



# Models for Identifying Target Units in Data Envelopment Analysis: Comparison and Extension

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# **Models for Identifying Target Units in Data Envelopment Analysis: Comparison and Extension**

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## **Abstract**

In addition to the efficiency scores, Data Envelopment Analysis also provides the inefficient DMUs with information on reference units which lie on the efficient frontier. Reference units are traditionally found by projecting the inefficient DMUs radially to the efficient surface. However, there may exist some other efficient solutions, which are better targets. This means that it is possible to find an input-output mix that is more suitable for example from a managerial point of view.

The purpose of this paper is to compare and contrast existing models for searching the most preferred reference units as targets for inefficient units, and based on that to present a dynamic and interactive way to provide these targets in Data Envelopment Analysis based on Multiple Objective Linear Programming.

**Keywords:** Data Envelopment Analysis, Multiple Objective Linear Programming, Targets, Preferences

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# Models for Identifying Target Units in Data Envelopment Analysis: Comparison and Extension

Tarja Joro

## 1. Introduction

Data Envelopment Analysis (DEA) by Charnes, Cooper and Rhodes [1978] is a method for evaluating the *relative efficiency* of comparable entities referred to as Decision Making Units (DMU). DEA forms a *production possibility frontier*, or an *efficient surface*. If a DMU lies on the surface, i.e. there is no other DMU that can either produce the same outputs by consuming less inputs (input oriented DEA) or produce more outputs by consuming the same amount of inputs (output oriented DEA), it is referred to as an efficient unit, otherwise inefficient. DEA also provides efficiency scores and reference units for inefficient DMUs. The efficiency score tells the percentage by which a DMU should decrease its inputs (input oriented DEA) or increase its outputs (output oriented DEA) in order to become efficient. Reference units are hypothetical units on the efficient surface which can be regarded as target units for inefficient units. They are in the traditional DEA obtained by projecting an inefficient DMU radially to the efficient surface. The production theoretical argument for this principle is that the DMU preserves its current input-output mix.

However, from a managerial point of view it is possible that some other solution on the efficient surface might be a more preferable target, i.e. there exists an input-output mix that is more suitable for the inefficient unit than the one obtained through radial projection. One line of research in DEA concentrates on finding these targets for inefficient DMUs (Golany [1988], Thanassoulis and Dyson [1992] and Zhu [1996]). The DMU can use the targets as goals or benchmarking units when working its way toward efficiency.

In Golany [1988] a set of hypothetical reference units is generated and presented to the DMU. The DMU may choose one of them to be a target, or new reference units may be generated. In Thanassoulis and Dyson [1992] the DMU may articulate its preferences as a set of preference weights over improvements for different input-output levels, or as an ideal target (not necessarily optimal nor feasible). The target corresponding to the preference weights or the ideal target is then calculated. In the model by Zhu [1996], DMU articulates its preferences as weights reflecting the relative degree of desirability of the potential adjustments of current input or output levels.



In Golany [1988] and in the preference weight model by Thanassoulis and Dyson [1992] it is possible to choose a target from among the reference units that dominate the unit under consideration. In these cases some of the output (or input) levels of the target are better while no input (or output) levels are worse. In Zhu [1996] and in the ideal target model by Thanassoulis and Dyson [1992] it is possible to choose also a target not dominating the unit under consideration. This means that in the target either the usage of some inputs is bigger, or the production of some outputs is smaller, and thus the input-output mix is to be reorganized to a greater extent than when choosing a dominating target.

In this paper we present an approach that enables the search of the target in a dynamic and interactive way with existing software. The approach is based on methods developed for Multiple Objective Linear Programming (MOLP). There are several suitable methods (see Steuer [1986] for an excellent survey), here we use the Reference Point approach (Wierzbicki [1986]), Reference Direction approach (Korhonen and Laakso [1986]) and its Dynamic Extension, Pareto Race (Korhonen and Wallenius [1988]). In Pareto Race it is possible to move along the efficient surface and scan all hypothetical and existing efficient reference units.

The rest of the paper is organized as follows. Section 2 reviews the basic DEA models. Section 3 reviews the existing approaches to finding targets, and the 4th section introduces our approach. The 5th section concludes the paper.

## 2. Basic DEA Models

Assume we have  $n$  DMUs each consuming  $m$  inputs and producing  $p$  outputs. Let  $\mathbf{X} \in \mathfrak{R}_+^{m \times n}$  and  $\mathbf{Y} \in \mathfrak{R}_+^{p \times n}$  be the matrices, consisting of nonnegative elements, containing the observed input and output measures for the DMUs. We further assume that there are no duplicated units in the data set. We denote by  $\mathbf{x}_j$  (the  $j$ th column of  $\mathbf{X}$ ) the vector of inputs consumed by DMU $j$ , and by  $x_{ij}$  the quantity of input  $i$  consumed by DMU $j$ . A similar notation is used for outputs. Furthermore, we denote  $\mathbf{1} = [1, \dots, 1]^T$ . The subscript zero refers to the unit under consideration. We use here output oriented CCR formulations with constant returns to scale (Charnes, Cooper and Rhodes [1978]) and output oriented BCC formulations with variable returns to scale (Banker, Charnes and Cooper [1984]):

