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Abstract

The proposed approach to the insurance of regionally distributed property against high risk catastrophes is based on finding statistically robust coverages of the insurance companies. Such coverages guarantee that all companies survive no matter what scenario of the catastrophe from a given set of scenarios takes place. We describe a sequential algorithm that computes the minimum of the companies' premiums and finds optimal coverages. A step of the algorithm is interpreted as searching a minimum-premium coverage that eliminates a current aggregate risk. The latter aggregates the risks of all companies with respect to all admissible catastrophe scenarios in a "fair" manner: the higher is the individual risk, the greater is its contribution to the aggregate risk. To justify the convergence of the algorithm we suggest a new global optimization procedure for a class of nonconvex minimization problems.

Contents

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Guaranteed Optimization in Insurance of Catastrophic Risks

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1 Introduction

In Ermolieva, et al. (1997) and Ermoliev, et al. (1998), a stochastic optimization method was proposed for finding safe coverages of companies insuring property against catastrophes. The method allows to lower the expected risk of insolvency through repeated modeling the events without knowing the exact probabilistic distribution of damages.

In this paper we suggest a complementary nonprobabilistic approach to planning rational distributions of the coverages between the companies insuring regional property against some kind of high risk catastrophes. The idea is to find a distribution that guarantees that all companies survive under all admissible scenarios of the catastrophe from a given set of scenarios. The admissible scenarios are generated beforehand as those having most severe consequences or a given level of likelihood. We focus on finding the minimal premium associated with this type of distributions of coverages (the insurance optimization problem) and describe an algorithm that step by step approaches the minimal premium and finds an associated optimal distribution.

In section 2 we present the basic model and pose the insurance optimization problem. Section 3 describes the solution algorithm. Section 4 gives numerical illustrations of the solution processes. In section 5, we employ a modified constraint aggregation technique of Ermoliev, et al. (1997) to justify the proposed solution algorithm for a class of nonconvex optimization problems which extends the insurance optimization problem.

2 Insurance optimization problem

Let us imagine a geographic region, G, which is split into a number of cells, G_i, $i = 1,\ldots,N$. Each cell G_i carries property whose total cost is D_i. A group of insurance companies (we refer to them as companies $1,\ldots,M$) insures the property in region G against some catastrophic events (earthquakes, floods, tornados, etc.) so that the whole property in G is distributed between the companies. We denote by x_{ij} the share of property (coverage) in cell G_i which is insured by company j. Obviously,

$$x_{ij} \geq 0, \quad \sum_{j=1}^{M} x_{ij} = 1. \tag{2.1}$$

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We will deal with the *distribution matrix*

$$X = \begin{pmatrix} x_{11} & \dots & x_{1M} \\ & \dots & \\ x_{N1} & \dots & x_{NM} \end{pmatrix}. \tag{2.2}$$

Let K_j be the starting capital of company j and c_{ij} be the transaction cost which company j pays for the right to insure a unit of property in cell G_i. We assume that the premium for a unit of the insured property, p, is the same for all companies, and in each cell G_i only the full damage (which costs D_i) is insured.

A catastrophe may damage several cells G_i. The collection of the numbers, i, of all damaged cells, G_i, represents a *catastrophe scenario.* In region G several catastrophe scenarios are admissible. Let us denote by $\mathcal{I}$ the set of all admissible catastrophe scenarios. We define the *risk* of company j under scenario I to be the difference between the company's expenditure and income:

$$r_j^I(p, X) = \sum_{i \in I} D_i x_{ij} + \sum_{i=1}^{N} c_{ij} x_{ij} - \left(K_j + \sum_{i=1}^{N} p x_{ij} \right). \tag{2.3}$$

Here we indicate the dependence on the premium, p, and the distribution matrix, X (2.2). These parameters will be viewed as variable controls, whereas the costs D_i, initial capitals K_j and transaction costs c_{ij} will be fixed positive constants. The inequality

$$r_j^I(p, X) \le 0 \tag{2.4}$$

reflects the fact that company j survives under scenario I. A pair of control variables, (p, X), *guarantees survival* of all companies under all admissible scenarios if (2.4) holds for all $j = 1, \dots, M$ and all $I \in \mathcal{I}$. We will study the following *insurance optimization problem*: Find the minimum of the premium p for which there exists a distribution matrix X such that (p, X) guarantees survival of all companies under all admissible scenarios. In standard notations of optimization theory the problem reads:

$$\text{minimize } p, \tag{2.5}$$

$$r_j^I(p, X) \le 0 \quad (j = 1, \dots, M,\ I \in \mathcal{I}), \tag{2.6}$$

$$p \ge 0, \quad X \in \mathcal{X}; \tag{2.7}$$

here $\mathcal{X}$ is the set of all distribution matricies, i.e., matricies X (2.2) satisfying (2.1).

