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# Foreword

One of the main problems with mathematical modelling is always the lack of knowledge about the real world to be modelled. There are several quite natural approaches for modelling uncertainty, vagueness and lack of knowledge. However, it is usually not easy to find modelling approaches which are intuitively clear as well as analytically tractable.

In the present paper, lack of knowledge is represented by interval values for coefficients of the objective function in a linear programming model. This modelling approach is natural and straightforward. In the present paper, the authors enhance the analytical tractability of this model by introducing a new way of generating solutions, which are frequently more preferable than the ones obtained by previous treatments.

## Abstract

In this paper, we focus on a treatment of a linear programming problem with an interval objective function. From the viewpoint of the achievement rate, a new solution concept, a maximin achievement rate solution is proposed. Nice properties of this solution are shown: a maximin achievement rate solution is necessarily optimal when a necessarily optimal solution exists, and if not, then it is still a possibly optimal solution. An algorithm for a maximin achievement rate solution is proposed based on a relaxation procedure together with a simplex method. A numerical example is given to demonstrate the proposed solution algorithm.



**Keywords:** Interval programming, linear programming, optimization, linear fractional programming

# Contents

# An Achievement Rate Approach to Linear Programming Problems with an Interval Objective Function

*Masahiro Inuiguchi, Masatoshi Sakawa**

## 1 Introduction

When a real world problem is formulated as a mathematical programming problem, we are sometimes faced by difficulties in the determination of coefficients. Even in such cases, it often occurs that the coefficients are known roughly, but not exactly. Interval programming approaches have been proposed in order to deal with such ambiguities of coefficients in mathematical programming problems (see [1], [5]-[10] and [12]). In interval programming problems, the ambiguous coefficients are represented by intervals of possible true values. Since an interval programming problem is an ill-posed problem, to solve the problem, it should be specified how to treat the objective functions and constraints with interval coefficients.

In this paper, we focus on the treatment of an interval objective function. Various treatments of interval objective functions have been proposed in the literature (see [1], [5]-[10] and [12]). In these proposals, we can find two major approaches: the optimizing approach and the satisfycing approach.

In the former approach, i.e., the optimizing approach, the concept of optimality or efficiency is extended to the interval objective function case. For example, Bitran [1] pointed out that two kinds of efficient solutions can be defined to a multiple objective linear programming problem with interval objective coefficients. In the first a solution is required to be efficient for all values of the interval objective coefficients. This is called a necessarily efficient solution in [6] in analogy with possibility theory [3]. In the other concept, a solution is required to be efficient for at least one combination of parameters in the interval objective coefficients. This is called a possibly efficient solution. Bitran proposed a testing method for the necessary efficiency of a given feasible basic solution. Inuiguchi and Kume [6] proposed a testing method for the possible efficiency of a given feasible solution. A similar attempt has been made for a single objective linear programming problem with an interval objective function (see [8]). In the single objective case, the necessarily and possibly efficient solutions are called the necessarily and possibly optimal solutions, respectively. A necessarily optimal solution is the most reasonable solution, but does not exist in many cases. However, a possibly optimal solution always exists if the feasible solution set is bounded.

In the latter approach, i.e., the satisfycing approach, the interval objective function is treated as optimizing the lower and upper bounds of the interval objective function value or minimizing the width of the interval objective function value (see [5], [9] and [10]).

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These approaches do not attempt to extend the concept of optimality or efficiency to the interval function case directly. The lower and upper bounds of the interval objective function value indicate the worst (pessimistic) and best (optimistic) objective function values. The width of the interval objective function value reflects the dispersion of the objective function value. The lower bound, upper bound, and width are nothing but criteria given by the decision maker to make a satisfactory decision. In this sense, these approaches are regarded as satisfycing approaches.

One interesting approach is an intermediate-like approach between the optimizing and satisfycing approaches. In an intermediate-like approach, a possibly optimal solution is obtained based on a suitable criterion considering all possibly optimal solutions. The solution is necessarily optimal when a necessarily optimal solution exists. As such a criterion, the minimax regret criterion has been proposed in [7]. A solution algorithm based on the relaxation procedure has also been proposed to obtain a minimax regret solution. However, other criteria for intermediate-like approaches have not been proposed yet.

In this paper, the maximin achievement rate criterion is proposed as a criterion for an intermediate-like approach. In the minimax regret criterion, it is assumed that the decision maker's decision depends on differences in the objective function value between the selected solution and the possibly optimal solutions. However, as will be seen in the next section, when the range of possibly optimal values is large relatively to the minimum possibly optimal value, the decision maker may be interested in the ratios of the objective function values between the selected solution and the possibly optimal solutions rather than in differences. From this point of view, the maximin achievement rate criterion is proposed. In the maximin achievement rate criterion, it is assumed that the decision maker's decision depends on the ratios of the objective function values between the selected solution and the possibly optimal solutions.

In Section 2, first, a linear programming problem with an interval objective function is described and the concepts of possibly optimal solutions and necessarily optimal solutions are introduced. Then the minimax regret criterion is reviewed and the attractiveness of the maximin achievement rate criterion is discussed. In Section 3, the maximin achievement rate criterion is introduced to a linear programming problem with an interval objective function and the maximin achievement rate problem is formulated. It is shown that a possibly optimal solution is obtained based on the maximin achievement rate criterion with considering all possibly optimal solutions and is also necessarily optimal when a necessarily optimal solution exists. In Section 4, a computation method for obtaining a maximin achievement rate solution is discussed. A solution algorithm based on a relaxation procedure together with the simplex method is proposed. In Section 5, a numerical example is given to illustrate the proposed solution algorithm.

# 2 Preliminaries

## 2.1 Possibly and necessarily optimal solutions

In this paper, the following linear programming problem with an interval objective function is treated:

$$\max_{\boldsymbol{x} \in X} \boldsymbol{\gamma x}, \tag{1}$$

$$X = \{\boldsymbol{x} \mid A\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq 0\} \subseteq \boldsymbol{R}^n, \tag{2}$$

where $A$ is an $m \times n$ matrix. $\boldsymbol{x}$ and $\boldsymbol{b}$ are $n$- and $m$-dimensional column vectors, respectively. $\boldsymbol{\gamma} = (c_1, c_2, \dots, c_n)$ and $c_i$ is a possibilistic variable restricted by an interval $[l_i, u_i]$. The interval $[l_i, u_i]$ shows the range in which the true coefficient $c_i$ possibly lies. Namely, the true coefficients are not known exactly, but the possible range is known. For the sake of simplicity, let

$$\theta = \{\boldsymbol{c} = (c_1, c_2, \dots, c_n) \mid l_i \le c_i \le u_i, i = 1, 2, \dots, n\}, \tag{3}$$

$\boldsymbol{l} = (l_1, l_2, \dots, l_n)$ and $\boldsymbol{u} = (u_1, u_2, \dots, u_n)$. $\theta$ shows the possible range of $\boldsymbol{\gamma}$. In this paper, $X$ is assumed to be bounded[1]. Moreover, we assume either of the following cases:

(a) $\max_{\boldsymbol{x} \in X} \boldsymbol{lx} > 0$,

(b) $\max_{\boldsymbol{x} \in X} \boldsymbol{ux} < 0$.

