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FOREWORD

Among the many concepts of derivatives of set-valued maps introduced so far, we can distinguish (at least) two classes: "graphical derivative" and "mutations". "Graphical derivatives" are **local**: they are defined at every point (x, y) of the **graph** of a map $F : X \rightsquigarrow Y$.
"Mutations" are **global**, in the sense that they are defined at every point x of the domain F.
The problem arose whether there were some connections between those radically different types of derivatives. The purpose of this paper is to provide one formula linking them. Not only does this formula relate the two concepts of derivatives of set-valued maps, but it enjoys many applications, some of them being mentioned in this paper.

Mutations and contingent derivatives of set-valued maps: How they are related

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1 Introduction

Among the many concepts of derivatives of set-valued maps introduced so far, we can distinguish (at least) two classes: "graphical derivative" and "mutations". "Graphical derivatives" introduced in [1,2,3] are local: they are defined at every point (x, y) of the graph of a map $F : X \rightsquigarrow Y$. Their graphs are "tangent cones" to the graph of F. We can take for instance the "contingent cone" $T_K(x)$ to K at x introduced by Bouligand in 1931 and defined by:

$$T_K(x) = \left\{ v \mid \liminf_{h \to 0+} \frac{d(x + hv, K)}{h} = 0 \right\}$$

where $d(x, K) = \inf_{y \in K} d(x, y)$ and d is a distance of X.
In this case, we obtain the "contingent derivative" $DF(x, y) : X \rightsquigarrow Y$ of F at $(x, y) \in \text{Graph}(F)$ defined by

$$\text{Graph}(DF(x, y)) = T_{\text{Graph}(F)}(x, y)$$

If $u \in X$ is a given direction, then $v \in DF(x, y)(u)$ if and only if

$$\liminf_{h \to 0+,\, u' \to u} d\left(v, \frac{F(x + hu') - y}{h}\right) = 0$$

Another equivalent formulation states the $DF(x, y)$ is the "graphical upper limit" of the difference quotients $u \rightsquigarrow \dfrac{F(x + hu) - y}{h}$. (See [8] for more details and a bibliography).
"Mutations" introduced in [5,6] are global, in the sense that they are defined at every point x of the domain F. The mutation $\overset{\circ}{F}(x)(u)$ at x in the direction u is a Lipschitz set-valued maps $\Phi : X \rightsquigarrow X$ with compact convex values such that

$$\lim_{h \to 0+} \frac{\mathbf{d}(\vartheta_\Phi(h, F(x)), F(x + hu))}{h} = 0$$

where $\mathbf{d}$ denotes the Hausdorff distance between compact sets, $\vartheta_\Phi(h, K)$ the reachable map from K at time h associated with Φ. This concept of mutation has been motivated by shape optimization [10,11,31], visual robotics [12] and mathematical morphology [30, 27,24,15].
The problem arose whether there were some connections between those radically different types of derivatives. The purpose of this paper is to provide one formula linking

them: *if F is Lipschitz with compact convex values, then for any mutation $\Phi \in \overset{\circ}{F}(x)(u)$, $\forall y \in F(x)$,*

$$DF(x,y)(u) = \Phi(y) + T_{F(x)}(y)$$

where $T_K(y)$ denotes the contingent cone to K at y.
Not only does this formula relate the two concepts of derivatives of set-valued maps, but it enjoys many applications, some of them being mentioned in this paper.
Before going further, we have to mention that this formula was already proved in [26,28] for special set-valued maps arising in mathematical morphology, whose mutations are constant compact convex subsets B, called "structuring elements". In this case, the solution to the mutational equation $\overset{\circ}{F} \ni B$ starting at K is the "dilation tube" $F(t) := K + tB$. Indeed, it was shown that:

$$\forall x \in F(t), \quad T_{F(t)}(x) + B = DF(t,x)(1)$$

When Φ is Lipschitz with convex compact values, the solution to the mutational equation $\overset{\circ}{F} \ni \Phi$ starting at K is the reachable map $\vartheta_\Phi(t,K)$ from K. The above formula provides the contingent derivative of the reachable map :

$$D\vartheta_\Phi(\cdot,K)(t,x)(1) = \Phi(x) + T_{\vartheta_\Phi(t,K)}(x)$$

When $t = 0$ and $K = \{x\}$ is reduced to a point, we obtain the formula for the infinitesimal generator of a set-valued semi-group obtained in [16, Frankowska]:

$$D\vartheta_\Phi(\cdot,x)(0,x)(1) = \Phi(x)$$

where an error estimate is also provided.
By taking for K the hypograph of a function (resp. the graph of a set-valued map), this formula allows to derive the formula of the "contingent" infinitesimal generator of the semi-group of (nonlinear) operators $\mathcal{U}_\Phi(t)$ on the space $\mathcal{C}(X)$ of continuous functions defined by:

$$(\mathcal{U}_\Phi(t)W)(x) = \sup_{y \in \vartheta_{-\Phi}(t,x)} W(y)$$

(the Koopman operators associated with the Lipschitz map with compact convex values). The infinitesimal generator of a semigroup is the "derivative" at $t = 0$ of the map $\mathcal{U}_\Phi(t)$ in some sense. In order to use the strong derivatives, one is forced to restrict the function W to the class of functions such that $t \mapsto \mathcal{U}_\Phi(t)W$ is differentiable, called the domain of the infinitesimal generator. For instance, when W is differentiable, we obtain:

$$\Lambda W(x) = \sup_{v \in \Phi(x)} \left\langle \frac{D}{Dx} W(x), -v \right\rangle$$

By using "contingent hypoderivatives", we do not need anymore to make this restriction, and we show that the above formula holds true when W is only continuous (or even, upper semicontinuous) and when derivatives are replaced by contingent hypoderivatives.
For constant set-valued maps $\Phi(x) := B$, where B is regarded as a structural element as in mathematical morphology, we find an Hamilton-Jacobi equation obtained in this way in [26].