Let us note that the same type of problems arises when premiums depend on i and have the structure $p_i = p\gamma_i$, where γ_i, $i = 1, \dots, N$, are given numbers

We assume that there exists a pair (p, X) satisfying the constraints (2.6), (2.7). Then the insurance optimization problem (2.5)–(2.7) has a solution. By p_* we denote the *optimal premium*, i.e., the minimum value in the problem (2.5)–(2.7). For every solution of (2.5)–(2.7), (p_*, X_*), we call X_* an *optimal distribution matrix.* For the set of all optimal distribution matricies we use the notation $\mathcal{X}_*$.

3 Solution algorithm

We propose the following sequential algorithm for solving the insurance optimization problem (2.5)–(2.7).

At step 0 we set $p^1 = 0$ and fix an arbitrary distribution matrix X^1. At step k ($k = 1, \ldots$) we transform the pair (p^k, X^k) into (p^{k+1}, X^{k+1}). We define p^{k+1} as the first component of

$$(p^{k+1}, U^{k+1}), \text{a solution of the problem} \tag{3.1}$$

$$\text{minimize } p, \tag{3.2}$$

$$p \geq p^k, \tag{3.3}$$

$$\sum_{I \in \mathcal{I}} \sum_{j=1}^{M} r_j^I(p^k, X^k)_+ r_j^I(p, U) \leq 0, \tag{3.4}$$

$$U \in \mathcal{X}; \tag{3.5}$$

here

$$r_j^I(p^k, X^k)_+ = \max\{0, r_j^I(p^k, X^k)\}.$$

Next, we compute X^{k+1} from

$$X^{k+1} = X^k + \tau_{k+1}(U^{k+1} - X^k) \tag{3.6}$$

where

$$\tau_{k+1} = \arg\min_{0 \leq \tau \leq 1} \left(\sum_{I \in \mathcal{I}} \sum_{j=1}^{M} r_j^I(p^k, X^k + \tau(U^{k+1} - X^k))_+^2 \right). \tag{3.7}$$

Remark 3.1 Since $r_j^I(U, p)$ are negative for large p (see (2.3)), the inequality (3.4) holds for large p; therefore the feasible set of the problem (3.2)–(3.5) is nonempty and the problem has a solution. The algorithm is defined correctly.

A general result presented in the next section in Theorem 5.1 implies the following:

Proposition 3.1 *Let $p^1 = 0$, X^1 be an arbitrary distribution matrix and (p^k, X^k) ($k = 2, \ldots$) be defined by the algorithm (3.1)–(3.5), (3.6), (3.7). Then p^k converges to the optimal premium and X^k converges to the set of optimal distribution matricies:*

$$\lim_{k \to \infty} p^k = p_*, \quad \lim_{k \to \infty} \text{dist}(X^k, \mathcal{X}_*) = 0.$$

Here and in what follows

$$\text{dist}(X^k, \mathcal{X}_*) = \inf\{|X^k - X_*| : X_* \in \mathcal{X}_*\}$$

($X \mapsto |X|$ is a fixed matrix norm).

The algorithm (3.1)–(3.5), (3.6), (3.7) represents a plain *risk aggregation* strategy which is implemented sequentially, round by round. In round k, the companies update the premium p^k and distribution matrix X^k following the next "rules of the game". They assume that the updated premium p^{k+1} will not be smaller than p^k. For all candidate premiums $p \geq p^k$ and distribution matricies U the companies analyze the *aggregate risk*

$$R^k(p, U) = \sum_{I \in \mathcal{I}} \sum_{j=1}^{M} r_j^I(X^k, p^k)_+ r_j^I(U, p).$$

The latter aggregates the hypothetical risk of company j under scenario I, $r_j^I(U, p)$, with the "fair" weight $r_j^I(X^k, p^k)_+$ (which equals the current positive risk $r_j^I(X^k, p^k)$ if company

j does not survive under scenario I at (p^k, X^k), and zero otherwise). The companies choose p^{k+1} as the minimum premium for which there is a distribution matrix U^{k+1} such that the aggregate risk $R^k(p^{k+1}, U^{k+11})$ is nonpositive. This is a plain interpretation of (3.1)–(3.5). The formulas (3.6), (3.7) are interpreted as follows: for constructing the new distribution matrix, X^{k+1}, the companies shift the matrix X^k towards U^{k+1} to a degree in which the total cooperative risk is minimal.

