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Foreword

A dynamical model for an evolutionary nonantagonistic (nonzero sum) game between two populations is considered. A scheme of a dynamical Nash equilibrium in the class of feedback (discontinuous) controls is proposed. The construction is based on solutions of auxiliary antagonistic (zero-sum) differential games. A method for approximating the corresponding value functions is developed. The method uses approximation schemes for constructing generalized (minimax, viscosity) solutions of first order partial differential equations of Hamilton-Jacobi type. A numerical realization of a grid procedure is described. Questions of convergence of approximate solutions to the generalized one (the value function) are discussed, and estimates of convergence are pointed out. The method provides equilibrium feedbacks in parallel with the value functions. Implementation of grid approximations for feedback control is justified. Coordination of long- and short-term interests of populations and individuals is indicated. A possible relation of the proposed game model to the classical replicator dynamics is outlined.

Contents

A Differential Model for a 2×2-Evolutionary Game Dynamics

A. M. Tarasyev[*]

Introduction

We consider a nonantagonistic dynamical game of two large groups (populations) of individuals. The dynamical system describing population evolution is motivated by differential (see [Isaacs, 1965]) and evolutionary game-theoretical models (see [Friedman, 1991], [Young, 1993]) relevant to problems of economic change (see [Nelson and Winter, 1982]) and population dynamics (see [Hofbauer, Sigmund, 1988]). For a particular class of 2×2 deterministic evolutionary game dynamics, an approach to analyse populations' behaviors via methods of the theory of differential games was proposed in [Kryazhimskii, 1994]. In the present paper we develop some aspects of this approach for a stochastic 2×2 evolutionary game dynamics. Namely, we focus on finding equilibrium populations' behaviors within the totally centralized regulation pattern. The model is reduced to a closed-loop differential game ([Krasovskii, Subbotin, 1988], [Krasovskii, 1985], [Kleimenov, 1993]) and analysed via methods of the theory of generalized (minimax, viscosity) solutions of Hamilton-Jacobi equations ([Crandall, Lions, 1983, 1984], [Subbotin, 1980, 1991]).

It is supposed that at each time instant, individuals of each population are divided into two parts playing different strategies. Individuals from different populations meet pairwise randomly and get their current payoffs determined by a combination of their strategies. Populations' goals are to maximize "long-term" payoffs represented as intergals of mathematical expectations for current payoffs, with an appropriate discount.

The right-hand side of the considered dynamical system depends on control parameters making individuals change their current strategies in accordance with a chosen feedback.

The nonantagonistic game in question consists in constructing Nash equilibrium feedbacks with respect to the "long-term" dynamical payoff functionals.

We consider the problem within the framework of the theory of positional differential games ([Krasovskii, Subbotin, 1988]). Following to [Kleimenov, 1993] we compose a Nash equilibrium with the help of solutions of auxiliary antagonistic (zero-sum) differential games. Solutions of these antagonistic games are based on algorithms of building the value functions. It is known ([Crandall, Lions, 1983, 1984], [Subbotin, 1980, 1991]) that the value function is the generalized solution of the Bellman-Isaacs equation being a first-order partial differential equation of Hamilton-Jacobi type. To construct value functions we use appropriate approximation schemes ([Dolcetta, 1983], [Souganidis, 1985], [Tarasyev, 1993], [Adiatulina, Tarasyev, 1987], [Bardi, Osher], [Subbotin, Tarasyev, Ushakov]). The corresponding numerical procedure is reduced to the method of contraction operators. Along with information of the value functions, the method provides equilibrium feedbacks.

Stress once again that a solution is obtained within the centralized scheme implying that long-term-equilibrium behaviors can, generally, contradict to short-term interests

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of individuals. We conclude the paper with a discussion of possible problem settings combining long- and short-term principles for constructing dynamical Nash equilibria. In particular, we consider the possibility of linking the proposed dynamical Nash equilibrium approach with the classical replicator dynamics (see [Hofbauer, Sigmund, 1988]).

1 The Model of Game Dynamics

1.1 Dynamics, Payoffs, Player's Preferences

We consider the following dynamical system which describes game interaction of two populations of individuals. We can assume, for example, that one of this population is an aggregate of sellers and another is an aggregate of buyers. For clearness of arguments suppose that individuals of populations can choose at each moment of time one of two simple actions (strategies): buyers can "buy" or "not buy", sellers can sell at "high price" or "low price". Actions of individuals of the first population are denoted by index i: index $i = 1$ corresponds to action "buy", index $i = 2$ corresponds to action "not buy". Analogously, actions of individuals of the second population are denoted by index j: index $j = 1$ corresponds to "high price", index $j = 2$ corresponds to "low price".