This formula is also closely related to the study of contingent solutions of Hamilton-Jacobi equations governing the evolution of the valued function of a differential inclusion obtained in [17,18,21].
The same formula applied to characteristic functions of sets allows to estimate shape derivatives of volume functionals when single-valued transitions are replaced by set-valued ones, fulfilling a need in mathematical morphology.
Replacing epigraphs of functions by graphs of set-valued maps, the formula provides in the same way partial differential inclusions governing the evolution of a set-convolutions of maps introduced in [29].
This formula allows also to answer a question posed by Laurent Najman about the viability of solutions on a tube $K(t)$ governed by a mutational equation. We shall prove that under adequate conditions that for any K_0 and for any $x_0 \in K_0$, there exists a solution to the differential inclusion $x' \in F(x)$ starting at x_0 which is viable in the sense that

$$\forall\, t \geq 0, \;\; x(t) \in K(t)$$

where $K(t)$ is a solution to the mutational equation $\overset{\circ}{K} \ni \Phi(K)$ starting at K_0 if and only if for every compact set K and every $x \in K$,

$$0 \;\in\; \Phi(K)(x) + T_K(x) - F(x)$$

In other words, this theorem states a consistency condition between a differential inclusion and a mutational equation, which has to be compared with the theorem on invariant manifolds of mutational equations in [6].

2 Contingent Derivatives of Set-Valued Maps

We recall that

$$v \in T_K(x) \iff (\exists h_n \to 0^+ \text{ and } \exists v_n \to v \text{ such that } \forall n \in \mathbb{N}, \;\; x + h_n v_n \in K)$$

By coming back to the original point of view proposed by Fermat (1637), we are able to define geometrically derivatives of set-valued maps by means of tangent cones to their graphs. We first recall that a set-valued map F from X to Y is characterized by its *graph* denoted Graph(F), which is the subset in the product space $X \times Y$ defined by: $\text{Graph}(F) = \{(x,y) \mid y \in F(x)\}$.

Definition 2.1 *Let $F : X \rightsquigarrow Y$ be a set-valued map. The* contingent derivative *$DF(x,y)$ of F at $(x,y) \in \text{Graph}(F)$ is the set-valued map from X to Y defined by:*

$$\text{Graph}(DF(x,y)) = T_{\text{Graph}(F)}(x,y)$$

In particular, if $f : X \mapsto Y$ is a single valued function, we put $Df(x) = Df(x, f(x))$. See [8] for a detailed description of differential calculus of set-valued map.
We denote by $D'F(x,y)(u)$ the "lop-sided contingent derivative" defined by: $v \in D'F(x,y)(u)$, if there exist sequences $h_n \to 0^+$ and $v_n \to v$ such that

$$\forall n \geq 0, \;\; y + h_n v_n \in F(x + h_n u)$$

The inclusion $D'F(x,y) \subset DF(x,y)$ always holds true, and equality $D'F(x,y) = DF(x,y)$ is true whenever F is Lispchitz.

3 Mutations of Set-Valued Maps

Let X and Y be finite dimensional vector spaces. The topic of this section is to present the "mutational calculus" of set-valued maps $F : X \rightsquigarrow Y$ at a point $x \in X$.
Mutations will be chosen in the space $\mathrm{LIP}(Y, Y)$ of Lipschitz set-valued maps $\Phi : Y \mapsto Y$ with compact convex values. This space of mutations contains in particular

1. Lipschitz single-valued maps $\varphi : Y \mapsto Y$, used in shape optimization,
2. compact convex subsets B, used in mathematical morphology, called "structuring elements" when they contain the origin.

When $\Phi \in \mathrm{LIP}(Y, Y)$, differential inclusion

$$x'(t) \in \Phi(x(t)) \tag{1}$$

does have a solution for any initial state x_0, thanks to

Theorem 3.1 (Filippov) *Assume that $\Phi : Y \rightsquigarrow Y$ is λ-Lipschitz with nonempty closed values. Let $y(\cdot)$ be a given absolutely continuous function such that $t \rightarrow d(y'(t), \Phi(y(t)))$ is integrable (for the measure $e^{-\lambda s}ds$).*
Then there exists a solution $x(\cdot)$ to differential inclusion (1) such that, for all $t \geq 0$,

$$\|x(t) - y(t)\| \leq e^{\lambda t}\left(\|x_0 - y(0)\| + \int_0^t d(y'(s), \Phi(y(s)))e^{-\lambda s}ds\right) \tag{2}$$

(see Theorem 5.3.2 of [4] and [21] for instance).
We associate with any Lipschitz set-valued map $\Phi \in \mathrm{LIP}(Y, Y)$ the set $\mathcal{S}(x)$ of solutions to the differential inclusion $x'(t) \in \Phi(x(t))$ starting from x and the reachable map ϑ_Φ defined by

$$\vartheta_\Phi(h, x) := \{x(h)\}_{x(\cdot)\in\vartheta(x)}$$

where $x(\cdot)$ range over $\mathcal{S}(x)$.
When the images of Φ are compact and convex, one can prove that the images $\mathcal{S}(x)$ are compact in the space $\mathcal{C}(0, \infty, Y)$ of continuous functions supplied with the compact convergence (see Theorem 3.5.2 of [4] for instance).
The reachable map $t \rightsquigarrow \vartheta_\Phi(t, x)$ enjoys the semi-group property: $\forall t, s \geq 0,\ \vartheta_\Phi(t+s, x) = \vartheta_\Phi(t, \vartheta_\Phi(s, x))$.
We observe that

$$(\vartheta_\Phi(t, \cdot))^{-1} := \vartheta_{-\Phi}(t, \cdot)$$

Indeed, if $y \in \vartheta_\Phi(t, x)$, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in \Phi(x)$ starting at x such that $y = x(t)$. We set $y(s) := x(t - s)$ if $s \in [0, t]$ and we choose any solution $y(\cdot)$ to the differential inclusion $y' \in -\Phi(y)$ starting at x at time t for $s \geq t$. Then such a function $y(\cdot)$ is a solution to the differential inclusion $y' \in -\Phi(y)$ starting at y and satisfying $y(t) = x$. This shows that $x \in \vartheta_{-\Phi}(t, y)$.
For more details on the differential inclusion theory see [4,7,21].
We then define the morphological transition on the family of closed subsets of Y by

$$\vartheta_\Phi(h, K) := \{\vartheta_\Phi(h, x)\}_{x\in K}$$

Definition 3.2 *Let X and Y be finite dimensional vector spaces and $F : X \rightsquigarrow Y$ be a set-valued map. The mutation $\overset{\circ}{F}(x)(u)$ at x in the direction u is the set of set-valued maps Φ from X to* $\mathrm{Lip}(Y,Y)$ *satisfying*

$$\Phi \in \overset{\circ}{F}(x)(u) \text{ if and only if } \lim_{h\to 0+} \frac{d\!\!l(F(x+hu), \vartheta_\Phi(h, F(x)))}{h} = 0$$

where $d\!\!l$ denotes the Hausdorff distance between compact sets.