This assumption[2] shows that, because of the constraint $\boldsymbol{x} \ge 0$, the signs of the optimal values of linear programming problems $\max_{\boldsymbol{x} \in X} \boldsymbol{cx}$ are the same for all $\boldsymbol{c} \in \theta$. In what follows, we will discuss the case of (a), since the concept of a maximin achievement rate solution is easy to understand in this case. The similar results can also be deduced in the case of (b)[3] (see Appendix).

Let $S(\boldsymbol{c})$ be a set of optimal solutions to a linear programming problem with the objective coefficient vector $\boldsymbol{c}$, i.e.,

$$S(\boldsymbol{c}) = \left\{ \boldsymbol{y} \in X \mid \boldsymbol{cy} = \max_{\boldsymbol{x} \in X} \boldsymbol{cx} \right\}. \tag{4}$$

The following two kinds of optimal solution sets to the problem (1) have been proposed in [6].

$$NS = \bigcap_{\boldsymbol{c} \in \theta} S(\boldsymbol{c}) \tag{5}$$

$$\Pi S = \bigcup_{\boldsymbol{c} \in \theta} S(\boldsymbol{c}) \tag{6}$$

An element of $NS$ is a feasible solution optimal for all $\boldsymbol{c} \in \theta$ and called 'a necessarily optimal solution'. On the other hand, an element of $\Pi S$ is a feasible solution optimal for at least one $\boldsymbol{c} \in \theta$ and called 'a possibly optimal solution'. A necessarily optimal solution is the most rational solution, but does not exist in many cases. Usually we have many possibly optimal solutions and we must select a final solution even if a possibly optimal solution set $\Pi S$ is obtained.

In order to cope with such defects of necessarily and possibly optimal solutions, a minimax regret solution has been proposed in [7]. A minimax regret solution coincides

[1] This assumption is introduced to guarantee the convergence of the algorithm discussed in what follows. This assumption can be relaxed to $\max_{\boldsymbol{x} \in X} \boldsymbol{ux} < +\infty$ the case of (a), and to $\max_{\boldsymbol{x} \in X} \boldsymbol{lx} > -\infty$ in the case of (b)

[2] In some real world application, neither (a) nor (b) holds, i.e., $\max_{\boldsymbol{x} \in X} \boldsymbol{lx} \le 0 \le \max_{\boldsymbol{x} \in X} \boldsymbol{ux}$ holds. In this case, we can add (resp. subtract) a sufficiently large number $M > 0$ to $\max_{\boldsymbol{x} \in X} \boldsymbol{lx}$ (resp. from $\max_{\boldsymbol{x} \in X} \boldsymbol{ux}$) to be $\max_{\boldsymbol{x} \in X} \boldsymbol{lx} + M > 0$ (resp. $\max_{\boldsymbol{x} \in X} \boldsymbol{ux} - M < 0$). By using such a number $M$, we can compute a maximin achievement rate solution by the proposed method. However, we must take care on the determination of $M$, since the maximin achievement rate solution changes depending on $M$.

[3] In this case, the 'max' and 'min' operations are replaced with the 'min' and 'max' operations, respectively. Thus, the solution corresponding to a maximin achievement rate solution in the case (a) is called 'a minimax achievement rate solution'.

with a necessarily optimal solution when a necessarily optimal solution exists. Considering all possibly optimal solutions, a minimax regret solution is obtained by a relaxation procedure together with a simplex method. The minimax regret solution is defined in the following manner.

## 2.2 Minimax regret solution

Assume we know the true objective function coefficient vector $\boldsymbol{c}$ after the determination of the solution of the problem (1) as $\boldsymbol{x}$. Under this assumption, the regret of this determination can be expressed by

$$r(\boldsymbol{x}, \boldsymbol{c}) = \max_{\boldsymbol{y} \in X} (\boldsymbol{cy} - \boldsymbol{cx}). \tag{7}$$

The regret $r(\boldsymbol{x}, \boldsymbol{c})$ shows the difference between the optimal value with the objective function coefficient vector $\boldsymbol{c}$ and the true objective value $\boldsymbol{cx}$ with respect to $\boldsymbol{x}$.

When the true objective function coefficient vector is unknown, the worst (maximum) regret of the determination of the solution as $\boldsymbol{x}$ can be defined by

$$R(\boldsymbol{x}) = \max_{\boldsymbol{c} \in \Gamma} r(\boldsymbol{x}, \boldsymbol{c}). \tag{8}$$

Problem (1) can now be formulated as the problem of minimizing the maximum regret $R(\boldsymbol{x})$, i.e.,

$$\min_{\boldsymbol{x} \in X} R(\boldsymbol{x}). \tag{9}$$

From (7) and (8), the problem (9) is rewritten as

$$\min_{\boldsymbol{x} \in X} \max_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{y} \in X}} (\boldsymbol{cy} - \boldsymbol{cx}). \tag{10}$$

The optimal solution to the problem (10) is called 'a minimax regret solution'.

In the minimax regret solution, it is assumed that the decision maker's decision depends on differences in the objective function value between the selected solution and the possibly optimal solutions. However, when the range of possibly optimal values is relatively wide, the decision maker may be interested in the ratios of the objective function values between the selected solution and the possibly optimal solutions rather than in differences. This is because the regret becomes relatively large comparing with the lower bound of optimal values. Let us consider the following example.

**Example** Let us consider the following linear programming problem with an interval objective function:

$$\begin{array}{rl} \text{maximize} & c_1 x_1 + c_2 x_2, \\ \text{subject to} & 3x_1 + x_2 \le 31, \\ & x_1 + 2x_2 \le 57, \\ & x_1 \ge 0, \ x_2 \ge 0, \end{array}$$

where the possibilistic vector $\boldsymbol{\gamma} = (c_1, c_2)$ is restricted by a set

$$\theta = \{\boldsymbol{c} = (c_1, c_2) \mid 1 \le c_1 \le 2, \ 0 \le c_2 \le 1\}.$$

The range of possibly optimal values to this problem is given by $[10\frac{1}{3}, 30]$. The width of this range, $19\frac{2}{3}$, is large relatively to the minimum possibly optimal value $10\frac{1}{3}$. The minimax regret solution is obtained as $(x_1, x_2) = (5\frac{2}{3}, 14)$. The maximum regret is given as $9\frac{1}{3}$. For this solution, checking all possibly optimal values and the corresponding regret values, we obtain the maximum regret rate as $\frac{14}{31} \cong 45.16129\%$[4]. This value is relatively large. In this case, the decision maker may be interested in the improvement of the maximum regret rate, i.e., the minimum achievement rate, where the sum of the regret rate and the achievement rate is 1. From this point of view, we introduce the maximin achievement rate criterion. To the above problem, we obtain the maximin achievement rate solution as $(x_1, x_2) = (6\frac{67}{149}, 11\frac{130}{149})$ and the maximum regret rate as $\frac{56}{149} \cong 37.5839\%$.

As shown in the above example, the decision maker may be interested in the improvement of the maximum regret rate, i.e., the minimum achievement rate. In the subsequent section, we propose a maximin achievement rate solution.

