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Abstract

The paper introduces a local cooperation pattern for repeated bimatrix games: the players choose a mutually acceptable strategy pair in every next round. A mutually acceptable strategy pair provides each player with a payoff no smaller than that expected, in average, at a historical distribution of players' actions recorded up to the latest round. It may happen that at some points mutually acceptable strategy pairs do not exist. A game round at such "still" points indicates that at least one player revises his/her payoffs and switches from a normal behavior to abnormal. We consider payoff switches associated with altruistic and aggressive behaviors, and define measures of all combinations of normal, altruistic and aggressive behaviors on every game trajectory. These behavior measures serve as criteria for the global analysis of game trajectories. Given a class of trajectories, one can identify the measures of desirable and undesirable behaviors on each trajectory and select optimal trajectories, which carry the minimum measure of undesirable behaviors. In the paper, the behavior analysis of particular classes of trajectories in the repeated Prisoner's Dilemma is carried out.

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Introduction

Altruism and aggression are extreme modes of interaction. When two players find actions profitable for both, one may view their behavior as desirable or normal. When they do not find such actions (and are still forced to interact), at least one of them loses. If the player 1 loses, the player 2 either wins, or loses, too. Player 2 wins if player 1 goes for a compromise, i.e., adopts (temporarily) the interest of player 2 and acts so as to help this player. Player 2 loses if player 1 acts against his/her interest (which may in particular be driven by a desire to move to a "better" state where normal behavior is feasible again). In the first case player 1 acts as an altruist with respect to player 2. In the second case player 1 acts as an aggressor with respect to player 2. Certainly, player 2 may also adopt altruism or aggression with respect to player 1. Accordingly, different combinations of players' behaviors may occur.

This informal classification of behaviors lies in the base of our study. We do not pretend to give an explanation of players' motives when they act normally, altruistically, or aggressively (we slightly touch this issue when we consider a problem of designing optimal behaviors in section 4). Our goal is to describe a game-theoretical method for identifying players' behaviors in one-round interactions and show how this method can be used in the analysis of multi-round interactions.

Our model operates under the informational conditions of fictitious play. The fictitious play dynamics proposed by Brown (1951) and Robinson (1951) is a round-by-round process of updating strategies in a nonzero sum bimatrix game. In every round, each player chooses a strategy, which gives him/her the largest expected payoff on the historical distribution of the strategies of the other player. In our setting, the players update their actions basing on the historical distributions of the strategies of both players.

The fictitious play dynamics was analyzed and generalized in different aspects. Fudenberg and Kreps (1993) viewed a (modified) Brown-Robinson procedure as a model of rational behavior and proved its convergence for 2×2 bimatrix games with a unique mixed Nash equilibrium. Kaniovski and Young (1995) gave an economic interpretation of a stochastically perturbed fictitious play dynamics and showed its convergence to Nash equilibria for general 2×2 bimatrix games; a further step in this direction was made in Kaniovski, et. al. (1997). Gaunersdorfer and Hofbauer (1995) analyzed the asymptotics of the fictitious play trajectories for a class of three-strategy bimatrix games and found connections with the replicator game dynamics (see Hofbauer and Sigmund, 1988). Smale (1980) considerably extended the frames of fictitious play by introducing (in the context of the repeated Prisoner's Dilemma game) a class of general strategy updating rules fed back with the historical distributions of payoffs. This approach was generalized in Benaim and Hirsch (1994).

We define strategy updating rules through the comparison of the current payoffs with those expected on the historical distributions of players' strategies. Different preferences in comparison are associated with different behaviors. The strategy updating rules are to a certain extend close to that used in fictitious play. There are two essential differences, however. First, all strategies, for which the payoffs are no smaller than the average payoffs on the historical strategy distributions are viewed as acceptable (recall that fictitious play admits strategies maximizing the average payoffs). Second, the proposed decisionmaking pattern is cooperative: every new strategy pair must be acceptable for each player, in other words, whenever an acceptable strategy pair is chosen, no one of the players loses (in the fictitious play dynamics the players update their strategies independently).

If in some round the players find an acceptable strategy pair and act so that no one of them loses, their behavior in this round is qualified as normal. Situations where at least one player loses arise when normal behavior is changed due to a change of the acceptable strategy pairs, or, equivalently, the payoff matricies. In this paper, we assume that player's payoff matrix can be changed to either the payoff matrix of the other player, or that taken with the opposite sign. In the first case the player identifies himself/herself with his/her rival and adopts altruism. In the second case the player identifies himself/herself with his/her rival's opponent and adopts aggression. It is important that every one-round transition, which is not normal, can be identified as a combination of altruistic and/or aggressive behaviors. In this context, our approach develops Kleimenov (1997, 1998) where the idea of identifying behaviors through switches in payoffs was proposed for nonzero sum differential games and population evolutionary games.

Our basic analytic tool is a measure of a given behavior on arbitrary game trajectory. The measure is defined as, roughly, the number of rounds, in which the given behavior is registered (as long as a one-round behavior is, generally, identified not uniquely, the minimum and maximum measures are introduced). We use the behavior measures for the estimation of the proportions of desirable and not desirable behaviors on the game trajectories. Namely, we consider a problem of behavior assessment and a problem of optimal behavior. Dealing with the problem of behavior assessment, we estimate the measures of desirable and not desirable behaviors on the trajectories generated by a given strategy updating rule. We focus, in particular, on the assessment of normal (desirable) and aggressive (not desirable) behaviors on the trajectories driven by the fictitious play dynamics. Dealing with the problem of optimal behavior, we minimize the measure of not desirable behaviors over a given class of game trajectories. In particular, we focus on the problem of minimizing the measure of abnormal behavior.

The paper is organized as follows. Our general method is presented in section 1. In the rest of the paper we apply the method to the analysis of the repeated Prisoner's Dilemma, in which the players choose between cooperation and defection. This game is often used for modeling socially desirable behaviors (see, e.g., Smale, 1980; Axelrod, 1984; Nowak and Sigmund, 1994). In section 2 we characterize the trajectories driven by different combinations of players' basic behaviors (normal, altruistic and aggressive) in the repeated Prisoner's Dilemma. In section 3 this characterization is used for the estimation of the measures of normal and aggressive behaviors on the fictitious play trajectories (on which the players never cooperate). We state that fictitious play may exhibit normal behavior and exclude aggression if mutual defection has a relatively high payoff, namely, two rounds of mutual defection provide a higher payoff than a round of cooperation versus defection and a round of defection versus cooperation. In the opposite situation normal behavior is eliminated and aggressive behavior dominates on many trajectories.

In section 4 we solve the problem of minimizing the measure of abnormal behavior on the trajectories convergent to the point of mutual cooperation. All optimal trajectories (moving in the space of empirical frequencies of cooperation and defection) embark on a "cooperation road" in a finite round and then develop cooperatively. In a neighborhood of the "road" all other behaviors except altruism of a "more cooperative" player are admissible. Beyond the neighborhood normal behavior is eliminated. Moreover, in this domain mutual defection is (under some circumstances) admissible, whereas mutual cooperation is not. An intuitive explanation is that it is "too early" to adopt mutual cooperation when one of the players is much "less cooperative" in the past.

The technical material for sections 1, 2, 3 and 4 is presented in Appendix 1 (section 5), Appendix 2 (section 6), Appendix 3 (section 7) and Appendix 4 (section 8), respectively.

1 Cooperative game dynamics

1.1 Cooperative repeated game

We consider a repeated two-player game. The player 1 has n strategies numbered $1,\dots,n$, and player 2 has m strategies numbered $1,\dots,m$. The players choose their strategies sequentially in rounds 1,2,... . The empirical frequency of a strategy i of player 1 in round k is the ratio $x_k^i = n_k^i/k$ where n_k^i is the number of rounds $r \le k$, in which player 1 chooses i. Similarly, the empirical frequency of a strategy j of player 2 in round k is the ratio $y_k^j = m_k^j/k$ where m_k^j is the number of rounds $r \le k$, in which player 2 chooses j. The empirical frequency vectors $x_k = (x_k^1,\dots,x_k^n)$ and $y_k = (y_k^1,\dots,y_k^m)$ belong to the $n-1$-dimensional simplex S_{n-1} and the $m-1$-dimensional simplex S_{m-1}, respectively; recall that the $p-1$-dimensional simplex S_{p-1} is the set of all p-dimensional vectors $x = (x^1\dots,x^p)$ with nonnegative coordinates whose sum is equal to 1. We shall call $S = S_{n-1}\times S_{m-1}$ the *state space* of the repeated game. Elements of S will be called *states*. Note that all states (x_k, y_k) admissible in round k cover a finite subset of S.

Following the pattern of fictitious play, we assume that in each round k the players observe the current state (x_k, y_k) and choose a strategy pair (i_{k+1}, j_{k+1}) for the next round. The number of rounds $r \le k+1$, in which player 1 chooses strategy i changes as follows: $n_{k+1}^i = n^i + 1$ if $i = i_{k+1}$ and $n_{k+1}^i = n^i$ if $i \ne i_{k+1}$. Hence, for the empirical frequency vector of player 1 we have:

$$
\begin{aligned}
x_{k+1}^{i_{k+1}} &= \frac{n_k^{i_{k+1}}+1}{k+1} = \frac{n_k^{i_{k+1}}}{k} - \frac{n_k^{i_{k+1}}}{k(k+1)} + \frac{1}{k+1},\\
x_{k+1}^{i} &= \frac{n_k^{i}}{k+1} = \frac{n_k^{i}}{k} - \frac{n_k^{i}}{k(k+1)} \quad (i \ne i_{k+1}),
\end{aligned}
$$

or

$$x_{k+1}^{i_{k+1}} = x_k^{i_{k+1}} - \frac{x_k^{i_{k+1}}+1}{k+1}, \tag{1.1}$$

$$x_{k+1}^{i} = x_k^{i} - \frac{x_k^{i}}{k+1} \quad (i \ne i_{k+1}). \tag{1.2}$$

Similarly,

$$y_{k+1}^{j_{k+1}} = y_k^{j_{k+1}} - \frac{y_k^{j_{k+1}}+1}{k+1}, \tag{1.3}$$

$$y_{k+1}^{j} = y_k^{j} - \frac{y_k^{j}}{k+1} \quad (j \ne j_{k+1}). \tag{1.4}$$

A finite or infinite sequence $t = ((x_k, y_k))$ in S $(k = k_0, k_0+1\dots)$ will be called a *trajectory* if for all indecies $k = k_0, k_0+1,\dots$ (except the final one provided t is finite) the equalities

(1.1) – (1.4) hold with some strategy pairs (i_{k+1}, j_{k+1}); the indecies k are identified with game rounds; the state (x_{k_0}, y_{k_0}) will be called the *initial* state of t; we shall also say that (x_{k_0}, y_{k_0}) *gives rise to* t in round k_0, and t *originates* from (x_{k_0}, y_{k_0}) in round k_0. We define the *length* of a trajectory t to be the difference between its final and initial rounds if t is finite and ∞ if t is infinite. A trajectory $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ will be called *stationary* if $(x_k, y_k) = (x_{k_0}, y_{k_0})$ for all $k \geq k_0$.

We consider the following rule for updating strategies. In round k, each player identifies a set of strategy pairs acceptable for him/her in round $k+1$. If the players find a strategy pair acceptable for both, they choose it for (i_{k+1}, j_{k+1}). If the players' acceptable sets do not intersect, (x_k, y_k) is the final state on the trajectory.

Let us specify the structure of the acceptable sets and introduce the associated trajectories. Let f_{ij}, and g_{ij} be *payoffs* to player 1 and player 2, respectively, for a strategy pair (i, j). The *expected payoffs* (briefly, the *payoffs*) to players 1 and 2 at a state (x_k, y_k) are defined by

$$f(x_k, y_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_k^i y_k^j f_{ij}, \tag{1.5}$$

$$g(x_k, y_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_k^i y_k^j g_{ij}, \tag{1.6}$$

respectively. In round k, a player views a strategy pair (i_{k+1}, j_{k+1}) as acceptable for round $k+1$ if his/her payoff at this strategy pair (i.e., $f_{i_{k+1}, j_{k+1}}$ for player 1 and $g_{i_{k+1}, j_{k+1}}$ for player 2) is no smaller than his/her payoff at the state (x_k, y_k).

Later, we shall admit changes in the payoff functions (and associate them with switches in players' behavior). Therefore, we formally define the acceptability of strategy pairs not only with respect to the original payoff functions f and g but also with respect to arbitrary "surrogate" payoff functions. We understand a *surrogate payoff function* as a scalar function φ on S, which has the same structure as f and g:

$$\varphi(x_k, y_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} x_k^i y_k^j \varphi_{ij}.$$

Given a surrogate payoff function φ, we call a strategy pair (i_{k+1}, j_{k+1}) *φ-acceptable* if $\varphi_{i_{k+1}, j_{k+1}} \geq \varphi(x_k, j_k)$. The set of all strategy pairs φ-acceptable at the state (x_k, y_k) will be denoted by $A_\varphi(x_k, y_k)$.

Given a pair of surrogate payoff functions, (φ, ψ), a trajectory $t = ((x_k, y_k))$ described by (1.1) – (1.4) will be called a *(φ, ψ)-trajectory* if in every round k (except the final one provided t is finite) the newly chosen strategy pair (i_{k+1}, j_{k+1}) is φ-acceptable and ψ-acceptable at (x_k, y_k), i.e., $(i_{k+1}, j_{k+1}) \in A_\varphi(x_k, y_k) \cap A_\psi(x_k, y_k)$.

The set of all states (x_k, y_k) such that the intersection $A_\varphi(x_k, y_k) \cap A_\psi(x_k, y_k)$ is nonempty will be called the *(φ, ψ)-active domain.* Every state from the (φ, ψ)-active domain will be called *(φ, ψ)-active.* By definition every (φ, ψ)-active state in every round gives rise to a (φ, ψ)-trajectory whose length is no smaller than 1. Every (φ, ψ)-active state, which gives rise to an infinite (φ, ψ)-trajectory in every round will be called *(φ, ψ)-kernel-active.* The set of all (φ, ψ)-kernel-active states will be called the *(φ, ψ)-kernel-active domain.* A state will be called *stationary (φ, ψ)-kernel-active* if in every round it gives rise to a single infinite (φ, ψ)-trajectory, and the latter is stationary. The set of all (φ, ψ)-kernel-active states, which are not stationary, will be called the *nonstationary (φ, ψ)-kernel-active domain.* In every round treated as initial, every state beyond the (φ, ψ)-active domain gives rise to a single (φ, ψ)-trajectory whose length is 0; we shall call such states (φ,ψ)-*still.* The set of all (φ,ψ)-still states will be called the (φ,ψ)-*still domain.* A (φ,ψ)-trajectory will be called *nonextendible* if it is either infinite, or finite and its final state is (φ,ψ)-still.

The next proposition describes a simple class of stationary (φ,ψ)-kernel-active states. We shall call a strategy pair (i_*,j_*) (φ,ψ)-*Pareto maximal* if there does not exist a strategy pair (i^*,j^*) such that $\varphi_{i^*j^*}\geq\varphi_{i_*j_*}$, $\psi_{i^*j^*}\geq\psi_{i_*j_*}$, and at least one of these inequalities is strict.

Proposition 1.1 *Let a strategy pair (i_*,j_*) be (φ,ψ)-Pareto maximal and there do not exist a strategy pair $(i^*,j^*)\neq(i_*,j_*)$ such that $\varphi_{i^*,j^*}=\varphi_{i_*,j_*}$ and $\psi_{i^*,j^*}=\psi_{i_*,j_*}$. Then a state (x_*,y_*) defined by*

$$x_*^{i_*}=1,\quad x_*^i=0\quad(i\neq i_*),\quad y_*^{j_*}=1,\quad y_*^j=0\quad(j\neq j_*)\tag{1.7}$$

is stationary (φ,ψ)-kernel-active.

A proof is given in Appendix 1.

Let us provide a characterization of the nonstationary (φ,ψ)-kernel-active states in a special case where there is a strategy pair (φ,ψ)-acceptable at every (φ,ψ)-active state.

Proposition 1.2 *Let there be a strategy pair (i_*,j_*) (φ,ψ)-acceptable at every nonstationary (φ,ψ)-active state, and a state (x_*,y_*) be defined by (1.7). Then a state $(x,y)\neq(x_*,y_*)$ is nonstationary (φ,ψ)-kernel active if and only if the closed segment with the end points (x,y) and (x_*,y_*) is contained in the (φ,ψ)-active domain.*

A proof is given in Appendix 1.

1.2 Normal behavior

By definition the surrogate payoffs φ and ψ do not decrease along the (φ,ψ)-trajectories. In particular, the actual payoffs f and g do not decrease along the (f,g)-trajectories. In this sense, the (f,g)-trajectories represent *normal behavior* beneficial for both players. We identify every (f,g)-trajectory as *normal.* We also identify the *active domain of normal behavior*, G^{00}, the *kernel-active domain of normal behavior*, G^{00}_∞, the *nonstationary kernel-active domain of normal behavior*, $\mathcal{G}^{00}_\infty$, and the *still domain of normal behavior*, $G^{00}_\emptyset$, with the (f,g)-active domain, the (f,g)-kernel-active domain, the nonstationary (f,g)-kernel-active domain, and the (f,g)-still domain, respectively. Stationary (f,g)-kernel-active states will be called *stationary for normal behavior.*

1.3 Basic behaviors

When a state of the game is in the still domain of normal behavior, $G^{00}_\emptyset$, the players are unable to make a new round via normal behavior. In order to make a new round, at least one player must change the behavior. We shall understand a change in behavior as a switch from the original payoff function to a surrogate one. Player's switch to a surrogate payoff function means that this player replaces the strategy pairs acceptable with respect to his/her original payoff function by those acceptable with respect to the surrogate one.

We shall consider altruistic and aggressive behaviors. When switching to altruistic behavior a player identifies his/her interest with his/her partner's. In this situation, the player replaces his/her original payoff function by his/her partner's. When switching to aggressive behavior, the player views himself/herself as partner's opponent and changes his/her payoff function for his/her partner's taken with the opposite sign. Combinations of individual behaviors generate joint behaviors; we will qualify them as 1-altruistic, 2-altruistic, 1-altruistic-2-aggressive, 1-aggressive-2-altruistic, and aggressive. These behaviors, together with normal behavior, will be called *basic*.