In other words, the mutation $\overset{\circ}{F}(x)(u)$ is a set of Lipschitz maps $\Phi : E \mapsto E$ such that

$$\vartheta_\Phi(h, F(x)) \subset B(F(x+hu), \beta(h)h) \ \& \ F(x+hu) \subset B(\vartheta_\Phi(h, F(x)), \beta(h)h)$$

where $\beta(h)$ converges to 0 with h.
In particular, for tubes $t \rightsquigarrow F(t) \subset Y$, we shall set

$$\Phi \in \overset{\circ}{F}(t) \text{ if and only if } \lim_{h\to 0+} \frac{d\!\!l(F(t+h), \vartheta_\Phi(h, F(t)))}{h} = 0$$

In [6], extensions of the Peano Theorem and the Cauchy-Lipschitz Theorem about the existence (and the uniqueness) of a solution $K(t)$ to mutational equations

$$\overset{\circ}{K}(t) \ni \Phi(K(t))(\cdot)$$

starting at an initial closed set K_0 has been proved under adequate assumptions, and the Filippov Theorem has been adapted in [13] to mutational inclusions

$$\overset{\circ}{K}(t) \cap \Phi(K(t))(\cdot) \neq \emptyset$$

3.1 The Formula

Theorem 3.3 *Assume that F is a set-valued map from X to Y with closed values. Let $\Phi \in \overset{\circ}{F}(x)(u)$ be a mutation of F at x in the direction u. Then:*

$$\forall y \in F(x), \quad D'F(x,y)(u) = \Phi(y) + T_{F(x)}(y)$$

Proof

1. We shall prove that if Φ is Lipschitz with closed values, property

$$\vartheta_\Phi(h, F(x)) \subset F(x+hu) + h\varepsilon_n B \tag{3}$$

implies inclusion

$$\forall y \in F(x), \quad \Phi(y) + T_{F(x)}(y) \subset D'F(x,y)(u) \tag{4}$$

Indeed, let $\varphi \in \Phi(y)$ and $w \in T_{F(x)}(y)$ be given. Then there exist $h_n \to 0^+$, $w_n \to w$ such that $y + h_n w_n \in F(x)$ for all n. Set $z_n(\tau) := y + h_n w_n + \tau\varphi$. Thanks to the Filippov theorem, there exists a solution $y_n(\cdot)$ to the differential inclusion $y' \in \Phi(y)$ starting at $y + h_n w_n$ at $\tau = 0$ and satisfying $\forall \tau \in [0, h]$,

$$\|y_n(\tau) - y - h_n w_n - \tau\varphi\| \leq h_n \mu_n$$

where

$$\mu_n := \frac{1}{h_n}\int_0^{h_n} e^{\lambda(h_n - s)} d(\varphi, \Phi(y + h_n w_n + s\varphi))ds \to 0$$

Then there exists e_n with $\|e_n\| \leq \mu_n$ such that

$$y + h_n w_n + h_n\varphi - h_n e_n = y_n(h_n) \in \vartheta_\Phi(h_n, F(x))$$

By (3), there exists f_n with $\|f_n\| \leq \varepsilon_{h_n}$ such that

$$y_n(h_n) - h_n f_n \in F(x + h_n u)$$

This implies that

$$y + h_n(w_n + \varphi - e_n - f_n) \in F(x + h_n u_n)$$

Since $w_n \to w$, $e_n \to 0$ and $f_n \to 0$, we deduce that $w + \varphi \in D'F(x, y)(u)$.

2. We shall prove that if Φ is upper semicontinuous with closed convex images and linear growth (such a map is called Marchaud), property

$$F(x + hu) \subset \vartheta_\Phi(h, F(x)) + h\varepsilon_n B \tag{5}$$

implies that

$$\forall y \in F(x), \quad D'F(x,y)(u) \subset \Phi(y) + T_{F(x)}(y) \tag{6}$$

Let v be fixed in $D'F(x, y)(u)$. There exist sequences $h_n \to 0^+$ and $v_n \to v$ such that $y + h_n v_n \in F(x + h_n u)$ for all $n \geq 0$. By (5), there exist $y_n \in F(x)$ and f_n such that $\|f_n\| \leq \varepsilon_n$, and such that

$$y + h_n v_n - h_n f_n \in \vartheta_\Phi(h_n, y_n) \tag{7}$$

where $y_n \in F(x)$. Since $y_n \in \vartheta_{-\Phi}(h_n, y + h_n v_n - h_n f_n)$ and since $y + h_n v_n - h_n f_n$ remains in a compact set, so does y_n. The Convergence Theorem (see Theorem 2.4.4 of [4] for instance) implies that a subsequence (again denoted by) $y_n(\tau)$ of solutions to the differential inclusion $y' \in \Phi(y)$ starting at y_n when $\tau = 0$ converges uniformly on $[0, 1]$ to a solution $y(\cdot)$. By letting $n \to \infty$ in (7), we infer that $y(0) = y$. Furthermore, we can write

$$y + h_n v_n = y_n + \frac{h_n}{h_n}\int_0^{h_n} y_n'(\tau)d\tau + h_n f_n$$

We observe that since Φ is upper semi-continuous, a subsequence of

$$\varphi_n := \frac{1}{h_n}\int_0^{h_n} y_n'(\tau)d\tau \in \frac{1}{h_n}\int_0^{h_n} \Phi(y_n(\tau))d\tau$$

converges to some $\varphi \in \overline{co}(\Phi(y)) = \Phi(y)$, since for every $\varepsilon > 0$, there exist N such that, for $n \geq N$, $\forall \tau \in [0, h_n]$, $\Phi(y_n(\tau)) \subset \Phi(y) + \varepsilon B$. Hence $v_n - \varphi_n = \dfrac{y_n - y}{h_n}$ converge to $v - \varphi$. Since $y_n \in F(x)$, we infer that $v - \varphi \in T_{F(x)}(y)$. □