Let us consider an arbitrary pair composed by individuals of different populations. This pair is interpreted as a situation (i, j) in the current game generated by strategy i of a player from the first population and by strategy j of a player from the second population. Assume that payoff of players of the first population is determined by coefficients a_{ij} of the payoff matrix $A = \{a_{ij}\}$. Analogously, payoff of players of the second population in situation (i, j) is determined by coefficients b_{ij} of the payoff matrix $B = \{b_{ij}\}$.

Let us assume that the first population consists of N individuals and at the instant of time t one part of them $N_1(t)$ plays the first strategy and another part $N_2(t)$ plays the second strategy. Of course, $N = N_1(t) + N_2(t)$. Similarly, assume that the second population consists of M individuals and at the moment t one part $M_1(t)$ plays the first strategy and another part $M_2(t)$ plays the second strategy, $M = M_1(t) + M_2(t)$.

Let us suppose that dynamics of the process in which individuals change their strategies from one to another is described by the multistep system of equations

$$\begin{aligned}
N_1(t+\delta) &= N_1(t) - n_{12}(t)\delta + n_{21}(t)\delta \\
N_2(t+\delta) &= N_2(t) + n_{12}(t)\delta - n_{21}(t)\delta \\
M_1(t+\delta) &= M_1(t) - m_{12}(t)\delta + m_{21}(t)\delta \\
M_2(t+\delta) &= M_2(t) + m_{12}(t)\delta - m_{21}(t)\delta \qquad (1.1)
\end{aligned}$$

The peculiarity of such dynamics consists in the fact that the number of individuals in populations which can change their strategies at the moment t is proportional to the time step δ. More precisely,

$n_{12}(t)\delta$	is the number of individuals of the first population which change their strategies from the first to the second, $0 \le n_{12}(t) \le N_1(t)$;
$n_{21}(t)\delta$	is the number of individuals of the first population which change their strategies from the second to the first, $0 \le n_{21}(t) \le N_2(t)$;
$m_{12}(t)\delta$	is the number of individuals of the second population which change their strategies from the first to the second, $0 \le m_{12}(t) \le M_1(t)$;
$m_{21}(t)\delta$	is the number of individuals of the second population which change their strategies from the second to the first, $0 \le m_{21}(t) \le M_2(t)$.

The fact that at the moment t only a part of individuals in population proportional to the time step δ can change their strategies has the following interpretations. For example, such inertia of behaviour of population can be explained if we assume that only "small" part of individuals is active in changing of their behaviour. We can give another explanation if assume that there are some restrictions ("queues") in case when "large" group of individuals change actions.

On the other hand we make rather natural assumption when suppose that the number $n_{ik}(t)$ or $m_{jl}(t)$ of individuals which potentially may wish to change their actions (but not obligatory change, because the number of those who change is equal to $n_{ik}(t)\delta$ or $m_{jl}(t)\delta$) satisfies the restrictions

$$\begin{array}{lll} 0 \le n_{ik} \le N_i(t), & i,k = 1,2, & i \ne k \\ 0 \le m_{jl} \le M_j(t), & j,l = 1,2, & j \ne l \end{array}$$

Let us suppose that at the moment t players of different populations compose pairs randomly with equal probabilities. The probability of the fact that the randomly chosen pair plays the situation (i,j) is determined by the formula

$$p_{ij}(t) = \frac{N_i(t)M_j(t)}{NM} \tag{1.2}$$

It is easy to verify standard relations for probabilities $p_{ij}(t)$

$$p_{ij}(t) \ge 0, \quad \sum_{i,j} p_{ij}(t) = 1, \quad i,j = 1,2 \tag{1.3}$$

Let us pass from the multistep dynamical system (1.1) which connects quantities $N_i(t+\delta)$ and $M_j(t+\delta)$ with quantities $N_i(t)$ and $M_j(t)$ to the system which connects probabilities $p_{ij}(t+\delta)$ and $p_{ij}(t)$.