The 1-*altruistic* behavior implies that player 1 acts altruistically and player 2 acts normally. This behavior is modeled by the (g, g)-trajectories. We call the (g, g)-trajectories 1-*altruistic*. We define the *active domain of* 1-*altruistic behavior* G^{+0}, the *kernel-active domain of* 1-*altruistic behavior*, G_{∞}^{+0}, the *nonstationary kernel-active domain of* 1-*altruistic behavior*, $\mathcal{G}_{\infty}^{+0}$, and the *still domain of* 1-*altruistic behavior*, $G_{\emptyset}^{+0}$, to be the (g, g)-active domain, the (g, g)-kernel-active domain, the nonstationary (g, g)-kernel-active domain and the (g, g)-still domain, respectively. Stationary (g, g)-kernel-active states will be called *stationary for* 1-*altruistic behavior.*

Symmetrically, the 2-*altruistic* behavior implies that player 1 acts normally and player 2 acts altruistically. We call the (f, f)-trajectories 2-*altruistic* and define the *active domain of* 2-*altruistic behavior*, G^{0+}, the *kernel-active domain of* 2-*altruistic behavior*, G_{∞}^{0+}, the *kernel-active domain of* 2-*altruistic behavior*, $\mathcal{G}_{\infty}^{0+}$, and the *still domain of* 2-*altruistic behavior*, $G_{\emptyset}^{0+}$, to be the (f, f)-active domain, the (f, f)-kernel-active domain and the (f, f)-still domain, respectively. Stationary (f, f)-kernel-active states will be called *stationary for* 2-*altruistic behavior.*

The 1-*altruistic*-2-*aggressive* behavior implies that player 1 acts altruistically and player 2 acts aggressively. This behavior is modeled by the $(g, -f)$-trajectories. We call the $(g, -f)$-trajectories 1-*altruistic*-2-*aggressive* and define the *active domain of* 1-*altruistic*-2-*aggressive behavior*, G^{+-}, the *kernel-active domain of* 1-*altruistic*-2-*aggressive behavior*, G_{∞}^{+-}, the *nonstationary kernel-active domain of* 1-*altruistic*-2-*aggressive behavior*, $\mathcal{G}_{\infty}^{+-}$, and the *still domain of* 1-*altruistic*-2-*aggressive behavior*, $G_{\emptyset}^{+-}$, to be the $(g, -f)$-active domain, the $(g, -f)$-kernel-active domain, the nonstationary $(g, -f)$-kernel-active domain and the $(g, -f)$-still domain, respectively. Stationary $(g, -f)$-kernel-active states will be called *stationary for* 1-*altruistic*-2-*aggressive behavior.*

The 1-*aggressive*-2-*altruistic* behavior implies that player 1 acts aggressively and and player 2 acts altruistically. This behavior is modeled by the $(-g, f)$-trajectories. We call the $(-g, f)$-trajectories 1-*aggressive*-2-*altruistic* and define the *active domain of* 1-*aggressive*-2-*altruistic behavior*, G^{-+}, the *kernel-active domain of* 1-*aggressive*-2-*altruistic behavior*, G_{∞}^{-+}, the *nonstationary kernel-active domain of* 1-*aggressive*-2-*altruistic behavior*, $\mathcal{G}_{\infty}^{-+}$, and the *still domain of* 1-*aggressive*-2-*altruistic behavior*, $G_{\emptyset}^{-+}$, to be the $(-g, f)$-active domain, the $(-g, f)$-kernel-active domain, the nonstationary $(-g, f)$-kernel-active domain and the $(-g, f)$-still domain, respectively. Stationary $(g, -f)$-kernel-active states will be called *stationary for* 1-*aggressive*-2-*altruistic behavior.*

The *aggressive* behavior implies that both players act aggressively. This behavior is modeled by the $(-g, -f)$-trajectories. We call the $(-g, -f)$-trajectories *aggressive.* We define the *active domain of aggressive behavior*, G^{--}, the *kernel-active domain of aggressive behavior*, G_{∞}^{--}, the *nonstationary kernel-active domain of aggressive behavior*, $\mathcal{G}_{\infty}^{--}$, and the *still domain of aggressive behavior*, $G_{\emptyset}^{--}$, to be the $(-g, -f)$-active domain, the $(-g, -f)$-kernel-active domain, the nonstationary $(-g, -f)$-kernel-active domain and the $(-g, -f)$-still domain, respectively. Stationary $(-g, -f)$-kernel-active states will be called *stationary for aggressive behavior.*

The normal trajectories, the 1-altruistic trajectories, the 2-altruistic trajectories, the 1-altruistic-2-aggressive trajectories, the 1-aggressive-2-altruistic trajectories, and the aggressive trajectories will further be called *basic.*

In a similar manner, one may introduce 1-aggressive-2-normal and 1-normal-2- aggressive behaviors. These behaviors imply that the players act as antagonists and have extremely narrow active domains. For example, a state (x_k, y_k) belongs to the active domain of the 1-aggressive-2-normal behavior if and only if there is a strategy pair (i_{k+1}, j_{k+1})

such that $g(x_k, y_k) = g_{i_{k+1},j_{k+1}}$. Generally, such states fill exceptional manifolds in the state space. By this reason we exclude the antagonistic behaviors from our considerations.

1.4 Universality of basic trajectories

The next observation follows straightforwardly from the definition of the φ-acceptable strategy pairs. A strategy pair (i_{k+1}, j_{k+1}), which is not φ-acceptable at (x_k, y_k), is $-\varphi$-acceptable at (x_k, y_k). This observation allows to state the "universality" of the basic trajectories: each trajectory is represented as a chain of basic subtrajectories.

Let us give relevant definitions. A trajectory $t = ((x_k, y_k))$ $(k = k_0, k_0 + 1, \ldots)$ will be said to be a *subtrajectory* of a trajectory $\bar{t} = ((\bar{x}_k, \bar{y}_k))$ $(k = \bar{k}_0, \bar{k}_0 + 1, \ldots)$ if $k_0 \geq \bar{k}_0$ and $(x_k, y_k) = (\bar{x}_k, \bar{y}_k)$ $(k = k_0, k_0 + 1, \ldots)$. A finite or infinite sequence of trajectories, (t_s) $(s = 1, 2, \ldots)$, will be called a *chain of subtrajectories* of a trajectory $\bar{t} = ((\bar{x}_k, \bar{y}_k))$ $(k = \bar{k}_0, \bar{k}_0 + 1, \ldots)$ if every t_s is a subtrajectory of $\bar{t}$, and the subtrajectories $t_1, t_2, \ldots$ cover $\bar{t}$; a more accurate formulation of the latter requirement is as follows:

(i) the initial round of t_1 is $\bar{k}_0$,

(ii) if t_s is finite and not final, the final round of t_s coincides with the initial round of t_{s+1}, and

(iii) the sum of the lengths of the subtrajectories $t_1, t_2, \ldots$ is equal to the length of $\bar{t}$.

Proposition 1.3 *For every trajectory there is a chain of its basic subtrajectories.*

The proof is given in Appendix 1. In fact we state that every trajectory is "chained" into subtrajectories of three types: 2-altruistic, 1-altruistic-2-aggressive, and aggressive. Other combinations of "chaining" behavior types can easily be identified. In particular, the following trajectory types "chain" every trajectory: normal, 1-altruistic-2-aggressive, 1-aggressive-2-altruistic, and aggressive.

1.5 Measures of basic behaviors

Basing on Proposition 1.3, we shall introduce measures of basic behaviors on a given trajectory. Let t be a trajectory and (t_s) be its chain of basic subtrajectories. We define the (t_s)-*measure of normal behavior* on t to be the sum of the lengths of all normal subtrajectories t_s; this sum may in particular be infinite. We define the *maximum* and *minimum measures of normal behavior* on t as, respectively, the maximum and minimum of the (t_s)-measures of normal behavior on t over all chains (t_s) of basic subtrajectories of t. Similarly, we define the (t_s)-measures, the maximum measures and the minimum measures of other basic behaviors on t.

Let us define the maximum and minimum measures of a class of basic behaviors on a trajectory t. Let B be a subclass of basic behaviors and (t_s) $(s = 1, 2 \ldots)$ be a chain of basic subtrajectories of t. If there is no subtrajectory t_s whose basic behavior belongs to B, we define the (t_s)-*measure* of B on t as zero. Let there be t_s whose basic behavior belongs to B. Let F be the set of all subtrajectories t_s from the chain (t_s) such that some basic behavior on t_s belongs to B. We define the (t_s)-*measure* of B on t to be the sum of the lengths of all $t_s \in F$. The *maximum measure* of B on t is the supremum of the (t_s)-measures of B on t over all chains (t_s) of basic subtrajectories of t, and the *minimum measure* of B on t is the infimum of the (t_s)-measures of B on t over all chains (t_s) of basic subtrajectories of t.

1.6 Behavior assessment

The maximum and minimum measures of classes of basic behaviors provide natural criteria for the estimation of the actual proportions of desired and undesired behaviors arising under a chosen strategy updating rule. We suggest the next general formulation of a problem of behavior assessment.

Problem of behavior assessment. *Given a class of trajectories, T, and a class of basic behavior types, B, find the maximum (minimum) measure of B on every trajectory from T.*

The problem may take various specific forms depending on the classes T and B. Recall that when the players exhibit normal behavior, neither of them loses in average payoff, and when they exhibit aggressive behavior, neither of them wins in average payoff. Therefore, normal behavior is mostly desirable and aggressive behavior is mostly undesirable. The assessment of these behaviors is of special interest. Let us formulate problems of the assessment of normal and aggressive behaviors on the fictitious play trajectories.

Following Brown (1951) and Robinson (1951), we shall say that a trajectory $((x_k, y_k))$ is a *fictitious play trajectory* if in each round k strategies i_{k+1} and j_{k+1} for round $k+1$ are chosen as best replies of players 1 and 2 to partner's empirical frequencies, i.e., i_{k+1} is a maximizer to $f^i(y_k) = \sum_{j=1}^m f_{ij} y_k^j$ over all $i = 1, 2, \ldots, n$ and j_{k+1} is a maximizer to $g^j(x_k) = \sum_{i=1}^n g_{ij} x_k^i$ over all $j = 1, 2, \ldots, m$. Letting T to be the set of all infinite fictitious play trajectories and $B = \{\text{normal}\}$, we arrive at the next specification of the general problem of behavior assessment.

Problem of the assessment of normal behavior on the fictitious play trajectories. *Find the maximum measure of normal behavior on every infinite fictitious play trajectory.*

Setting $B = \{\text{aggressive}\}$, we get the next formulation.

Problem of assessment of aggressive behavior on the fictitious play trajectories. *Find the minimum measure of aggressive behavior on every infinite fictitious play trajectory.*

In section 3 we shall solve these problems for the repeated Prisoner's Dilemma.

1.7 Behavior optimization

The behavior assessment is intended to reconstruct the structure of given trajectories; in this sense, the problem of behavior assessment falls in the category of inverse problems. A primary problem, in this context, will be a problem of the design of trajectories. Let us consider such a problem, in which the measures of basic behaviors serve as optimality criteria.

Let the players start from a state (x_*, y_*) in round k_0, and let there be a set of trajectories originating from (x_*, y_*) in round k_0, which are viewed by each player as favorable in the long run; we shall call these trajectories *desired*. For example, the players may treat as desirable all trajectories convergent to a Pareto point in the original static game. Let B be a class of basic behaviors viewed as undesirable. Let $\mu(t)$ denote the minimum measure of B on a trajectory t. We shall treat $\mu(t)$ as t's index of optimality. The less is $\mu(t)$, the less rounds with undesirable behaviors are on t. A problem of designing an optimal desired trajectory arises.

Let us give its accurate formulation. Denote by $\mu_{\min}$ the minimum of $\mu(t)$ over all trajectories from T. We call $\mu_{\min}$ the *minimum measure of B* on T. A trajectory t from T such that $\mu^{00}(t) = \mu_{\min}$ will be called *B-optimal* in T.

Problem of optimal behavior. *Given B, a class of undesirable basic behaviors, and T, a class of desired trajectories, find the minimum measure of B on T and describe all trajectories B-optimal in T.*

Let us consider a special case where $B = \{$all behaviors except normal$\}$. In this case $\mu(t)$ is the *minimum measure of abnormal behavior* on a trajectory t, and $\mu_{\min}$ is the *minimum measure of abnormal behavior* on T. Briefly, we shall call trajectories, which are B-optimal in T, *optimal.* The problem of optimal behavior is specified then into a problem of minimizing the measure of abnormal behavior.

Problem of minimizing the measure of abnormal behavior. *Given T, a class of desired trajectories, find the minimum measure of abnormal behavior on T and describe all optimal trajectories from T.*

In section 4 we shall solve this problem for a class of desired trajectories in the repeated Prisoner's Dilemma.

2 Cooperative dynamics in repeated Prisoner's Dilemma

2.1 Preliminaries

In the Prisoner's Dilemma, the players choose between *cooperation*, C, and *defection*, D. We identify C as strategy 1 and D as strategy 2. Indicies 1 and 2 in the notation of the payoffs, f_{ij} and g_{ij} $(i, j = 1, 2)$, will be, accordingly, replaced by C and D. The game is symmetric:

$$f_{CC} = g_{CC}, \quad f_{DD} = g_{DD}, \quad f_{CD} = g_{DC}, \quad f_{DC} = g_{CD},$$

and the next relations hold:

$$f_{DC} > f_{CC} > f_{DD} > f_{CD}, \quad 2f_{CC} > f_{CD} + f_{DC}. \tag{2.1}$$

In the repeated Prisoner's Dilemma, every empirical frequency vector $(z_k^1, z_k^2) \in S_1$ is uniquely determined by its z_k^1 component ($z_k^2 = 1 - z_k^1$). We shall operate with these components only. Thus, a pair $(x_k, y_k) = (x_k^1, y_k^1) \in [0,1] \times [0,1]$ will always be understood as $((x_k^1, x_k^2), (y_k^1, y_k^2)) \in S_1 \times S_1$. In this sense, the state space $S_1 \times S_1$ will be identified with the square $[0,1] \times [0,1]$. We keep calling $[0,1] \times [0,1]$ the state space; as earlier, we denote it by S; and call its elements states.

The payoffs to players 1 and 2 at a state (x, y) are given by (see (1.5), (1.6))

$$\begin{aligned} f(x,y) &= cxy - c_1 x - c_2 y + f_{DD}, \\ g(x,y) &= cxy - c_2 x - c_1 y + f_{DD}, \end{aligned}$$

where

$$c = f_{CC} - f_{CD} - f_{DC} + f_{DD}, \quad c_1 = f_{DD} - f_{CD}, \quad c_2 = f_{DD} - f_{DC}. \tag{2.2}$$

Note that (2.1) implies

$$c_1 > 0, \quad c_2 < 0, \quad c_2 < c < c_1. \tag{2.3}$$

For a strategy pair $(i, j) \in \{(C,C), (C,D), (D,C), (D,D)\}$ and a surrogate payoff function φ we denote by $H_{ij}(\varphi)$ the set of all states, for which (i, j) is φ-acceptable. We describe $H_{ij}(\varphi)$ using functions

$$h_{CC}(x) = \frac{f_{CC} - f_{DD} + c_1 x}{cx - c_2}, \tag{2.4}$$

$$h_{DD}(x) = \frac{c_1 x}{cx - c_2} \tag{2.5}$$

defined on $[0,1]$ (note that for $x \in [0,1]$ the denominator in (2.4) and (2.5) is positive due to (2.3)). The next lemma proved in Appendix 2 lists properties of h_{CC} and h_{DD}, which are used in our analysis.

Lemma 2.1 *The functions h_{CC} and h_{DD} are strictly convex if $c < 0$, linear if $c = 0$ and strictly concave if $c > 0$, and the following relations hold:*

$$h_{CC}(1) = 1, \quad h_{CC}(0) > 0, \quad h_{DD}(0) = 0,, \tag{2.6}$$

$$h_{CC}(x) > h_{DD}(x) \geq 0, \quad h_{DD}(x) > 0 \quad (x > 0), \tag{2.7}$$

$$h_{DD}(x) < x \quad (x \in (0,1]) \quad \text{if} \quad c_1 + c_2 \leq 0,, \tag{2.8}$$

$$h_{DD}(x) \geq x \quad (x \in (0, (c_1 + c_2)/c]),$$

$$h_{DD}(x) < x \quad (x \in ((c_1 + c_2)/c, 1]) \quad \text{if} \quad c_1 + c_2 > 0,$$

$$h'_{CC}(x) > 0, \quad h'_{DD}(x) > 0,$$

$$h'_{CC}(1) = \frac{c_1 - c}{c - c_2} = \frac{f_{DC} - f_{CC}}{f_{CC} - f_{CD}} < 1, \tag{2.9}$$

$$h'_{DD}(0) = -\frac{c_1}{c_2}.$$

The next equalities hold:

$$H_{CC}(f) = \{(x,y) \in S : y \leq h_{CC}(x)\}, \tag{2.10}$$

$$H_{DD}(f) = \{(x,y) \in S : y \leq h_{DD}(x)\}, \tag{2.11}$$

$$H_{CD}(f) = \{(C,D)\}, \tag{2.12}$$

$$H_{DC}(f) = S. \tag{2.13}$$

Indeed, by definition (C,C) is acceptable at (x,y) if $f(x,y) \leq f_{CC}$, which is equivalent to $y \leq h_{CC}(x)$ (here we refer to (2.4) and take into account that $cx - c_2 > 0$, see (2.3)). Thus we arrive at (2.10). Similarly we obtain (2.11). By (2.1) $f(x,y) > f_{CD}$ for $(x,y) \neq (C,D)$ and $f(C,D) = f_{CD}$; similarly, $f(x,y) < f_{DC}$ for $(x,y) \neq (D,C)$ and $f(D,C) = f_{DC}$. Hence we get (2.12) and (2.13). Similar arguments give

$$H_{CC}(g) = \{(x,y) \in S : x \leq h_{CC}(y)\}, \tag{2.14}$$

$$H_{DD}(g) = \{(x,y) \in S : x \leq h_{DD}(y)\}, \tag{2.15}$$

$$H_{CD}(g) = S, \tag{2.16}$$

$$H_{DC}(f) = \{(D,C)\}. \tag{2.17}$$

For every $E \subset S$ we denote by $\bar{E}$ the closure of $S \setminus E$. Obviously,

$$H_{ij}(-f) = \bar{H}_{ij}(f), \quad H_{ij}(-f) = \bar{H}_{ij}(f) \quad ((i,j) = (C,C),(C,D),(D,C),(D,D)).$$

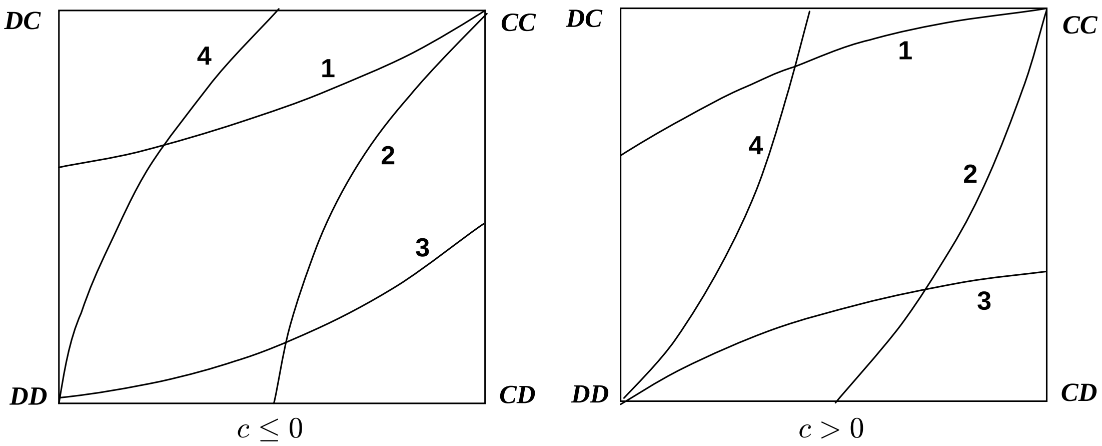


Figure 2.1: the bordering curves, which may separate active and still domains for the basic behaviors. The curves have the equations $y = h_{CC}(x)$ (curve 1), $x = h_{CC}(y)$ (curve 2), $y = h_{DD}(x)$ (curve 3), $x = h_{DD}(y)$ (curve 4). In all figures given below neither the bordering curves, nor the corner points (C,C), (C,D), (D,C), (D,D) are indicated specially.

2.2 Characterization of basic trajectories

The relations given in the previous subsection imply that for all basic behaviors the borders between the active and still domains (if these are nonempty) go along the curves $y = h_{CC}(x)$, $y = h_{DD}(x)$, $x = h_{CC}(y)$ and $x = h_{DD}(y)$; the curves are schematically shown in Figure 2.1.

Moreover, an accurate analysis of the sets $H_{ij}(f)$ and $H_{ij}(g)$ ($(i,j) = (C,C), (C,D), (D,C), (D,D)$) and Lemma 2.1 yields a description of all characteristic domains (active, kernel-active, etc.) for all basic behaviors. The analysis is given in in Appendix 2. The structure of basic trajectories is described in Propositions 6.4, 6.5, 6.6 (normal trajectories), 6.8 (2-altruistic trajectories), 6.10. (1-altruistic trajectories) 6.12 (1-aggressive-2-altruistic trajectories), 6.14. (1-altruistic-2-aggressive trajectories) and 6.18, 6.19, 6.20. (aggressive trajectories).

The structure of basic trajectories is shown schematically in Figures 2.2 - 2.7.

Let us comment Figures 2.2 – 2.7.

Figure 2.2: normal trajectories. In case of $c \leq 0$ the nonstationary kernel-active domain, $\mathcal{G}_\infty^{00}$, is essentially smaller than the active domain; $\mathcal{G}_\infty^{00}$ lies between the straight lines tangent to the two bordering curves at the "north-east" corner point, (C,C). In case of $c > 0$ and $c_1 + c_2 \leq 0$, $\mathcal{G}_\infty^{--}$, coincides with the active domain minus the corner points. The arrows originating from states in the active and kernel-active domains point to the strategy pairs (corner points) admissible in these states for the players acting normally. The normal trajectories move towards these corner points in every round. The "south-west" corner point, (D,D), is "half-stationary". A trajectory originating from (D,D) can either stay in this point forever, or abandon it in some round; in the latter case the rest of the trajectory is nonstationary.