# 3 A Maximin Achievement Rate Solution and Its Properties

As described in Subsection 2.2, assume we know the true objective function coefficient vector $\boldsymbol{c}$ after the determination of the solution of the problem (1) as $\boldsymbol{x}$. Under this assumption, the ratio of the objective function value $\boldsymbol{cx}$ for the solution $\boldsymbol{x}$ to the optimal value can be represented as

$$ra(\boldsymbol{x}, \boldsymbol{c}) = \frac{\boldsymbol{cx}}{\max_{\boldsymbol{y} \in X} \boldsymbol{cy}}. \tag{11}$$

$ra(\boldsymbol{x}, \boldsymbol{c})$ shows the achievement rate of $\boldsymbol{x}$ to the optimal value. Since we assume the case (a), we have $0 < ra(\boldsymbol{x}, \boldsymbol{c}) \leq 1$ and the closer to 1 $ra(\boldsymbol{x}, \boldsymbol{c})$ is, the better the solution $\boldsymbol{x}$ is.

When the true objective function coefficient vector is unknown, the worst (minimum) achievement rate of the determination of the solution as $\boldsymbol{x}$ can be defined by

$$Ra(\boldsymbol{x}) = \min_{\boldsymbol{c} \in \theta} ra(\boldsymbol{x}, \boldsymbol{c}). \tag{12}$$

The problem (1) can be formulated as the problem maximizing the minimum achievement rate $Ra(\boldsymbol{x})$, i.e.,

$$\max_{\boldsymbol{x} \in X} Ra(\boldsymbol{x}). \tag{13}$$

[4]For example, in the case of $(c_1, c_2) = (2, 0)$, we obtain the optimal value $20\frac{2}{3}$ at the optimal solution $(x_1, x_2) = (10\frac{1}{3}, 0)$ and an objective function value $11\frac{1}{3}$ at the maximin regret solution $(x_1, x_2) = (5\frac{2}{3}, 14)$. Thus, the regret is

$$20\frac{2}{3} - 11\frac{1}{3} = 9\frac{1}{3}.$$

Hence, we obtain the regret rate as

$$9\frac{1}{3} \div 20\frac{2}{3} = \frac{14}{31} \cong 45.1629\%$$

From (7), (8) and the assumption (a), (13) can be transformed as

$$\begin{aligned}
\max_{\boldsymbol{x}\in X} Ra(\boldsymbol{x}) &\iff \max_{\boldsymbol{x}\in X} \min_{\boldsymbol{c}\in\theta} \frac{\boldsymbol{cx}}{\max_{\boldsymbol{y}\in X} \boldsymbol{cy}} \\
&\iff \max_{\boldsymbol{x}\in X} \min_{\boldsymbol{c}\in\theta} \frac{\boldsymbol{cx}}{\max_{\substack{\boldsymbol{y}\in X \\ \boldsymbol{cy}>0}} \boldsymbol{cy}} \\
&\iff \max_{\boldsymbol{x}\in X} \min_{\substack{\boldsymbol{c}\in\theta \\ \boldsymbol{y}\in X \\ \boldsymbol{cy}>0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}},
\end{aligned} \tag{14}$$

where there is no guarantee for $\boldsymbol{cy} > 0$ for any $\boldsymbol{y} \in X$ so that, introducing a constraint $\boldsymbol{cy} > 0$ based on the assumption (a), the 'max' in the denominator is changed to the 'min' of the fraction. The optimal solution to the problem (14) is called 'a maximin achievement rate solution'.

Here, let us discuss the properties of a maximin achievement rate solution. Under the assumption (a), we have $0 < Ra(\boldsymbol{x}) \leq 1$. If $Ra(\boldsymbol{x}) = 1$ holds, we have $\boldsymbol{cx} = \max_{\boldsymbol{y}\in X} \boldsymbol{cy}$ for all $\boldsymbol{c} \in \theta$, i.e., the maximin achievement rate solution $\boldsymbol{x}$ is a necessarily optimal solution. Thus, we have the following theorem.

[**Theorem 1**] Let $\boldsymbol{x}^*$ be an optimal solution to (13). If $Ra(\boldsymbol{x}^*) = 1$ holds, then there exists a necessarily optimal solution to the problem (1) and $\boldsymbol{x}^*$ is a necessarily optimal solution. Conversely, if a necessarily optimal solution to the problem (1) exists, then we have $Ra(\boldsymbol{x}^*) = 1$ and $\boldsymbol{x}^*$ is a necessarily optimal solution.

From Theorem 1, a maximin achievement rate solution is necessarily optimal when a necessarily optimal solution exists.

The fact that a maximin achievement rate solution is possibly optimal can be proved by using the following lemma proved in [7].

[**Lemma 1**] Consider the following multi-objective linear programming problem:

$$v - \max_{\boldsymbol{x}\in X} (\boldsymbol{c}^1\boldsymbol{x}, \boldsymbol{c}^2\boldsymbol{x}, \ldots, \boldsymbol{c}^q\boldsymbol{x}), \tag{15}$$

where $\boldsymbol{c}^j$, $j = 1, 2, \ldots, q$ are all elements of a set $\Delta$ composed of all extreme points of $\theta$. Namely,

$$\Delta = \{\boldsymbol{c} = (c_1, c_2, \ldots, c_n) \mid c_i = l_i \text{ or } c_i = u_i,\ i = 1, 2, \ldots, n\} = \{\boldsymbol{c}^1, \boldsymbol{c}^2, \ldots, \boldsymbol{c}^q\}. \tag{16}$$

A solution is possibly optimal to the problem (1) if and only if, it is weakly efficient to the problem (15). A solution is necessarily optimal to the problem (1) if and only if it is completely optimal to the problem (15).

[**Theorem 2**] A maximin achievement rate solution is a possibly optimal solution to the problem (1).

(**Proof**) Let $\boldsymbol{x}^*$ be a maximin achievement rate solution. Suppose it is not a possibly optimal solution, then it is not a weakly efficient solution to the problem (15) by Lemma 1. Thus, there exists a feasible solution $\boldsymbol{x}$ such that $\boldsymbol{c}^j\boldsymbol{x} > \boldsymbol{c}^j\boldsymbol{x}^*, j = 1, 2, \ldots, q$. Namely, $\sum_{j=1}^q \lambda_j \boldsymbol{c}^j\boldsymbol{x} > \sum_{j=1}^q \lambda_j \boldsymbol{c}^j\boldsymbol{x}^*$ holds for all $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$ such that $\sum_{j=1}^q \lambda_j = 1$ and

$\lambda_j \geq 0,\ j = 1, 2, \ldots, q$. Since $\boldsymbol{c}^j,\ j = 1, 2, \ldots, q$ are all extreme points of $\theta$, this inequality can be rewritten as $\boldsymbol{cx} > \boldsymbol{cx}^*$, for all $\boldsymbol{c} \in \theta$. Thus we have

$$\min_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{y} \in X \\ \boldsymbol{cy} > 0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}} < \min_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{y} \in X \\ \boldsymbol{cy} > 0}} \frac{\boldsymbol{cx}^*}{\boldsymbol{cy}}.$$

This contradicts the fact that $\boldsymbol{x}^*$ is a maximin achievement rate solution, i.e., an optimal solution to the problem (13). Hence, a maximin achievement rate solution is a possibly optimal solution. (Q.E.D.)

Moreover, the problem (13) is transformed to an easier problem as shown in what follows. Given $\boldsymbol{c} \in \theta$, an optimal solution $\boldsymbol{y}$ to a linear programming problem

$$\max_{\boldsymbol{y} \in X} \boldsymbol{cy} \tag{16}$$

belongs to a set $S(\boldsymbol{c})$ defined by (4). From (6), $\Pi S$ is composed of $S(\boldsymbol{c})$'s for all $\boldsymbol{c} \in \theta$. Thus, the problem (14) is equivalent to

$$\max_{\boldsymbol{x} \in X} \min_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{y} \in \Pi S \\ \boldsymbol{cy} > 0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}}. \tag{17}$$

This shows that it is sufficient to consider the possibly optimal solution set $\Pi S$ as the region of $\boldsymbol{y}$ instead of the feasible set $X$. From (17), a maximin achievement rate solution is a compromise solution obtained by considering all possibly optimal solutions.