Let us compose, for example, the corresponding dynamical equation for the probability $p_{11}(t+\delta)$. We have

$$\begin{aligned} p_{11}(t+\delta) &= \frac{N_1(t+\delta)M_1(t+\delta)}{NM} \\ &= \frac{(N_1(t) - n_{12}(t)\delta + n_{21}(t)\delta)(M_1(t) - m_{12}(t)\delta + m_{21}(t)\delta)}{NM} \\ &= \frac{N_1(t)M_1(t)}{NM} - \end{aligned}$$

$$\frac{N_1(t)M_1(t)}{NM}\frac{m_{12}(t)}{M_1(t)}\delta + \frac{N_1(t)M_2(t)}{NM}\frac{m_{21}(t)}{M_2(t)}\delta -$$
$$\frac{N_1(t)M_1(t)}{NM}\frac{n_{12}(t)}{N_1(t)}\delta + \frac{N_2(t)M_1(t)}{NM}\frac{n_{21}(t)}{N_2(t)}\delta +$$
$$\frac{(-n_{12}(t)+n_{21}(t))(-m_{12}(t)+m_{21}(t))}{NM}\delta^2$$

Taking into account notations for probabilities $p_{ij}(t)$ we obtain the equation

$$p_{11}(t+\delta) - p_{11}(t) = -p_{11}(t)v_1\delta + p_{12}(t)v_2\delta - p_{11}(t)u_1\delta + p_{21}(t)u_2\delta + \phi(t)\delta^2 \tag{1.4}$$

Here

$$\begin{aligned} u_1 = u_1(t) &= \frac{n_{12}(t)}{N_1(t)} \\ u_2 = u_2(t) &= \frac{n_{21}(t)}{N_2(t)} \\ v_1 = v_1(t) &= \frac{m_{12}(t)}{M_1(t)} \\ v_2 = v_2(t) &= \frac{m_{21}(t)}{M_2(t)} \end{aligned} \tag{1.5}$$

$$\begin{array}{ll} 0 \le u_i \le 1, & i = 1,2 \\ 0 \le v_j \le 1, & j = 1,2 \end{array} \tag{1.6}$$

$$|\phi(t)| \le 1$$

Dividing equation (1.4) into $\delta > 0$ and passing to limit when $\delta \downarrow 0$ we come to the differential equation

$$\dot{p}_{11}(t) = -p_{11}(t)u_1(t) + p_{21}(t)u_2(t) - p_{11}(t)v_1(t) + p_{12}(t)v_2(t)$$

Analogously one can deduce differential equations for $\dot{p}_{12}(t)$, $\dot{p}_{21}(t)$, $\dot{p}_{22}(t)$.

Let us write differential equations which describe the motion of the considered dynamical system using standard notations

$$x_1 = p_{11}, \quad x_2 = p_{12}, \quad x_3 = p_{21}, \quad x_4 = p_{22} \tag{1.7}$$

We obtain the following bilinear system of differential equations with respect to probabilities x_1, x_2, x_3, x_4

$$\begin{aligned} \dot{x}_1 &= -x_1u_1 + x_3u_2 - x_1v_1 + x_2v_2 = f_1(x,u,v) \\ \dot{x}_2 &= -x_2u_1 + x_4u_2 + x_1v_1 - x_2v_2 = f_2(x,u,v) \\ \dot{x}_3 &= x_1u_1 - x_3u_2 - x_3v_1 + x_4v_2 = f_3(x,u,v) \\ \dot{x}_4 &= x_2u_1 - x_4u_2 + x_3v_1 - x_4v_2 = f_4(x,u,v) \end{aligned} \tag{1.8}$$

Here

$$x = (x_1, x_2, x_3, x_4), \quad u = (u_1, u_2), \quad v = (v_1, v_2)$$

1.2 Properties of Dynamical System

Let us turn our attention to some properties of control dynamical system (1.8). This dynamics conserves the following properties of probabilities.

Lemma 1.1 *If*

$$x_1(0) + x_2(0) + x_3(0) + x_4(0) = 1 \tag{1.9}$$

then

$$x_1(t) + x_2(t) + x_3(t) + x_4(t) = 1 \quad \forall t \tag{1.10}$$

Proof. Actually

$$f_1(x, u, v) + f_2(x, u, v) + f_3(x, u, v) + f_4(x, u, v) = 0$$

and, hence,

$$\dot{x}_1(t) + \dot{x}_2(t) + \dot{x}_3(t) + \dot{x}_4(t) = 0 \quad \forall t$$

We obtain

$$x_1(t) + x_2(t) + x_3(t) + x_4(t) = c \quad \forall t$$

From (1.9) we have $c = 1$.