Now we shall represent the algorithm in a more explicit form. Let us specify the basic inequality (3.4). Denote for brevity

$$\lambda_j^{I,k} = r_j^I(p^k, X^k)_+. \tag{3.8}$$

Setting

$$U = \begin{pmatrix} u_{11} & \dots & u_{1M} \\ & \dots & \\ u_{N1} & \dots & u_{NM} \end{pmatrix}$$

and substituting (2.3) (with x_{ij} replaced by u_{ij}), into (3.4), we rewrite the latter in the form

$$\sum_{I\in\mathcal{I}}\sum_{j=1}^{M}\lambda_j^{I,k}\left[\sum_{i\in I} D_i u_{ij} + \sum_{i=1}^{N} c_{ij}u_{ij} - \left(K_j + \sum_{i=1}^{N} p u_{ij}\right)\right] \le 0,$$

equivalently,

$$\sum_{I\in\mathcal{I}}\sum_{j=1}^{M}\lambda_j^{I,k}\left[\sum_{i=1}^{N}(\sigma_i(I)D_i u_{ij} + c_{ij} - p)u_{ij}\right] \le \sum_{I\in\mathcal{I}}\sum_{j=1}^{M}\lambda_j^{I,k}K_j$$

where

$$\sigma_i(I) = \begin{cases} 1, & i \in I \\ 0, & i \notin I \end{cases}.$$

Changing in the left hand side the order of the sums in I, j, and i we get the equivalent inequality

$$\sum_{i=1}^{N}\sum_{j=1}^{M}\sum_{I\in\mathcal{I}}\lambda_j^{I,k}(\sigma_i(I)D_i u_{ij} + c_{ij} - p)u_{ij} \le \sum_{I\in\mathcal{I}}\sum_{j=1}^{M}\lambda_j^{I,k}K_j.$$

A brief form of this inequality is

$$\sum_{i=1}^{N}\sum_{j=1}^{M}(\beta_{ij}^k - \gamma_j^k p)u_{ij} \le \alpha^k \tag{3.9}$$

where

$$\beta_{ij}^k = \sum_{I\in\mathcal{I}}\lambda_j^{I,k}(\sigma_i(I)D_i + c_{ij}),$$

$$\gamma_j^k = \sum_{I\in\mathcal{I}}\lambda_j^{I,k},$$

$$\alpha^k = \sum_{I\in\mathcal{I}}\sum_{j=1}^{M}\lambda_j^{I,k}K_j.$$

Thus, (3.9) is an equivalent form of (3.4). Observing (3.9), we easily find that the projections p of all feasible points (p, U) of the problem (3.2)–(3.5) cover the set

$$P^k = \left\{p \ge p^k : \min_{U\in\mathcal{X}}\sum_{i=1}^{N}\sum_{j=1}^{M}(\beta_{ij}^k - \gamma_j^k p)u_{ij} \le \alpha^k\right\}.$$

For a distribution matrix U we have

$$u_{ij} \geq 0, \quad \sum_{j=1}^{M} u_{ij} = 1.$$

Therefore

$$\min_{U \in \mathcal{X}} \sum_{i=1}^{N} \sum_{j=1}^{M} (\beta_{ij}^k - \gamma_j^k p) u_{ij} = \sum_{i=1}^{N} \min_{j=1,\ldots,M} (\beta_{ij}^k - \gamma_j^k p).$$

Note that the minimum over $U \in \mathcal{X}$ in the left hand side is reached at

$$U^k(p) = \begin{pmatrix} u_{11}^k(p) & \ldots & u_{1M}^k(p) \\ & \ldots & \\ u_{N1}^k(p) & \ldots & u_{NM}^k(p) \end{pmatrix}$$

where

$$u_{ij}^k(p) = \begin{cases} 1, & j = j^k(p) \\ 0, & j \neq j^k(p) \end{cases} \tag{3.10}$$

and $j^k(p)$ is a minimizer of $\beta_{ij}^k - \gamma_j^k p$:

$$\beta_{ij^k(p)}^k - \gamma_{j^k(p)}^k p = \min_{j=1,\ldots,M} (\beta_{ij}^k - \gamma_j^k p). \tag{3.11}$$

Hence,

$$P^k = \left\{ p \geq p^k : \sum_{i=1}^{N} \min_{j=1,\ldots,M} (\beta_{ij}^k - \gamma_j^k p) \leq \alpha^k \right\}. \tag{3.12}$$