Output-Oriented CCR Primal (CCR <sub>p</sub> - O)	Output-Oriented BCC Primal (BCC <sub>p</sub> - O)
$\max Z_C = \theta + \varepsilon(I^T s^+ + I^T s^-)$ <p>s.t. (2.1a)</p> $\mathbf{Y}\lambda - \theta \mathbf{y}_0 - s^+ = \mathbf{0}$ $\mathbf{X}\lambda + s^- = \mathbf{x}_0$ $\lambda, s^-, s^+ \geq \mathbf{0}$ $\varepsilon > 0$	$\max Z_B = \theta + \varepsilon(I^T s^+ + I^T s^-)$ <p>s.t. (2.1b)</p> $\mathbf{Y}\lambda - \theta \mathbf{y}_0 - s^+ = \mathbf{0}$ $\mathbf{X}\lambda + s^- = \mathbf{x}_0$ $I^T \lambda = 1$ $\lambda, s^-, s^+ \geq \mathbf{0}$ $\varepsilon > 0$

A DMU is efficient iff  $Z_C^* = Z_B^* = 1$  and all slack variables  $s^-, s^+$  equal zero; otherwise it is inefficient (Charnes et al. 1994).

For the purposes of the approaches presented later in this paper, we introduce a combined model that simultaneously minimizes inputs and maximizes outputs. In DEA, the efficiency of a DMU is traditionally determined either maximizing outputs subject to given input levels or minimizing inputs subject to given output levels. The Additive model which was the first model considering simultaneously both input minimization and output maximization was introduced as early as 1985 (Charnes, Cooper, Golany, Seiford, and Stutz [1985]). Other models considering simultaneous input minimization and output maximization exist (see, for example, Warwick DEA-User Manual, Thanassoulis and Dyson [1992], Zhu [1996], Joro, Korhonen and Wallenius [1998] and Halme, Joro, Korhonen, Salo and Wallenius [1998]).

Combined CCR Primal (CCR <sub>p</sub> - C)	Combined BCC Primal (BCC <sub>p</sub> - C)
$\max F_C = \theta + \varepsilon(I^T s^+ + I^T s^-)$ <p>s.t. (2.2a)</p> $\mathbf{Y}\lambda - \theta \mathbf{y}_0 - s^+ = \mathbf{y}_0$ $\mathbf{X}\lambda + \theta \mathbf{x}_0 + s^- = \mathbf{x}_0$ $\lambda, s^-, s^+ \geq \mathbf{0}$ $\varepsilon > 0$	$\max F_B = \theta + \varepsilon(I^T s^+ + I^T s^-)$ <p>s.t. (2.2a)</p> $\mathbf{Y}\lambda - \theta \mathbf{y}_0 - s^+ = \mathbf{y}_0$ $\mathbf{X}\lambda + \theta \mathbf{x}_0 + s^- = \mathbf{x}_0$ $I^T \lambda = 1$ $\lambda, s^-, s^+ \geq \mathbf{0}$ $\varepsilon > 0$

Define the set

$$\Lambda = \begin{cases} \{\lambda \mid \lambda \in \mathfrak{R}_+^n\}, & \text{for a CCR model} \\ \{\lambda \mid \lambda \in \mathfrak{R}_+^n \text{ and } I^T \lambda = 1\}, & \text{for a BCC model} \end{cases}$$

and  $T = \{ (y, x) \mid y = Y\lambda, x = X\lambda, \lambda \in \Lambda \}$ . All efficient DMUs lie on the efficient frontier, which is defined as a subset of points of set  $T$  satisfying the efficiency condition below.

**Definition 1.** A solution  $(Y\lambda^*, X\lambda^*) = (y^*, x^*)$  of problem (2.2) is *efficient* iff there does not exist another  $(y, x) \in T$  such that  $y \geq y^*, x \leq x^*$ , and  $(y, x) \neq (y^*, x^*)$ .

**Definition 2.** A point  $(y^*, x^*) \in T$  is *weakly efficient* iff there does not exist another  $(y, x) \in T$  such that  $y > y^*$  and  $x < x^*$ .

The efficiency of DMUs can also be determined using the following Multiple Objective Linear Programming (MOLP) model:

$$\begin{aligned} \max \quad & Y\lambda \\ \min \quad & X\lambda \\ \text{s.t.} \quad & \lambda \in \Lambda. \end{aligned} \tag{2.3}$$

Like any multiple criteria model, model (2.3) has no unique solution. Its solutions are defined analogously to the efficient solutions in DEA. One possible, currently popular way to perform the search for solutions on the efficient frontier of a MOLP-problem is to use *the achievement (scalarizing) function* (ASF) suggested by Wierzbicki [1980]. This leads to the following formulations: (Joro et al. [1998]).