Figure 2.3: 2-altruistic trajectories. The nonstationary kernel-active domain, $\mathcal{G}_\infty^{0+}$, covers the whole state space except of its "north-east" corner point, (D,C), which is stationary. The arrows originating from states point to the strategy pairs (corner points) admissible in these states when player 1 acts normally and player 2 altruistically. The 2-altruistic trajectories move towards these corner points in every round.

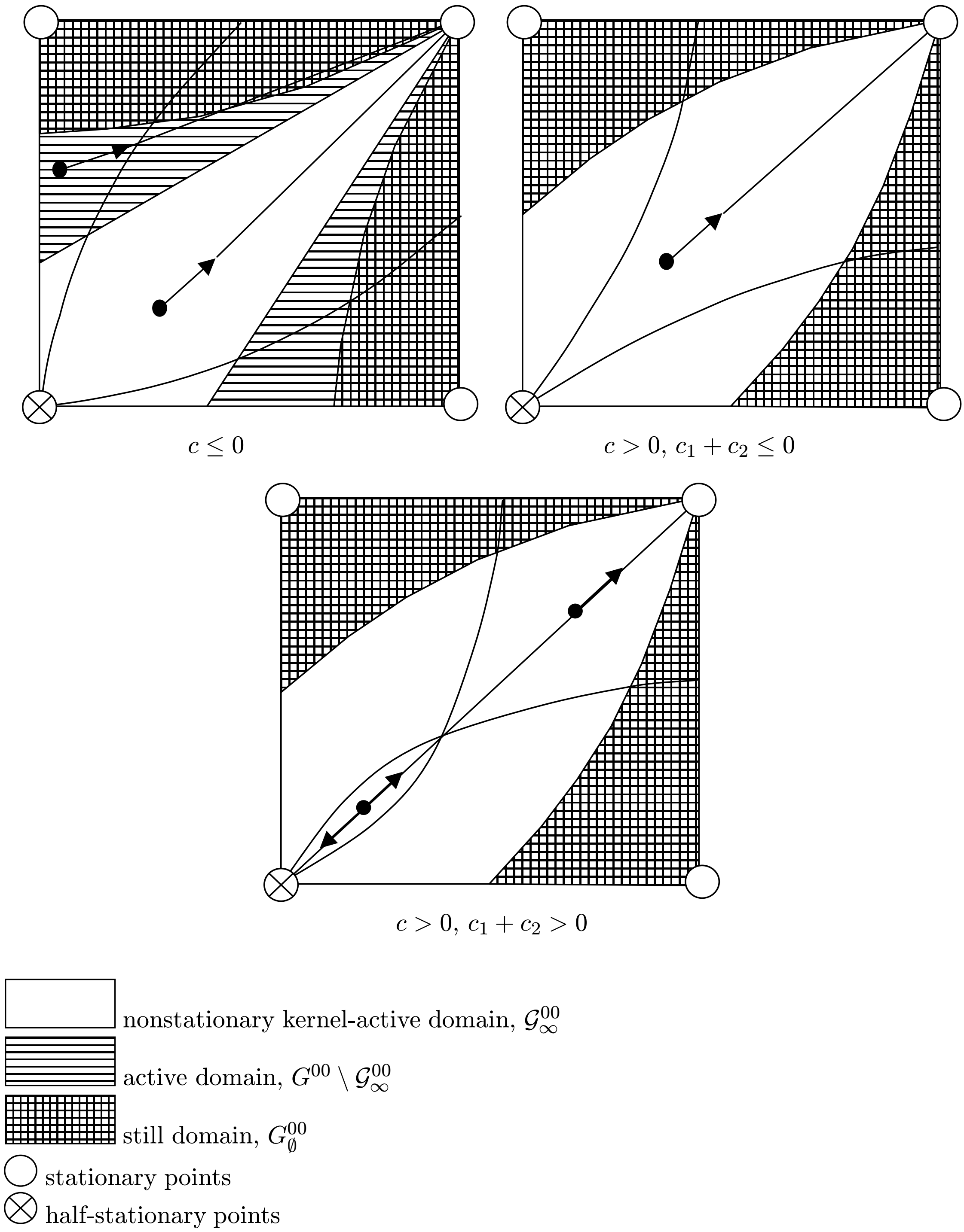
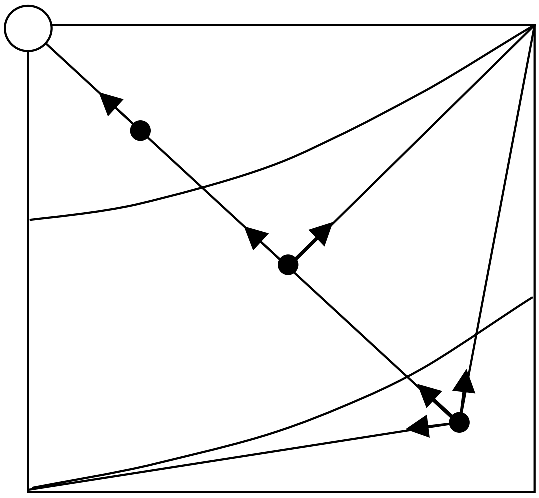


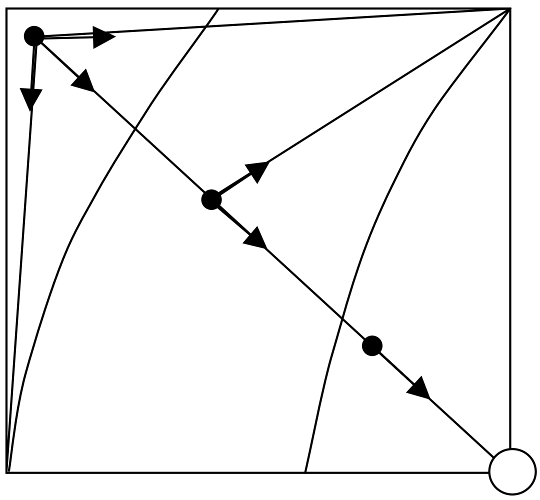
Figure 2.2: normal trajectories



nonstationary kernel-active domain, $\mathcal{G}_\infty^{0+}$

stationary point

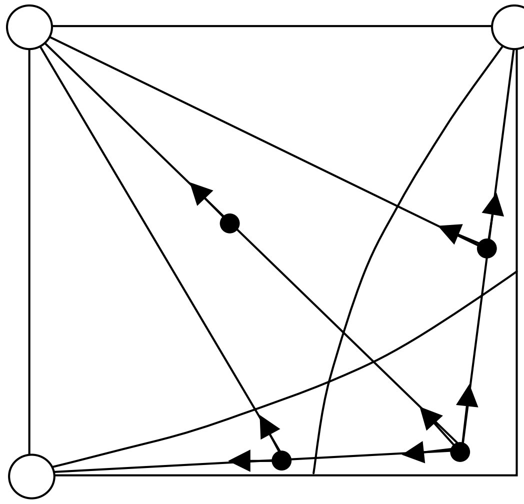
Figure 2.3: 2-altruistic trajectories



nonstationary kernel-active domain, $\mathcal{G}_\infty^{+0}$

stationary point

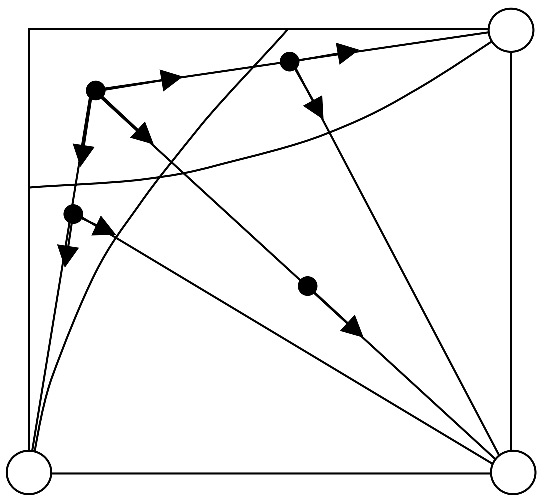
Figure 2.4: 1-altruistic trajectories



nonstationary kernel-active domain, $\mathcal{G}_\infty^{-+}$

stationary points

Figure 2.5: 1-aggressive-2-altruistic trajectories



nonstationary kernel-active domain, $\mathcal{G}_\infty^{+-}$

stationary points

Figure 2.6: 1-altruistic-2-aggressive trajectories

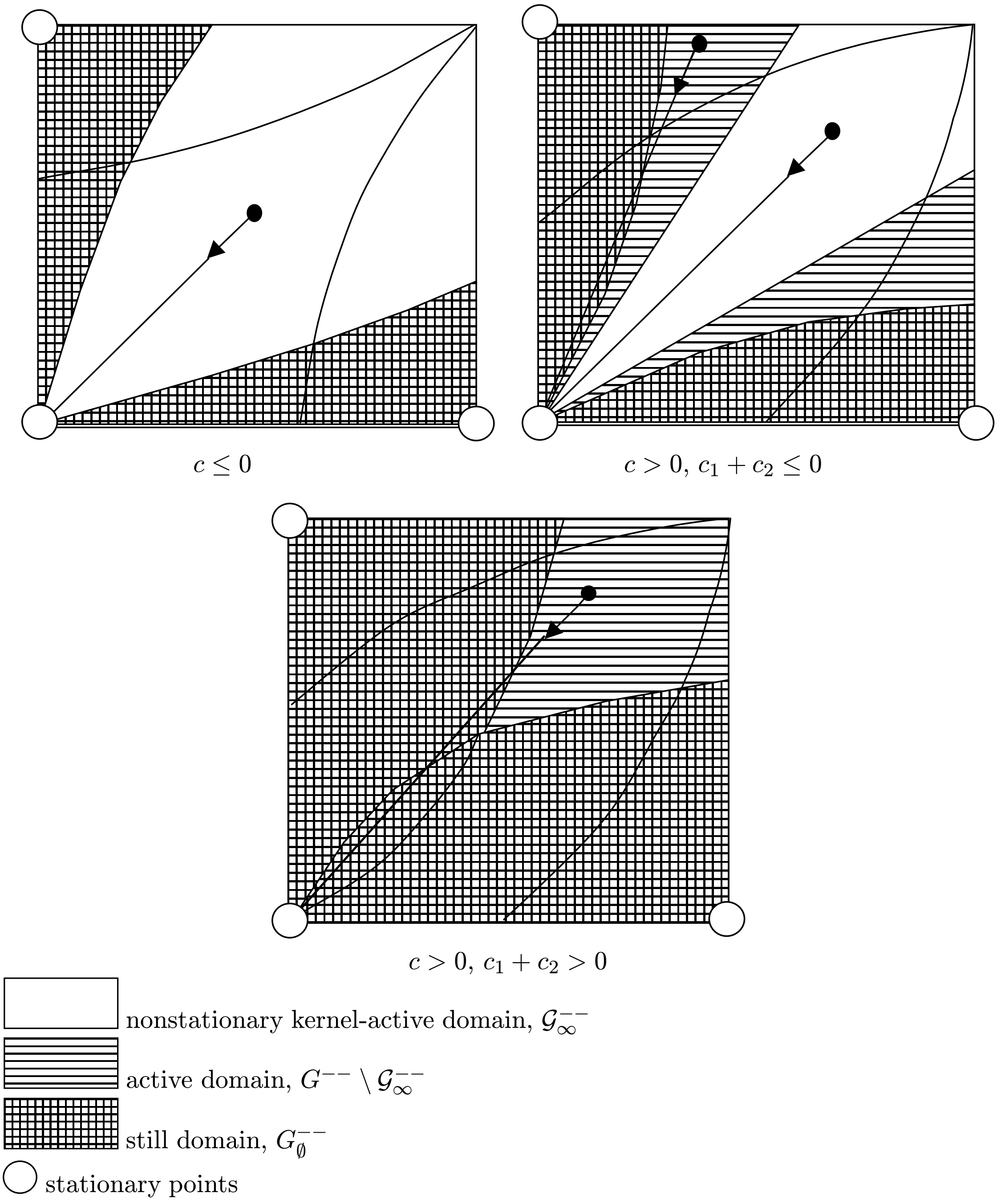


Figure 2.7: aggressive trajectories

Figure 2.4: 1-altruistic trajectories. The nonstationary kernel-active domain, $\mathcal{G}_\infty^{+0}$, covers the whole state space except of its "south-east" corner point, (C, D), which is stationary. The arrows originating from states point to the strategy pairs (corner points) admissible in these states when player 1 acts altruistically and player 2 normally. The 1-altruistic trajectories move towards these corner points in every round.

Figure 2.5: 1-aggressive-2-altruistic trajectories. The nonstationary kernel-active domain, $\mathcal{G}_\infty^{-+}$, covers the whole state space except of the three corner points, (D, D), (D, C) and (C, C), which are stationary. The arrows originating from states point to the strategy pairs (corner points) admissible in these states when player 1 acts aggressively and player 2 altruistically. The 1-aggressive-2-altruistic trajectories move towards these corner points in every round.

Figure 2.6: 1-altruistic-2-aggressive trajectories. The nonstationary kernel-active domain, $\mathcal{G}_\infty^{+-}$, covers the whole state space except of the three corner points, (D, D), (C, D) and (C, C), which are stationary. The arrows originating from states point to the strategy pairs (corner points) admissible in these states when player 1 acts altruistically and player 2 aggressively. The 1-altruistic-2-aggressive trajectories move towards these corner points in every round.

Figure 2.7: aggressive trajectories. In case of $c \le 0$ the nonstationary kernel-active domain, $\mathcal{G}_\infty^{--}$, coincides with the active domain minus the three stationary points. In case of $c \le 0$ and $c_1 + c_2 \le 0$, $\mathcal{G}_\infty^{--}$ lies between the straight lines tangent to the two bordering curves at the "south-west" corner point, (D, D). In case of $c \le 0$ and $c_1 + c_2 > 0$, $\mathcal{G}_\infty^{--}$ is empty. The arrows originating from states in the active and kernel-active domains point to the strategy pairs (corner points) admissible for the aggressive players in these states. The aggressive trajectories move towards these corner points in every round. In the first two cases all nonextendable aggressive trajectories are infinite and in the last case all of them are finite.

3 Behavior assessment of fictitious play trajectories

3.1 Fictitious play

In this section we give a behavior assessment of the fictitious play trajectories (see subsection 1.6) in the repeated Prisoner's Dilemma. The argument refers to the characterizations of basic trajectories, given in section 2.

Recall that the fictitious play dynamics arises when each player chooses the best replies to the empirical frequencies of partner's strategies. In the repeated Prisoner's Dilemma, the average payoff to player 1 in round k is $f_C(y_k) = f_{CC} y_k + f_{CD}(1 - y_k)$ if player 1 chooses C, and $f_D(y_k) = f_{DC} y_k + f_{DD}(1 - y_k)$ if he/she chooses D. A best reply of player 1 to y_k provides a greater average payoff. Using (2.2) and (2.3), we easily find that $f_D(y_k) > f_C(y_k)$. Hence, D is a single best reply of player 1 at (x_k, y_k). Similarly, D is a single best reply of player 2 at (x_k, y_k). Therefore, a nonextendable fictitious play trajectory originating from $(x_*, y_*) \ne (D, D)$ is infinite and moves towards (D, D) in each round. A nonextendable fictitious play trajectory originating from (D, D) is infinite and stationary.

3.2 Assessment of normal and aggressive behaviors

Introduce the sets

$$E_1 = \left\{ (x, y) \in S : y \le -\frac{c_1}{c_2} x \right\},$$

$$E_2 = \left\{(x,y) \in S : x \le -\frac{c_1}{c_2}y\right\},$$

$$E = H_{DD}(f) \cap H_{DD}(g).$$

Note that if $c_1 + c_2 > 0$, then $\emptyset \ne E \subset E_1 \cap E_2$ (see Proposition 6.3, 3).

The next propositions present a solution of the problem of the assessment of normal and aggressive behaviors on the fictitious play trajectories.

Proposition 3.1 *The following statements hold true:*

1) *if $c_1 + c_2 \le 0$, then the maximum measure of normal behavior is zero on every nonstationary infinite fictitious play trajectory,*

2) *if $c_1 + c_2 > 0$, then*
(i) *all infinite fictitious play trajectories originating from $S \setminus (E_1 \cup E_2)$ have the zero maximum measure of normal behavior,*
(ii) *all infinite fictitious play trajectories originating from $E_1 \cup E_2$ have the infinite maximum measure of normal behavior, and*
(iii) *all infinite fictitious play trajectories originating from $E \subset E_1 \cup E_2$ are normal.*

Proposition 3.2 *The following statements hold true:*

1) *if $c \le 0$, then*
(i) *all infinite fictitious play trajectories originating from G^{--} are aggressive,*
(ii) *all infinite fictitious play trajectories originating from $[\bar{E}_1 \cap \bar{E}_2] \setminus G^{--}$ have the infinite minimum measure of aggressive behavior, and*
(iii) *all infinite fictitious play trajectories originating from $E_1 \cup E_2$ have the zero minimum measure of aggressive behavior,*

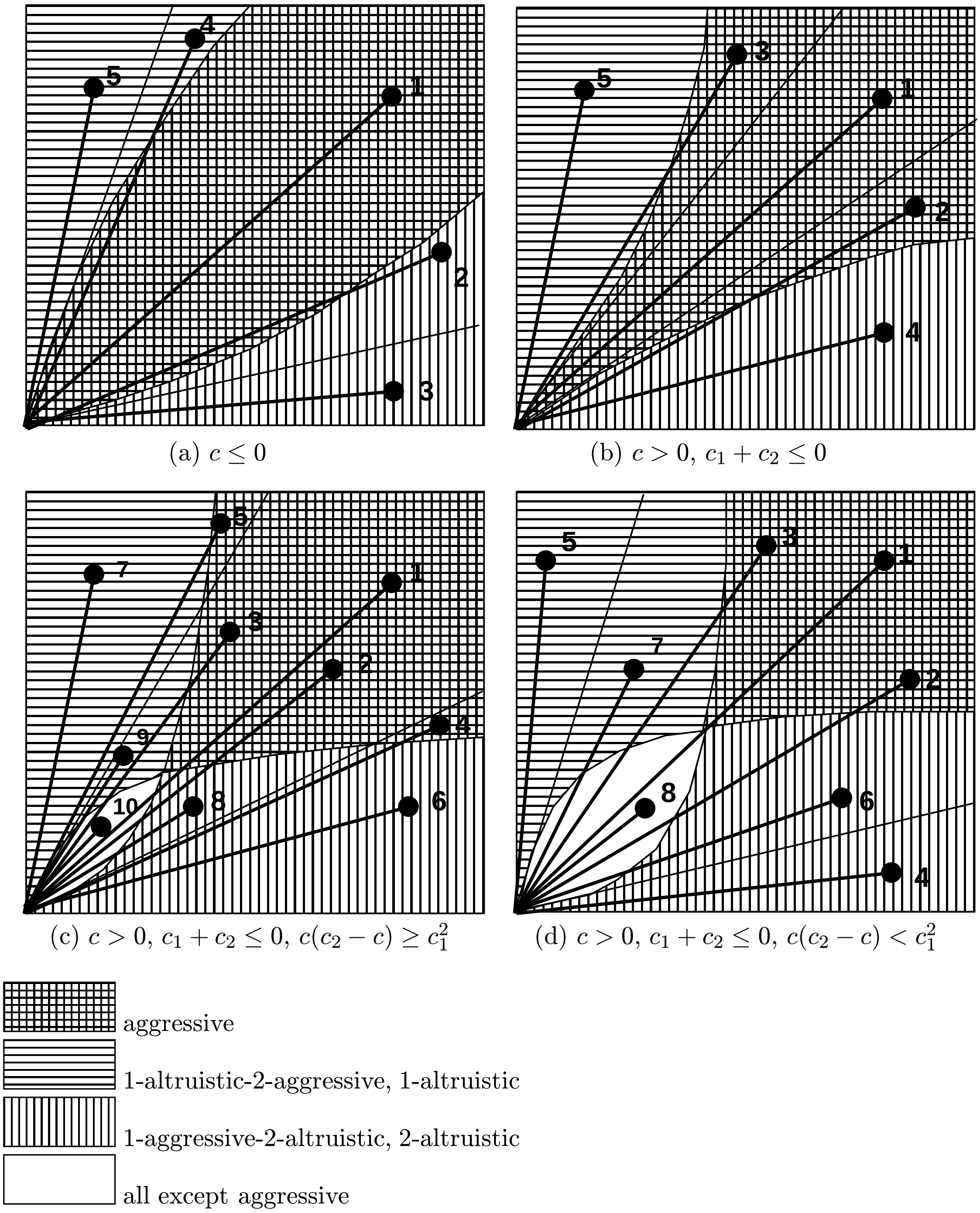
2) *if $c > 0$ and $c_1 + c_2 \le 0$, then*
(i) *all infinite fictitious play trajectories originating from $\bar{E}_1 \cap \bar{E}_2$ are aggressive,*
(ii) *all infinite fictitious play trajectories originating from $G^{--} \setminus [\bar{E}_1 \cap \bar{E}_2]$ have finite nonzero minimum measures of aggressive behavior, and*
(iii) *all infinite fictitious play trajectories originating from $S \setminus G^{--}$ have the zero minimum measure of aggressive behavior,*

3) *if $c > 0$ and $c_1 + c_2 > 0$, then*
(i) *all infinite fictitious play trajectories originating from G^{--} have finite nonzero minimum measures of aggressive behavior, and*
(ii) *all infinite fictitious play trajectories originating from $S \setminus G^{--}$ have the zero minimum measure of aggressive behavior.*

The analysis of fictitious play trajectories, which leads to Propositions 3.1 and 3.2 is based on Propositions 6.1 – 6.3, 6.9, 6.7, 6.11, 6.13 and 6.15 – 6.17. The results of this analysis are schematically shown in Figure 3.1. Exact formulations are given in Appendix 3 in Propositions 7.1, 7.2, 7.3, and 7.4.