According to the fundamental theorem [4] of linear programming, there is an optimal basic feasible solution $s_{\boldsymbol{c}}$ in $S(\boldsymbol{c})$ when $S(\boldsymbol{c})$ is non-empty. Letting

$$\Pi B = \bigcup_{\boldsymbol{c} \in \theta} \{s_{\boldsymbol{c}}\}, \tag{18}$$

then the problem (17) can be written as

$$\max_{\boldsymbol{x} \in X} \min_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{y} \in \Pi B \\ \boldsymbol{cy} > 0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}}. \tag{19}$$

This indicates that it is sufficient to consider all possibly optimal basic feasible solutions in $\Pi B$ as the region of $\boldsymbol{y}$.

# 4 A Solution Algorithm Based on a Relaxation Procedure

## 4.1 An algorithm based on a Relaxation Procedure

From the boundedness of $X$, $\Pi B$ has a finite number of elements. Thus, $\Pi B$ can be expressed as

$$\Pi B = \{\boldsymbol{y}^1, \boldsymbol{y}^2, \ldots, \boldsymbol{y}^p\}. \tag{20}$$

All elements of $\Pi B$, say $\boldsymbol{y}^j$, $j = 1, 2, \ldots, p$, are easily obtained by parametric linear programming techniques (see [12]-[14]). Introducing $\boldsymbol{y}^j$'s and an auxiliary variable $r$, the maximin achievement rate problem (19) can be represented as

$$\begin{array}{rl} \text{maximize} & r, \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \displaystyle\min_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{c}\boldsymbol{y}^j > 0}} \frac{\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{c}\boldsymbol{y}^j} \geq r, j = 1, 2, \ldots, p, \\ & \boldsymbol{x} \geq 0, \end{array} \tag{21}$$

or equivalently, without using min in the constraints,

$$\begin{array}{rl} \text{maximize} & r, \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \dfrac{\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{c}\boldsymbol{y}^j} \geq r, \ \forall \boldsymbol{c} \in \theta \cap \{\boldsymbol{c} \mid \boldsymbol{c}\boldsymbol{y}^j > 0\}, \ j = 1, 2, \ldots, p, \\ & \boldsymbol{x} \geq 0. \end{array} \tag{21'}$$

Defining a function (sub-problem) $\phi(\boldsymbol{x}, \boldsymbol{y})$ as

$$\phi(\boldsymbol{x}, \boldsymbol{y}) \triangleq \min_{\substack{\boldsymbol{c} \in \theta \\ \boldsymbol{c}\boldsymbol{y} > 0}} \frac{\boldsymbol{c}\boldsymbol{x}}{\boldsymbol{c}\boldsymbol{y}}, \tag{22}$$

we have the following solution algorithm for the maximin achievement rate problem:

[Step 1] Let $\boldsymbol{c}^1 = \boldsymbol{l}$ and $\boldsymbol{z}^1$ be a $\boldsymbol{y}$ maximizing $\boldsymbol{l}\boldsymbol{y}$ under the constraint $\boldsymbol{y} \in \Pi B$.

[Step 2] Set $k = 2$, $r^1 = 999999$ (a sufficiently large number) and $\boldsymbol{x}^1 = \boldsymbol{z}^1$.

[Step 3] Calculate $\phi(\boldsymbol{x}^{k-1}, \boldsymbol{y})$ for all $\boldsymbol{y} \in \Pi B$. Let $\boldsymbol{z}^k$ be a $\boldsymbol{y}$ minimizing $\phi(\boldsymbol{x}^{k-1}, \boldsymbol{y})$. Let $\boldsymbol{c}^k$ be an optimal solution to the sub-problem $\phi(\boldsymbol{x}^{k-1}, \boldsymbol{z}^k)$.

[Step 4] If $\phi(\boldsymbol{x}^{k-1}, \boldsymbol{y}) \geq r^{k-1} - \varepsilon$ holds, then terminate the algorithm. In this case, $\boldsymbol{x}^k$ is an approximation of a maximin achievement rate solution.

[Step 5] Solve the following linear programming problem:

$$\begin{array}{rl} \text{maximize} & r, \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \dfrac{\boldsymbol{c}^j\boldsymbol{x}}{\boldsymbol{c}^j\boldsymbol{z}^j} \geq r, j = 1, 2, \ldots, k, \\ & \boldsymbol{x} \geq 0. \end{array} \tag{23}$$

Let $(\boldsymbol{x}^k, r^k)$ be an optimal solution. Return to Step 3 with $k = k + 1$.

Here, $\varepsilon$ is a predetermined sufficiently small positive number. The smaller $\varepsilon$ is, the better approximate solution we obtain. Since $\boldsymbol{c}^j \in \theta$, $\boldsymbol{z}^j \in \Pi B$ and $\boldsymbol{c}^j\boldsymbol{z}^j > 0$ hold from the definitions at Step 3, the problem (17) can be regarded as a relaxed problem of (15) where infinitely many constraints, $\boldsymbol{c}\boldsymbol{x}/\boldsymbol{c}\boldsymbol{y}^j \geq r$, $\forall \boldsymbol{c} \in \theta \cap \{\boldsymbol{c} \mid \boldsymbol{c}\boldsymbol{y}^i > 0\}$, are relaxed to $k$

constraints, $\boldsymbol{c}^j\boldsymbol{x}/\boldsymbol{c}^j\boldsymbol{z}^j \geq r$, $j = 1, 2, \ldots, k$. This means that this algorithm is based on a relaxation procedure. It can be shown, in the same manner as done for the conventional min-max problem in [11], that the algorithm terminates in a finite number of iterations for any $\varepsilon > 0$.

In what follows, let us discuss a computation method of $\phi(\boldsymbol{x}^k, \boldsymbol{y})$ for all $\boldsymbol{y} \in \Pi B$ at Step 3.

## 4.2 A computation method at Step 3

Since a constraint $\boldsymbol{c} \in \theta$ can be represented as $\boldsymbol{l} \leq \boldsymbol{c} \leq \boldsymbol{u}$, the subproblem (22) can be rewritten as

$$\begin{array}{rl} \text{minimize} & \dfrac{\boldsymbol{cx}}{\boldsymbol{cy}}, \\ \text{subject to} & \boldsymbol{l} \leq \boldsymbol{c} \leq \boldsymbol{u}, \\ & \boldsymbol{cy} > 0, \end{array} \tag{24}$$

where $\boldsymbol{c}$ is the decision variable. This type of problem is known as a linear fractional programming problem [2]. Since we have $\boldsymbol{cy} > 0$, it is possible to solve this problem by transforming it to a linear programming problem (see [2]),

$$\begin{array}{rl} \text{minimize} & \boldsymbol{dx}, \\ \text{subject to} & \boldsymbol{l}t \leq \boldsymbol{d} \leq \boldsymbol{u}t, \\ & \boldsymbol{dy} = 1, \\ & t \geq 0. \end{array} \tag{25}$$

where an auxiliary variable $t$ represents the reciprocal of $\boldsymbol{cy}$ and a vector $\boldsymbol{d}$ corresponds to $t\boldsymbol{c}$. Let $(\boldsymbol{d}^*, t^*)$ be an optimal solution to the problem (25). The optimal value and solution of the problem (24) can be obtained as $\boldsymbol{d}^*\boldsymbol{x}$ and $\boldsymbol{c}^* = \boldsymbol{d}^*/t^*$, respectively. The constraint $\boldsymbol{cy} > 0$ of (24) is satisfied by the constraint $\boldsymbol{dy} = 1$ of (25).