For every $p \in P^k$ the pair $(p, U^k(p))$ is feasible in the problem (3.2)–(3.5). Hence, (p^{k+1}, U^{k+1}) determined by

$$p^{k+1} = \min P^k, \tag{3.13}$$

$$U^{k+1} = U^k(p^{k+1}) \tag{3.14}$$

is a solution of the problem (3.2)–(3.5). If $\gamma_j^k = 0$ for all $j = 1, \ldots, M$, then $P^k = [p^k, \infty)$, $p^{k+1} = p^k$, $U^{k+1} = U^k$, and (see (3.6)) $X^{k+1} = X^k$; consequently, $(p^l, X^l) = (p^k, X^k)$ for all $l \geq k$ and by Proposition 3.1 (p^k, X^k) is a solution of the insurance optimization problem (2.5)–(2.7). If $\gamma_j^k > 0$ for some j, then $P^k = [p^{k+1}, \infty)$ and $p^{k+1} > p^k$.

Proposition 3.1 takes the next more specific form.

Proposition 3.2 *Let $p^1 = 0$, X^1 be an arbitrary distribution matrix and (p^k, X^k) ($k = 2, \ldots$) be defined by the algorithm (3.13), (3.14), (3.6), (3.7). Then p^k converges to the optimal premium and X^k converges to the set of optimal distribution matricies:*

$$\lim_{k \to \infty} p^k = p_*, \quad \lim_{k \to \infty} \text{dist}(X^k, \mathcal{X}_*) = 0.$$

Moreover, if for some k

$$\gamma_j^k = \sum_{I \in \mathcal{I}} r_j^I (p^k, X^k)_+ = 0 \quad (j = 1, \ldots, M),$$

then p^k is the optimal premium and X^k an optimal distribution: $p^k = p_*$, $X^k \in \mathcal{X}_*$.

9	8	22	21	26	20
10	10	25	24	22	20
10	25	31	19	15	18
14	15	17	14	15	16
8	15	16	7	8	9
7	16	14	10	10	9

Figure 1: Property values.

The formulas (3.13), (3.14), (3.6), (3.7) specify the risk aggregation strategy. In particular, (3.14) shows that when the premium p^{k+1} is found, a distribution matrix U^{k+1} minimizing the aggregate risk $R^k(U, p^{k+1})$ is decomposable over the cells G_i; namely U^{k+1} prescribes that all property in cell G_i is insured by a single company $j^k(p)$ (see (3.10)) which has the minimum relative over-scenario risk, $\beta_{ij}^k - \gamma_j^k p$, in cell G_i.

The algorithm (3.13), (3.14), (3.6), (3.7) is of relatively low numerical complexity. The operations (3.6) and (3.14) are explicit. An implicit operation is finding p^{k+1}, (3.13). A simplest way to find p^{k+1} is to fix some step size $\delta > 0$ and verify sequentially if $p_s^k = p^k + \delta s$ $(s = 0, 1, \ldots)$ lie in the set P^k (3.12). If $p_0^k = p^k \in P^k$, then (p^k, X^k) is a solution of the insurance optimization problem and we stop the algorithm. Otherwise p^{k+1} lies between $p_{s_*-1}^k$ and $p_{s_*}^k$ where $p_{s_*}^k$ is the first point belonging to P^k. Then one can either set approximately $p^{k+1} = p_s^k$, or specify p^{k+1} by trying the central point, $p_{s_*1}^k$, of the interval $[p_{s_*-1}^k, p_{s_*}^k]$, and then continuing with $p_{s_*2}^k$, $p_{s_*3}^k$,... until reaching a desired accuracy.

4 Numerical illustrations

Consider a regional model with 6×6 cells $(N = 36)$. Property values in the cells, D_i, are shown in Fig. 1. The region is insured by a network of four companies $(M = 4)$. The initial capitals of the companies are $K_1 = K_2 = 100$, $K_3 = 120$, $K_4 = 130$. There are 8 admissible scenarios $(\mathcal{I} = \{I_1, \ldots, I_8\})$ shown in Fig. 2. The initial insurance contract diversification is uniform, i. e., $x_{ij}^1 = 1/M = 0.25$ $(i = 1, \ldots, 36,\ j = 1, \ldots, 4)$. Transaction costs for all companies and all cells are identical: $c_{ij} = 10$ $(i = 1, \ldots, 36,\ j = 1, \ldots, 4)$.