Reference Point Model Primal (REF <sub>p</sub> )	Reference Point Model Dual (REF <sub>d</sub> )
$\begin{aligned} \max \quad & \sigma + \varepsilon(I^T s^+ + I^T s^-) \\ \text{s.t.} \quad & Y\lambda - \sigma w^y - s^+ = g^y \\ & X\lambda + \sigma w^x + s^- = g^x \\ & \lambda \in \Lambda \\ & \lambda, s^-, s^+ \geq 0 \\ & \varepsilon > 0 \end{aligned} \tag{2.4a}$	$\begin{aligned} \min \quad & v^T g^x - \mu g^y + \xi \\ \text{s.t.} \quad & -\mu^T Y + v^T X + \xi I^T \geq 0 \\ & \mu^T w^y + v^T w^x = 1 \\ & \xi = \begin{cases} 0, & \text{in a CCR model} \\ \text{free,} & \text{in a BCC model} \end{cases} \\ & \mu, v \geq \varepsilon I \\ & \varepsilon > 0 \end{aligned} \tag{2.4b}$

Vector  $g^x$  consists of aspiration levels for inputs and  $g^y$  of aspiration levels for outputs. Vectors  $w^x \geq 0$  and  $w^y \geq 0$  are the weighting vectors for inputs and outputs, respectively. (At least one of the weights has to be strictly positive, otherwise the value of  $\sigma$  is unbounded. Without  $\sigma$  at the objective function the projection into the efficient frontier would work even when all the weights are zero.)

The Reference Point model allows us also to perform DEA analysis in which both inputs and outputs are treated simultaneously. This kind of *combined* DEA formulation is obtained from (2.4 a and b) by setting  $w^x = g^x = x_0$  and  $w^y = g^y = y_0$ . In combined

models efficient DMUs receive an efficiency score of 0: the interpretation of an efficiency score is the percentage by which the inputs should be decreased and outputs increased to obtain efficiency. By modifying the constraint  $\mathbf{A}\lambda \leq \mathbf{b}$  it is possible to obtain the CCR model, the BCC model, the model with nonincreasing returns to scale, etc.

### 3. Literature Review on Target Models

Target models aim at assisting the inefficient DMU to find the most suitable target for itself. Such models seek to project the inefficient DMU onto the efficient frontier. The first approach that concentrates on producing preferred targets is IMOLP, Interactive MOLP procedure by Golany [1988]. In this approach the DMU is presented with  $k-1$  optimal targets. The DMU may use some of them as the most preferred target, or indicate which of them is the best, and proceed with the search. In Golany's model DMU may choose the preferred targets for outputs, the inputs are assumed to be the current inputs consumed by the DMU. In this paper we concentrate on models in which the DMU may choose both target inputs and outputs.

Thanassoulis and Dyson [1992] and Zhu [1996] have introduced models for setting targets in different situations. Thanassoulis and Dyson present target setting approaches based on a general weight-based preference structure, and on an ideal target for situations where some input or output is given a pre-emptive right to improve and for situations where some input or output levels are exogenously fixed. The approach presented by Zhu is based on a general weight-based preference structure. The notation used in this section is unified to correspond to the notation used in section 2 to make the comparison easier. We present here the CCR models, but the formulations can be used as BCC models by adding the  $\lambda$ -restriction.

#### 3.1. Weight-based general preference structure by Thanassoulis and Dyson

In this model the inputs and outputs are divided into two categories: those that the DMU wishes to improve, and those that it wishes to maintain at their current levels. The  $\mathbf{Y}$  and  $\mathbf{X}$  matrices containing the input and output information of different DMUs are correspondingly divided into two matrices:  $\mathbf{Y}_I$  and  $\mathbf{X}_I$  contain the information on the inputs and outputs to be improved, and  $\mathbf{Y}_C$  and  $\mathbf{X}_C$  the information on those to be maintained at the current level. Vectors  $\mathbf{y}_{0I}$ ,  $\mathbf{x}_{0I}$ ,  $\mathbf{y}_{0C}$  and  $\mathbf{x}_{0C}$  contain the corresponding information on the DMU under consideration.  $\mathbf{w}^x \geq \mathbf{0}$  and  $\mathbf{w}^y \geq \mathbf{0}$  (at least one of the weights has to be strictly positive) are user-specified weighting vectors attached to coefficient vectors  $\mathbf{z}$  and  $\mathbf{p}$ . Here  $\mathbf{z}$  refers to the vector of factors by which the different outputs are increased and  $\mathbf{p}$  to the vector of proportions to which the inputs are decreased.

By varying the weights  $\mathbf{w}^x$  and  $\mathbf{w}^y$  the DMU may search the efficient surface in the input-output space. As a result the DMU obtains coefficient vectors  $\mathbf{p}$  and  $\mathbf{z}$  that define the target corresponding to the preference weights used. By multiplying the current

components of inputs and output vectors with the components of vectors  $\mathbf{p}$  and  $\mathbf{z}$  the DMU obtains the target solution.