Let us comment Figure 3.1. Infinite fictitious play trajectories go along straight lines and converge to (D, D).

In case (a) five trajectories illustrate the typical situations described in statements 1 – 5 of Proposition 7.1; the numbers of the trajectories are those of the associated statements. Trajectory 1 is aggressive. Trajectory 2 (respectively, 4) starts with a finite number of 2-altruistic or 1-aggressive-2-altruistic (respectively, 1-altruistic or 1-altruistic-2-aggressive) rounds and then develops aggressively. Trajectory 3 (respectively, 5) are 2-altruistic and 1-aggressive-2-altruistic (respectively, 1-altruistic and 1-altruistic-2-aggressive).



(a) $c \leq 0$

(b) $c > 0$, $c_1 + c_2 \leq 0$

(c) $c > 0$, $c_1 + c_2 \leq 0$, $c(c_2 - c) \geq c_1^2$

(d) $c > 0$, $c_1 + c_2 \leq 0$, $c(c_2 - c) < c_1^2$

Figure 3.1: infinite fictitious play trajectories

In case (b) five trajectories illustrate the typical situations described in statements 1 – 5 of Proposition 7.2. Trajectory 1 is aggressive. Trajectory 2 (respectively, 3) starts with a finite number of aggressive rounds and then develops 2-altruistically or 1-aggressively-2-altruistically (respectively, 1-altruistically or 1-altruistically-2-aggressively). Trajectory 4 (respectively, 5) are 2-altruistic and 1-aggressive-2-altruistic (respectively, 1-altruistic and 1-altruistic-2-aggressive).

In case (c) ten trajectories illustrate the typical situations described in statements 1 – 10 of Proposition 7.3. Trajectory 1, which starts on the diagonal, is aggressive. Trajectory 2 (respectively, 3) starts with a finite number of aggressive rounds, few rounds goes 2-altruistically or 1-aggressively-2-altruistically (respectively, 1-altruistically or 1-altruistically-2-aggressively), enters the white "linse" adjoining the "south-west" corner point and exhibits there every basic behavior except aggressive. Trajectory 4 (respectively, 5) starts with a finite number of aggressive rounds and switches to 2-altruistic or 1-aggressive-2-altruistic (respectively, 1-altruistic or 1-altruistic-2-aggressive) behavior. Trajectory 6 (respectively, 7) is 2-altruistic or 1-aggressive-2-altruistic (respectively, 1-altruistic or 1-altruistic-2-aggressive). Trajectory 8 (respectively, 9) starts with a finite number of 2-altruistic or 1-aggressive-2-altruistic (respectively, 1-altruistic or 1-altruistic-2-aggressive) rounds, enters the white "linse" and exhibits there every basic behavior except aggressive. Trajectory 10, which starts in the white "linse", exhibits every basic behavior except aggressive.

In case (d) eight trajectories illustrate the typical situations described in statements 1 – 8 of Proposition 7.4. Trajectory 1 starts on the diagonal; it is aggressive. Trajectory 2 (respectively, 3) starts with a finite number of aggressive rounds, few rounds goes 2-altruistically or 1-aggressively-2-altruistically (respectively, 1-altruistically or 1-altruistically-2-aggressively), and exhibits every basic behavior except aggressive within the white "linse". Trajectory 4 (respectively, 5) is 2-altruistic or 1-aggressive-2-altruistic (respectively, 1-altruistic or 1-altruistic-2-aggressive). Trajectory 6 (respectively, 7) starts with a finite number of 2-altruistic or 1-aggressive-2-altruistic (respectively, 1-altruistic or 1-altruistic-2-aggressive) rounds and exhibits every basic behavior except aggressive within the white "linse". Trajectory 8 starts in the white "linse" and exhibits every basic behavior except aggressive.

Statement 1) of Proposition 3.1 follows from Propositions 7.1 and 7.2 (see Figure 3.1, (a) and (b)) and statement 2) from Propositions 7.3 and 7.4. (see Figure 3.1, (c) and (d)). Statement 1) of Proposition 3.2 follows from Proposition 7.1 (see Figure 3.1, (a)) and statement 2) from Propositions 7.2, 7.3 and 7.4. (see Figure 3.1, (b), (c) and (d)).

Propositions 3.1 and 3.2 indicate that the lower is the sum $c_1 + c_2 = 2f_{DD} - f_{CD} - f_{DC}$ (see (2.2)), the less fictitious play trajectories exhibit normal behavior and the more fictitious play trajectories exhibit aggressive behavior.

4 Optimal paths to cooperation

4.1 Problem of optimal behavior

The more frequently the strategy pair (C, C) is chosen in the repeated Prisoner's Dilemma, the less conflict are the interactions between the players. The trajectories, along which the frequency of (C, C) grows to infinity and dominates those of other strategy pairs, are mostly favorable for the players. We shall view such trajectories, which are obviously convergent to (C, C), as desirable.

Let us be more specific. Assume that the players start the repeated Prisoner's Dilemma from a fixed state (x_*, y_*) in round k_0. Referring to subsection 2.7, we define the desired trajectories as all those, which originate from (x_*, y_*) in round k_0 and converge to (C, C); T will denote the set of all desirable trajectories. In this section we shall solve the problem of minimizing the measure of abnormal behavior (see subsection 2.7). Namely, we shall find $\mu_{\min}$, the minimum measure of abnormal behavior on T, and describe all optimal trajectories from T.

Let us give a preliminary argument. If the initial state lies in the kernel-active domain of normal behavior, $(x_*, y_*) \in \mathcal{G}^{00}_\infty$, then by Proposition 6.4, 4), 5), there exists an infinite normal trajectory, t, moving towards (C, C) in every round. This trajectory is desirable, and the minimum measure of abnormal behavior on t is zero. Therefore t is optimal. Moreover, any other desirable trajectory has a nonzero measure of abnormal behavior on it and is therefore not optimal. If $(x_*, y_*) \notin \mathcal{G}^{00}_\infty$, a solution is less obvious. If (x_*, y_*) does not belong to the active domain of normal behavior, G^{00}, then every desirable trajectory starts with abnormal behavior. Which abnormal behavior should be in the start of an optimal trajectory? If (x_*, y_*) lies in the active but not kernel-active domain of normal behavior, $(x_*, y_*) \in G^{00} \setminus \mathcal{G}^{00}_\infty$, then a desirable trajectory may start with several basic behaviors – including normal. Should the players start with normal behavior? One can hardly give immediate intuitive answers to these questions.

4.2 Assumptions

In our analysis, we restrict ourselves to the case where $c \leq 0$. In the previous subsection we noted that if $(x_*, y_*) \in \mathcal{G}^{00}_\infty$, then an infinite normal trajectory $t \in T$ moving towards (C, C) in every round is a unique optimal trajectory and the minimum measure of abnormal behavior on it is zero. We leave aside this trivial situation and assume that $(x_*, y_*) \notin \mathcal{G}^{00}_\infty$. We also assume that (x_*, y_*) is above the diagonal, $x_* < y_*$ (a symmetric situation is treated similarly), and is not among the edge points (C, C), (D, D), (D, C), ($(x_*, y_*) \neq (C, D)$ is implied by the previous assumption). Finally, we assume that the initial round, k_0, is large enough.

Let us specify the latter assumption. By Proposition 6.1, 4), the kernel-active domain of normal behavior, $\mathcal{G}^{00}_\infty$, is the set of all $(x, y) \in S \setminus \{(C, C), (D, C), (C, D)\}$ satisfying the inequalities (6.10) with β and γ given by (6.11). On the square S, $\mathcal{G}^{00}_\infty$ looks like a diagonal-symmetric "road" towards (C, C); the "road" is bordered by the straight lines $y = \beta x + \gamma$ and $x = \beta y + \gamma$, which cross at (C, C) and represent the "north-west" and "south-east" boundaries of the "road". Let us consider a trajectory t^*, which originates from (x_*, y_*) in round k_0 and moves towards (C, D) in every round. The trajectory t^* moves "south-east" and crosses the "road". We require that t^* visits the "road". More accurately, we assume the next condition to be satisfied.

Crossing condition. The trajectory $t^* = ((x^*_k, y^*_k))$, which originates from (x_*, y_*) in round k_0 and moves towards (C, D) in every round, visits $\mathcal{G}^{00}_\infty$, i.e., $(x^*_k, y^*_k) \in \mathcal{G}^{00}_\infty$ in some round k.

The Crossing Condition is satisfied if the initial round, k_0, is sufficiently large. Indeed, all states on the trajectory t^* lie on the segment I_* with the end points (x_*, y_*) and (C, D). The segment I_* has evidently a solid intersection with the "road" $\mathcal{G}^{00}_\infty$; more accurately, $\mathcal{G}^{00}_\infty$ contains a subinterval $I \subset I_*$ of nonzero length. The distance between the states (x_k, y_k) and (x_{k+1}, y_{k+1}) on t is obviously no greater than $2^{1/2}/(k+1)$. Setting k_0 so large that $2^{1/2}/(k_0 + 1)$ is smaller than the length of I, we get that some state on t^* lies in I; hence, the Crossing Condition is satisfied.

4.3 Optimal trajectories

We shall say that a trajectory $t = ((x_k, y_k))$ *moves normally* (1-*altruistically*, etc.) in round k if its one-round subtrajectory $((x_k, y_k), (x_{k+1}, y_{k+1}))$ is normal (1-altruistic, etc.). Let F denote the class of all desirable trajectories $t = ((x_k, y_k))$, which visit $\mathcal{G}^{00}_\infty$ in some round s, i.e., $(x_s, y_s) \in \mathcal{G}^{00}_\infty$, and move normally (towards (C, C)) in every round $k \geq s$. Crossing Condition implies that F is nonempty. The minimal measure of abnormal behavior on every $t \in F$ is obviously finite. The next lemma is proved in Appendix 4.

Lemma 4.1 *All optimal trajectories lie in F.*

For every trajectory $t \in F$, we denote by $\nu_k(t)$ the number of all rounds $r = k_0, \ldots, k-1$, in which t moves not normally; we set $\nu_{k_0}(t) = 0$. We shall use the Bellman approach to characterize the optimal trajectories and the minimum measure of abnormal behavior, $\mu_{\min}$. A function $V : (k, x_k, y_k) \mapsto V(k, x_k, y_k) : \{k_0, k_0 + 1, \ldots\} \times S \mapsto \{0, 1, \ldots\}$ will be called a *Bellman function* if

(i) $V(k, x_k, y_k) = 0$ provided $(x_k, y_k) \in \mathcal{G}^{00}_\infty$,

(ii) for every trajectory $t = ((x_k, y_k)) \in F$

$$\nu_{k+1}(t) + V(k+1, x_{k+1}, y_{k+1}) \geq \nu_k(t) + V(k, x_k, y_k) \quad (k = k_0, \ldots, s_V(t) - 1)$$

where

$$s_V(t) = \min\{r = k_0, k_0 + 1, \ldots : V(r, x_r, y_r) = 0\}$$

((i) implies that the definition of $s_V(t)$ is correct), and

(iii) the set F_V of all $t = ((x_k, y_k)) \in F$ such that

$$\nu_k(t) + V(k, x_k, y_k) = V(k_0, x_{k_0}, y_{k_0}) \quad (k = k_0, \ldots, s_V(t))$$

and $(x_{s_V(t)}, y_{s_V(t)}) \in \mathcal{G}^{00}_\infty$ is nonempty.

Proposition 4.1 *Let V be a Bellman function. Then $\mu_{\min} = V(k_0, x_*, y_*)$ and F_V is the set of all optimal trajectories.*

The proposition is proved in Appendix 4.

Now our goal will be to find a Bellman function, V. Observing the definition of V, one may find it reasonable to identify $V(k, x_k, y_k)$ with the first round, in which an abnormal trajectory, τ, originating from (x_k, y_k) in round k can reach $\mathcal{G}^{00}_\infty$. We shall, generally, follow this intuition. Let us make two simplifying assumptions. First, we consider only the states (x_k, y_k) located above the diagonal. Second, we replace the requirement that τ visits $\mathcal{G}^{00}_\infty$ by a weaker requirement that it crosses the "north-west" border of $\mathcal{G}^{00}_\infty$ denoted further by L_0; recall that L_0 is described by the equation $y = \beta x + \gamma$. For (x_k, y_k) located above the diagonal a trajectory τ, which crosses L_0 within a minimum number of rounds, should most likely move in the direction "maximally orthogonal" to L_0. This happens when τ moves towards (C, D), the "south-east" corner of the square S.

Basing on this informal judgement, for every state (x_k, y_k) above the diagonal ($x_k < y_k$) and every round number $k \geq k_0$, we introduce the infinite trajectory $\tau = \tau(k, x_k, y_k) = ((\xi_r, \eta_r))$ originating from (x_k, y_k) in round k and moving towards (C, D), and define $p(k, x_k, y_k)$ to be the minimum round r, in which (ξ_r, η_r) lies "below" L_0, more accurately,

$$p(k, x_k, y_k) = \min\{r \geq k : \eta_r \leq \beta\xi_r + \gamma\}.$$

The length of τ before crossing L_0 is $p(k, x_k, y_k) - k$. Therefore, our guess is

$$V(k, x_k, y_k) = p(k, x_k, y_k) - k. \tag{4.1}$$

From the definition of $p(k, x_k, y_k)$ it follows straightforwardly that the function V given by (4.1) satisfies condition (i) from the definition of a Bellman function. Let us consider a trajectory $\omega \in F$, which moves towards (C, D) in rounds $k_0, \ldots, p(k_0, x_*, y_*) - 1$ (until L_0 is crossed) and moves normally in rounds $p(k_0, x_*, y_*), p(k_0, x_*, y_*) + 1, \ldots$; we shall call ω the *reference* trajectory. Obviously, the reference trajectory lies in the set F_V. We see that the function V given by (4.1) satisfies condition (iii) with $t = \omega$. Let us fix these observations.

Lemma 4.2 *The function V given by (4.1) satisfies conditions (i) and (iii) from the definition of a Bellman function; in particular, F_V contains the reference trajectory ω.*

In order to state that V given by (4.1) satisfies condition (ii), we study the index $p(k, x_k, y_k)$ in more detail. The next lemma, which is proved in Appendix 4, gives an explicit formula for this index. Below, for a real z, $[z]_+$ denotes the minimal nonnegative integer no smaller than z:

$$[z]_+ = \min\{q = 0, 1, \ldots : q \geq z\}.$$

Lemma 4.3

$$p(k, x_k, y_k) = [(\beta(1 - x_k) + y_k)k]_+. \tag{4.2}$$

The analysis of the formula (4.2) allows to estimate changes of $p(k, x_k, y_k)$ in all one-round transitions.

Lemma 4.4 *Let $t = ((x_k, y_k))$ be a trajectory, $p_k = p(k, x_k, y_k)$, and in some round k the state (x_k, y_k) lie "above" L_0, the "north-west" boundary of $\mathcal{G}^{00}_\infty$, i.e., $y_k \geq \beta x_k + \gamma$. The next statements hold true:*

(i) *if t moves towards (C, D) in round k, then $p_{k+1} = p_k$,*
(ii) *if t moves towards (C, C) in round k, then $p_{k+1} = p_k + 1$,*
(iii) *if t moves towards (D, C) in round k, then $p_{k+1} \in \{p_k + 1, p_k + 2\}$,*
(iv) *if t moves towards (D, D) in round k and*

$$[z_k + \beta]_+ = [z_k]_+ \tag{4.3}$$

where

$$z_k = (\beta(1 - x_k) + y_k)k, \tag{4.4}$$

then $p_{k+1} = p_k$,
(v) *if t moves towards (D, D) in round k and*

$$[z_k + \beta]_+ > [z_k]_+, \tag{4.5}$$

then $p_{k+1} = p_k + 1$.

The lemma is proved in Appendix 4.

Lemmas 4.2 and 4.4 easily imply the next key statement.

Proposition 4.2 *The function V given by (4.1) is a Bellman function.*

The proof is given in Appendix 4.

Proposition 4.2, 4.1 and Lemma 4.2 imply that the reference trajectory, ω, is optimal. Now our goal will be to describe all optimal trajectories.

Combining Propositions 4.2, 4.1 and Lemma 4.4, we easily select trajectories, which are not optimal. Namely, the following statement is proved in Appendix 4.

Corollary 4.1 *Let a trajectory $t = ((x_k, y_k)) \in F$ satisfy one of the next conditions in round $k \le s_V(t)$:*

(i) *t moves not normally towards (C, C) or towards (D, C),*

(ii) *t moves not normally towards (D, D) and for z_k given by (4.4) the inequality (4.5) holds.*

Then t is not optimal.

Let us denote by F^0 the set of all trajectories $t \in F$ that do not satisfy the nonoptimality conditions of Corollary 4.1. More accurately, F^0 is the set of all trajectories $t = ((x_k, y_k)) \in F$ such that in every round k one of the next conditions is satisfied:

(i) t moves normally (towards (C, C)),

(ii) t moves (not normally) towards (C, D),

(iii) t moves (not normally) towards (D, D) provided the equality (4.3) holds for z_k given by (4.4).

Our final statement is as follows.

Proposition 4.3 *The class of all optimal trajectories is F^0.*

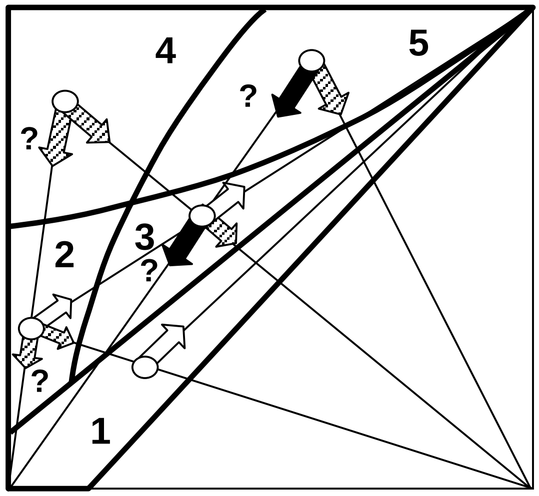
A proof given in Appendix 4 is based on Proposition 4.2, Lemma 4.4 and the next technical lemma, which is also proved in Appendix 4.

Lemma 4.5 *For every $t = ((x_k, y_k)) \in F^0$ and every round $k \ge k_0$ such that $k < s_V(t)$, the trajectory $\tau(k, x_k, y_k) = ((\xi_r^k, \eta_r^k))$ originating from (x_k, y_k) in round k and moving towards (C, D) visits $\mathcal{G}_\infty^{00}$ in round $p_k = p(k, x_k, y_k)$, i.e., $(\xi_{p_k}^k, \eta_{p_k}^k) \in \mathcal{G}_\infty^{00}$.*

4.4 Optimal behavior

Let us specify, how must the players behave when driving an optimal trajectory $t = ((x_k, y_k)) \in F^0$ (see Proposition 4.3), given that the initial state (x_*, y_*) is located to the "north-west" of the "cooperation road" $\mathcal{G}_\infty^{00}$ and the initial round k_0 is sufficiently large (Crossing Condition is satisfied).