When $\Phi := \{\varphi\}$ is single-valued, we obtain

Corollary 3.4 *If $\varphi \in \overset{\circ}{F}(x)(u)$ is Lipschitz single valued, then*

$$\forall y \in F(x), \quad D'F(x,y)(u) \;=\; \varphi(y) + T_{F(x)}(y)$$

For tubes, we derive the following consequence:

Corollary 3.5 *Assume that $K(\cdot) : \mathbf{R} \rightsquigarrow Y$ is a tube with closed values. Let $\Phi \in \overset{\circ}{K}(t)$ be a mutation of K at x. Then:*

$$\forall y \in K(t), \quad DK(t,y)(1) \;=\; \Phi(y) + T_{K(t)}(y)$$

Indeed, one can check that for a tube, $D'K(t,x) = DK(t,x)$.

4 Derivatives of Inf-Convolutions

We associate with $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ and a Lipschitz map $\Phi \in \mathrm{LIP}(X,X)$ with compact convex values the inf-convolution V_Φ defined by

$$V_\Phi(t,x) \;:=\; \inf_{y \in \vartheta_{-\Phi}(t,x)} V(y)$$

This function satisfies $V_\Phi(0,x) = V(x)$. In the vocabulary of mathematical morphology, the function V_Φ can be regarded as the dilation of the function V by a varying structuring element $\Phi(x)$ (instead of a fixed structuring element).
The function V_Φ, up to a transformation of time, can be regarded as a value function of a control problem of Mayer type with a final condition at time T:

$$\inf_{y \in \vartheta_{\Phi}(T-t,x)} V(y)$$

Indeed, $V_{-\Phi}(T-t,x)$ is the minimal value of V of the final state $x(T)$ of solutions to the differential inclusion $x' \in \Phi(x)$ starting at x at time $t \in [0,T]$.

We recall that the contingent cone to the epigraph of an extended function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ at $(x, V(x))$ is the epigraph of an extended function denoted $D_\uparrow V(x)$:

$$\mathcal{E}p(D_\uparrow V(x)) = T_{\mathcal{E}p(V)}(x, V(x))$$

which is defined by[1]:

$$\forall\, u \in X, \quad D_\uparrow V(x)(u) \;=\; \liminf_{h \to 0+, u' \to u} (V(x+hu') - V(x))/h$$

We denote by $D_\uparrow V_\Phi(t,x)$ the epicontingent derivative of $(t,x) \mapsto V_\Phi(t,x)$ and by $\frac{D_\uparrow}{Dx} V_\Phi(t,x)$ the epicontingent derivative of $x \mapsto V_\Phi(t,x)$.
Let us set

[1] We can reformulate this formula below by saying that *the contingent epiderivative $D_\uparrow V(x)$ is the lower epilimit of the differential quotients*

$$u \rightsquigarrow \nabla_h V(x)(u) \;:=\; \frac{V(x+hu) - V(x)}{h}$$

Theorem 4.1 *Assume that $V : X \mapsto \mathbf{R}$ is lower semicontinuous. Then the function $V_\Phi(t,\cdot)$ satisfies*

$$D_\uparrow V_\Phi(t,x)(1,u) \;=\; \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(u-v)$$

In particular, by taking $u=0$, we derive the formula

$$D_\uparrow V_\Phi(t,x)(1,0) \;=\; \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(-v)$$

If V_Φ is differentiable, we infer that V_Φ is a solution to an Hamilton-Jacobi equation

$$\frac{dV_\Phi}{dt}(t,x) \;=\; -\sup_{v\in\Phi(x)} \left\langle \frac{DV_\Phi}{Dx}(t,x), v\right\rangle$$

When $t=0$, we infer from the initial condition $V_\Phi(0,x)=V(x)$ that

$$D_\uparrow V_\Phi(0,x)(1,u) \;=\; \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V(x)(u-v)$$

Proof — The function $V_\Phi(t,\cdot)$ satisfies

$$\mathcal{E}p\left(V_\Phi(t)\right) \;=\; \vartheta_{-\Phi\times\{0\}}\left(t, \mathcal{E}p(V)\right)$$

Setting $K(t) = \mathcal{E}p(V_\Phi(t,\cdot))$, we see that $\Phi(\cdot)\times\{0\}$ is a mutation of the tube $t \rightsquigarrow \mathcal{E}p(V_\Phi(t)) \subset X\times\mathbf{R}$.
Corollary 3.5 implies that

$$(\Phi(x)\times\{0\}) + T_{\mathcal{E}p(V_\Phi(t))}(x, V(t,x)) \;=\; D'K(t)(x, V(t,x))(1)$$

But we know that

$$T_{\mathcal{E}p(V_\Phi(t))}(x, V(t,x)) \;=\; \mathcal{E}p\left(\frac{D_\uparrow}{Dx} V_\Phi(t,x)\right)$$

Setting

$$\left(\frac{D_\uparrow}{Dx} V_\Phi(t,x) \oplus_\uparrow \Phi(x)\right)(u) \;:=\; \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(u-v)$$

we observe that

$$(\Phi(x)\times\{0\}) + \mathcal{E}p\left(\frac{D_\uparrow}{Dx} V_\Phi(t,x)\right) \;=\; \mathcal{E}p\left(\frac{D_\uparrow}{Dx} V_\Phi(t,x) \oplus_\uparrow \Phi(x)\right)$$

Therefore (u,λ) belongs to

$$(\Phi(x)\times\{0\}) + T_{\mathcal{E}p(V_\Phi(t))}(x, V(t,x))$$

if and only if

$$\lambda \;\geq\; \left(\frac{D_\uparrow}{Dx} V_\Phi(t,x) \oplus_\uparrow \Phi(x)\right)(u)$$