Mathematically the formulation is the following:

$$\begin{aligned}
\max \quad & \mathbf{w}^y \mathbf{z} - \mathbf{w}^x \mathbf{p} + \varepsilon (\mathbf{I}^T \mathbf{s}^+ + \mathbf{I}^T \mathbf{s}^-) & (3.1) \\
\text{s.t.} \quad & \mathbf{Y}_{0i} \mathbf{z} - \mathbf{Y}_i \boldsymbol{\lambda} & = \mathbf{0} \\
& \mathbf{Y}_c \boldsymbol{\lambda} - \mathbf{s}^+ & = \mathbf{y}_{0c} \\
& \mathbf{X}_{0i} \mathbf{p} - \mathbf{X}_i \boldsymbol{\lambda} & = \mathbf{0} \\
& \mathbf{X}_c \boldsymbol{\lambda} + \mathbf{s}^- & = \mathbf{x}_{0c} \\
& \mathbf{z} \geq \mathbf{1} \\
& \mathbf{p} \leq \mathbf{1} \\
& \mathbf{Y}_{0i} = \text{diag}(\mathbf{y}_{0i})^1 \\
& \mathbf{X}_{0i} = \text{diag}(\mathbf{x}_{0i}) \\
& \boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0} \\
& \varepsilon > 0
\end{aligned}$$

We modify the formulation to make it easier to compare it to the original DEA model and to the Reference Point model. We add slack variables to all equations, and substitute each component of vector  $\mathbf{z}$  with the corresponding component of vector  $(\mathbf{I} + \boldsymbol{\sigma}^y)$  and each component of  $\mathbf{p}$  with components of  $(\mathbf{I} - \boldsymbol{\sigma}^x)$ . Thus we obtain:

$$\begin{aligned}
\max \quad & \mathbf{w}^y \boldsymbol{\sigma}^y + \mathbf{w}^x \boldsymbol{\sigma}^x + \varepsilon (\mathbf{I}^T \mathbf{s}^+ + \mathbf{I}^T \mathbf{s}^-)^2 & (3.2) \\
\text{s.t.} \quad & \mathbf{Y}_i \boldsymbol{\lambda} - \mathbf{Y}_{0i} \boldsymbol{\sigma}^y - \mathbf{s}_i^+ & = \mathbf{y}_{0i} \\
& \mathbf{Y}_c \boldsymbol{\lambda} - \mathbf{s}_c^+ & = \mathbf{y}_{0c} \\
& \mathbf{X}_i \boldsymbol{\lambda} + \mathbf{X}_{0i} \boldsymbol{\sigma}^x + \mathbf{s}_i^- & = \mathbf{x}_{0i} \\
& \mathbf{X}_c \boldsymbol{\lambda} + \mathbf{s}_c^- & = \mathbf{x}_{0c} \\
& \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^x \geq \mathbf{0} \\
& \mathbf{Y}_{0i} = \text{diag}(\mathbf{y}_{0i}) \\
& \mathbf{X}_{0i} = \text{diag}(\mathbf{x}_{0i}) \\
& \boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0} \\
& \varepsilon > 0
\end{aligned}$$

where  $\mathbf{s}^+ = \begin{bmatrix} \mathbf{s}_i^+ \\ \mathbf{s}_c^+ \end{bmatrix}$  and  $\mathbf{s}^- = \begin{bmatrix} \mathbf{s}_i^- \\ \mathbf{s}_c^- \end{bmatrix}$ . Let us assume that the DMU wishes to improve all inputs

and outputs. In this case, the only difference between the above model and the combined DEA formulation is that instead of the scalar  $\sigma$  this model uses vectors  $\boldsymbol{\sigma}^y$  and

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<sup>1</sup> Notation  $\text{diag}(\mathbf{a})$  refers to a diagonal matrix which diagonal elements are the elements of vector  $\mathbf{a}$ , correspondingly.

<sup>2</sup> The constant term  $\mathbf{w}^y \mathbf{I} - \mathbf{w}^x \mathbf{I}$  is omitted from the object function.

$\sigma^x$ , and that the objective function is thus the weighted sum of the elements of these vectors.

Thanassoulis and Dyson have also introduced a model, in which one input or output is given a pre-emptive right to improve. Technically, the formulation is the same as above. The input or output to be improved is included in matrix  $\mathbf{Y}_I$  or  $\mathbf{X}_I$  and all other inputs and outputs are included in matrices  $\mathbf{Y}_C$  and  $\mathbf{X}_C$ . Also the model in which some of the inputs or outputs are exogenously fixed is technically the same, now the fixed inputs and outputs are included in  $\mathbf{Y}_C$  and  $\mathbf{X}_C$ .

### 3.2 Target setting with ideal targets by Thanassoulis and Dyson

The previous model is capable of finding find targets that dominate the current unit. To be able to set also targets that do not dominate the unit, i.e. targets in which some inputs are bigger and some outputs smaller than in the unit under consideration, Thanassoulis and Dyson introduced the following formulation. In this model the DMU in the first stage articulates an ideal target point  $(\mathbf{g}^y, \mathbf{g}^x)$ , not necessarily optimal or feasible. Then the DMU assigns user-specified preference weight vectors  $\mathbf{w}_U^x \geq \mathbf{0}$  and  $\mathbf{w}_O^x \geq \mathbf{0}$  (at least one of the weights has to be strictly positive) for the deviation vectors  $s_U^x, s_O^x$ , (under- and overachievements of inputs) and  $\mathbf{w}_U^y \geq \mathbf{0}$  and  $\mathbf{w}_O^y \geq \mathbf{0}$  for the deviation vectors  $s_U^y$  and  $s_O^y$  (under- and overachievements of outputs). Then the model minimizes the weighted sum of the deviations. The more undesirable the deviation from the ideal level is, the larger weight is attached to it. With the under- and overachievement vectors we can locate a feasible point  $(\mathbf{g}^{yf}, \mathbf{g}^{xf})$  as close as possible to the target point.