The algorithm (3.1)–(3.5), (3.6), (3.7) was executed on a computer. At the 21-th step, the iterational procedure stopped as the total risks for all companies, γ_j^{21}, reached zero. The history of evolution of the premium p^k and total risks $\gamma_1^k, \ldots \gamma_4^k$ $(k = 1, \ldots, 21)$ is shown in Table 1. As the results of the algorithm execution, the optimal premium, $p_* = 6.36$, was computed and an optimal distribution of (insurance) contracts was found. The optimal proportions of property insured by all companies in all cells are shown in Fig. 3. In each cell, we put four numbers representing an optimal share of property in this cell which is insured by companies 1, 2, 3, 4. Note that normally there are infinitely many optimal distributions for the same optimal premium. Indeed an optimal distribution is approached by solving problem (3.7) at every step of the algorithm. This problem has non-unique solution if the minimized function has constant (zero valued) pieces, which is actually takes place by the definition of $r_j^I(\cdot, \cdot)_+$. The rate of convergence of the algorithm

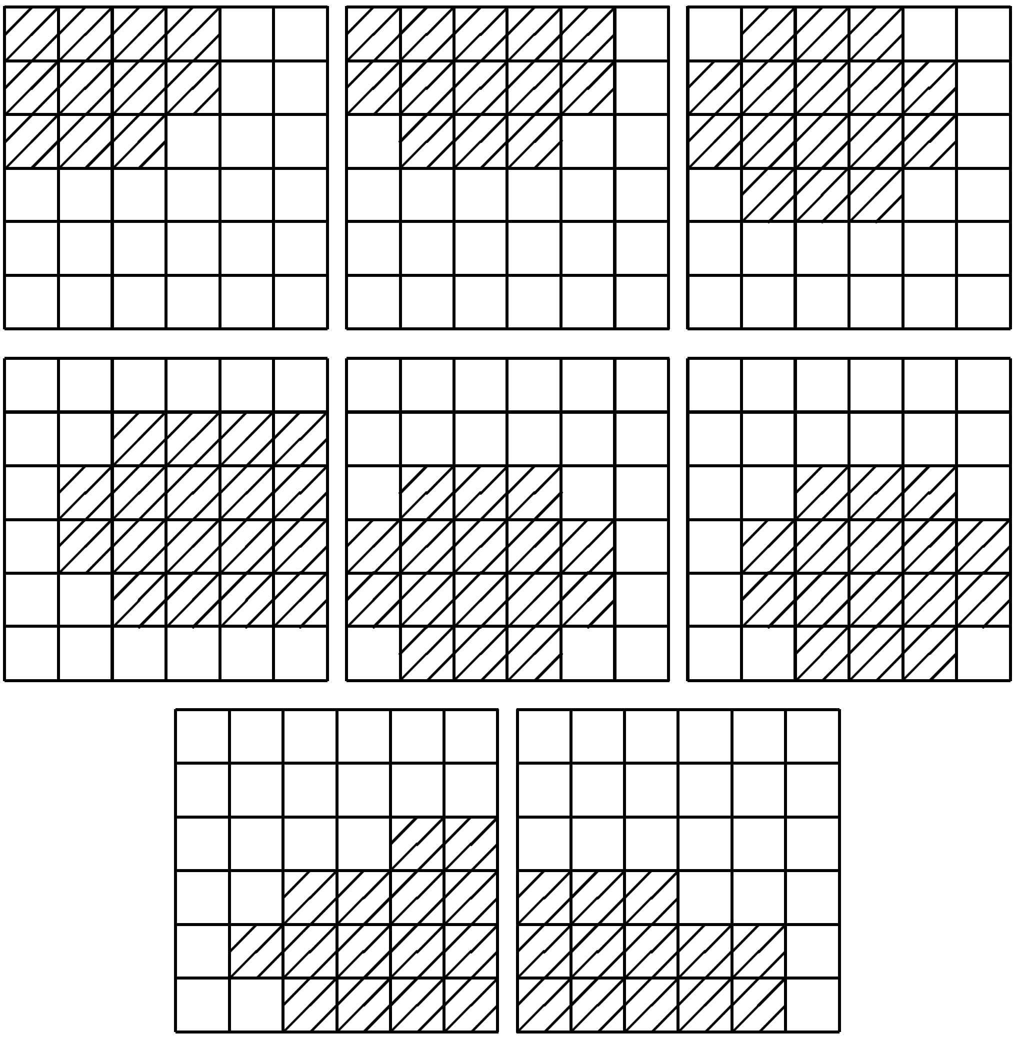

Figure 2: Admissible scenarios.