On the other hand, (u,λ) belongs to $D'K(t)(x,V(t,x))(1)$ if and only if there exist sequences $h_n \to 0+$, $u_n \to u$ and $\lambda_n \to \lambda$ such that

$$(x + h_n u_n, V_\Phi(t,x) + h_n\lambda_n) \;\in\; \mathcal{E}p\left(V_\Phi(t+h_n)\right)$$

i.e., such that

$$(t + h_n, x + h_n u_n, V_\Phi(t,x) + h_n\lambda_n) \in \mathcal{E}p(V_\Phi)$$

This means that

$$\lambda \geq D_\uparrow V_\Phi(t,x)(1,u)$$

This concludes the proof □

Remark — The proof of Theorem 3.3 implies actually that

1. If Φ is Lipschitz with closed values, then

$$D_\uparrow V_\Phi(t,x)(1,u) \leq \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(u-v)$$

2. If Φ is Marchaud, then

$$D_\uparrow V_\Phi(t,x)(1,u) \geq \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(u-v) \quad \square$$

When a function V is differentiable at x, its gradient $V'(x)$, being a continuous linear functional, is therefore an element $V'(x) \in X^\star$ of the dual of X. When V is no longer differentiable, we can still introduce *subgradients* of V at x, which are those continuous linear functionals $p \in X^\star$ satisfying

$$\forall\, v \in X, \;\; < p, v > \leq D_\uparrow V(x)(v)$$

which constitute the (possibly empty) closed convex subset

$$\partial_- V(x) := \{p \in X^\star \mid \forall\, v \in X, \;\; < p, v > \;\leq\; D_\uparrow V(x)(v)\}$$

called the *subdifferential* of V at x. We introduce the subdifferential

$$\begin{cases} \partial_- V_\Phi(t,x) \;=\; \{(p_t,p_x) \in \mathbf{R} \times X^\star \\ \text{such that} \\ \forall\, v \in X, \;\; p_t + \langle p_x, v\rangle \leq D_\uparrow V_\Phi(t,x)(1,v)\} \end{cases}$$

and the "partial subdifferential"

$$\partial_{-,x} V_\Phi(t,x) \;=\; \{q \in X^\star \mid \forall\, v \in X, \;\langle q, v\rangle \leq \frac{D_\uparrow}{Dx} V_\Phi(t,x)(v)\}$$

Therefore, using these dual concepts, we deduce the following

Proposition 4.2 *Assume that V is bounded and that Φ belongs to* LIP(X,X). *Then V_Φ is a solution to the two following conditions: for every $(p_t,p_x) \in \partial_- V_\Phi(t,x)$, then*

$$\begin{cases} i) \quad p_t + \sup_{v\in\Phi(x)}\langle p_x, v\rangle \;\leq\; 0 \\ \& \\ ii) \quad p_x \;\in\; \partial_{-,x} V_\Phi(t,x) \end{cases} \tag{8}$$

Property (8) means that V_Φ is a viscosity upper solution to the Hamilton-Jacobi equation

$$\frac{dU}{dt}(t,x) + \sup_{v\in\Phi(x)} \left\langle \frac{DU}{Dx}(t,x)), v \right\rangle = 0$$

Proof — Indeed, if $(p_t, p_x) \in \partial_- V_\Phi(t,x)$, then, for any $v \in X$,

$$\begin{cases} p_t + \langle p_x, v\rangle \le D_\uparrow V_\Phi(t,x)(1,v) \\ = \inf_{w\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(v-w) \end{cases}$$

We thus deduce that

$$\sup_{w\in\Phi(x),\, v\in X} \left[p_t + \langle p_x, w\rangle + \left(\langle p_x, v-w\rangle - \frac{D_\uparrow}{Dx} V_\Phi(t,x)(v-w) \right) \right] \le 0$$

It is enough to observe that

$$\sup_{z\in X} \left(\langle p_x, z\rangle - \frac{D_\uparrow}{Dx} V_\Phi(t,x)(z) \right) = \begin{cases} 0 & \text{if } p_x \in \partial_{-,x} V_\Phi(t,x) \\ +\infty & \text{if } p_x \notin \partial_{-,x} V_\Phi(t,x) \end{cases}$$

Since this latter situation is impossible, we infer that $p_x \in \partial_{-,x} V_\Phi(t,x)$ and that

$$\sup_{z\in X} \left(\langle p_x, z\rangle - \frac{D_\uparrow}{Dx} V_\Phi(t,x)(z) \right) = 0$$

Therefore,

$$\sup_{w\in\Phi(x)} [p_t + \langle p_x, w\rangle] \le 0 \quad \square$$

Remark — One can deduce from the minimax theorem that any solution V_Φ to (8) is conversely a solution to the "partial differential equation"

$$D_\uparrow V_\Phi(t,x)(1,u) = \inf_{v\in\Phi(x)} \frac{D_\uparrow}{Dx} V_\Phi(t,x)(u-v)$$

when we assume that the functions $v \mapsto D_\uparrow V_\Phi(t,x)(1,u)$ are convex and the subdifferential $\partial_- V_\Phi(t,x)$ is bounded. $\square$

5 Contingent Infinitesimal Generator of a Koopman Process

We associate now with a Lipschitz map $\Phi \in \mathrm{LIP}(X,X)$ with compact convex values the semi-group of (nonlinear) operators $\mathcal{U}_\Phi(t)$ on the space $\mathcal{C}(X)$ of continuous functions defined by:

$$(\mathcal{U}_\Phi(t)W)(x) = \sup_{y\in\vartheta_{-\Phi}(t,x)} W(y)$$

Indeed, since the set-valued map $x \rightsquigarrow \vartheta_\Phi(h,x)$ is continuous with compact values, the Maximum Theorem implies that $x \mapsto \mathcal{U}(t)W(x)$ is continuous. Actually, when W is upper semicontinuous and when Φ is Marchaud, $x \mapsto \mathcal{U}(t)W(x)$ is upper semicontinuous, so that the operators $\mathcal{U}_\Phi(t)$ operate on the cone of upper semicontinuous functions. When $\Phi = \{\varphi\}$, we recognize the Koopman operator defined by

$$(\mathcal{U}_\varphi(t)W)(x) = W(\vartheta_{-\varphi}(t,x))$$

It is a semigroup in the sense that

$$(\mathcal{U}_\Phi(t+s)W)(x) := (\mathcal{U}_\Phi(t)(\mathcal{U}_\Phi(s)W)(x))$$