Mathematically the model is the following:

$$\begin{aligned}
 \min \quad & \mathbf{w}_U^{xT} s_U^x + \mathbf{w}_{OV}^{xT} s_O^x + \mathbf{w}_U^{yT} s_U^y + \mathbf{w}_O^{yT} s_O^y & (3.3) \\
 \text{s.t.} \quad & \\
 & \mathbf{Y}\lambda + s_U^y - s_O^y = \mathbf{g}^y \\
 & \mathbf{X}\lambda + s_U^x - s_O^x = \mathbf{g}^x \\
 & \lambda, s_U^x, s_O^x, s_U^y, s_O^y, \mathbf{g}^y, \mathbf{g}^x \geq \mathbf{0}
 \end{aligned}$$

This first stage of the model projects the ideal target point  $(\mathbf{g}^y, \mathbf{g}^x)$  to a feasible point  $(\mathbf{g}^{yf}, \mathbf{g}^{xf})$ , where  $\mathbf{g}^{yf} = \mathbf{g}^y + s_O^{y*} - s_U^{y*}$  and  $\mathbf{g}^{xf} = \mathbf{g}^x + s_O^{x*} - s_U^{x*}$  and where  $s_O^{y*}, s_U^{y*}, s_O^{x*}$  and  $s_U^{x*}$  are the solution of the model (3.3). In the second stage the feasible point  $(\mathbf{g}^{yf}, \mathbf{g}^{xf})$  is, in case it is inefficient, projected to an efficient point. The corresponding model is

$$\begin{aligned}
 \max \quad & \mathbf{I}^T s_U + \mathbf{I}^T s_O & (3.4) \\
 \text{s.t.} \quad & \\
 & \mathbf{Y}\lambda - s_O = \mathbf{g}^{yf} \\
 & \mathbf{X}\lambda + s_U = \mathbf{g}^{xf} \\
 & \lambda, s_U, s_O, \mathbf{g}^{yf}, \mathbf{g}^{xf} \geq \mathbf{0}
 \end{aligned}$$

If  $s_U$  and  $s_O$  are zero,  $(\mathbf{g}^{yf}, \mathbf{g}^{xf})$  is efficient. Otherwise the final efficient target is  $(\mathbf{g}^{ye}, \mathbf{g}^{xe})$ , where  $\mathbf{g}^{ye} = \mathbf{g}^{yf} + s_O^*$  and  $\mathbf{g}^{xe} = \mathbf{g}^{xf} - s_U^*$  and where  $s_O^*$  and  $s_U^*$  are the solution of the model (3.4).

Let us assume that the DMU chooses  $\mathbf{g}^y = \mathbf{y}_0$  and  $\mathbf{g}^x = \mathbf{x}_0$ , and wishes to improve all inputs and all outputs. Since this ideal point is an existing DMU, the first stage will give  $(\mathbf{y}_0, \mathbf{x}_0)$  as  $(\mathbf{g}^{yf}, \mathbf{g}^{xf})$ . Thus for the second stage we obtain:

$$\begin{aligned}
 \max \quad & \mathbf{I}^T s_U + \mathbf{I}^T s_O & (3.5) \\
 \text{s.t.} \quad & \\
 & \mathbf{Y}\lambda - s_O = \mathbf{y}_0 \\
 & \mathbf{X}\lambda + s_U = \mathbf{x}_0 \\
 & \lambda, s_U, s_O \geq \mathbf{0}
 \end{aligned}$$

Notice that terms  $s_U, s_O$  from this model and  $\mathbf{y}_0\sigma^y, \mathbf{x}_0\sigma^x$  from the model (3.2) are equal, they both give the distance from point  $(\mathbf{y}_0, \mathbf{x}_0)$  to the final efficient target. Thus model (3.5) can be obtained from model (3.2) by choosing the weighting vectors to be  $\mathbf{w}^x = \mathbf{x}_0$  and  $\mathbf{w}^y = \mathbf{y}_0$ . Since all the inputs and outputs are to be improved in (3.2) all the slack variables are zero.

### 3.3. Weights-based preference structure by Zhu

In this model the DMU first specifies the preference weight vectors  $\mathbf{w}^x \geq \mathbf{0}$  and  $\mathbf{w}^y \geq \mathbf{0}$  reflecting the relative degree of desirability of the potential adjustments of current input and output levels. By varying them the DMU may search the efficient frontier. Like in the weights-based model by Thanassoulis and Dyson, also here DMU gets coefficient vectors  $\mathbf{z}$  and  $\mathbf{p}$  that define the target corresponding to the used preference weights as a result of the model. By multiplying the current inputs and outputs with vectors  $\mathbf{g}$  and  $\mathbf{h}$  the DMU obtains the target solution.