Step	Premium	Companies' total risks			
		1	2	3	4
1	0.00	389.250	389.250	229.250	150.000
2	0.06	338.250	338.250	178.250	283.649
3	0.13	300.826	300.826	278.019	240.129
4	2.70	115.619	115.619	81.579	102.138
5	2.76	104.009	104.009	98.618	91.407
6	4.29	33.234	33.234	27.484	30.849
7	4.39	28.760	28.760	28.058	27.211
8	5.52	8.481	7.543	6.276	6.484
9	5.59	7.093	6.211	6.237	5.594
10	5.65	6.103	5.473	5.359	5.628
11	5.72	5.489	4.862	5.046	4.841
12	5.78	4.874	4.250	4.316	4.472
13	5.84	4.312	3.878	3.649	3.749
14	5.91	3.713	3.280	3.293	2.977
15	5.97	3.063	2.632	2.519	2.724
16	6.04	2.471	2.041	2.140	1.960
17	6.10	1.867	1.439	1.418	1.564
18	6.17	1.294	0.867	0.979	0.823
19	6.23	0.697	0.272	0.265	0.403
20	6.30	0.066	0.000	0.019	0.000
21	6.36	0.000	0.000	0.000	0.000

Table 1: Algorithm execution: dynamics of premium and total risks.

depends on the precision in the calculation of p^k and τ^k at every step and on the threshold in total risks γ_j^k which is treated as the null risk value.

A detailed description of scenario-based insurance optimization technique for the case of seismic events is given in Digas (1998). Illustrations on generating admissible earthquake scenarios are also available there.

5 Justification via nonconvex constraint aggregation

The insurance optimization problem (2.5)–(2.7) is a special case of the next optimization problem:

$$\text{minimize } p, \tag{5.1}$$

$$h_s(p,x) \le 0 \quad (s = 1,\dots,m), \tag{5.2}$$

$$p \ge 0, \quad x \in Z. \tag{5.3}$$

Here Z is a convex compactum in R^n, and functions $h_s : (p,x) \mapsto h_s(p,x) : [0,\infty)\times Z \mapsto R^1$ $(s = 1,\dots,m)$ are continuous, convex in x and satisfy the condition

$$\lim_{p\to\infty} \sup \sup_{x\in Z} h_s(p,x) < 0. \tag{5.4}$$

This condition implies the existence of a solution of the problem (5.1)–(5.3). We shall denote W_* the set of all solutions of the problem (5.1)–(5.3) and use the notation

$$\text{dist}(p,x,W_*) = \inf\{|p-p^*| + |x-x_*| : (p_*,x_*) \in W_*\} \quad (p \ge 0, x \in Z).$$

0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290
0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.225 0.222 0.267 0.287
0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.225 0.222 0.267 0.287	0.225 0.222 0.267 0.287
0.221 0.222 0.267 0.290	0.225 0.222 0.267 0.287	0.225 0.222 0.267 0.287	0.221 0.222 0.267 0.290	0.225 0.222 0.267 0.287	0.225 0.222 0.267 0.287
0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.225 0.222 0.267 0.287	0.221 0.222 0.267 0.290	0.225 0.222 0.267 0.287	0.225 0.222 0.267 0.287
0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290	0.221 0.222 0.267 0.290

Figure 3: Optimal proportions of insured property for all cells.

The algorithm (3.1)–(3.5), (3.6), (3.7) for the special problem (2.5)–(2.7) is an application of the next algorithm for the general problem (5.1)–(5.3). In what follows, $\langle\cdot,\cdot\rangle$ is the scalar product in R^m,

$$h(p,x)_+ = (h_1(p,x)_+, \dots, (h_m(p,x)_+),$$

$$h_s(p,x)_+ = \max\{0, h_s(p,x)\},$$

$$h(p,u) = (h_1(p,u), \dots, (h_m(p,u)).$$

At step 0 of the algorithm we set $p^1 = 0$ and fix arbitrary $x^1 \in Z$. At step k $(k = 1, \dots)$ we find

$$(p^{k+1}, u^{k+1}), \text{a solution of the problem} \tag{5.5}$$

$$\text{minimize } p, \tag{5.6}$$

$$p \geq p^k, \tag{5.7}$$

$$\langle h(p^k, x^k)_+, h(p,u)\rangle \leq 0, \tag{5.8}$$

$$u \in Z, \tag{5.9}$$

and set

$$x^{k+1} = x^k + \tau_{k+1}(u^{k+1} - x^k) \tag{5.10}$$

where

$$\tau_{k+1} = \arg\min_{0\leq\tau\leq 1} \left(\sum_{s=1}^{m} r_j^I(p^k, x^k + \tau(u^{k+1} - x^k))_+^2 \right). \tag{5.11}$$

The algorithm (5.5)–(5.9), (5.10), (5.11) is a modification of a constraint aggregation algorithm proposed in Ermoliev, et al. (1997) for problems of convex programming. The problem (5.1)–(5.3) lies beyond the frames of convex programming and therefore the results of Ermoliev, et al. (1997) are not applicable. We shall prove the convergence of the nonconvex constraint aggregation algorithm (5.5)–(5.9), (5.10), (5.11) using a modified argument.