The players must behave normally in every round k, in which the state (x_k, y_k) lies on the "road" $\mathcal{G}_\infty^{00}$. In every round k, in which $(x_k, y_k) \notin \mathcal{G}_\infty^{00}$, the players must choose between modes (i), (ii) and (iii) described in the definition of F^0. In mode (i) the players behave normally. This mode is admissible if the state (x_k, y_k) lies in the active domain of normal behavior, G^{00} (see Proposition 6.4 and Figure 3.2, case $c \le 0$). In mode (ii) the players act (C, D). This mode is compatible with 1-altruistic behavior (player 1 behaves altruistically and player 2 normally) and 1-altruistic-2-aggressive behavior (player 1 behaves altruistically and player 2 aggressively). Mode (ii) is admissible for every location of $(x_k, y_k) \notin \mathcal{G}_\infty^{00}$ (see Propositions 6.10 and 6.14 and Figures 3.4 and 3.6). In mode (iii) the players act (D, D). This mode is compatible with 1-altruistic behavior and 1-altruistic-2-aggressive behavior if (x_k, y_k) lies in the domain $H_{DD}(g)$ (see Propositions 6.10 and 6.14 and Figures 3.4 and 3.6), and it is compatible with aggressive behavior if (x_k, y_k) lies in the active domain of aggressive behavior, G^{--} (see Proposition 6.18 and Figure 3.7, case $c \le 0$). The interiors of the domains $H_{DD}(g)$ and G^{--} do not intersect, and the union of these domains covers the whole space "above" the "north-west" border of the "road" $\mathcal{G}_\infty^{00}$. Therefore, mode (iii) is admissible for every location of the state $(x_k, y_k) \notin \mathcal{G}_\infty^{00}$ subject to the constraint that the value z_k (4.4) satisfies the equality (4.3). If the latter constraint is not satisfied, the choice of mode (iii) brings the players away from an optimal trajectory. The players must "look one round forward" (verify the constraint (4.3) before choosing mode (iii). A geometric characterization of the optimal behaviors is schematically shown in Figure 4.1.



normal
1-altruistic-2-aggressive, 1-altruistic
aggressive
? subject to constraint (4.3)

Figure 4.1: a geometric characterization of the optimal behaviors

We conclude with a comment to Figure 4.1. Due to Crossing Condition, the optimal trajectories never enter the area below the "cooperation road" bordered by the two straight lines. The whole square S without this unessential area is split into five domains. The arrows originating from each of these domains point to the strategy pairs admissible on the optimal trajectories. Domain 1 is the "road". The part of the active domain of normal behavior, which is located above the "road" and below the bordering curve crossing the "north-east" corner of the square (see Figure 2.1), is split into domains 2 and 3 by the bordering curve crossing the "south-west" corner of the square. In a similar manner the part of the still domain of normal behavior, which is located above the active domain of normal behavior, is split into domains 4 and 5. In domains 2 – 5 the arrows pointing to mutual defection, (D, D), are marked with "?", which reminds us that beyond the "road" the optimal behavior admits mutual defection only if the constraint (4.3) is fulfilled. In domains 2 and 4 mutual defection represents 1-altruistic and 1-altruistic-2-aggressive behaviors, and in domains 3 and 5 aggressive behavior. Everywhere above the "road" "maximum" altruism of player 1 (player 1 cooperates and player 2 defects) is admissible, and in domains 2 and 3, which adjoin the "road", mutual cooperation, (C, C), is admissible. It is interesting that in domains 4 and 5, which are far away from the "road", mutual cooperation is not admissible. An intuitive explanation is that in these domains it is "too early" to adopt mutual cooperation because the empirical frequency of cooperation of player 1 is too low compared to player 2 (player 1 was too less cooperative than player 2 in the past). Less intuitively clear is the fact that in these domains the optimal behavior (under some circumstances) is compatible with mutual defection.

5 Appendix 1. Cooperative game dynamics

5.1 Proof of Proposition 1.1

Clearly, (i_*, j_*) is a single strategy pair (φ, ψ)-acceptable at (x_*, y_*). Let $t = ((x_k, y_k))$ be an arbitrary nonextendable (φ, ψ)-trajectory originating from (x_*, y_*) in some round k_0. As long as (i_*, j_*) is a single strategy pair (φ, ψ)-acceptable at (x_*, y_*), the latter is (φ, ψ)-active and $(x_{k_0+1}, y_{k_0+1}) = (x_*, y_*)$. Now we easily show by induction that t is infinite and stationary. Thus, (x_*, y_*) is stationary (φ, ψ)-kernel-active. The proposition is proved.

5.2 Proof of Proposition 1.2

Necessity. Let $(x, y) \neq (x_*, y_*)$ be nonstationary (φ, ψ)-kernel active. Let E be the closed segment with the end points (x, y) and (x_*, y_*) and A stand for the (φ, ψ)-active domain. Suppose $E \not\subset A$. Since A is, obviously, closed, there is a point in E whose open neighborhood V (in S) does not intersect A. Taking into account that the state (x, y) is (φ, ψ)-kernel active and by assumption (i_*, j_*) is (φ, ψ)-acceptable at every (φ, ψ)-active state, we conclude that for every natural k_0 there is an infinite trajectory $t = ((x_k, y_k))$ originating from (x, y) in round k_0 and such that in each round k a strategy pair (i_{k+1}, j_{k+1}) in the state adjustment rule (1.1) – (1.4) is (i_*, j_*). By assumption (see (1.7)) $x_*^{i_*} = 1$, $x_*^i = 0$ $(i \neq i_*)$, and $y^{j_*} = 1$, $y^j = 0$ $(j \neq j_*)$. Hence, (1.1) – (1.4) imply

$$x_{k+1} = \left(1 - \frac{1}{k+1}\right) x_k + \frac{1}{k+1} x_*, \quad y_{k+1} = \left(1 - \frac{1}{k+1}\right) y_k + \frac{1}{k+1} y_*, \tag{5.1}$$

i.e., (x_{k+1}, y_{k+1}) lies on the segment with the end points (x_k, y_k) and (x_*, y_*). Now we easily state by induction that in every round k the state (x_k, y_k) lies on the segment E with the end points (x, y) and (x_*, y_*). Obviously,

$$|x_{k+1} - x_k| = \frac{1}{k+1}|x_* - x_k| \leq \frac{1}{k_0+1}|x_* - x_{k_0}|, \tag{5.2}$$

$$|y_{k+1} - y_k| = \frac{1}{k+1}|x_* - y_k| \leq \frac{1}{k_0+1}|y_* - y_{k_0}|. \tag{5.3}$$

Thus, (x_{k_0}, y_{k_0}) coincides with the E's end point (x, y), the points $(x_k, y_k) \in E$ converge to the E's end point (x_*, y_*), and the distance between (x_k, y_k) and (x_{k+1}, y_{k+1}) is arbitrarily small if k_0 is sufficiently large. Then, for a sufficiently large k_0, (x_k, y_k) is in the open interval $E \cap V$ in some round k. By the definition of the neighborhood V, this state is not (φ, ψ)-active, i.e., (φ, ψ)-still. Consequently, (x_k, y_k) is the final state of the trajectory t. We obtained that t is finite, whereas by assumption t is infinite. The contradiction proves that the segment E is contained in the (φ, ψ)-active domain A.

Sufficiency. Let $E \subset A$. For arbitrary natural k_0, let an infinite trajectory $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ originating from $(x, y) \neq (x_*, y_*)$ be defined by (1.1) – (1.4) where $(i_{k+1}, j_{k+1}) = (i_*, j_*)$. As in the previous argument, we easily arrive at (5.1), which shows that $(x_k, y_k) \in E$ for every $k \geq k_0$. Then for every $k \geq k_0$ (x_k, y_k) lies in the (φ, ψ)-active domain A. By assumption (i_*, j_*) is (φ, ψ)-acceptable at every (φ, ψ)-active state. Hence, (i_*, j_*) is (φ, ψ)-acceptable at (x_k, y_k). We obtained that t is an infinite (φ, ψ)-trajectory. Moreover, (5.1) and the fact that $(x, y) \neq (x_*, y_*)$ show that t is nonstationary. Thus, (x, y) is nonstationary (φ, ψ)-kernel-active. The proposition is proved.

5.3 Proof of Proposition 1.3

Let us consider arbitrary state (x_k, y_k) and arbitrary strategy pair (i_{k+1}, j_{k+1}). It is sufficient to show that (i_{k+1}, j_{k+1}) is (φ, ψ)-acceptable at (x_k, y_k) for some $(\varphi, \psi) \in \{(f, g), (g, g), (f, f), (g, -f), (-g, f), (-g, -f)\}$ (we shall see that in fact (φ, ψ) can be restricted to $\{(f, f), (g, -f)(-g, -f)\}$). If (i_{k+1}, j_{k+1}) is f-acceptable at (x_k, y_k) then (i_{k+1}, j_{k+1}) is (f, f)-acceptable (we shall no longer mention (x_k, y_k) in this proof). Let (i_{k+1}, j_{k+1}) be not f-acceptable. Then (i_{k+1}, j_{k+1}) is $-f$-acceptable. If (i_{k+1}, j_{k+1}) is g-acceptable, then (i_{k+1}, j_{k+1}) is $(g, -f)$-acceptable. If (i_{k+1}, j_{k+1}) is not g-acceptable then (i_{k+1}, j_{k+1}) is $-g$-acceptable; consequently, (i_{k+1}, j_{k+1}) is $(-f, -g)$-acceptable. The proposition is proved.

6 Appendix 2. Cooperative dynamics in repeated Prisoner's Dilemma

6.1 Proof of Lemma 2.1

Using (2.2), we get

$$h_{CC}(1) = 1. \tag{6.1}$$

Obviously

$$h_{CC}(0) > 0, \quad h_{DD}(0) = 0 \tag{6.2}$$

By (2.1) and (2.3) the numerator in (2.4) is greater than the numerator in (2.5), and the latter numerator is positive everywhere except 0. Hence,

$$h_{CC}(x) > h_{DD}(x) \geq 0, \quad h_{DD}(x) > 0 \quad (x > 0).$$

We have

$$h'_{CC}(x) = \frac{c_1(cx - c_2) - c(f_{CC} - f_{DD} + c_1 x)}{(cx - c_2)^2}.$$

The numerator is transformed as follows:

$$\begin{aligned} c_1(cx - c_2) - c(f_{CC} - f_{DD} + c_1 x) &= -c_1 c_2 - c(f_{CC} - f_{DD}) \\ &= -c_1 c_2 - c(c - c_1 - c_2) \\ &= (c_1 - c)(c - c_2) > 0; \end{aligned}$$

the inequality follows from (2.3). Hence,

$$h'_{CC}(x) = \frac{(c_1 - c)(c - c_2)}{(cx - c_2)^2} > 0, \tag{6.3}$$

in particular,

$$h'_{CC}(1) = \frac{c_1 - c}{c - c_2} = \frac{f_{DC} - f_{CC}}{f_{CC} - f_{CD}} < 1; \tag{6.4}$$

the inequality follows from (2.1). From (6.3) and (6.1) we get

$$h_{CC}(x) < 1 \quad (x \in [0, 1)). \tag{6.5}$$

Furthermore,

$$h''_{CC}(x) = -2c\frac{(c_1 - c)(c - c_2)}{(cx - c_2)^3}.$$

The ratio on the right is positive, i.e., $h''_{CC}(x)$ has the sign of $-c$. Hence, h_{CC} is strictly convex if $c < 0$, linear if $c = 0$ and strictly concave if $c > 0$.

If $c < 0$, then (6.4) and the convexity of h_{CC} imply $h'_{CC}(x) < 1$ for all $x \in [0, 1]$. Hence, in view of (6.1),

$$h_{CC}(x) > x \quad (x \in [0, 1)). \tag{6.6}$$

If $c \geq 0$, then (6.4) the inequality $h_{CC}(0) > 0$ (see (6.2)) and the concavity of h_{CC} again imply (6.6). Thus, (6.6) always holds true.

Let us turn to the function h_{DD}. We have

$$h'_{DD}(x) = \frac{c_1(cx - c_2) - cc_1x}{(cx - c_2)^2} = \frac{-c_1c_2}{(cx - c_2)^2} > 0,$$

$$h''_{DD}(x) = 2c\frac{c_1c_2}{(cx - c_2)^3}.$$

The ratio on the right is negative (see (2.2)), i.e., $h''_{DD}(x)$ has the sign of $-c$. Hence, h_{DD} is strictly convex if $c < 0$, linear if $c = 0$ and strictly concave if $c > 0$.

Let us identify points $x > 0$ such that $x \leq h_{DD}(x)$. The latter inequality is equivalent to

$$x \leq \frac{c_1x}{cx - c_2}$$

(see (2.5)) or

$$cx \leq c_2 + c_1. \tag{6.7}$$

Note that by (2.1) and (2.3)

$$c > c_1 + c_2. \tag{6.8}$$

Let $c_1 + c_2 \leq 0$. If $c \leq 0$, then $cx \geq c > c_1 + c_2$, and (6.7) does not hold. If $c > 0$, then $cx > 0 \geq c_1 + c_2$, and (6.7) does not hold either. Let $c_1 + c_2 > 0$. Then (6.7) is equivalent to $x \leq (c_2 + c_1)/c$. By (6.8) the right hand side is smaller than 1. We summarize as follows:

$$h_{DD}(x) < x \quad (x \in (0, 1]) \quad \text{if } c_1 + c_2 \leq 0,$$

$$h_{DD}(x) \geq x \quad (x \in (0, (c_1 + c_2)/c]),$$

$$h_{DD}(x) < x \quad (x \in ((c_1 + c_2)/c, 1]) \quad \text{if} \quad c_1 + c_2 > 0.$$

6.2 Characterization of normal trajectories

The segment $\{(x, y) \in S : x = y\}$ will further be called the *diagonal.*

Proposition 6.1 *Let $c < 0$. The next statements hold true:*

1) G^{00}*, the active domain of normal behavior, is given by*

$$G^{00} = [H_{CC}(f) \cap H_{CC}(g)] \cup \{(C, C), (D, D), (C, D), (D, C)\}, \tag{6.9}$$

2) (C, C) *and* (D, D) *are acceptable for normal behavior ((f, g)-acceptable) at* (D, D)*, and for every* $(x, y) \in G^{00} \setminus \{(D, D), (D, C), (C, D)\}$*,* (C, C) *is a single strategy pair acceptable for normal behavior at* (x, y)*,*

3) *the states* (C, C)*,* (D, C) *and* (C, D) *are stationary for normal behavior,*

4) $\mathcal{G}^{00}_{\infty}$*, the nonstationary kernel-active domain of normal behavior, is the set of all* $(x, y) \in S \setminus \{(C, C), (D, C), (C, D)\}$ *satisfying*

$$y \leq \beta x + \gamma, \quad x \leq \beta y + \gamma \tag{6.10}$$

where

$$\beta = h'_{CC}(1) = \frac{f_{DC} - f_{CC}}{f_{CC} - f_{CD}} \in (0,1), \quad \gamma = 1 - \beta = \frac{2f_{CC} - f_{CD} - f_{DC}}{f_{CC} - f_{CD}} \in (0,1), \quad (6.11)$$

5) $\mathcal{G}^{00}_\infty \cup \{(C,C)\}$ *contains the diagonal and is strictly contained in* $G^{00} \setminus \{(D,C),(C,D)\}$,

6) G^{00}_∞, *the kernel-active domain of normal behavior, is given by*

$$G^{00}_\infty = \mathcal{G}^{00}_\infty \cup \{(C,C),(D,C),(C,D)\} \quad (6.12)$$

and is strictly contained in G^{00}.

Proof. 1. Obviously,

$$G^{00} = [H_{CC}(f) \cap H_{CC}(g)] \cup [H_{DD}(f) \cap H_{DD}(g)] \cup [H_{DC}(f) \cap H_{DC}(g)] \cup [H_{CD}(f) \cap H_{CD}(g)]$$

Since $h_{CC}(x) > h_{DD}(x)$ (see (2.7)), by (2.10), (2.11), (2.14), (2.15) we have $H_{DD}(f) \subset H_{CC}(f)$ and $H_{DD}(g) \subset H_{CC}(g)$. Hence,

$$H_{DD}(f) \cap H_{DD}(g) \subset H_{CC}(f) \cap H_{CC}(g).$$

In view of (2.7)

$$\{(D,D),(C,C)\} \subset H_{CC}(f) \cap H_{CC}(g).$$

By (2.13), (2.17), (2.12), (2.16)

$$H_{DC}(f) \cap H_{DC}(g) = \{(D,C)\}, \quad H_{CD}(f) \cap H_{CD}(g) = \{(C,D)\}. \quad (6.13)$$

The observed relations yield

$$G^{00} = H_{CC}(f) \cap H_{CC}(g) \cap \{(C,C),(D,D),(C,D),(D,C)\}.$$

Statement 1 is proved.

2. Since $h_{CC}(0) \geq 0$ and $h_{DD}(0) = 0$ (see (2.7) and (2.6)), (C,C) and (D,D) are acceptable for normal behavior ((f,g)-acceptable) at (D,D). Let $(x,y) \in G^{00} \setminus \{(D,D),(D,C),(C,D)\}$. By statement 1 (C,C) is (f,g)-acceptable at (x,y). The relations (6.13) imply that (D,C) and (C,D) are not (f,g)-acceptable at (x,y). Suppose (D,D) is (f,g)-acceptable at (x,y). We have $c_1 + c_2 < 0$, since $c_1 + c_2 < c$ and $c \leq 0$ by assumption. Then, taking into account that $(x,y) \neq (D,D) = (0,0)$, and referring to (2.8), we get that $h_{DD}(x) \leq x$, and $h_{DD}(y) \leq y$, and at least one of these inequalities holds strictly. As long as (D,D) is (f,g)-acceptable at (x,y), we have $y \leq h_{DD}(x)$ and $x \leq h_{DD}(y)$. Hence, $y \leq x$ and $x \leq y$, and one of these inequalities holds strictly, which is not possible. A contradiction shows that (D,D) is not (f,g)-acceptable at (x,y). Statement 2 is proved.

3. The states (C,C), (D,C) and (C,D) are obviously (f,g)-Pareto maximal. By Proposition 1.1 they are stationary for normal behavior.

4. By statement 1 (C,C) is (f,g)-acceptable at every state from $\mathcal{G}^{00}_\infty$. Then by Proposition 1.2 a state $(x,y) \neq (C,C)$ belongs to $\mathcal{G}^{00}_\infty$ if and only if the closed segment F with the end points (x,y) and (C,C) is contained in G^{00}, or, equivalently (see statement 1) in $H_{CC}(f) \cap H_{CC}(g)$. Recall that the function h_{CC} is strictly concave. Therefore, F is contained in $H_{CC}(f)$ if and only if (x,y) is located below the line L which runs through (C,C) and has a slope determined by $h'_{CC}(1)$ (see (2.9)). An accurate condition is $y \leq \beta x + \gamma$

where β and γ are given by (6.11). Similarly, we get that F is contained in $H_{CC}(g)$ if and only if $x \le \beta y + \gamma$. Thus $F \subset H_{CC}(f) \cap H_{CC}(g)$ if and only if (6.10) holds. Statement 4 is proved.

5. By definition, the line L runs through (C, C), and due to (2.9) the slope of L is lower than that of daig(S), the diagonal of S. Consequently, L lies above daig(S); equivalently, daig$(S) \subset H_{CC}(f)$. Similarly, we obtain that daig$(S) \subset H_{CC}(g)$. Hence $\mathcal{G}^{00}_\infty$ $\cap\{(C, C)\}$ (see statement 1). Since h_{CC} is strictly concave, the line L (restricted to S) lies strictly below the graph of h_{CC} everywhere except the point (C, C) (see (2.6)). The states located strictly between the line L and the graph of h_{CC} do not belong to $H_{CC}(f)$, and, consequently, $\mathcal{G}^{00}_\infty$. Therefore $\mathcal{G}^{00}_\infty$ is strictly contained in $G^{00} \setminus \cap\{(C, C), (D, C), (C, D)\}$. Statement 5 is proved.