We recall that the contingent cone to the hypograph of an extended function $W : X \mapsto \mathbf{R} \cup \{-\infty\}$ at $(x, W(x))$ is the hypograph of an extended function denoted $D_\downarrow W(x)$:

$$\mathcal{H}yp(D_\downarrow W(x)) = T_{\mathcal{H}yp(W)}(x, W(x))$$

which is defined by:

$$\forall\, u \in X,\;\; D_\downarrow W(x)(u) = \limsup_{h \to 0+, u' \to u} (W(x+hu') - W(x))/h$$

The infinitesimal generator of a semigroup is the "derivative" at $t = 0$ of the map $\mathcal{U}_\Phi(t)$ in some sense. In order to use the strong derivatives, one is forced to restrict the function W to the class of functions such that $t \mapsto \mathcal{U}_\Phi(t)W$ is differentiable, called the domain of the infinitesimal generator.
By using contingent hypoderivatives, we do not need anymore to make this restriction. For that purpose, we define the contingent infinitesimal generator Λ of the Koopman semi-group by

$$\forall\, W \in \mathcal{C}(X), \forall\, x \in X,\;\; \Lambda W(x) := D_\downarrow \mathcal{U}_\Phi W(0,x)(1,0)$$

Theorem 4.1 provides a formula of the "contingent" infinitesimal generator of the Koopman semi-group:

Theorem 5.1 *Let $\Phi : X \rightsquigarrow X$ be Lipschitz with compact convex values, $\mathcal{U}_\Phi(t)$ be its associated Koopman operator. For any upper semicontinuous function W, the contingent hypoderivative of the Koopman transform $\mathcal{U}_\Phi(t)W$ satisfies:*

$$D_\downarrow \mathcal{U}_\Phi(t)W(x)(1,u) = \sup_{v \in \Phi(x)} \frac{D_\downarrow}{Dx} \mathcal{U}_\Phi(t)W(x)(u-v)$$

Its contingent infinitesimal generator Λ is equal to

$$\Lambda W(x) = \sup_{v \in \Phi(x)} \frac{D_\downarrow}{Dx} W(x)(-v)$$

In particular, if W is differentiable, then

$$\Lambda W(x) = \sup_{v \in \Phi(x)} \left\langle \frac{D}{Dx} W(x), -v \right\rangle$$

Remark — When Φ is set-valued, the operators $\mathcal{U}_\Phi(t)$ are no longer linear. However, we can associate with it, as in [9], the notion of Koopman process, which is a closed convex process (i.e., a set-valued continuous linear operator) : A closed convex process $\mathcal{F}$ is a set-valued map the graph of which is a closed convex cone, i.e., a closed map satisfying

$$\begin{cases} i) & \forall \lambda > 0, \ \mathcal{F}(\lambda W) = \lambda \mathcal{F}(W) \\ ii) & \forall W_1, W_2, \ \mathcal{F}(W_1) + \mathcal{F}(W_2) \subset \mathcal{F}(W_1 + W_2) \end{cases}$$

The Koopman process thus associates with any nonnegative function $W \in \mathcal{C}(X)$ the subset $\widehat{\mathcal{U}}_\Phi(t)(W)$ of nonnegative functions $V \in \mathcal{C}(X)$ satisfying

$$\forall x \in X, \quad \sup_{y \in \vartheta_{-\Phi}(t,x)} W(y) \leq V(x)$$

If W is nonnegative, we set $\widehat{\mathcal{U}}_\Phi(t)(W) = \emptyset$. We observe the following

Lemma 5.2 *The Koopman process $\widehat{\mathcal{U}}_\Phi(t)$ is a closed convex process containing.*

A closed convex process can be transposed. In [9], we have associated with Φ a closed convex process $\mathcal{F}_\Phi(t)$ on the space of regular measures. It maps a probability measure μ to the set $\mathcal{F}(t)\mu$ of probability measures ν defined by

$$\forall B \in \mathcal{B}, \quad \nu(B) \leq \mu(\vartheta_{-\Phi}(t, B))$$

We extend it as a set-valued map by setting

$$(\mathcal{F}_\Phi(\mu)(B)) := \begin{cases} \emptyset & \text{if } \mu \text{ is nonpositive,} \\ \{0\} & \text{if } \mu = 0 \\ \mu(X)\mathcal{F}_\Phi(\mu/\mu(X)) & \text{if } \mu \text{ is positive} \end{cases}$$

It has been proved that it is a closed convex process extending $\vartheta_\Phi(t,\cdot)$ in the sense that for Dirac measures, $\delta_y \in \mathcal{F}_\Phi(t)(\delta_x)$ if and only if $y \in \vartheta_\Phi(t,x)$.

Lemma 5.3 *The Koopman process $\widehat{\mathcal{U}}_\Phi$ contains the transpose $\mathcal{F}_\Phi^\star(t)$.*

Proof — Take $V \in \mathcal{F}_\Phi^\star(t)(W)$, so that $\int W\nu \leq \int V\mu$ for any $\nu \in \mathcal{F}_\Phi(t)(\mu)$. Since δ_y belongs to $\mathcal{F}_\Phi(t)(\delta_x)$ whenever $y \in \vartheta_\Phi(t,x)$, we infer that $W(y) \leq V(x)$ for any $y \in \vartheta_\Phi(t,x)$, so that $V \in \widehat{\mathcal{U}}_\Phi(t)(W)$. □

6 Applications to Shape Derivatives

Let $X := \mathbf{R}^n$ and consider the shape functional

$$J(K) := \int_K \alpha(x)dx$$

We shall estimate its contingent hypoderivative

$$D_\downarrow J(K)(\Phi) := \limsup_{h \to 0+, v \to 0} \frac{1}{h} \left(\int_{\vartheta_\Phi(h, K-v)} \alpha(x)dx - \int_K \alpha(x)dx \right)$$

of this shape functional. It is then natural to introduce the characteristic function χ_K of a subset K and to observe that

$$\chi_{\vartheta_\Phi(h,K-v)}(x) \;=\; \sup_{y\in\vartheta_{-\Phi}(h,x)} \chi_{K-v}(y) \;=\; \mathcal{U}_\Phi(h)\chi_K(x+hv)$$