Theorem 5.1 *Let $p^1 = 0$, $x^1 \in Z$ and (p^k, x^k) $(k = 2, \ldots)$ be defined by the algorithm (5.5)–(5.9), (5.10), (5.11). Then (p^k, x^k) converges to the solution set of the problem (5.1)–(5.3):*

$$\lim_{k\to\infty} \mathrm{dist}(p^k, x^k, W_*) = 0.$$

Proof. Let p_* be the minimum value in the problem (5.1)–(5.3). It is sufficient to show that for all k

$$p^k \le p_* \tag{5.12}$$

and

$$\lim_{k\to\infty} h_s(p^k, x^k)_+ = 0 \quad (s = 1, \ldots, m). \tag{5.13}$$

We prove (5.12) by induction. For $k = 1$ we have $0 = p^1 \le p_*$, i.e. (5.12) holds true. Suppose that (5.12) holds for some k. Let (p_*, x_*) be a solution of the problem (5.1)–(5.3). Since $h_s(p_*, x_*) \le 0$ $(s = 1, \ldots, m)$, the point (p_*, x_*) is feasible in the problem (5.6)–(5.9). Hence, for (p^{k+1}, x^{k+1}), a minimizer in the problem (5.6)–(5.9), we have $p^{k+1} \le p_*$. Therefore (5.12) holds for all k. Let us prove (5.13). Due to the convexity of $h_s(p, x)$ in x, for $\tau \in [0, 1]$

$$\begin{aligned} h_s(p^{k+1}, x^k + \tau(u^{k+1} - x^{k+1}) &\le (1-\tau)h_s(p^{k+1}, x^k) + \tau h_s(p^{k+1}, u^{k+1}) \\ &\le (1-\tau)h_s(p^{k+1}, x^k)_+ + \tau h_s(p^{k+1}, u^{k+1}). \end{aligned}$$

If the left hand side is positive, the right hand side is no smaller in absolute value. Hence,

$$h_s(p^{k+1}, x^k + \tau(u^{k+1} - x^{k+1})_+ \le |(1-\tau)h_s(p^{k+1}, x^k)_+ + \tau h_s(p^{k+1}, u^{k+1})|.$$

Therefore

$$\begin{gathered} |h(p^{k+1}, x^k + \tau(u^{k+1} - x^{k+1})_+|^2 \le \\ |(1-\tau)h(p^{k+1}, x^k)_+ + \tau h_s(p^{k+1}, u^{k+1})|^2 \le \\ (1-\tau)^2|h(p^{k+1}, x^k)_+|^2 + \\ 2(1-\tau)\tau\langle h(p^{k+1}, x^k)_+, h_s(p^{k+1}, u^{k+1})\rangle + \\ \tau^2|h(p^{k+1}, u^{k+1})|^2 \le \\ (1-\tau)^2|h(p^k, x^k)_+|^2 + 2(1-\tau)\tau\langle h(p^k, x^k)_+, h_s(p^{k+1}, u^{k+1})\rangle + \beta_k + K\tau^2 \le \\ (1-2\tau)|h(p^k, x^k)_+|^2 + 2(1-\tau)\tau\langle h(p^k, x^k)_+, h_s(p^{k+1}, u^{k+1})\rangle + \beta_k + 2K^2\tau^2 \end{gathered}$$

where

$$\begin{aligned} \beta_k = {} & ||h(p^{k+1}, x^k)_+|^2 - |h(p^k, x^k)_+|^2| + \\ & 2(1-\tau)\tau|\langle h(p^{k+1}, x^k)_+ - h(p^k, x^k)_+, h_s(p^{k+1}, u^{k+1})\rangle \end{aligned} \tag{5.14}$$

and

$$K \ge \sup\{|h(p, u)| : p \in [0, p_*],\ u \in Z\}.$$

By (5.10) and (5.11) the minimum of the left hand side over $\tau \in [0,1]$ equals $|h(p^{k+1}, x^{k+1})_+|^2$. Hence, the latter does not exceed the minimum of the right hand side which is reached at

$$\tau = \frac{|h(p^k, x^k)_+|^2}{2K^2}.$$

We get

$$\begin{aligned}|h(p^{k+1}, x^{k+1})_+|^2 &\leq \left(1 - 2\frac{|h(p^k, x^k)_+|^2}{2K^2}\right)|h(p^k, x^k)_+|^2 + \\ &\quad 2K^2\left(\frac{|h(p^k, x^k)_+|^2}{2K^2}\right)^2 + \beta_k \\ &= \left(1 - \frac{|h(p^k, x^k)_+|^2}{2K^2}\right)|h(p^k, x^k)_+|^2 + \beta_k.\end{aligned}$$

