.46	0.58	0.49
6	1.00	0.84	1.00	0.55
7	0.64	0.42	0.49	0.35
8	1.00	1.00	1.00	0.60
9	0.70	0.56	0.63	0.56
10	0.58	0.48	0.54	0.40
11	0.80	0.60	0.69	0.46
12	0.82	0.58	0.65	0.37
13	0.46	0.42	0.45	0.42
14	0.74	0.66	0.72	0.59
15	0.89	0.81	0.87	0.72
16	0.61	0.59	0.60	0.50
17	0.64	0.61	0.64	0.51
18	0.67	0.62	0.67	0.58
19	0.68	0.61	0.66	0.59
20	0.58	0.57	0.57	0.44
21	1.00	0.96	1.00	0.92
22	0.82	0.73	0.79	0.67
23	1.00	1.00	1.00	0.83
24	1.00	0.93	1.00	1.00
25	1.00	1.00	1.00	1.00
26	1.00	1.00	1.00	0.80
27	1.00	1.00	0.54	0.54
28	0.73	0.68	0.70	0.40

DMU_{27} , however, is CCR inefficient. In the CCR Value Efficiency Analysis we have used a linear combination of DMU_1 and DMU_{25} (the reference set of DMU_{27}) as MPS. To compute the Value Efficiency scores in the CCR model, we have defined free the lambda variables corresponding to DMU_1 and DMU_{25}. (Another possibilty would be to build a Pareto Race model with CCR assumptions and locate a new MPS.)

In both CCR and BCC models, when VEA rather than DEA is used the number of efficient DMUs is reduced. Some DMUs received a VEA score that was considerably lower than the corresponding DEA score, like DMUs 4, 5, 7, 11 and 12 in the BCC model, and DMUs 2, 6, 8, 11, 12, 26 and 28 in the CCR model. This is obviously because such DMUs lie on a different facet than the MPS.

Given the MPS, the Value Efficiency Analysis for the above problem was straightforward, although the data was demanding from a computational point of view apparently due to the different scales used in measuring inputs and outputs. It took a few hours to perform the calculations and analysis required for the Value Efficiency Analysis. This is because the VEA calculations were performed with LP software. However, a customized software that supports VEA is being developed. We emphasize that VEA calculations are no more demanding than DEA calculations. Original DEA is beset with the same difficulties as VEA.

6. Conclusions

Increasing competition and tightening government budgets in many countries are forcing private and public sector organizations to closely analyze their performance. Data Envelopment Analysis is an excellent tool for performance evaluation, but it has suffered from the difficulty of incorporating DM's preference information in the analysis. In this paper we have developed an operational procedure and the requisite theory for incorporating DM's preference information into DEA type efficiency analysis. Due to the well-known difficulties associated with the elicitation and use of importance weights for inputs and outputs, we have taken a different route. We model the DM's preferences via his/her Most Preferred Solution. Briefly, the DM is first supported by an interactive procedure in the search for the best input/output vector. Such a vector is a convex combination of the input/output vectors of the DMUs under consideration. Note that sensitivity analysis with respect to the choice of the MPS should be performed in each analysis. The DM is assumed to have a pseudoconcave value function at the moment he/she terminates the search, enabling us to use a linear approximation of the indifference contour of the value function at his/her Most Preferred Solution. When the linear approximation is not uniquely defined, our approximation will produce, in the spirit of DEA, the most optimistic efficiency score for each DMU. The formulation to calculate efficiency scores for each DMU, incorporating DM's preference information, reduces to a straightforward application of linear programming. Our efficiency scores can be interpreted as the relative difference in value between the Most Preferred Solution and the unit under investigation. It is most closely related to measuring classical overall efficiency. The model is immediately applicable and easily implemented for solving practical problems. Possible application areas in the private and public sectors are numerous.

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