6. Statement 6 follows from statements 5 and 3. The proof is completed.

Proposition 6.2 *Let $c \ge 0$ and $c_1 + c_2 \le 0$. The next statements hold true:*

1) *G^{00}, the active domain of normal behavior, is given by (6.9),*

2) *(C, C) and (D, D) are acceptable for normal behavior ((f, g)-acceptable) at (D, D), and for every $(x, y) \in G^{00} \setminus \{(D, D), (D, C), (C, D)\}$, (C, C) is a single strategy pair acceptable for normal behavior at (x, y),*

3) *the states (C, C), (D, C) and (C, D) are stationary for normal behavior,*

4) *$\mathcal{G}^{00}_\infty$, the nonstationary kernel-active domain of normal behavior, is given by*

$$\mathcal{G}^{00}_\infty = G^{00} \setminus \{(C, C), (D, D), (D, C), (C, D)\}. \tag{6.14}$$

5) *$\mathcal{G}^{00}_\infty \cup \{(C, C)\}$ coincides with $G^{00} \setminus \{(D, C), (C, D)\}$ and contains the diagonal,*

6) *G^{00}_∞, the kernel-active domain of normal behavior, coincides with G^{00}.*

Proof. 1. Statement 1 is proved identically with statement 1 of Proposition 6.1

2. The (f, g)-acceptability of (C, C) and (D, D) at (D, D) is proved like in the previous subsection. Let $(x, y) \in G^{00} \setminus \{(D, D), (D, C), (C, D)\}$. Like in the previous subsection we state that (C, C) is (f, g)-acceptable and (D, C) and (C, D) are not (f, g)-acceptable at (x, y). Suppose (D, D) is (f, g)-acceptable at (x, y). We have $c_1 + c_2 \le 0$ by assumption. Then, taking into account that $(x, y) \ne (D, D) = (0, 0)$, and referring to (2.8), we get $h_{DD}(x) \le x$, and $h_{DD}(y) \le y$; moreover, at least one of these inequalities holds strictly. Arguing as in the previous subsection, we arrive at a contradiction.

3. Statement 3 is proved identically with statement 3 of Proposition 6.1.

4. By statement 1 (C, C) is (f, g)-acceptable at every state from $\mathcal{G}^{00}_\infty$. Then by Proposition 1.2 a state $(x, y) \ne \{(C, C)$ belongs to $\mathcal{G}^{00}_\infty$ if and only if the closed segment F with the end points (x, y) and (C, C) is contained in G^{00}, or, equivalently (see statement 1) in $H_{CC}(f) \cap H_{CC}(g)$ Recall that the function h_{CC} is convex. Therefore $F \subset H_{CC}(f) \cap H_{CC}(g)$ for arbitrary $(x, y) \in G^{00} \setminus \{(C, C), (D, D), (D, C), (C, D)\}$. Statement 4 is proved.

5. Let $B = \mathcal{G}^{00}_\infty \cap \{(C, C)\}$. The equality $B = G^{00} \setminus \{(C, C), (D, C), (C, D)\}$ follows from statement 4. The fact that B contains the diagonal is proved like in the previous subsection.

6. Statement 6 follows from statements 4 and 3.

Proposition 6.3 *Let $c \ge 0$ and $c_1 + c_2 > 0$. The next statements hold true:*

1) *G^{00}, the active domain of normal behavior, is given by (6.9),*

2) (C,C) *and* (D,D) *are acceptable for normal behavior* ((f,g)*-acceptable*) *at every* $(x,y)\in E=H_{DD}(f)\cap H_{DD}(g)$, *and for every* $(x,y)\in G^{00}\setminus[\{(D,C),(C,D)\}\cup E]$, (C,C) *is a single strategy pair acceptable for normal behavior at* (x,y),

3) *the set* $E\setminus\{(D,D)\}$ *is nonempty and is strictly contained in* $\mathcal{G}^{00}_{\infty}\setminus\{(D,D)\}$,

4) *the states* (C,C), (D,C) *and* (C,D) *are stationary for normal behavior,*

5) $\mathcal{G}^{00}_{\infty}$, *the nonstationary kernel-active domain of normal behavior, is given by (6.14),*

6) $\mathcal{G}^{00}_{\infty}\cap\{(C,C)\}$ *coincides with* $G^{00}\setminus\{(C,C),(D,C),(C,D)\}$ *and contains the diagonal,*

7) G^{00}_{∞}, *the kernel-active domain of normal behavior, coincides with* G^{00}.

Proof. All statements except statement 2 are proved identically with those of Proposition 6.1 Let us prove statement 2. Since $h_{DD}(x)<h_{CC}(x)$ (see (2.7)), $E=H_{DD}(f)\cap H_{DD}(g)$ is a subset of $H_{CC}(f)\cap H_{CC}(g)$. Therefore (and due to statement 1) (C,C) and (D,D) are (f,g)-acceptable at every $(x,y)\in E$. Let $(x,y)\in G^{00}\setminus$ $[\{(D,C),(C,D)\}\cup E]$. Like in the previous subsection we state that (C,C) is (f,g)-acceptable and (D,C) and (C,D) are not (f,g)-acceptable at (x,y). Finally, (D,D) is not (f,g)-acceptable at (x,y), since $(x,y)\notin E$.

Propositions 6.1, 6.2 and 6.3 easily lead to the next characterizations of the nonextendable normal trajectories. We shall say that a trajectory $t=((x_k,y_k))$ *moves* towards a point $(\bar{x},\bar{y})\in S$ in round k if, first, k is not the final round of t, and, second, (x_{k+1},y_{k+1}) lies on the segment with the end points (x_k,y_k) and $(\bar{x},\bar{y})$, and does not coincide with (x_k,y_k).

Proposition 6.4 *Let* $c<0$ *and* $t=((x_k,y_k))$ $(k=k_0,\ldots)$ *be a nonextendable normal trajectory originating from a state* (x_*,y_*). *The next statements hold true:*

1) *if* $(x_*,y_*)\notin G^{00}$, *then* (x_*,y_*) *is in the still domain of normal behavior, and* t *has the length* 0,

2) *if* $(x_*,y_*)\in\{(C,C),(D,C),(C,D)\}$, *then* t *is infinite and stationary,*

3) *if* $(x_*,y_*)\in G^{00}\setminus[\{(C,C),(D,C),(C,D)\}\cup G^{00}_{\infty}]$, *then* t *has a finite length greater than* 0 *and moves towards* (C,C) *in every round,*

4) *if* $(x_*,y_*)\in G^{00}_{\infty}\setminus\{(D,D)\}$, *then* t *is infinite and moves towards* (C,C) *in every round,*

5) *if* $(x_*,y_*)=(D,D)$, *then either*
(i) t *is infinite, and stationary, or*
(ii) t *is infinite, its initial finite subtrajectory* t_* *is stationary, and* t *moves towards* (C,C) *in every round* $k\geq k_*$ *where* k_* *is the final round of* t_*.

Proposition 6.5 *Let* $c\geq 0$, $c_1+c_2\leq 0$ *and* $t=((x_k,y_k))$ $(k=k_0,\ldots)$ *be a nonextendable normal trajectory originating from a state* (x_*,y_*). *The next statements hold true:*

1) *if* $(x_*,y_*)\notin G^{00}$, *then* (x_*,y_*) *is in the still domain of normal behavior, and* t *has the length* 0,

2) *if* $(x_*,y_*)\in\{(C,C),(D,C),(C,D)\}$, *then* t *is infinite and stationary,*

3) *if* $(x_*,y_*)\in G^{00}\setminus\{(C,C),(D,D),(D,C),(C,D)\}$, *then* t *is infinite and moves towards* (C,C) *in every round,*

4) *if* $(x_*,y_*)=(D,D)$, *then either*
(i) t *is infinite, and stationary, or*

(ii) *t is infinite, its initial finite subtrajectory t_* is stationary, and t moves towards (C,C) in every round $k \geq k_*$ where k_* is the final round of t_*.*

Proposition 6.6 *Let $c \geq 0$, $c_1 + c_2 > 0$ and $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable normal trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) \notin G^{00}$, then (x_*, y_*) is in the still domain of normal behavior, and t has the length* 1,

2) *if $(x_*, y_*) \in \{(C,C),(D,C),(C,D)\}$, then t is infinite and stationary,*

3) *if $(x_*, y_*) \in G^{00} \setminus \{(C,C),(D,D),(D,C),(C,D)\}$, then t is infinite, moves towards either (C,C) or (D,D) in every round k such that $(x_k, y_k) \in E = H_{DD}(f) \cap H_{DD}(g)$, and moves towards (C,C) in every round k such that $(x_k, y_k) \notin E$.*

6.3 Characterization of 2-altruistic and 1-altruistic trajectories

In the next subsections we characterize the abnormal basic trajectories. We omit proofs, which are similar with those in the previous subsection.

Proposition 6.7 *The next statements hold true:*

1) *G^{0+}, the active domain of 2-altruistic behavior, and G^{0+}_{∞}, the kernel-active domain of 2-altruistic behavior, coincide with the state space S,*

2) *at every $(x,y) \notin H_{CC}(f)$, (D,C) is a single strategy pair acceptable for 2-altruistic behavior ((f,f)-acceptable), at every $(x,y) \in H_{CC}(f) \setminus H_{DD}(f)$, the strategy pairs acceptable for 2-altruistic behavior are (D,C) and (C,C), at every $(x,y) \in H_{DD}(f) \setminus \{(C,D)\}$ the strategy pairs acceptable for 2-altruistic behavior are (D,C), (C,C) and (D,D), and at (C,D) the strategy pairs acceptable for 2-altruistic behavior are (D,C), (C,C), (D,D), and (C,D),*

3) *the state (D,C) is stationary for 2-altruistic behavior,*

4) $\mathcal{G}^{0+}_{\infty}$, the nonstationary kernel-active domain of 2-altruistic behavior, is $S \setminus \{(D,C)\}$.

Proposition 6.7 leads to the next characterization of the nonextendable 2-altruistic trajectories.

Proposition 6.8 *Let $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable 2-altruistic trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) = (D,C)$, then t is infinite and stationary,*

2) *if $(x_*, y_*) \notin \{(D,C),(C,D)\}$, then t is infinite and nonstationary, moreover in each round k*
(i) *t moves towards (D,C) if $(x_k, y_k) \notin H_{CC}(f)$,*
(ii) *t moves towards (D,C) or (C,C) if $(x_k, y_k) \in H_{CC}(f) \setminus H_{DD}(f)$,*
(iii) *t moves towards (D,C), (C,C) or (D,D) if $(x_k, y_k) \in H_{DD}(f) \setminus \{(C,D)\}$,*

3) *if $(x_*, y_*) = (C,D)$, then either*
(i) *t is infinite and stationary, or*
(ii) *t is infinite, its initial finite subtrajectory t_* is stationary, and in every round $k \geq k_*$, where k_* is the final round of t_*, conditions* (i) – (iii) *of statement 2 are satisfied.*

Let us provide symmetric assertions on 1-altruistic behavior.

Proposition 6.9 *The next statements hold true:*

1) G^{+0}, *the active domain of 1-altruistic behavior, and* G^{+0}_{∞}, *the kernel-active domain of 1-altruistic behavior, coincide with the state space* S,

2) *at every* $(x,y) \notin H_{CC}(g)$, (C,D) *is a single strategy pair acceptable for 1-altruistic behavior ((g, g)-acceptable), at every* $(x,y) \in H_{CC}(g) \setminus H_{DD}(g)$, *the strategy pairs acceptable for 2-altruistic behavior are* (C,D) *and* (C,C), *at every* $(x,y) \in H_{DD}(g) \setminus \{(D,C)\}$ *the strategy pairs acceptable for 1-altruistic behavior are* (C,D), (C,C) *and* (D,D), *and at* (D,C) *the strategy pairs acceptable for 1-altruistic behavior are* (C,D), (C,C), (D,D), *and* (D,C),

3) *the state* (C,D) *is stationary for 1-altruistic behavior,*

4) $\mathcal{G}^{+0}_{\infty}$, *the nonstationary kernel-active domain of 1-altruistic behavior, is* $S \setminus \{(C,D)\}$.

Proposition 6.10 *Let* $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ *be a nonextendable 1-altruistic trajectory originating from a state* (x_*, y_*). *The next statements hold true:*

1) *if* $(x_*, y_*) = (C,D)$, *then* t *is infinite and stationary,*

2) *if* $(x_*, y_*) \notin \{(D,C),(C,D)\}$, *then* t *is infinite and nonstationary, moreover in each round* k

(i) t *moves towards* (C,D) *if* $(x_k, y_k) \notin H_{CC}(g)$,
(ii) t *moves towards* (C,D) *or* (C,C) *if* $(x_k, y_k) \in H_{CC}(g) \setminus H_{DD}(g)$,
(iii) t *moves towards* (C,D), (C,C) *or* (D,D) *if* $(x_k, y_k) \in H_{DD}(f) \setminus \{(D,C)\}$,

3) *if* $(x_*, y_*) = (D,C)$, *then either*
(i) t *is infinite and stationary, or*
(ii) t *is infinite, its initial finite subtrajectory* t_* *is stationary, and in every round* $k \geq k_*$, *where* k_* *is the final round of* t_*, *conditions* (i) – (iii) *of statement 2 are satisfied.*

6.4 Characterization of 1-aggressive-2-altruistic and 1-altruistic-2-aggressive trajectories

Proposition 6.11 *The next statements hold true:*

1) G^{-+}, *the active domain of 1-aggressive-2-altruistic behavior, and* G^{-+}_{∞}, *the kernel-active domain of 1-aggressive-2-altruistic behavior, coincide with the state space* S,

2) *at every* $(x,y) \in \bar{H}_{DD}(f) \cap H_{CC}(g)$, (D,C) *is a single strategy pair acceptable for 1-aggressive-2-altruistic behavior ((−g, f)-acceptable), at every* $(x,y) \in H_{DD}(f) \setminus H_{CC}(g)$, *the strategy pairs acceptable for 1-aggressive-2-altruistic behavior are* (D,C) *and* (D,D), *at every* $(x,y) \in H_{CC}(g) \setminus H_{DD}(f)$ *the strategy pairs acceptable for 1-aggressive-2-altruistic behavior are* (D,C) *and* (C,C), *at every* $(x,y) \in [\bar{H}_{CC}(g) \cap H_{DD}(f)] \setminus \{(C,D\}$ *the strategy pairs acceptable for 1-aggressive-2-altruistic behavior are* (D,C), (C,C) *and* (D,D), *and at* (C,D) *the strategy pairs acceptable for 1-aggressive-2-altruistic behavior are* (D,C), (C,C) *and* (D,D) *and* (C,D),

3) *the states* (D,C), (C,C) *and* (D,D) *are stationary for 1-aggressive-2-altruistic behavior,*

4) $\mathcal{G}^{-+}_{\infty}$, *the nonstationary kernel-active domain of 1-aggressive-2-altruistic behavior, is* $S \setminus \{(D,C),(C,C),(D,D)\}$.

Proposition 6.11 leads to the next characterization of the nonextendable 1-aggressive-2-altruistic trajectories.

Proposition 6.12 *Let $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable 1-aggressive-2-altruistic trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) \in \{(D, C), (C, C), (D, D)\}$, then t is infinite and stationary,*

2) *if $(x_*, y_*) \notin \{(D, C), (C, C), (D, D)\}$, then t is infinite and nonstationary, moreover, in each round k*
(i) *t moves towards (D, C) if $(x_k, y_k) \in \bar{H}_{DD}(f) \cap H_{CC}(g)$,*
(ii) *t moves towards (D, C) or (D, D) if $(x_k, y_k) \in H_{DD}(f) \setminus H_{CC}(g)$,*
(iii) *t moves towards (D, C) or (C, C) if $(x_k, y_k) \in H_{CC}(g) \setminus H_{DD}(f)$, and*
(iv) *t moves towards (D, C), (C, C) of (D, D) if*
$(x_k, y_k) \in [\bar{H}_{CC}(g) \cap H_{DD}(f)] \setminus \{(D, C\}$,

3) *if $(x_*, y_*) = (C, D)$, then either*
(i) *t is infinite and stationary, or*
(ii) *t is infinite, its initial finite subtrajectory t_* is stationary, and in every round $k \geq k_*$, where k_* is the final round of t_*, conditions* (i) – (iv) *of statement 2 are satisfied.*

Now we provide symmetric assertions on 1-altruistic-2-aggressive behavior.

Proposition 6.13 *The next statements hold true:*

1) *G^{+-}, the active domain of 1-altruistic-2-aggressive behavior, and G^{+-}_{∞}, the kernel-active domain of 1-altruistic-2-aggressive behavior, coincide with the state space S,*

2) *at every $(x, y) \in \bar{H}_{DD}(g) \cap H_{CC}(f)$, (C, D) is a single strategy pair acceptable for 1-altruistic-2-aggressive behavior ($(f, -g)$-acceptable), at every $(x, y) \in H_{DD}(g) \setminus H_{CC}(f)$, the strategy pairs acceptable for 1-altruistic-2-aggressive behavior are (C, D) and (D, D), at every $(x, y) \in H_{CC}(f) \setminus H_{DD}(g)$ the strategy pairs acceptable for 1-altruistic-2-aggressive behavior are (C, D) and (C, C), at every $(x, y) \in [\bar{H}_{CC}(f) \cap H_{DD}(g)] \setminus \{(D, C)\}$ the strategy pairs acceptable for 1-altruistic-2-aggressive behavior are (C, D), (C, C) and (D, D), and at (D, C) the strategy pairs acceptable for 1-altruistic-2-aggressive behavior are (C, D), (C, C) and (D, D) and (D, C),*

3) *the states (C, D) (C, C) and (D, D) are stationary for 1-altruistic-2-aggressive behavior,*

4) *$\mathcal{G}^{+-}_{\infty}$, the nonstationary kernel-active domain of 1-altruistic-2-aggressive behavior, is $S \setminus \{(C, D), (C, C), (D, D)\}$.*

Proposition 6.13 leads to the next characterization of the nonextendable 1-altruistic-2-aggressive trajectories.

Proposition 6.14 *Let $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable 1-altruistic-2-aggressive trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) \in \{(C, D), (C, C), (D, D)\}$, then t is infinite and stationary,*

2) *if $(x_*, y_*) \notin \{(C, D), (C, C), (D, D)\}$, then t is infinite and nonstationary, moreover in each round k*
(i) *t moves towards (C, D) if $(x_k, y_k) \in \bar{H}_{DD}(g) \cap H_{CC}(f)$,*
(ii) *t moves towards (C, D) or (D, D) if $(x_k, y_k) \in H_{DD}(g) \setminus H_{CC}(f)$,*
(iii) *t moves towards (C, D) or (C, C) if $(x_k, y_k) \in H_{CC}(f) \setminus H_{DD}(g)$, and*

(iv) *t moves towards (C,D), (C,C) of (D,D) if $(x_k,y_k) \in [H_{CC}(f) \cap H_{DD}(g)] \setminus \{(C,D)\}$,*

3) *if $(x_*,y_*) = (D,C)$, then either*

(i) *t is infinite and stationary, or*

(ii) *t is infinite, its initial finite subtrajectory t_* is stationary, and in every round $k \geq k_*$, where k_* is the final index of t_*, conditions* (i) – (iv) *of statement 2 are satisfied.*

6.5 Characterization of aggressive trajectories

Proposition 6.15 *Let $c \leq 0$. The next statements hold true:*

1) *G^{--}, the active domain of aggressive behavior, and G_∞^{--}, the kernel-active domain of aggressive behavior, coincide, are nonempty and given by*

$$G^{--} = G_\infty^{--} = \bar{H}_{DD}(f) \cap \bar{H}_{DD}(g),$$

2) *at every $(x,y) \in G^{--}$, (D,D) is a single strategy pair acceptable for aggressive behavior ($(-g,-f)$-acceptable),*

3) *the states (D,D), (D,C) and (C,D) are stationary for aggressive behavior,*

4) *$\mathcal{G}_\infty^{--}$, the nonstationary kernel-active domain of aggressive behavior, is $G^{--} \setminus \{(D,D)\}$.*

Proposition 6.16 *Let $c > 0$ and $c_1 + c_2 \leq 0$. The next statements hold true:*

1) *G^{--}, the active domain of aggressive behavior, is nonempty and given by*

$$G^{--} = \bar{H}_{DD}(f) \cap \bar{H}_{DD}(g), \tag{6.15}$$

2) *at every $(x,y) \in G^{--}$, (D,D) is a single strategy pair acceptable for aggressive behavior,*

3) *the states (D,D), (D,C) and (C,D) are stationary for aggressive behavior,*

4) *$\mathcal{G}_\infty^{--}$, the nonstationary kernel-active domain of aggressive behavior, is nonempty and described as the set of all states $(x,y) \neq (D,D)$ such that*

$$y \geq -\frac{c_1}{c_2}x, \quad x \geq -\frac{c_1}{c_2}y,$$

5) *G_∞^{--}, the kernel-active domain of aggressive behavior, is given by $G_\infty^{--} = \mathcal{G}_\infty^{--} \cup \{(D,D),(C,D),(D,C)\}$.*

Proposition 6.17 *Let $c > 0$ and $c_1 + c_2 > 0$. The next statements hold true:*

1) *G^{--}, the active domain of aggressive behavior, is nonempty and given by (6.15),*

2) *at every $(x,y) \in G^{--}$, (D,D) is a single strategy pair acceptable for aggressive behavior,*

3) *the states (D,D), (D,C) and (C,D) are stationary for aggressive behavior,*

4) *G_∞^{--}, the kernel-active domain of aggressive behavior, is $\{(D,D),(D,C),(C,D)\}$,*

5) *$\mathcal{G}_\infty^{--}$, the nonstationary kernel-active domain of aggressive behavior, is empty.*

Propositions 6.15 – 6.17 lead to the next characterizations of the nonextendable aggressive trajectories.