Introducing the notation

$$\epsilon_k = |h(p^{k+1}, x^{k+1})_+|^2$$

we rewrite the obtained inequality as

$$\epsilon_{k+1} \leq (1 - \alpha\epsilon_k)\epsilon_k + \beta_k \tag{5.15}$$

where

$$\alpha = \frac{1}{K^2}.$$

We shall complete the proof by showing that

$$\lim_{k\to\infty} \epsilon_k = 0. \tag{5.16}$$

which is equivalent to (5.13). By (5.5)–(5.9) and (5.12) the sequence (p^k) is growing and bounded. Hence, $\lim_{k\to\infty} |p_{k+1} - p^k| = 0$. Therefore, in view of the continuity of h and (5.14), we have

$$\lim_{k\to\infty} \beta_k = 0. \tag{5.17}$$

Now we shall state that the sequence ϵ_k has a limit. Suppose the contrary:

$$\epsilon^- = \liminf_{k\to} \epsilon_k < \limsup_{k\to} \epsilon_k = \epsilon^+.$$

Take a small $\sigma > 0$ and choose k_0 so that

$$\beta_k < \sigma \quad (k \geq k_0). \tag{5.18}$$

Fix $k_1 > k_0$ such that

$$\epsilon_{k_1} < \epsilon^- + \frac{\epsilon^+ - \epsilon^-}{4}.$$

Let

$$k_2 = \min\left\{k > k_1 : \epsilon_k > \epsilon^- + \frac{\epsilon^+ - \epsilon^-}{2}\right\}. \tag{5.19}$$

By (5.15)

$$\epsilon_{k_2} \leq \epsilon_{k_2-1} + \beta_{k_2-1}$$

implying

$$\epsilon_{k_2-1} \geq \epsilon_{k_2} - \beta_{k_2-1} > \epsilon^- + \frac{\epsilon^+ - \epsilon^-}{2} - \sigma$$

(see (5.18)). Then using (5.15) and (5.18), we get

$$\begin{aligned}\epsilon_{k_2} &\leq (1-\alpha\epsilon_{k_2-1})\epsilon_{k_2-1}+\beta_{k_2-1}\\ &\leq \left[1-\alpha\left(\epsilon^-+\frac{\epsilon^+-\epsilon^-}{2}-\sigma\right)\right]\left(\epsilon^-+\frac{\epsilon^+-\epsilon^-}{2}\right)+\sigma\\ &= \left(\epsilon^-+\frac{\epsilon^+-\epsilon^-}{2}\right)-\alpha\left(\epsilon^-+\frac{\epsilon^+-\epsilon^-}{2}-\sigma\right)\left(\epsilon^-+\frac{\epsilon^+-\epsilon^-}{2}\right)+\sigma.\end{aligned}$$

Choosing σ small enough in advance, we get that the right hand side does not exceed its first term, and therefore

$$\epsilon_{k_2} \leq \epsilon^- + \frac{\epsilon^+-\epsilon^-}{2}$$

which contradicts (5.19). The contradiction proves that there is the limit

$$\bar{\epsilon} = \lim_{k\to\infty} \epsilon_k.$$

Suppose $\bar{\epsilon} > 0$. Take a small $\sigma > 0$ and k such that

$$|\epsilon_k - \bar{\epsilon}| < \sigma, \quad |\epsilon_{k+1} - \bar{\epsilon}| < \sigma. \tag{5.20}$$

The first inequality in (5.20) together with (5.15) and (5.18) implies

$$\begin{aligned}\epsilon_{k+1} &\leq [1-\alpha(\bar{\epsilon}-\sigma)(\bar{\epsilon}+\sigma)+\sigma\\ &\leq \bar{\epsilon}-\alpha(\bar{\epsilon}-\sigma)^2+2\sigma.\end{aligned}$$

Choosing σ sufficiently small, we estimate the right hand side from above by

$$\bar{\epsilon} - \frac{\alpha}{2}\bar{\epsilon}^2 \leq \bar{\epsilon} - \sigma.$$

Hence,

$$\epsilon_{k+1} < \bar{\epsilon} - \sigma,$$

which contradicts the second inequality in (5.20). Thus (5.16) holds true. The proof is completed.

6 References

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