We thus deduce from Theorem 5.1 that

$$D_\downarrow \chi_{\vartheta_\Phi(t,K)}(x)(1,u) \;=\; \sup_{v\in\Phi(x)} \frac{D_\downarrow}{Dx}\chi_{\vartheta_\Phi(t,K)}(x)(u-v)$$

In particular, for $u=0$, we obtain

$$D_\downarrow \chi_{\vartheta_\Phi(t,K)}(x)(1,0) \;=\; \sup_{v\in\Phi(x)} \frac{D_\downarrow}{Dx}\chi_{\vartheta_\Phi(t,K)}(x)(-v)$$

If we assume now that the differential quotients are bounded by an integrable function β:

$$\forall\, t\in[0,1],\quad \sup_{\|u\|\le 1}\left|\frac{\chi_{\vartheta_\Phi(t,K)}(x)-\chi_{K-u}(x)}{t}\right| \;\le\; \beta(t)$$

the Fatou-Lebesgue Theorem implies that

$$\int_{\mathbf{R}^n} \limsup_{h\to 0+,v\to 0} \frac{\chi_{\vartheta_\Phi(h,K-v)}(x)-\chi_K(x)}{h}\alpha(x)dx \;\le\; \limsup_{h\to 0+,v\to 0}\int_{\mathbf{R}^n} \frac{\chi_{\vartheta_\Phi(h,K-v)}(x)-\chi_K(x)}{h}\alpha(x)dx$$

Using the above equations, we infer that

$$\int_{\mathbf{R}^n} \sup_{v\in\Phi(x)} \frac{D_\downarrow}{Dx}\chi_K(x)(-v)\alpha(x)dx \;\le\; D_\downarrow J(K)(\Phi)$$

When the set-valued map Φ is a constant compact convex subset B, we obtain the estimate

$$\int_{\mathbf{R}^n} \sup_{v\in B} \frac{D_\downarrow}{Dx}\chi_K(x)(-v)\alpha(x)dx \;\le\; D_\downarrow J(K)(B)$$

When $\Phi := \{\varphi\}$ is single valued and smooth, we deduce that

$$\int_{\mathbf{R}^n} \frac{D_\downarrow}{Dx}\chi_K(x)(-\varphi(x))\alpha(x)dx \;\le\; D_\downarrow J(K)(\varphi)$$

7 Partial Differential Equation Governing Transforms of Set-Valued Maps

Let us consider a set-valued maps $F : X \rightsquigarrow Y$ and associate with $\Phi \in \mathrm{LIP}(X,X)$ and $\Psi \in \mathrm{LIP}(Y,Y)$ the set-valued map $F_{\Phi,\Psi}(t,.) : X \rightsquigarrow Y$ defined by

$$F_{\Phi,\Psi}(t,x) \;:=\; \vartheta_\Psi(t, F(\vartheta_{-\Phi}(t,x)))$$

The interesting particular case is obtained when $\Psi := 0$, since

$$F_{\Phi,0}(t,x) \;:=\; F(\vartheta_{-\Phi}(t,x))$$

Theorem 7.1 *Assume that the graph of $F : X \rightsquigarrow Y$ is closed. Then the map $F_{\Phi,\Psi}(t,\cdot)$ is solution to the partial differential inclusion*

$$D_{t,x}F_{\Phi,\Psi}(t,x,y)(1,u) \ = \ \bigcup_{v\in\Phi(x)} (D_x F_{\Phi,\Psi}(t,x)(u-v)) + \Psi(y)$$

Proof — Indeed, we consider $K(t) = \text{Graph}(F_{\Phi,\Psi}(t))$ and we observe that

$$\text{Graph}(F_{\Phi,\Psi}(t)) \ = \ \vartheta_{\Phi\times\Psi}(t, \text{Graph}(F))$$

By Corollary 3.5, we have $\forall (x,y) \in \text{Graph}(F_{\Phi,\Psi}(t)),.)$

$$\Phi(x)\times\Psi(y) + T_{\text{Graph}(F_{\Phi,\Psi}(t))}(x,y) = D\text{Graph}(F_{\Phi,\Psi})(t)(x,y)(1)$$

We recall that $T_{\text{Graph}(F_{\Phi,\Psi}(t))}(x,y) = \text{Graph}(D_x F_{\Phi,\Psi}(t)(x,y))$. □

For instance, when the maps F, Φ and Ψ are single-valued and differentiable, we obtain:

$$\frac{d}{dt}f_{\varphi,\psi}(0,x) \ = \ -\left\langle \frac{d}{dx}f(x), \varphi(x)\right\rangle + \psi(f(x))$$

8 Viability on Tubes Governed by Mutational Equations

Corollary 3.5 allows to characterize a viability problem in tubes evolving according a mutational equation.

Let us consider a tube $K(\cdot) : \mathbf{R}_+ \rightsquigarrow X$, solution to a mutational equation

$$\overset{\circ}{K}(t) \ni \Phi(K(t))(\cdot)$$

and a solution $x(\cdot) : \mathbf{R}_+ \mapsto X$ to a differential inclusion

$$x'(t) \ \in \ F(x(t))$$

Theorem 8.1 *Assume that the set-valued map $F : \mathbf{R}_+ \times X \rightsquigarrow X$ is Marchaud and that $\Phi : X \ \mapsto \text{LIP}(X,X)$ is a continuous map,* bounded *in the sense that*

$$\forall\, K, \ \ \|\Phi(K;\cdot)\|_\Lambda \ \le \ c$$

The two conditions are equivalent:

1. *For every compact set K and every $x \in K$,*

$$0 \ \in \ \Phi(K)(x) + T_K(x) - F(x)$$

2. *For any K_0 and for any $x_0 \in K_0$, there exists a solution to the differential inclusion $x' \in F(x)$ starting at x_0 which is viable in the sense that*

$$\forall\, t \ge 0, \ x(t) \in K(t)$$

where $K(t)$ is a solution to the mutational equation $\overset{\circ}{K} \ni \Phi(K)$ starting at K_0.