Proposition 6.18 *Let $c \le 0$ and $t = (x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable aggressive trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) \notin G^{--}$, then t has the length 0,*

2) *if $(x_*, y_*) \in \{(D,D),(D,C),(C,D)\}$, then t is infinite and stationary,*

3) *if $(x_*, y_*) \in G^{--} \setminus \{(D,D),(D,C),(C,D)\}$, then t is infinite, nonstationary, and in each round moves towards (D,D).*

Proposition 6.19 *Let $c > 0$, $c_1 + c_2 \le 0$ and $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable aggressive trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) \notin G^{--}$, then t has the length 0,*

2) *if $(x_*, y_*) \in \{(D,D),(D,C),(C,D)\}$, then t is infinite and stationary,*

3) *if $(x_*, y_*) \in G^{--} \setminus [G^{--}_{\infty} \cup \{(D,D),(D,C),(C,D)\}]$, then t is finite, nonstationary, and in each round moves towards (D,D).*

4) *if $(x_*, y_*) \in \mathcal{G}^{--}_{\infty}$ then t is infinite, nonstationary, and in each round moves towards (D,D).*

Proposition 6.20 *Let $c > 0$, $c_1 + c_2 > 0$ and $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable aggressive trajectory originating from a state (x_*, y_*). The next statements hold true:*

1) *if $(x_*, y_*) \notin G^{--}$, then t has the length 0,*

2) *if $(x_*, y_*) \in \{(D,D),(D,C),(C,D)\}$, then t is infinite and stationary,*

3) *if $(x_*, y_*) \in G^{--} \setminus \{(D,D),(D,C),(C,D)\}$, then t is finite, nonstationary, and in each round moves towards (D,D).*

7 Appendix 3. Behavior assessment of fictitious play trajectories

7.1 Analysis of fictitious play trajectories

The next statements follow from Propositions 6.1 – 6.3, 6.9, 6.7, 6.11, 6.13 and 6.15 – 6.17. The proofs are elementary. Note in advance that in all propositions given below, the sets mentioned in statements 1, 2, ... are nonempty; we do not repeat this in the formulations. The sets E_1, E_2 and E are defined in subsection 3.2.

Proposition 7.1 *Let $c \le 0$ and $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be the infinite fictitious play trajectory originating from a state $(x_*, y_*) \ne (D,D)$. The next statements hold true:*

1) *if $(x_*, y_*) \in G^{--}$ (see Proposition 6.15, 1)), then the trajectory t is aggressive and every basic behavior except aggressive has the zero maximum measure on t,*

2) *if $(x_*, y_*) \in H_{DD}(f) \setminus E_1$, then*
(i) *the trajectory t is not basic,*
(ii) *the minimum measure of aggressive behavior on t is infinite,*
(iii) *every basic behavior except aggressive, 2-altruistic and 1-aggressive-2-altruistic has the zero minimum measure on t, and the maximum and minimum measures on t of each of the latter two basic behaviors is nonzero and finite,*

3) *if* $(x_*, y_*) \in H_{DD}(f) \cap E_1$, *then the trajectory* t *is 2-altruistic and 1-aggressive-2-altruistic, and every other basic behavior has the zero maximum measure on* t,

4) *if* $(x_*, y_*) \in H_{DD}(g) \setminus E_2$, *then*
(i) *the trajectory* t *is not basic,*
(ii) *the minimum measure of aggressive behavior on* t *is infinite,*
(iii) *every basic behavior except aggressive, 1-altruistic and 1-altruistic-2-aggressive has the zero maximum measure on* t, *and the maximum and minimum measures on* t *of each of the latter two basic behaviors is nonzero and finite,*

5) *if* $(x_*, y_*) \in H_{DD}(g) \cap E_2$, *then the trajectory* t *is 1-altruistic and 1-altruistic-2-aggressive, and every other basic behavior has the zero maximum measure on* t.

Proposition 7.2 *Let* $c > 0$, $c_1 + c_2 \leq 0$ *and* $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ *be the infinite fictitious play trajectory originating from a state* $(x_*, y_*) \neq (D, D)$. *The next statements hold true:*

1) *if* $(x_*, y_*) \in \mathcal{G}_\infty^{--} = \bar{E}_1 \cap \bar{E}_2$ *(see Proposition 6.16, 4)), then the trajectory* t *is aggressive and every basic behavior except aggressive has the zero maximum measure on* t,

2) *if* $(x_*, y_*) \in G^{--} \cap E_1$, *then*
(i) *the trajectory* t *is not basic,*
(ii) *every basic behavior except aggressive, 2-altruistic and 1-aggressive-2-altruistic has the zero maximum measure on* t,
(iii) *the minimum and maximum measures on* t *of aggressive behavior are nonzero, finite and identical,*
(iv) *the minimum and maximum measures on* t *of 2-altruistic behavior are, respectively, zero and infinity,*
(v) *the minimum and maximum measures on* t *of 1-aggressive-2-altruistic behavior are, respectively, zero and infinity,*

3) *if* $(x_*, y_*) \in G^{--} \cap E_2$, *then*
(i) *the trajectory* t *is not basic,*
(ii) *every basic behavior except aggressive, 1-altruistic and 1-altruistic-2-aggressive has the zero maximum measure on* t,
(iii) *the minimum and maximum measures on* t *of aggressive behavior are nonzero, finite and identical,*
(iv) *the minimum and maximum measures on* t *of 1-altruistic behavior are, respectively, zero and infinity,*
(v) *the minimum and maximum measures on* t *of 1-altruistic-2-aggressive behavior are, respectively, zero and infinity,*

4) *if* $(x_*, y_*) \in H_{DD}(f)$, *then*
(i) *the trajectory* t *is 2-altruistic and 1-aggressive-2-altruistic,*
(ii) *every basic behavior except 2-altruistic and 1-aggressive-2-altruistic has the zero maximum measure on* t,
(iii) *the minimum measures on* t *of 2-altruistic and 1-aggressive-2-altruistic behaviors are zero,*

5) *if* $(x_*, y_*) \in H_{DD}(g)$, *then*
(i) *the trajectory* t *is 1-altruistic and 1-altruistic-2-aggressive,*
(ii) *every basic behavior except 1-altruistic and 1-altruistic-2-aggressive has the zero maximum measure on* t,
(iii) *the minimum measures on* t *of 1-altruistic and 1-altruistic-2-aggressive behaviors are zero.*

Proposition 7.3 *Let $c > 0$, $c_1 + c_2 > 0$, $c_2(c_2 - c) \geq c_1^2$ and $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ be a nonextendable fictitious play trajectory originating from a state $(x_*, y_*) \neq (D, D)$. The next statements hold true:*

1) *if $(x_*, y_*) \in G^{--} \cap E_1 \cap E_2$ and $x_* = y_*$, then*
(i) *the trajectory t is not basic,*
(ii) *the minimum and maximum measures on t of aggressive behavior are nonzero, finite and identical,*
(iii) *for every basic behavior except aggressive, its maximum measure on t is infinite and its minimum measure on t is zero,*

2) *if $(x_*, y_*) \in G^{--} \cap E_1 \cap E_2$ and $x_* > y_*$, then*
(i) *the trajectory t is not basic,*
(ii) *the maximum and minimum measures on t of aggressive behavior are nonzero, finite and identical,*
(iii) *the maximum measures on t of 2-altruistic and 1-aggressive-2-altruistic behaviors are infinite, and the minimum measures on t of these behaviors are finite and identical,*
(iv) *the maximum and minimum measures on t of normal behavior are, respectively, infinite and zero,*

3) *if $(x_*, y_*) \in G^{--} \cap E_1 \cap E_2$ and $x_* < y_*$, then*
(i) *the trajectory t is not basic,*
(ii) *the maximum and minimum measures on t of aggressive behavior are nonzero, finite and identical,*
(iii) *the maximum measures on t of 1-altruistic and 1-altruistic-2-aggressive behaviors are infinite, and the minimum measures of these behaviors on t are finite and identical,*
(iv) *the maximum and minimum measures on t of normal behavior are, respectively, infinite and zero,*

4) *if $(x_*, y_*) \in G^{--} \setminus E_1$, then*
(i) *the trajectory t is not basic,*
(ii) *the maximum and minimum measures on t of aggressive behavior are nonzero, finite and identical,*
(iii) *the minimum measures on t of 2-altruistic and 1-aggressive-2-altruistic behaviors are infinite,*
(iv) *the maximum measure on t of normal behavior is zero,*

5) *if $(x_*, y_*) \in G^{--} \setminus E_2$, then*
(i) *the trajectory t is not basic,*
(ii) *the maximum and minimum measures on t of aggressive behavior are nonzero, finite and identical,*
(iii) *the minimum measures on t of 1-altruistic and 1-altruistic-2-aggressive behaviors are infinite,*
(iv) *the maximum measure on t of normal behavior is zero,*

6) *if $(x_*, y_*) \in S \setminus [E_1 \cup G^{--}]$, then*
(i) *the trajectory t is 2-altruistic and 1-aggressive-2-altruistic,*
(ii) *the maximum measure on t of every basic behavior except 2-altruistic and 1-aggressive-2-altruistic is zero,*

7) *if $(x_*, y_*) \in S \setminus [E_2 \cup G^{--}]$, then*
(i) *the trajectory t is 1-altruistic and 1-altruistic-2-aggressive,*
(ii) *the maximum measure on t of every basic behavior except 1-altruistic and 1-altruistic-2-aggressive is zero,*

8) *if* $(x_*, y_*) \in E_1 \setminus [G^{--} \cup E]$*, then*
(i) *the trajectory* t *is not basic,*
(ii) *the maximum measure on* t *of aggressive behavior is zero,*
(iii) *the maximum measures on* t *of 2-altruistic and 1-aggressive-2-altruistic behaviors are infinite, and the minimum measures on* t *of these behaviors are finite and identical,*
(iv) *the maximum and minimum measures on* t *of normal behavior are, respectively, infinite and zero,*

9) *if* $(x_*, y_*) \in E_2 \setminus [G^{--} \cup E]$*, then*
(i) *the trajectory* t *is not basic,*
(ii) *the maximum measure on* t *of aggressive behavior is zero,*
(iii) *the maximum measures on* t *of 1-altruistic and 1-altruistic-2-aggressive behaviors are infinite, and the minimum measures on* t *of these behaviors are finite and identical,*
(iv) *the maximum and minimum measures on* t *of normal behavior are, respectively, infinite and zero,*

10) *if* $(x_*, y_*) \in E$*, then*
(i) *the trajectory* t *is normal, 2-altruistic, 1-aggressive-2-altruistic, 1-altruistic and 1-altruistic-2-aggressive,*
(ii) *the minimum measure on* t *of every basic behavior is zero,*
(iii) *the maximum measure on* t *of aggressive behavior is zero.*

Proposition 7.4 *Let* $c > 0$*,* $c_1 + c_2 > 0$*,* $c_2(c_2 - c) < c_1^2$ *and* $t = ((x_k, y_k))$ $(k = k_0, \ldots)$ *be the infinite fictitious play trajectory originating from a state* $(x_*, y_*) \neq (D, D)$*. The next statements hold true:*

1) *if* $(x_*, y_*) \in G^{--}$ *and* $x_* = y_*$*, then*
(i) *the trajectory* t *is not basic,*
(ii) *the minimum and maximum measures on* t *of aggressive behavior are nonzero, finite and identical,*
(iii) *for every basic behavior except aggressive, its maximum measure on* t *is infinite and its minimum measure on* t *is zero,*

2) *if* $(x_*, y_*) \in G^{--}$ *and* $x_* > y_*$*, then*
(i) *the trajectory* t *is not basic,*
(ii) *the maximum and minimum measures on* t *of aggressive behavior are nonzero, finite and identical,*
(iii) *the maximum measures on* t *of 2-altruistic and 1-aggressive-2-altruistic behaviors are infinite, and the minimum measures on* t *of these behaviors are finite and identical,*
(iv) *the maximum and minimum measures on* t *of normal behavior are, respectively, infinite and zero,*

3) *if* $(x_*, y_*) \in G^{--}$ *and* $x_* < y_*$*, then*
(i) *the trajectory* t *is not basic,*
(ii) *the maximum and minimum measures on* t *of aggressive behavior are nonzero, finite and identical,*
(iii) *the maximum measures on* t *of 1-altruistic and 1-altruistic-2-aggressive behaviors are infinite, and the minimum measures of these behaviors on* t *are finite and identical,*
(iv) *the maximum and minimum measures on* t *of normal behavior are, respectively, infinite and zero,*

4) *if* $(x_*, y_*) \in S \setminus [E_1 \cup G^{--}]$*, then*
(i) *the trajectory* t *is 2-altruistic and 1-aggressive-2-altruistic,*

(ii) *the maximum measure on t of every basic behavior except 2-altruistic and 1-aggressive-2-altruistic is zero,*

5) *if* $(x_*, y_*) \in S \setminus [E_2 \cup G^{--}]$, *then*
(i) *the trajectory t is 1-altruistic and 1-altruistic-2-aggressive,*
(ii) *the maximum measure on t of every basic behavior except 1-altruistic and 1-altruistic-2-aggressive is zero,*

6) *if* $(x_*, y_*) \in E_1 \setminus [G^{--} \cup E]$, *then*
(i) *the trajectory t is not basic,*
(ii) *the maximum measure on t of aggressive behavior is zero,*
(iii) *the maximum measures on t of 2-altruistic and 1-aggressive-2-altruistic behaviors are infinite, and the minimum measures on t of these behaviors are finite and identical,*
(iv) *the maximum and minimum measures on t of normal behavior are, respectively, infinite and zero,*

7) *if* $(x_*, y_*) \in E_2 \setminus [G^{--} \cup E]$, *then*
(i) *the trajectory t is not basic,*
(ii) *the maximum measure on t of aggressive behavior is zero,*
(iii) *the maximum measures on t of 1-altruistic and 1-altruistic-2-aggressive behaviors are infinite, and the minimum measures on t of these behaviors are finite and identical,*
(iv) *the maximum and minimum measures on t of normal behavior are, respectively, infinite and zero,*

8) *if* $(x_*, y_*) \in E$, *then*
(i) *the trajectory t is normal, 2-altruistic, 1-aggressive-2-altruistic, 1-altruistic and 1-altruistic-2-aggressive,*
(ii) *the minimum measure on t of every basic behavior is zero,*
(iii) *the maximum measure on t of aggressive behavior is zero.*

8 Appendix 4. Optimal paths to cooperation

8.1 Proof of Lemma 4.1

Let $t \notin F$. Then t either never visits $\mathcal{G}_\infty^{00}$, or visits $\mathcal{G}_\infty^{00}$ in round s and moves not normally in some round $k \geq s$. Let us prove that t is not optimal.

Assume, first, that t never visits $\mathcal{G}_\infty^{00}$. Then the minimum measure of abnormal behavior on t, i.e., the number of rounds, in which t moves not normally, is infinite. Indeed, if it is finite, then t moves normally in all rounds after some round r. Hence, the subtrajectory of t, which starts in round r, is infinite and normal. The starting state for this subtrajectory is necessarily in the kernel-active domain of normal behavior, $\mathcal{G}_\infty^{00}$. Therefore, t visits $\mathcal{G}_\infty^{00}$ in round r, which contradicts the assumption. The contradiction proves that the minimum measure of abnormal behavior on t is infinite; t is not optimal since every trajectory from F has a finite minimum measure of abnormal behavior.

Now let t visit $\mathcal{G}_\infty^{00}$ in round s and move not normally in some round $k \geq s$. The trajectory, which follows t in rounds $k_0, \ldots, s$ and moves normally in all rounds $k \geq s$, lies in F and obviously has a smaller minimum measure of abnormal behavior than t. Therefore, t is not optimal.

8.2 Proof of Proposition 4.1

We shall make use of conditions (ii) and (iii) from the definition of a Bellman function V. Given a trajectory $t = ((x_k, y_k)) \in F$, we shall use simplified notations $\nu_k = \nu_k(t)$,

$V_k = V(k, x_k, y_k)$ and $s = s_V(t)$. For arbitrary $t \in F$ by (ii)

$$\nu_s = \nu_s + V_s \geq \nu_{k_0} + V_{k_0} = V_{k_0},$$

By definition $\mu = \mu(t)$ is the number of rounds, in which t moves not normally. Obviously, $\mu \geq \nu_s$. Consequently, $\mu \geq V_{k_0}$.

For $t \in F_V$ by (iii) we have $\nu_s = V_{k_0}$ and $(x_s, y_s) \in \mathcal{G}^{00}_\infty$. By the definition of F, t moves normally in all rounds $k \geq s$. Hence, $\mu = \nu_s = V_0$. Therefore, t is optimal.

Let $t \in F \setminus F_V$. Then either $\nu_s > V_{k_0}$, or $\nu_s = V_{k_0}$ and $(x_s, y_s) \notin \mathcal{G}^{00}_\infty$. If $\nu_s > V_{k_0}$, then t is obviously not optimal. Let $\nu_s = V_{k_0}$ and $(x_s, y_s) \notin \mathcal{G}^{00}_\infty$. Consider the infinite subtrajectory τ of t, which starts in round s. Since $(x_s, y_s) \notin \mathcal{G}^{00}_\infty$, τ is not normal, hence, t moves not normally in some round $k \geq s$. Consequently, $\mu \geq \nu_s + 1 = V_{k_0} + 1$, which implies that t is not optimal.

8.3 Proof of Lemma 4.3

Recall that the straight line L_0 serving for the "north-west" boundary of $\mathcal{G}^{00}_\infty$ is described by the equation $y = \beta x + \gamma$ (see Proposition L_0 6.1, 4)). Let (x_k, y_k) lie "below" L_0, i.e., $y_k \leq \beta x_k + \gamma$. Then $p(k, x_k, y_k) = 0$ by definition. The right hand side in (4.2) is obviously zero. Hence, (4.2) holds.