However, at Step 3, we must solve the sub-problem (25) for all $\boldsymbol{y} \in \Pi B$. Thus, this requires a formidable computation effort. If post-optimization techniques are available, we will considerably cut down on the effort, since we do not need to recalculate an initial feasible basic solution by the first phase of the two-phase method for each $\boldsymbol{y} \in \Pi B$.

Suppose $\boldsymbol{lx} > 0$, we have $\boldsymbol{cx} > 0$ for all $\boldsymbol{c} \in \theta$ from the non-negativity constraint $\boldsymbol{x} \geq 0$. Thus, the objective function value of (24), $\boldsymbol{cx}/\boldsymbol{cy}$, always becomes positive. In this case, the optimal solution to the problem (24) can be obtained by solving the following problem with the objective function defined by the reciprocal of the objective function of (24):

$$\begin{array}{rl} \text{maximize} & \dfrac{\boldsymbol{cy}}{\boldsymbol{cx}}, \\ \text{subject to} & \boldsymbol{l} \leq \boldsymbol{c} \leq \boldsymbol{u}, \\ & \boldsymbol{cy} > 0. \end{array} \tag{26}$$

This problem is also a linear fractional programming problem and solved by transforming it to the following linear programming problem:

$$\begin{array}{rl} \text{maximize} & \boldsymbol{dy}, \\ \text{subject to} & \boldsymbol{l}t \leq \boldsymbol{d} \leq \boldsymbol{u}t, \\ & \boldsymbol{dx} = 1, \\ & t \geq 0. \end{array} \tag{27}$$

Let $(\hat{\boldsymbol{d}}, \hat{t})$ be an optimal solution to the problem (27). The optimal value and solution to the problem (24) can be obtained as $1/\hat{\boldsymbol{d}}\boldsymbol{y}$ and $\boldsymbol{c}^* = \hat{\boldsymbol{d}}/\hat{t}$, respectively. Since $t > 0$ holds for all feasible solutions $(\boldsymbol{d}, t)$ of (27) [5], $\hat{\boldsymbol{d}}\boldsymbol{y} > 0$ (i.e., $\boldsymbol{c}^*\boldsymbol{y} > 0$) holds under the assumption (a).

Since $\boldsymbol{y}$ appears only in the objective function of (27), a post-optimization technique with respect to change of the objective coefficient vector is available to solve the sub-problem (25) for all $\boldsymbol{y} \in \Pi B$. Thus, Step 3 can be performed easily.

As discussed above, if $\boldsymbol{l}\boldsymbol{x} > 0$ holds, Step 3 can be performed easily by the use of a post-optimization technique. In the next subsection, let us show that $\boldsymbol{l}\boldsymbol{x}^k > 0$ always holds in the algorithm proposed in Subsection 4.1.

## 4.3 On the satisfaction of $\boldsymbol{l}\,\boldsymbol{x}^k > 0$

Since $\boldsymbol{z}^1$ satisfy $\boldsymbol{l}\boldsymbol{z}^1 = \max_{\boldsymbol{x}\in X} \boldsymbol{l}\boldsymbol{x}$ from the definition at Step 1, we have $\boldsymbol{c}^1\boldsymbol{z}^1 > 0$ under the assumption (a). Using this fact, the validity of $\boldsymbol{l}\boldsymbol{x}^k = \boldsymbol{c}^1\boldsymbol{x}^k > 0$, for any $k$ can be shown as follows by a reductio ad absurdum. Suppose $\boldsymbol{l}\boldsymbol{x}^k \le 0$, we have $\boldsymbol{c}^1\boldsymbol{x}^k/\boldsymbol{c}^1\boldsymbol{z}^1 = \boldsymbol{l}\boldsymbol{x}^k/\boldsymbol{l}\boldsymbol{z}^1 \le 0$. From the constraints of (23), this yields $r^k \le \boldsymbol{c}^1\boldsymbol{x}/\boldsymbol{c}^1\boldsymbol{z}^1 \le 0$. On the other hand, let

$$r^0 = \min_{j=1,2,\ldots,k} \frac{\boldsymbol{c}^j\boldsymbol{z}^1}{\boldsymbol{c}^j\boldsymbol{z}^j}.$$

From $\boldsymbol{z}^1 \in X$, a solution $(\boldsymbol{z}^1, r^0)$ is a feasible solution of the problem (23). Under the assumption (a), $\boldsymbol{c}^j\boldsymbol{z}^j > \boldsymbol{c}^j\boldsymbol{z}^1 \ge \boldsymbol{l}\boldsymbol{z}^1 > 0$ hold for $j = 1, 2, \ldots, k$. Thus, we have $r^0 > 0 \ge r^k$. This contradicts the optimality of the solution $(\boldsymbol{x}^k, r^k)$. Hence, $\boldsymbol{l}\boldsymbol{x}^k > 0$ holds for all $k$.

The above discussion guarantees that Step 3 can be executed by a post-optimization technique in linear programming.

# 5 A Numerical Example

In order to illustrate the solution algorithm proposed in Section 4, let us consider the following linear programming problem with an interval objective function:

$$\begin{array}{rl}
\text{maximize} & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 + c_6x_6 + c_7x_7 + c_8x_8, \\
\text{subject to} & x_1 + 3x_2 - 4x_3 + x_4 - x_5 + x_6 + 2x_7 + 4x_8 \le 40, \\
& 5x_1 + 2x_2 + 4x_3 - x_4 - 3x_5 + 7x_6 + 2x_7 + 7x_8 \le 84, \\
& 4x_2 - x_3 - x_4 - 3x_5 + x_8 \le 18, \\
& -3x_1 - 4x_2 + 8x_3 + 2x_4 + 3x_5 - 4x_6 + 5x_7 - x_8 \le 100, \\
& 12x_1 + 8x_2 - x_3 + 4x_4 + x_6 + x_7 \le 40, \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 \ge 12, \\
& 8x_1 - 12x_2 - 3x_3 + 4x_4 - x_5 \le 30, \\
& -5x_1 - 6x_2 + 12x_3 + x_4 - x_7 + x_8 \le 100, \\
& x_j \ge 0,\ j = 1, 2, \ldots, 8,
\end{array} \tag{28}$$

where a set of objective coefficient vectors, $\theta$, which restricts a possibilistic variable vector $\gamma = (c_1, c_2, \ldots, c_8)$, is given as

$$\theta = \{\boldsymbol{c} = (c_1, c_2, \ldots, c_8) \mid (0, 1, -1, -1, -3, 0, 0, 1) \le \boldsymbol{c} \le (1, 1, 1, 1, -1, 1, 1, 1)\}. \tag{29}$$

[5] Suppose $t = 0$, then we have $\boldsymbol{d} = 0$ from the first constraint of (27). This contradicts the second constraint of (27). Hence, we have $t > 0$.