Mathematically the model is the following:

$$\begin{aligned}
 \max \quad & \mathbf{w}^{yT} \mathbf{g} - \mathbf{w}^{xT} \mathbf{h} + \varepsilon (\mathbf{I}^T \mathbf{s}^+ + \mathbf{I}^T \mathbf{s}^-) & (3.6) \\
 \text{s.t.} \quad & \\
 & \mathbf{Y}\lambda - \mathbf{s}^+ = \mathbf{Y}_0 \mathbf{g} \\
 & \mathbf{X}\lambda + \mathbf{s}^- = \mathbf{X}_0 \mathbf{h} \\
 & \mathbf{g} \geq \mathbf{1} \\
 & \mathbf{h} \leq \mathbf{1} \\
 & \mathbf{Y}_0 = \text{diag}(\mathbf{y}_0) \\
 & \mathbf{X}_0 = \text{diag}(\mathbf{x}_0) \\
 & \lambda, \mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0} \\
 & \varepsilon > 0
 \end{aligned}$$

Where  $\mathbf{I}^T \mathbf{w}^y - \mathbf{I}^T \mathbf{w}^x = 1$ . The original model presented by Zhu had the term  $\mathbf{I}^T \mathbf{w}^y - \mathbf{I}^T \mathbf{w}^x$  as the subtractor in the objective function. Again if we substitute components of  $\mathbf{g}$  with the corresponding component of vector  $(\mathbf{I} + \boldsymbol{\sigma}^y)$  and components of  $\mathbf{p}$  with components of  $(\mathbf{I} - \boldsymbol{\sigma}^x)$  we obtain

$$\begin{aligned}
 \max \quad & \mathbf{w}^{yT} \boldsymbol{\sigma}^y + \mathbf{w}^{xT} \boldsymbol{\sigma}^x + \varepsilon (\mathbf{I}^T \mathbf{s}^+ + \mathbf{I}^T \mathbf{s}^-)^3 & (3.7) \\
 \text{s.t.} \quad & \\
 & \mathbf{Y} \boldsymbol{\lambda} - \mathbf{Y}_0 \boldsymbol{\sigma}^y - \mathbf{s}^+ = \mathbf{y}_0 \\
 & \mathbf{X} \boldsymbol{\lambda} + \mathbf{X}_0 \boldsymbol{\sigma}^x + \mathbf{s}^- = \mathbf{x}_0 \\
 & \mathbf{I}^T \mathbf{w}^y - \mathbf{I}^T \mathbf{w}^x = 1 \\
 & \mathbf{Y}_0 = \text{diag}(\mathbf{y}_0) \\
 & \mathbf{X}_0 = \text{diag}(\mathbf{x}_0) \\
 & \boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0} \\
 & \varepsilon > 0
 \end{aligned}$$

The difference between this model and model (3.1) is that here the nonnegativity constraint for  $\boldsymbol{\sigma}^y$  and  $\boldsymbol{\sigma}^x$  has been relaxed, and the model can also give targets that do not dominate the current unit.

### 3.4. Illustrative example

Below the approaches are applied to the following simple example with both CCR and BCC assumptions. In the example, only DMU<sub>3</sub> is CCR efficient, and all DMUs except DMU<sub>6</sub> are BCC efficient.

Unit	Input	Output
DMU1	3	1
DMU2	4	4
DMU3	6	7
DMU4	9	9
DMU5	13	10
DMU6	10	3

<sup>3</sup> Also in this model the constant term  $\mathbf{w}^{yT} \mathbf{I} - \mathbf{w}^{xT} \mathbf{I} (= 1)$  is omitted from the object function.