Proof — Theorem 11.1.3 of [4] states that the second property holds true if and only of

$$\forall\, t \geq 0,\ \ \forall\, x \in K(t),\ \ 0\ \in\ DK(t,x)(1) - F(x)$$

Since the evolution of $K(t)$ is governed by a mutational equation, then $\Phi(K(t))(\cdot)$ is a mutation of $K(t)$ so that, by Corollary 3.5, we know that

$$DK(t,x)(1)\ =\ \Phi(K(t))(x) + T_{K(t)}(x)$$

This concludes the proof. □

References

[1] J.-P. Aubin. (1981) *Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions and differential inclusions*, Advances in Mathematics, Supplementary studies, Ed. Nachbin L., 160-232

[2] J.-P. Aubin. (1982) *Comportement lipschitzien des solutions de problèmes de minimisation convexes*, Comptes-Rendus de l'Académie des Sciences, PARIS, 295, 235-238

[3] J.-P. Aubin (1984) *Lipchitz Behavior .of Solutions to Convex Minimization Problems*, Mathematics of Operations Research, 8, 87-111

[4] J.-P. Aubin. (1991) *Viability theory.* Birkhauser, Systems and Control : Foundations and Applications.

[5] J.-P. Aubin. (1992) *A Note on Differential Calculus in Metric Spaces and its Applications to the Evolution of Tubes*, Bulletin of the Polish Academy of Sciences, 40, 151-162

[6] J.-P. Aubin. (1993) *Mutational Equations in Metric Spaces*, Set-Valued Analysis, 1, 3-46

[7] J.-P. Aubin and A. Cellina. (1984) *Differential Inclusions (Set-valued maps and viability theory).* Springer-Verlag.

[8] J.-P. Aubin and H. Frankowska. (1990) *Set-Valued Analysis.* Birkhauser.

[9] J.-P. Aubin, H. Frankowska & A. Lasota (1991) *Poincaré's Recurrence Theorem for Set-Valued Dynamical Systems*, Annales Polonici Mathematici, 85-91

[10] J. Céa. (1981) Numerical methods of Shape Optimal Design. In E.J. Haug and Céa, editors, *Optimization of Distribued Parameter Structures*, volume II, pages 1049–1087. Sijhoff and Noorhoff, Alphen aan den Rijn, The Netherlands.

[11] J. Céa. (1981) Problem of Shape Optimal Design. In E.J. Haug and Céa, editors, *Optimization of Distribued Parameter Structures*, volume II, pages 1005–1048. Sijhoff and Noorhoff, Alphen aan den Rijn, The Netherlands.

[12] L. Doyen. (1993) *Évolution, contrôle et optimisation de formes.* Thèse de Doctorat, Université Paris-Dauphine, Paris, France.

[13] L. Doyen. Filippov and invariance theorems for mutational inclusions of tubes. *Journal of Set-Valued Analysis*, (to appear).

[14] L. Doyen. Inverse Function Theorems and Shape Optimization. *S.I.A.M Journal on Control and Optimization*, (to appear).

[15] L. Doyen, L. Najman, and J. Mattioli. Mutational Equations of Morphological Dilation Tubes. In J. Serra, editor, *Mathematical Morphology and its Application to Signal Processing II*, ISMM'94, Fontainebleau, France, september 1994. École des Mines de Paris. submitted.

[16] H. Frankowska. (1987) *Local controllability and infinitesimal generators of semi-groups of set-valued maps*, SIAM J. on Control and Optimization, 25, 412-432

[17] H. Frankowska. (1987) *L'équation d'Hamilton-Jacobi contingente*, Comptes Rendus de l'Académie des Sciences, PARIS, Série 1, 304, 295-298

[18] H. Frankowska. (1989) *Optimal trajectories associated to a solution of contingent Hamilton-Jacobi Equations*, Applied Mathematics and Optimization, 19, 291-311

[19] H. Frankowska. (1989) *Nonsmooth solutions of Hamilton-Jacobi-Bellman Equations*, Proceedings of the International Conference Bellman Continuum, Antibes, France, June 13-14, 1988, Lecture Notes in Control and Information Sciences, Springer Verlag

[20] H. Frankowska. (1993) *Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equation*, SIAM J. on Control and Optimization, Janvier

[21] H. Frankowska. *Control of nonlinear systems and differetial inclusions.* Birkhäuser, to appear.

[22] A. Lasota & M.C. Mackey (1984) *Globally asymptotic properties of proliferating cell populations*, J. Math. Biology, 19, 43-62

[23] A. Lasota & M.C. Mackey (1985) PROBABILISTIC PROPERTIES OF DETERMINISTIC SYSTEMS, Cambridge University Press

[24] G. Matheron. (1975) *Random Sets and Integral Geometry.* John Wiley and Sons, New York, 1975.

[25] J. Mattioli. (1993) Differential Inclusions for Mathematical Morphology. In *SPIE: Image Algebra and Morphological Image Processing IV*, volume 2030, pages 12–23, San Diego.

[26] J. Mattioli. (1993) Differential Relations of Morphological Operators. In *Mathematical Morphology and its applications to Signal Processing*, pages 162–167, Barcelona, Spain.

[27] J. Mattioli. (1993) *Problèmes inverses et relations différentielles en morphologie mathématique.* Thèse de Doctorat, Université Paris Dauphine, Paris, France.

[28] J. Mattioli. (1993) Relations différentielles d'opérations de la morphologie mathématique. *C.R. Acad. Sci. Paris*, t. 316, Série I:879–884.

[29] J. Mattioli. Filtering set-valued and vectorial images with set-convolutions of maps. *Set-Valued Analysis*, (submitted).

[30] M. Schmitt and J. Mattioli. (1994) *Morphologie Mathématique.*

[31] J. Sokolowski and J.-P. Zolesio. (1992) *Introduction to Shape Optimization.* Springer-Verlag, Berlin.