Table 1 47 possible optimal basic solutions

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
|---|---|---|---|---|---|---|---|---|
| $y^1$ | 0 | 2.4615 | 2 | 0 | 0 | 0 | 0 | 10.1538 |
| $y^2$ | 0 | 2.8000 | 0 | 0 | 0 | 4.400 | 0 | 6.8000 |
| $y^3$ | 0 | 3.9268 | 0 | 0 | 0 | 8.5854 | 0 | 2.2927 |
| $y^4$ | 0 | 6.3905 | 11.2235 | 0 | 0 | 0.0993 | 0 | 3.6614 |
| $y^5$ | 0 | 6.3808 | 11.2301 | 0 | 0 | 0 | 0.1841 | 3.7071 |
| $y^6$ | 0 | 5.6333 | 11.5249 | 0 | 0 | 0 | 6.4587 | 1.9595 |
| $y^7$ | 0 | 3.5289 | 4.5392 | 0 | 0 | 0 | 16.3081 | 3.7385 |
| $y^8$ | 0 | 2.8016 | 1.7899 | 0 | 0 | 0 | 19.3774 | 0 |
| $y^9$ | 0 | 0 | 0.1575 | 5.6693 | 0 | 0 | 17.4803 | 0 |
| $y^{10}$ | 0 | 0 | 0 | 5.7143 | 0 | 0 | 17.1429 | 0 |
| $y^{11}$ | 0 | 0 | 0 | 0.0000 | 0 | 0 | 20.0000 | 0 |
| $y^{12}$ | 0 | 0 | 0 | 4.4545 | 0 | 8.8182 | 13.3636 | 0 |
| $y^{13}$ | 0 | 0 | 0 | 0 | 0 | 7.3333 | 16.3333 | 0 |
| $y^{14}$ | 0 | 0 | 1.7875 | 0 | 0 | 4.9500 | 21.1000 | 0 |
| $y^{15}$ | 1.4075 | 0 | 1.8491 | 0 | 0 | 3.9296 | 21.0296 | 0 |
| $y^{16}$ | 0 | 2.4810 | 0 | 0 | 0 | 7.7468 | 12.4051 | 0 |
| $y^{17}$ | 1.4627 | 0 | 0 | 0 | 0 | 6.3582 | 16.0896 | 0 |
| $y^{18}$ | 3.2803 | 0 | 0 | 0 | 0 | 0.6360 | 0 | 9.0209 |
| $y^{19}$ | 3.3568 | 0 | 0.2810 | 0 | 0 | 0 | 0 | 9.4418 |
| $y^{20}$ | 1.1027 | 0 | 2.7946 | 7.3906 | 0 | 0 | 0 | 10.6713 |
| $y^{21}$ | 0 | 0 | 5.0000 | 11.2500 | 0 | 0 | 0 | 10.7500 |
| $y^{22}$ | 0 | 0 | 6.5366 | 11.6342 | 0 | 0 | 0 | 9.9268 |
| $y^{23}$ | 0 | 0 | 4.1571 | 10.6178 | 0 | 0 | 1.6858 | 10.6597 |
| $y^{24}$ | 0 | 0.0997 | 4.0025 | 10.8011 | 0 | 0 | 0 | 11.2274 |
| $y^{25}$ | 0 | 0.1534 | 0 | 7.9603 | 0 | 6.9313 | 0 | 6.1620 |
| $y^{26}$ | 1.4592 | 0 | 0 | 4.5817 | 0 | 4.1633 | 0 | 7.4490 |
| $y^{27}$ | 0 | 0 | 0 | 7.5000 | 0 | 7.3519 | 2.6481 | 4.9630 |
| $y^{28}$ | 0 | 0 | 2.2128 | 4.2534 | 0 | 5.8007 | 19.3987 | 0 |
| $y^{29}$ | 0 | 0 | 8.3091 | 8.4958 | 0 | 6.1215 | 8.2044 | 0 |
| $y^{30}$ | 1.6941 | 0 | 0.5177 | 0 | 0 | 0 | 20.1883 | 0 |
| $y^{31}$ | 0 | 0 | 4.8368 | 8.6309 | 0 | 0 | 10.3133 | 7.5225 |
| $y^{32}$ | 2.1442 | 0 | 2.9932 | 0 | 0 | 0 | 17.2626 | 3.8258 |
| $y^{33}$ | 0 | 2.3348 | 2.8892 | 0 | 0 | 3.8702 | 20.3412 | 0 |
| $y^{34}$ | 0 | 0 | 7.3007 | 10.4012 | 0 | 0 | 5.6958 | 7.6867 |
| $y^{35}$ | 0 | 5.3287 | 11.5942 | 0 | 0 | 1.8071 | 7.1580 | 0 |
| $y^{36}$ | 0 | 6.4024 | 11.2192 | 0 | 0.0585 | 0 | 0 | 3.7849 |
| $y^{37}$ | 0 | 6.3669 | 11.1953 | 0.0652 | 0 | 0 | 0 | 3.7929 |
| $y^{38}$ | 0.0251 | 6.3641 | 11.2128 | 0 | 0 | 0 | 0 | 3.7565 |
| $y^{39}$ | 0 | 3.6471 | 0 | 0 | 0 | 6.2353 | 4.5883 | 3.4117 |
| $y^{40}$ | 0 | 3.5037 | 0 | 1.5772 | 0 | 5.6618 | 0 | 5.5625 |
| $y^{41}$ | 1.1334 | 2.8349 | 0 | 0 | 0 | 3.7199 | 0 | 6.6605 |
| $y^{42}$ | 1.5818 | 2.6000 | 0 | 0 | 0 | 0.2182 | 0 | 7.6000 |
| $y^{43}$ | 0 | 2.6000 | 0 | 0 | 0 | 1.8000 | 0 | 7.6000 |
| $y^{44}$ | 0 | 4.1365 | 3.9795 | 0 | 0 | 0 | 10.8875 | 5.4334 |
| $y^{45}$ | 0 | 3.7396 | 3.2083 | 3.3230 | 0 | 0 | 0 | 9.5729 |
| $y^{46}$ | 1.7309 | 2.5947 | 1.5279 | 0 | 0 | 0 | 0 | 9.1492 |
| $y^{47}$ | 0 | 6.3509 | 11.1930 | 0 | 0 | 0 | 0 | 3.7895 |

In the interval function representation, the objective function of (28) can be represented as

$$[0,1]x_1 + x_2 + [-1,1]x_3 + [-1,1]x_4 + [-3,-1]x_5 + [0,1]x_6 + [0,1]x_7 + x_8. \qquad (30)$$

In the problem (28), even the signs of some coefficients of the objective function are not clear. Thus, one can expect that the range of the possibly optimal values are wide. Indeed, the range of the possibly optimal values is obtained as $[10.6154, 31.6655]$. Since the upper bound is almost three times as large as the lower bound, this range can be regarded as wide.