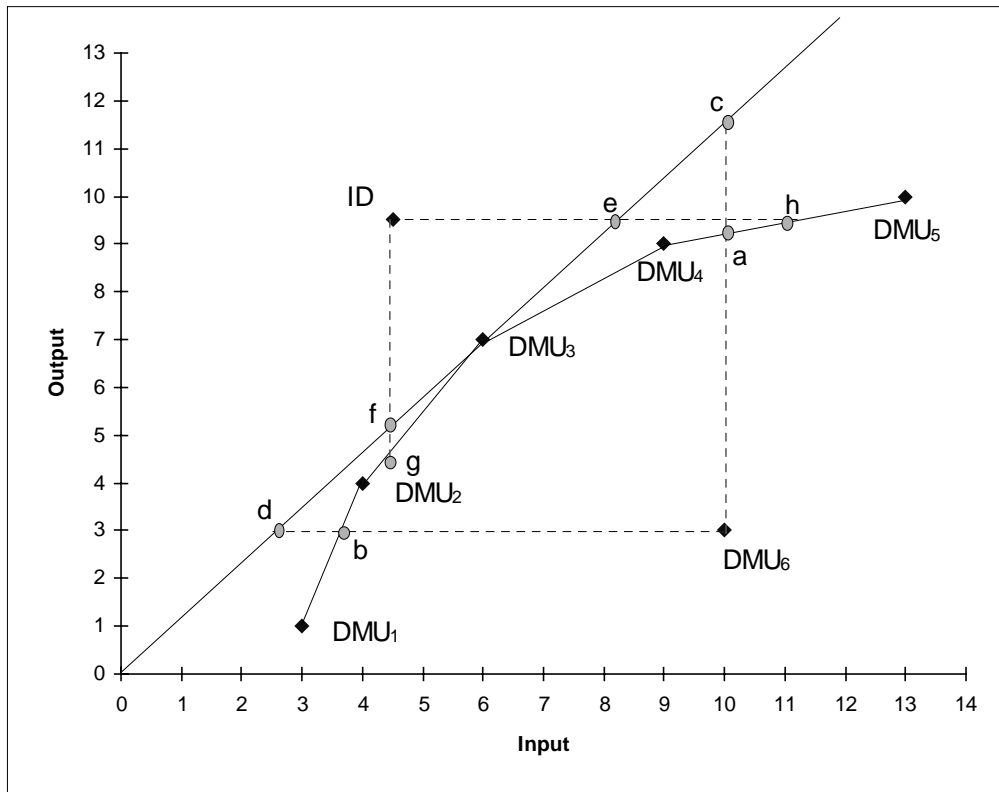


Figure 1

Table 1 and Figure 1 show the targets that different models provide for  $DMU_6$ . TD1 refers to the first and TD 2 to the second model by Thanassoulis and Dyson [1992] In TD2 ID is assumed to be the ideal target  $(g^y, g^x)$ <sup>4</sup>. ZHU refers to the model by Zhu [1996].

**TABLE 1: Targets obtained from different models**

Model	Orientation	Possible targets
TD1	CCR	c, d
	BCC	a, $DMU_4$ , $DMU_3$ , $DMU_2$ , b
TD2	CCR	e, f
	BCC	h, $DMU_4$ , $DMU_3$ , $DMU_2$ , g
ZHU	CCR	origin, infinity
	BCC	all efficient DMUs

Which target is obtained from the possible ones depends on the weights. The obtained targets are either existing units or solutions that lie on the edge of the part of the efficient frontier dominating the target (or being dominated by the target). This means that these models are generally not capable of providing a target that is located in the relative interior of a facet. Theoretically it is possible to obtain degenerated solutions, i.e. to have for example the whole line segment between  $DMU_4$  and  $DMU_3$  as a target. Since with CCR assumptions Zhu's model in this oversimplified example does not have

<sup>4</sup> With  $DMU_6$  as  $(g^y, g^x)$  TD2 would obtain the same targets as TD1.

any finite extreme points other than the origin, the target provided is either the origin or infinity.

If we want to guarantee that in all situations targets other than origin and infinity are received, it is possible to use the following model:

$$\begin{aligned}
\max \quad & \mathbf{w}^{yT} \boldsymbol{\sigma}^y + \mathbf{w}^{xT} \boldsymbol{\sigma}^x + \varepsilon (\mathbf{I}^T \mathbf{s}^+ + \mathbf{I}^T \mathbf{s}^-) & (3.7) \\
\text{s.t.} \quad & \\
& \mathbf{Y}_1 \boldsymbol{\lambda} - \mathbf{Y}_{01} \boldsymbol{\sigma}^y - \mathbf{s}^+ = \mathbf{y}_1^{\min} \\
& \mathbf{Y}_c \boldsymbol{\lambda} \quad \quad - \mathbf{s}^+ = \mathbf{y}_{0c} \\
& \mathbf{X}_1 \boldsymbol{\lambda} + \mathbf{X}_{01} \boldsymbol{\sigma}^x + \mathbf{s}^- = \mathbf{x}_1^{\max} \\
& \mathbf{X}_c \boldsymbol{\lambda} \quad \quad + \mathbf{s}^- = \mathbf{x}_{0c} \\
& \boldsymbol{\sigma}^y, \boldsymbol{\sigma}^x \geq \mathbf{0} \\
& \mathbf{Y}_{01} = \text{diag}(\mathbf{y}_{01}) \\
& \mathbf{X}_{01} = \text{diag}(\mathbf{x}_{01}) \\
& \boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0} \\
& \varepsilon > 0
\end{aligned}$$

Here  $(\mathbf{y}_1^{\min}, \mathbf{x}_1^{\max})$  refers to a vector that contains the minimum values of outputs and maximum values of inputs to be improved from the data set. If all inputs and outputs are to be improved, this model allows us to scan all the extreme solutions dominating  $(\mathbf{y}_1^{\min}, \mathbf{x}_1^{\max})$ . In the BCC model this is not a restriction, the model allows one to scan the whole efficient frontier. In the CCR model it restricts us to the part of the efficient frontier that dominates the data set. It can be argued that it is probable that the realistic targets are to be found from this area.

## 4. Reference Point Approach

In this paper we suggest the use of the Reference Point Approach instead of the weighted sums. The Reference Direction approach uses the *achievement scalarizing function* (ASF) suggested by Wierzbicki [1980]. By ASF it is possible to obtain also solutions or targets that are not existing DMUs.

$$\begin{aligned}
\max \quad & \sigma + \varepsilon (\mathbf{I}^T \mathbf{s}^+ + \mathbf{I}^T \mathbf{s}^-) & (4.1) \\
\text{s.t.} \quad & \\
& \mathbf{Y} \boldsymbol{\lambda} - \sigma \mathbf{w}^y - \mathbf{s}^+ = \mathbf{g}^y \\
& \mathbf{X} \boldsymbol{\lambda} + \sigma \mathbf{w}^x + \mathbf{s}^- = \mathbf{g}^x \\
& \boldsymbol{\lambda}, \mathbf{s}^-, \mathbf{s}^+ \geq \mathbf{0} \\
& \varepsilon > 0
\end{aligned}$$

Let us investigate what kind of targets this model gives in the example when we vary the weighting vector  $(\mathbf{w}^y, \mathbf{w}^x)$ . As vector  $(\mathbf{g}^x, \mathbf{g}^y)$  we have used vector  $(\mathbf{y}_0, \mathbf{x}_0)$ , i.e. the inputs and outputs of DMU<sub>6</sub>.