Let (x_k, y_k) lie "above" L_0, i.e., $y_k > \beta x_k + \gamma$. Consider the trajectory $\tau = \tau(k, y_k, y_k) = ((\xi_r, \eta_r))$. Recall that τ is infinite, originates from (x_k, y_k) in round k and moves ("south-east") towards (C, D). All states on the trajectory τ lie on the straight line L_1 running through the points (x_k, y_k) and $(C, D) = (1, 0)$. The line L_1 has the equation

$$y = \frac{y_k}{1 - x_k}(1 - x).$$

Let (a, b) be the point, in which the lines L_0 and L_1 intersect. Obviously,

$$s = p(k, x_k, y_k)$$

is the first round, in which τ visits the stripe $E = \{(x, y) \in S : y \leq b\}$. Let us compute s. We have

$$\eta_{k+1} = y_k - \frac{y_k}{k+1} = \frac{k}{k+1} y_k,$$

$$\eta_{k+2} = \frac{k+1}{k+2}\eta_{k+1} = \frac{k}{k+2} y_k.$$

Continuing in this manner, we get

$$\eta_{k+q} = \frac{k}{k+q} y_k.$$

By definition s is the minimum of all r such that $\eta_r \leq b$, or

$$\frac{k}{r} y_k \leq b.$$

Obviously,

$$s = \left[\frac{y_k}{b} k\right]_+.$$

Since $(a, b) \in L_1$, we have

$$b = \frac{y_k}{1 - x_k}(1 - a).$$

Hence,

$$s = \left[\frac{1-x_k}{1-a}k\right]_+. \tag{8.1}$$

The next computations give an expression for $1-a$:

$$b = \frac{y_k}{1-x_k}(1-a) = \beta a + \gamma,$$

$$\frac{y_k}{1-x_k} - \gamma = \left(\frac{y_k}{1-x_k} + \beta\right) a,$$

$$a = \frac{\frac{y_k}{1-x_k} - \gamma}{\frac{y_k}{1-x_k} + \beta},$$

$$1-a = \frac{\frac{y_k}{1-x_k} + \beta - \frac{y_k}{1-x_k} + \gamma}{\frac{y_k}{1-x_k} + \beta} = \frac{(\beta+\gamma)(1-x_k)}{\beta(1-x_k)+y_k} = \frac{1-x_k}{\beta(1-x_k)+y_k}$$

(recall that $\beta+\gamma=1$; see (6.11)). Substituting into (8.1), we get

$$s = [(\beta(1-x_k)+y_k)k]_+. \tag{8.2}$$

The formula (4.2) is proved.

8.4 Proof of Lemma 4.4

1. Let t move towards $(C,D)=(1,0)$ in round k. Then

$$\begin{aligned} x_{k+1} &= \frac{k}{k+1}x_k + \frac{1}{k+1}, \\ y_{k+1} &= \frac{k}{k+1}y_k, \end{aligned}$$

The first equality implies

$$1-x_{k+1} = \frac{k}{k+1}(1-x_k). \tag{8.3}$$

Then

$$\begin{aligned} z_{k+1} &= (\beta(1-x_{k+1})+y_{k+1})(k+1) \\ &= \left(\beta\frac{k}{k+1}(1-x_k) + \frac{k}{k+1}y_k\right)(k+1) = z_k; \end{aligned}$$

here and in what follows z_k is defined by (4.4). Therefore, referring to (4.2), we get

$$p_{k+1} = [z_{k+1}]_+ = [z_k]_+ = p_k.$$

Statement (i) is proved.

2. Let t move towards $(C,C)=(1,1)$ in round k. Then

$$\begin{aligned} x_{k+1} &= \frac{k}{k+1}x_k + \frac{1}{k+1}, \\ y_{k+1} &= \frac{k}{k+1}y_k + \frac{1}{k+1}. \end{aligned}$$

Again we have (8.3). Then

$$\begin{aligned} z_{k+1} &= (\beta(1-x_{k+1})+y_{k+1})(k+1) \\ &= \left(\beta\frac{k}{k+1}(1-x_k) + \frac{k}{k+1}y_k + \frac{1}{k+1}\right)(k+1) = z_k+1. \end{aligned}$$

By (4.2)

$$p_{k+1} = [z_{k+1}]_+ = [z_k]_+ + 1 = p_k + 1.$$

Statement (ii) is proved.

3. Let t move towards $(D, C) = (0, 1)$ in round k. Then

$$\begin{aligned} x_{k+1} &= \frac{k}{k+1} x_k, \\ y_{k+1} &= \frac{k}{k+1} y_k + \frac{1}{k+1}. \end{aligned}$$

The first equality implies

$$1 - x_{k+1} = \frac{k}{k+1}(1 - x_k) + \frac{1}{k+1}. \tag{8.4}$$

Then

$$\begin{aligned} z_{k+1} &= (\beta(1 - x_{k+1}) + y_{k+1})(k+1) \\ &= \left(\beta \frac{k}{k+1}(1 - x_k) + \frac{k}{k+1} y_k + \frac{\beta + 1}{k+1}\right)(k+1) = z_k + \beta + 1. \end{aligned}$$

By (4.2)

$$p_{k+1} = [z_{k+1}]_+ \in \{[z_k]_+ + 1, [z_k]_+ + 2\} = \{p_k + 1, p_k + 2\}.$$

Statement (iii) is proved.

4. Let t move towards $(D, D) = (0, 0)$ in round k. Then

$$\begin{aligned} x_{k+1} &= \frac{k}{k+1} x_k, \\ y_{k+1} &= \frac{k}{k+1} y_k. \end{aligned}$$

Again we have (8.4) Then

$$\begin{aligned} z_{k+1} &= (\beta(1 - x_{k+1}) + y_{k+1})(k+1) \\ &= \left(\beta \frac{k}{k+1}(1 - x_k) + \frac{k}{k+1} y_k + \frac{\beta}{k+1}\right)(k+1) = z_k + \beta. \end{aligned}$$

If $[z_k + \beta]_+ = [z_k]_+$, then by (4.2)

$$p_{k+1} = [z_{k+1}]_+ = [z_k + \beta]_+ = p_k.$$

If $[z_k + \beta]_+ > [z_k]_+$, then, obviously, $[z_k + \beta]_+ = [z_k]_+ + 1$, and by (4.2)

$$p_{k+1} = [z_{k+1}]_+ = [z_k + \beta]_+ = p_k + 1.$$

Statements (iv) and (v) are proved.

8.5 Proof of Proposition 4.2

In view of Lemma 4.2, it is sufficient to prove that V satisfies condition (ii) in the definition of a Bellman function. Take a trajectory $t = ((x_k, y_k)) \in F$ and set $\nu_k = \nu_k(t)$, $V_k = V(k, x_k, y_k)$, $p_k = p(k, x_k, y_k)$ and $s = s_V(t)$. We must show that

$$\nu_{k+1} + V_{k+1} \geq \nu_k + V_k \tag{8.5}$$

for all $k = k_0, \dots, s-1$. Assume that t moves normally in round k. Then $\nu_{k+1} = \nu_k$. Since t moves necessarily towards (C, C) in round k, by Lemma 4.4, (ii) we have $p_{k+1} = p_k + 1$. Hence,

$$\nu_{k+1} + V_{k+1} = \nu_{k+1} + p_{k+1} - (k+1) = \nu_k + p_k - k = \nu_k + V_k,$$

and (8.5) is satisfied. Now assume that t moves not normally in round k. Then $\nu_{k+1} = \nu_k + 1$. By Lemma 4.4, (i) – (v), $p_{k+1} \geq p_k$. Hence,

$$\nu_{k+1} + V_{k+1} = \nu_{k+1} + p_{k+1} - (k+1) \geq \nu_k + 1 + p_k - (k+1) = \nu_k + V_k.$$

Again, (8.5) is satisfied.

8.6 Proof of Corollary 4.1

Let V be a Bellman function given by (4.1). For the trajectory $t = ((x_k, y_k))$ we set $p_k = p(k, x_k, y_k)$, $\nu_k = \nu_k(t)$, $V_k = V(k, x_k, y_k)$. Since $k \leq s_V(t)$, the state (x_k, y_k) lies "above" the "north-west" boundary of $\mathcal{G}_\infty^{00}$; hence, the conditions of Lemma 4.4 are satisfied. By Lemma 4.4 we conclude that if t moves in round k towards one of the points (C, C), (D, C), or towards (D, D) and in the latter case the inequality (4.5) holds, then $p_{k+1} \geq p_k + 1$. If, besides, t moves not normally in this round, we have $\nu_{k+1} = \nu_k + 1$. Hence,

$$\begin{aligned} \nu_{k+1} + V_{k+1} &= \nu_{k+1} + p_{k+1} - (k+1) \\ &\geq \nu_k + 1 + p_k + 1 - (k+1) = \nu_k + p_k - k + 1 > \nu_k + V_k. \end{aligned}$$

Therefore, $t \notin F_V$ (see condition (iii) in the definition of a Bellman function). By Proposition 4.1 F_V is the set of all optimal trajectories. Consequently, t is not optimal.

8.7 Proof of Proposition 4.3

Let V be a Bellman function given by (4.1) (see Proposition 4.2). Due to Proposition 4.1 it is sufficient to show that $F^0 = F_V$. By Corollary 4.1 all trajectories from $F \setminus F^0$ are not optimal. All desirable trajectories beyond F are not optimal by Lemma 4.1. Therefore, for stating the equality $F^0 = F_V$ we must prove the inclusion $F^0 \subset F_V$. Take arbitrary $t = ((x_k, y_k)) \in F^0$. We shall use notations $p_k = p(k, x_k, y_k)$, $\nu_k = \nu_k(t)$, $V_k = V(k, x_k, y_k)$, $s = s_V(t)$. Let us prove that

$$\nu_{k+1} + V_{k+1} = \nu_k + V_k \tag{8.6}$$

for $k = k_0, \dots, s$, and

$$(x_s, y_s) \in \mathcal{G}_\infty^{00}. \tag{8.7}$$

These relations yield that $t \in F_V$. Take a round number $k \leq s$. By the definition of F^0 in round k one of the next conditions is satisfied:

(i) t moves normally (towards (C, C)),

(ii) t moves (not normally) towards (C, D),

(iii) t moves (not normally) towards (D, D) provided the equality (4.3) holds for z_k given by (4.4).

Since $t \in F$, the state (x_k, y_k) lies "above" L_0, the "north-west" boundary of $\mathcal{G}_\infty^{00}$; hence, the conditions of Lemma 4.4 are satisfied. If (i) holds, then $\nu_{k+1} = \nu_k$ (t moves normally in round k) and by Lemma 4.4 $p_{k+1} = p_k + 1$. In this case

$$\begin{aligned} \nu_{k+1} + V_{k+1} &= \nu_{k+1} + p_{k+1} - (k+1) \\ &= \nu_k + p_k + 1 - (k+1) = \nu_k + p_k - k = \nu_k + V_k, \end{aligned}$$

and (8.6) holds. If (ii) or (iii) hold, then $\nu_{k+1} = \nu_k + 1$ (t moves not normally in round k) and by Lemma 4.4 $p_{k+1} = p_k$. In this case we have

$$\begin{aligned} \nu_{k+1} + V_{k+1} &= \nu_{k+1} + p_{k+1} - (k+1) \\ &= \nu_k + 1 + p_k - (k+1) = \nu_k + p_k - k = \nu_k + V_k, \end{aligned}$$

and (8.6) holds again.

A proof of (8.7) is based on Lemma 4.5. If $s = k_0$, then (8.7) holds due to Crossing Condition. Let $s > k_0$. By the definition of s we have $V_{s-1} = p_{s-1} - (s-1) > 0$. Hence, by the definition of p_k we conclude that, first, the state (x_{s-1}, y_{s-1}) lies "above" L_0 ($y_{s-1} > \beta x_{s-1} + \gamma$), and, second, $p_{s-1} = s$. In round $s-1$ cases (i), (ii) and (iii) are admissible. In case (i) (x_s, y_s) cannot lie "below" L_0, therefore, this case cannot take place.

Let (ii) take place. Then (x_s, y_s) follows (x_{s-1}, y_{s-1}) on the trajectory τ, which originates from (x_{s-1}, y_{s-1}) in round $s-1$ and moves towards (C, D). By Lemma 4.5 the trajectory τ visits $\mathcal{G}^{00}_\infty$ in round $p_{s-1} = s$, which is equivalent to (8.7).

Let (iii) take place. Then (x_s, y_s) follows (x_{s-1}, y_{s-1}) on the trajectory τ, which originates from (x_{s-1}, y_{s-1}) in round $s-1$ and moves towards (D, D). This trajectory never abandons $\mathcal{G}^{00}_\infty$ after crossing L_0. Hence, we have (8.7) again. The proposition is proved.

8.8 Proof of Lemma 4.5

We use induction in the round number, k. For $k = k_0$ the statement follows from Crossing Condition. Assume that the statement is true for some $k \geq k_0$, i.e., for every trajectory $t = ((x_q, y_q)) \in F^0$ and every round $q \leq k$ such that $q < s_V(t)$ the trajectory $\tau(q, x_q, y_q) = ((\xi^q_r, \eta^q_r))$ visits $\mathcal{G}^{00}_\infty$ in round $p_q = p(q, x_q, y_q)$, i.e., $(\xi^q_{p_q}, \eta^q_{p_q}) \in \mathcal{G}^{00}_\infty$.

Fix arbitrary trajectory $t = ((x_q, y_q)) \in F^0$ such that $k+1 < s_V(t)$. To complete the proof, we must show that the trajectory $\tau(k+1, x_{k+1}, y_{k+1}) = ((\xi^{k+1}_r, \eta^{k+1}_r))$ visits $\mathcal{G}^{00}_\infty$ in round $p_{k+1} = p(k+1, x_{k+1}, y_{k+1})$, i.e.,

$$(\xi^{k+1}_{p_{k+1}}, \eta^{k+1}_{p_{k+1}}) \in \mathcal{G}^{00}_\infty. \tag{8.8}$$

By the definition of F^0, in round k the trajectory t moves towards (C, D), towards (C, C), or towards (D, D).

Let t move towards (C, D) in round k. Then the trajectory $\tau(k+1, x_{k+1}, y_{k+1})$ is the subtrajectory of $\tau(k, x_k, y_k)$, which starts in round $k+1$, and $p_{k+1} = p_k$. By assumption $\tau(k, x_k, y_k)$ visits $\mathcal{G}^{00}_\infty$ in round p_k. Hence, $\tau(k+1, x_{k+1}, y_{k+1})$ visits $\mathcal{G}^{00}_\infty$ in round $p_k = p_{k+1}$.

Let t move towards (C, C) in round k. By assumption the trajectory $\tau(k, x_k, y_k) = ((\xi^k_r, \eta^k_r))$ visits $\mathcal{G}^{00}_\infty$ in round p_k, i.e.,

$$(\xi^k_{p_k}, \eta^k_{p_k}) \in \mathcal{G}^{00}_\infty. \tag{8.9}$$

Let a one-round trajectory σ^* originate from $((\xi^k_{p_k}, \eta^k_{p_k}))$ in round p_k and move (normally) towards (C, C). Let (ξ^*, η^*) be the state on σ^* in round $p_k + 1$. Due to (8.9)

$$(\xi^*, \eta^*) \in \mathcal{G}^{00}_\infty. \tag{8.10}$$

We shall show that the trajectory $\tau(k+1, x_{k+1}, y_{k+1}) = ((\xi^{k+1}_r, \eta^{k+1}_r))$ meets (ξ^*, η^*) in round $p_k + 1$:

$$(\xi^{k+1}_{p_k+1}, \eta^{k+1}_{p_k+1}) = (\xi^*, \eta^*). \tag{8.11}$$

This fact implies $p_{k+1} \le p_k$. Hence, taking into account the definition of $p_{k+1} = p(k+1, x_{k+1}, y_{k+1})$, we easily deduce that the point $(\xi^{k+1}_{p_{k+1}}, \eta^{k+1}_{p_{k+1}})$ lies on a segment between $(\xi^*, \eta^*) \in \mathcal{G}^{00}_\infty$ and some point on L_0, the "north-west" boundary of $\mathcal{G}^{00}_\infty$. Since $\mathcal{G}^{00}_\infty$ is convex, we get (8.8), which finalizes the proof.

Let us show (8.11). For each $r = k+1, \ldots, p_k$ introduce a one-round trajectory σ^*_r originating from (ξ^k_r, η^k_r) in round r and moving towards (C, C). Obviously, $\sigma^*_{p_k} = \sigma^*$. Let $(\xi^*_{r+1}, \eta^*_{r+1})$ be the state on σ^*_r in round $r+1$. We shall prove that

$$(\xi^{k+1}_{r+1}, \eta^{k+1}_{r+1}) = (\xi^*_{r+1}, \eta^*_{r+1}) \tag{8.12}$$

for all $r = k+1, \ldots, p_k$. For $r = p_k$ (8.12) gives (8.11). We use induction in r. Let $r = k+1$. Our goal is to state the equality

$$(\xi^{k+1}_{k+2}, \eta^{k+1}_{k+2}) = (\xi^*_{k+2}, \eta^*_{k+2}). \tag{8.13}$$

We have

$$\begin{aligned}
x_{k+1} &= \frac{k}{k+1} x_k + \frac{1}{k+1}, \\
y_{k+1} &= \frac{k}{k+1} y_k + \frac{1}{k+1}, \\
\xi^{k+1}_{k+2} &= \frac{k+1}{k+2} x_{k+1} + \frac{1}{k+2}, \\
\eta^{k+1}_{k+2} &= \frac{k+1}{k+2} y_{k+1},
\end{aligned}$$

$$\begin{aligned}
\xi^{k}_{k+1} &= \frac{k}{k+1} x_k + \frac{1}{k+1}, \\
\eta^{k}_{k+1} &= \frac{k}{k+1} y_k, \\
\xi^*_{k+2} &= \frac{k+1}{k+2} \xi^k_{k+1} + \frac{1}{k+2}, \\
\eta^*_{k+2} &= \frac{k+1}{k+2} \eta^k_{k+1} + \frac{1}{k+2}.
\end{aligned}$$

Obviously, $\xi^{k+1}_{k+2} = \xi^*_{k+2}$. Furthermore,

$$\begin{aligned}
\eta^{k+1}_{k+2} &= \frac{k+1}{k+2} \eta^{k+1}_{k+1} \\
&= \frac{k+1}{k+2} \left(\frac{k}{k+1} y_k + \frac{1}{k+1} \right) = \frac{k}{k+2} y_k + \frac{1}{k+2},
\end{aligned}$$

and

$$\begin{aligned}
\eta^*_{k+2} &= \frac{k+1}{k+2} \eta^k_{k+1} + \frac{1}{k+2} \\
&= \frac{k+1}{k+2} \frac{k}{k+1} y_k + \frac{1}{k+2} = \frac{k}{k+2} y_k + \frac{1}{k+2} = \eta^{k+1}_{k+2}.
\end{aligned}$$

The equality (8.13) is proved. Now we assume that (8.12) holds for some $r < p_k$ and prove that $(\xi^{k+1}_{r+2}, \eta^{k+1}_{r+2}) = (\xi^*_{r+2}, \eta^*_{r+2})$. The proof is identical to that given for the equality (8.13).

Let, finally, t move towards (D, D) in round k. By assumption we have (8.9) Consider a one-round trajectory σ^* originating from $(\xi^k_{p_k}, \eta^k_{p_k})$ in round p_k and moving towards (D, D). Let (ξ^*, η^*) be the state on σ^* in round $p_k + 1$. The point (ξ^*, η^*) lies on the segment with the endpoints $(\xi^k_{p_k}, \eta^k_{p_k})$ and (D, D). Both of them belong to $\mathcal{G}^{00}_\infty$. Since $\mathcal{G}^{00}_\infty$ is convex, we have (8.10). The rest of the proof is identical to that given previously for the case where t moves towards (C, C) in round k. The lemma is proved.

9 Conclusion

We showed how a cooperative interpretation of strategy updating in repeated games leads to a clear classification of players' behaviors in round-to-round transitions. The players are behaving normally as long as each of them is winning, i.e., getting no less than what is being expected at the latest historical distribution of players' strategies. As soon as at least one of the players loses, altruistic and aggressive behaviors come in. The numbers of rounds, in which different combinations of players' behaviors are registered, serve as behavior measures for every game trajectory. A problem of behavior assessment and a problem of optimal behavior show how these measures work in the global analysis of game trajectories.

A global behavior analysis of the repeated Prisoner's Dilemma revealed several non-trivial phenomena. We observed that the players who defect in each round may in fact exhibit different behaviors at different stages of the repeated game. For such players, the domain of normal behavior is very often empty, whereas the domain of mutually aggressive behavior may be large or small but never vanishes.

For the players who minimize the measure of abnormal behavior on the trajectories approaching mutual cooperation, mutual cooperation is allowed only if the players are historically "nearly identical", i.e., their historical frequencies of cooperation lie close to each other (this happens in a neighborhood of the diagonal of the state square). It is not so surprising that in situations where the payers' historical frequencies are strongly disbalanced (locations far away from the diagonal) the "less cooperative" player should, typically, behave altruistically and cooperate versus his/her rival's defection. More striking is the observation that in the "disbalanced" situations both optimizing players may sometimes behave as mutually aggressive defectors, whereas mutual cooperation is not acceptable for them.

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