Table 2 The minimum achievement rate and the maximum regret

| | the minimum achievement rate | the maximum regret |
|---|---|---|
| $\boldsymbol{x}^{\mathrm{w}}$ | 0.516660 | 13.5807 |
| $\boldsymbol{x}^{\mathrm{m}}$ | 0.426846 | 12.0861 |

Let us compute a maximin achievement rate solution of the problem (28). In order to examine the boundedness of the feasible area $X$, a function $x_1+x_2+x_3+x_4+x_5+x_6+x_7+x_8$ is maximized under $\boldsymbol{x} \in X$. The maximum value is obtained as 140.889, hence, the boundedness of $X$ is confirmed. Maximizing a function $\boldsymbol{lx}$ under $\boldsymbol{x} \in X$, we obtain the maximum value as 10.6154. Thus, the assumption (a) holds. By a parametric linear programming technique, we obtain the set $\Pi B$ consisting of 47 basic feasible solutions as listed in Table 1. Setting $\varepsilon = 0.000001$, we get the following maximin achievement rate solution by the proposed algorithm:

$$\boldsymbol{x}^{\mathrm{w}} = (0.026142, 3.817153, 2.576039, 1.408137, 0, 1.628976, 4.463591, 6.715565)^t. \qquad (31)$$

The minimum achievement rate is obtained as 0.516660. The iteration process to obtain this solution is shown in Figure 1.

For the purpose of comparing the proposed maximin achievement rate solution with the minimax regret solution, a minimax regret solution is computed and obtained as follows by the algorithm proposed in [7]:

$$\boldsymbol{x}^{\mathrm{m}} = (0, 3.9548, 3.5372, 1.4008, 0, 0.1837, 6.1122, 7.1189)^t. \qquad (32)$$

The minimum achievement rates and the maximum regrets with respect to the maximin achievement rate solution $\boldsymbol{x}^{\mathrm{w}}$ and the minimax regret solution $\boldsymbol{x}^{\mathrm{m}}$ are listed in Table 2. The minimum achievement rate with respect to the minimax regret solution, 0.426846, is too small since this value means that we cannot achieve even half of the value of the optimal value in the worst case. From this point of view, the decision maker might have interest in maximizing the minimum achievement rate and select the maximin achievement rate solution.

# 6 Conclusions

A new solution concept, i.e., a maximin achievement rate solution, has been proposed for linear programming problems with an interval objective function. It has been shown

**Step 1** $\boldsymbol{z}^1 = (0, 2.4615, 2, 0, 0, 0, 0, 10.1538)^t$.
$\boldsymbol{c}^1 = \boldsymbol{l} = (0, 1, -1, -1, -3, 0, 0, 1)$.

**Step 2** $k = 2$, $r^1 = 999999$ and $\boldsymbol{x}^1 = \boldsymbol{z}^1$.

**Step 3** $\boldsymbol{z}^2 = (0, 0, 2.2128, 4.2534, 0, 5.8007, 19.3987, 0)^t$.
$\boldsymbol{c}^2 = (1, 1, -1, 1, -1, 1, 1, 1)$.

**Step 4** $\phi(\boldsymbol{x}^1, \boldsymbol{z}^2) = 0.389695 < 999999 - 0.000001 = r^1 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^2 = (0, 3.630707, 0, 0, 0, 6.25649, 4.69785, 3.363922)^t$.
$r^2 = 0.65892$ and $k = 3$. Go to Step 3.

**Step 3** $\boldsymbol{z}^3 = (0, 0, 6.5366, 11.6342, 0, 0, 0, 9.9268)^t$.
$\boldsymbol{c}^3 = (1, 1, 1, 1, -1, 0, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^2, \boldsymbol{z}^3) = 0.24894 < 0.65892 - 0.000001 = r^2 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^3 = (0, 3.216977, 1.673966, 3.268789, 0, 2.862995, 0, 7.728287)^t$.
$r^2 = 0.565458$ and $k = 4$. Go to Step 3.

**Step 3** $\boldsymbol{z}^4 = (1.6941, 0, 0.5177, 0, 0, 0, 20.1883, 0)^t$.
$\boldsymbol{c}^4 = (1, 1, -1, -1, -1, 0, 1, 1)$.

**Step 4** $\phi(\boldsymbol{x}^3, \boldsymbol{z}^4) = 0.280955 < 0.565458 - 0.000001 = r^3 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^4 = (0, 3.887159, 3.4255, 1.483617, 0, 0.350575, 6.070235, 7.043515)^t$.
$r^4 = 0.564705$ and $k = 5$. Go to Step 3.

**Step 3** $\boldsymbol{z}^5 = (0, 3.9268, 0, 0, 0, 8.5854, 0, 2.2927)^t$.
$\boldsymbol{c}^5 = (1, 1, -1, -1, -1, 1, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^4, \boldsymbol{z}^5) = 0.42858 < 0.564705 - 0.000001 = r^4 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^5 = (0.287152, 3.675478, 2.794914, 1.20531, 0, 0.780540, 4.343485, 7.298313)^t$.
$r^5 = 0.543148$ and $k = 6$. Go to Step 3.

**Step 3** $\boldsymbol{z}^6 = (0, 0.1534, 0, 7.9603, 0, 6.9313, 0, 6.162)^t$.
$\boldsymbol{c}^6 = (0, 1, -1, 1, -1, 1, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^5, \boldsymbol{z}^6) = 0.47931 < 0.543148 - 0.000001 = r^5 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^6 = (0.661368, 3.342359, 2.061282, 0.92735, 0, 1.200624, 2.475964, 7.619195)^t$.
$r^6 = 0.520029$ and $k = 7$. Go to Step 3.

**Step 3** $\boldsymbol{z}^7 = (0, 0, 1.7875, 0, 0, 4.95, 21.1, 0)^t$.
$\boldsymbol{c}^7 = (0, 1, -1, -1, -1, 1, 1, 1)$.

**Step 4** $\phi(\boldsymbol{x}^6, \boldsymbol{z}^7) = 0.480145 < 0.520029 - 0.000001 = r^6 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^7 = (0.083871, 3.774961, 2.533958, 1.360274, 0, 1.584199, 4.302525, 6.794389)^t$.
$r^2 = 0.517747$ and $k = 8$. Go to Step 3.

**Step 3** $\boldsymbol{z}^8 = (0, 0, 6.5366, 11.6342, 0, 0, 0, 9.9268)^t$.
$\boldsymbol{c}^8 = (0, 1, 1, 1, -1, 0, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^7, \boldsymbol{z}^8) = 0.514762 < 0.517747 - 0.000001 = r^7 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^8 = (0, 3.837788, 2.602606, 1.423149, 0, 1.639906, 4.567799, 6.674601)^t$.
$r^8 = 0.517416$ and $k = 9$. Go to Step 3.

**Step 3** $\boldsymbol{z}^9 = (0, 0.0997, 4.0025, 10.8011, 0, 0, 0, 11.2274)^t$.
$\boldsymbol{c}^9 = (1, 1, -1, 1, -1, 0, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^8, \boldsymbol{z}^9) = 0.514901 < 0.517416 - 0.000001 = r^8 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^9 = (0.021638, 3.821902, 2.590281, 1.410665, 0, 1.615737, 4.497014, 6.713338)^t$.
$r^9 = 0.517346$ and $k = 10$. Go to Step 3.

**Step 3** $\boldsymbol{z}^{10} = (0, 0.0997, 4.0025, 10.8011, 0, 0, 0, 11.2274)^t$.
$\boldsymbol{c}^{10} = (0, 1, -1, 1, -1, 0, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^9, \boldsymbol{z}^{10}) = 0.516152 < 0.517346 - 0.000001 = r^9 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^{10} = (0.041183, 3.807552, 2.579148, 1.399389, 0, 1.593905, 4.433075, 6.748328)^t$.
$r^{10} = 0.517283$ and $k = 11$. Go to Step 3.