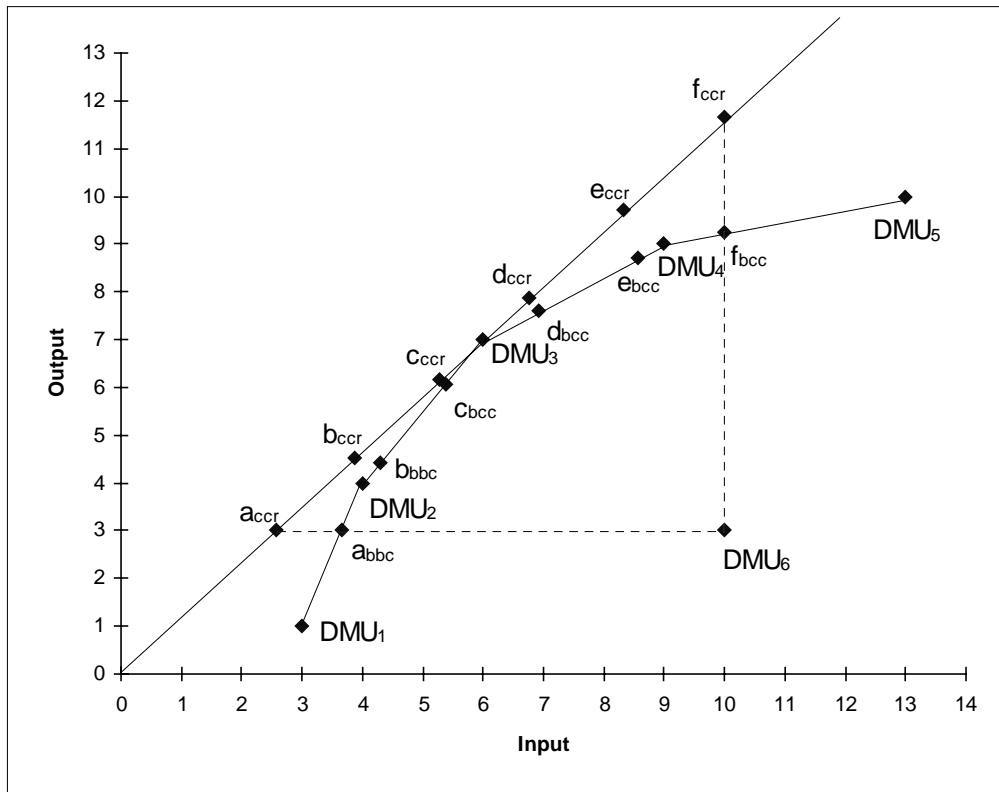


Figure 2

Six (6) targets have been calculated (from a to f) both in the CCR and the BCC context (see Figure 2). The targets are obtained by weighting input and output with vectors  $(1, 0)$ ,  $(0.8, 0.2)$ ,  $(0.6, 0.4)$ ,  $(0.4, 0.6)$ ,  $(0.2, 0.8)$  and  $(0, 1)$ . To be able to scan the whole part of the efficient frontier dominating the data set, we suggest the use of vector  $(y^{\min}, x^{\max})$  as vector  $(g^x, g^y)$  as in (3.7).

In the Reference Point approach the efficient frontier can be scanned either by changing the weighting vectors  $w^x$  and  $w^y$  or the aspiration level vectors  $g^x$  and  $g^y$ . In the above example we changed the weighting vector, but in computational applications it is more effective to fix the weighting vector and change the righthandside parameters, i.e. the aspiration level vector.

If we replace in (4.1)  $g^x$  and  $g^y$  with  $g^x + t^x d^x$  and  $g^y + t^y d^y$ , we obtain the Reference Direction formulation (Korhonen and Laakso [1986]). The Reference Direction formulation can be solved with the parametric linear programming technique: thus it is possible to provide the user with information in real time even when the problem is of realistic size. This guarantees true interactivity. The dynamic extension of the Reference Direction approach, the Pareto Race (Korhonen and Wallenius [1988]) is implemented in VIG-software. Korhonen [1997] has discussed the use of the Reference Direction approach in DEA.

## 5. Conclusion

In this paper we have demonstrated how the search of the most preferred target can be carried out with the Reference Point Approach. The advantage in using the Reference Point Approach is that we can also find targets that are not extreme point solutions, but lie in the relative interior of some facet.

The Reference Direction model, an extension of the Reference Point model, is implemented as Pareto Race in VIG-software. Thus it is possible to search the targets in a dynamic and interactive way with existing software.

The target setting extensions of DEA provide interesting possibilities to expand the use of DEA from *ex post evaluation* to *ex ante planning*. Another way of utilizing the most preferred target is *Value Efficiency Analysis* (Halme et al. [1998]). The Value Efficiency Analysis (VEA) is a way to incorporate preference information in DEA without using weight restrictions. In VEA it is assumed that some DM, who has control over the DMUs, sets the most preferable values for input and output variables.

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