**Step 3** $\boldsymbol{z}^{11} = (1.4075, 0, 1.8491, 0, 0, 3.9296, 21.0296, 0)^t$.
$\boldsymbol{c}^{11} = (1, 1, -1, -1, -1, 1, 1, 1)$.

**Step 4** $\phi(\boldsymbol{x}^{10}, \boldsymbol{z}^{11}) = 0.515773 < 0.517283 - 0.000001 = r^0 - \varepsilon$. Continue.

**Step 5** $\boldsymbol{x}^{11} = (0.026142, 3.817153, 2.576039, 1.408137, 0, 1.628976, 4.463591, 6.715565)^t$.
$r^{11} = 0.51666$ and $k = 12$. Go to Step 3.

**Step 3** $\boldsymbol{z}^{12} = (0, 0.0997, 4.0025, 10.8011, 0, 0, 0, 11.2274)^t$.
$\boldsymbol{c}^{12} = \boldsymbol{l} = (0, 1, -1, 1, -1, 0, 0, 1)$.

**Step 4** $\phi(\boldsymbol{x}^{11}, \boldsymbol{z}^{12}) = 0.5166 \geq 0.5166 - 0.000001 = r^{11} - \varepsilon$. Terminate.

The solution is obtained as
$\boldsymbol{x}^{11} = (0.026142, 3.817153, 2.576039, 1.408137, 0, 1.628976, 4.463591, 6.715565)^t$.

Figure 1 An iteration process of the proposed solution algorithm that a maximin achievement rate solution is a possibly optimal solution obtained by considering all possibly optimal solutions and coincides with a necessarily optimal solution when there exists a necessarily optimal solution. A solution algorithm for a maximin achievement rate solution has been proposed based on a relaxation procedure together with the simplex method. In order to illustrate the proposed solution algorithm, a simple numerical example is given and a maximin achievement rate solution is compared with a minimax regret solution.

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# Appendix

## A Solution Algorithm under the Assumption (b)

Under the assumption (b), the smaller $ra(\boldsymbol{x}, \boldsymbol{c})$ of (12), the better the solution $\boldsymbol{x}$ is. Thus, the worst achievement rate can be defined as

$$Ra(\boldsymbol{x}) = \max_{\boldsymbol{c}\in\theta} ra(\boldsymbol{x}, \boldsymbol{c}). \tag{A1}$$

From the same viewpoint as the problem (13), the linear programming problem with an interval objective function, (1), can be formulated as

$$\min_{\boldsymbol{x}\in X} \max_{\substack{\boldsymbol{c}\in\theta \\ \boldsymbol{y}\in X \\ \boldsymbol{cy}<0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}}. \tag{A2}$$

In this problem, since 'min' and 'max' operations are replaced with 'max' and 'min' operations, the solution solves the problem (A2) is called 'a minimax achievement rate solution' instead of a maximin achievement rate solution.

Even in this case, we have theorems corresponding to Theorems 1 and 2 and the problem (A2) is equivalent to

$$\min_{\boldsymbol{x}\in X} \max_{\substack{\boldsymbol{c}\in\theta \\ \boldsymbol{y}\in\Pi S \\ \boldsymbol{cy}<0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}}, \tag{A3}$$

or using $\Pi B$ of (13),

$$\min_{\boldsymbol{x}\in X} \max_{\substack{\boldsymbol{c}\in\theta \\ \boldsymbol{y}\in\Pi B \\ \boldsymbol{cy}<0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}}. \tag{A4}$$

Introducing an auxiliary variable $r$, the problem (A4) can be rewritten as

$$\begin{array}{rl} \text{minimize} & r, \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \displaystyle\max_{\substack{\boldsymbol{c}\in\theta \\ \boldsymbol{cy}^j<0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}^j} \le r,\ j = 1, 2, \ldots, p, \\ & \boldsymbol{x} \ge 0. \end{array} \tag{A5}$$

where $\boldsymbol{y}^j$, $j = 1, 2, \ldots, p$ are elements of $\Pi B$. Let

$$\psi(\boldsymbol{x}, \boldsymbol{y}) \triangleq \max_{\substack{\boldsymbol{c}\in\theta \\ \boldsymbol{cy}<0}} \frac{\boldsymbol{cx}}{\boldsymbol{cy}}, \tag{A6}$$

for $\boldsymbol{x} \in X$ and $\boldsymbol{y} \in \Pi B$. The problem (A5) can be solved by the following algorithm:

[Step 1] Let $\boldsymbol{c}^1 = \boldsymbol{u}$ and $\boldsymbol{z}^1$ be a $\boldsymbol{y}$ maximizing $\boldsymbol{uy}$ under the constraint $\boldsymbol{y} \in \Pi B$.

[Step 2] Set $k = 2$, $r^1 = 0$ and $\boldsymbol{x}^1 = \boldsymbol{z}^1$.

[Step 3] Calculate $\psi(\boldsymbol{x}^{k-1}, \boldsymbol{y})$ for all $\boldsymbol{y} \in \Pi B$. Let $\boldsymbol{z}^k$ be a $\boldsymbol{y}$ maximizing $\psi(\boldsymbol{x}^{k-1}, \boldsymbol{y})$. Let $\boldsymbol{c}^k$ be an optimal solution to the sub-problem $\psi(\boldsymbol{x}^{k-1}, \boldsymbol{z}^k)$.

[Step 4] If $\psi(\boldsymbol{x}^{k-1}, \boldsymbol{z}^k) \leq r^{k-1} + \varepsilon$ holds, then terminate the algorithm. In this case, $\boldsymbol{x}^k$ is an approximation of a minimax achievement rate solution is obtained as $\boldsymbol{x}^k$.

[Step 5] Solve the following linear programming problem:

$$\begin{array}{rl} \text{minimize} & r, \\ \text{subject to} & A\boldsymbol{x} = \boldsymbol{b}, \\ & \dfrac{\boldsymbol{c}^j \boldsymbol{x}}{\boldsymbol{c}^j \boldsymbol{z}^j} \leq r, j = 1, 2, \ldots, k, \\ & \boldsymbol{x} \geq 0. \end{array} \tag{A7}$$

Let $(\boldsymbol{x}^k, r^k)$ be an optimal solution. Return to Step 3 with $k = k + 1$.

When calculate $\psi(\boldsymbol{x}^k, \boldsymbol{y})$ for all $\boldsymbol{y} \in \Pi B$ at Step 3, we can apply a post-optimization technique with respect to the change of objective coefficient vector to the following linear programming problem,

$$\begin{array}{rl} \text{minimize} & \boldsymbol{d}\boldsymbol{y}, \\ \text{subject to} & -\boldsymbol{u}t \leq \boldsymbol{d} \leq -\boldsymbol{l}t, \\ & \boldsymbol{d}\boldsymbol{x}^k = 1, \\ & t \geq 0, \end{array} \tag{A8}$$

where $\psi(\boldsymbol{x}^k, \boldsymbol{y})$ and the optimal solution to the sub-problem $\psi(\boldsymbol{x}^k, \boldsymbol{y})$ are obtained as $\hat{\boldsymbol{d}}\boldsymbol{y}$ and $\boldsymbol{c}^* = -\hat{\boldsymbol{d}}/\hat{t}$, respectively, where $(\hat{\boldsymbol{d}}, \hat{t})$ is the optimal solution to the problem (A8).