Lemma 1.2 *If*

$$x_1(0)x_4(0) - x_2(0)x_3(0) = 0 \tag{1.11}$$

then

$$x_1(t)x_4(t) - x_2(t)x_3(t) = 0 \quad \forall t \tag{1.12}$$

Proof. Let us note that (1.11) takes place for our model because

$$x_1(0)x_4(0) - x_2(0)x_3(0) = \frac{N_1M_1N_2M_2}{NM} - \frac{N_1M_2N_2M_1}{NM} = 0$$

Let

$$z(t) = x_1(t)x_4(t) - x_2(t)x_3(t), \quad z(0) = 0$$

We have

$$\begin{aligned} \dot{z} = \dot{x}_1x_4 + x_1\dot{x}_4 - \dot{x}_2x_3 - x_2\dot{x}_3 = \\ (-x_1x_4 + x_2x_3)(u_1 + u_2 + v_1 + v_2) = \\ -z(u_1 + u_2 + v_1 + v_2) \end{aligned}$$

Hence

$$z(t) = z(0)\exp(-\int_0^t (u_1(s) + u_2(s) + v_1(s) + v_2(s))ds)$$

Since $z(0) = 0$ then $z(t) \equiv 0$.

Thus the relations

$$x_1 + x_2 + x_3 + x_4 = 1 \tag{1.13}$$

$$x_1x_4 - x_2x_3 = 0 \tag{1.14}$$

are first integrals for dynamical system (1.8).

One can prove also that system (1.8) conserves the following properties of probabilities.

Lemma 1.3 *If*

$$0 \le x_i(0) \le 1 \tag{1.15}$$

then

$$0 \le x_i(t) \le 1 \quad \forall t \quad i = 1,2,3,4 \tag{1.16}$$

Proof. We prove this fact below for the reduced system.

1.3 Reduced Dynamical Systems

Let us note that since there are exist two first integrals (1.13),(1.14) for dynamical system (1.8) then its order can be reduced from the fourth to the second. We shall make this reduction in convenient way for us. To this end we introduce the following variables

$y_1 = x_1 + x_2$	is the probability of the fact that a player from the first population holds the first strategy
$y_2 = x_3 + x_4$	is the probability of the fact that a player from the first population holds the second strategy
$y_3 = x_1 + x_3$	is the probability of the fact that a player from the second population holds the first strategy
$y_4 = x_2 + x_4$	is the probability of the fact that a player from the second population holds the second strategy

It is obvious that

$$y_1 + y_2 = 1, \quad y_3 + y_4 = 1 \tag{1.17}$$

Using (1.13),(1.14) one can prove also that

$$y_1 y_3 = x_1, \quad y_1 y_4 = x_2, \quad y_2 y_3 = x_3, \quad y_2 y_4 = x_4 \tag{1.18}$$

Actually, we have, for example,

$$y_1 y_3 = (x_1 + x_2)(x_1 + x_3) = x_1(x_1 + x_2 + x_3) + x_2 x_3 = x_1(x_1 + x_2 + x_3 + x_4) = x_1$$

From dynamical system (1.8) we obtain the following control system with respect to probabilities y_1, y_2, y_3, y_4

$$\begin{aligned} \dot{y}_1 &= -y_1 u_1 + y_2 u_2 \\ \dot{y}_2 &= y_1 u_1 - y_2 u_2 \\ \dot{y}_3 &= -y_3 v_1 + y_4 v_2 \\ \dot{y}_4 &= y_3 v_1 - y_4 v_2 \end{aligned} \tag{1.19}$$

Introducing notations $y_1 = x$, $y_3 = y$ we obtain from (1.17) that $y_2 = 1 - x$, $y_4 = 1 - y$. Substituting these relations to (1.19) we come to the following bilinear system of differential equations with respect to probabilities x and y

$$\begin{aligned} \dot{x} &= -x u_1 + (1 - x) u_2 \\ \dot{y} &= -y v_1 + (1 - y) v_2 \end{aligned} \tag{1.20}$$

Let us remind that controls u_1, u_2, v_1, v_2 are pure numbers here. They are determined by relations (1.5) and satisfies restrictions (1.6). The extreme values of controls $u_i = 0$ or $u_i = 1, i = 1,2$ and $v_j = 0$ or $v_j = 1, j = 1,2$ can be enterpreted as signals to individuals of corresponding populations to change or not to change their actions. For example, the value $u_1 = 0$ signals to individuals of the first population to hold the first strategy, not to change actions, and the value $u_1 = 1$ signals about necessity to alter the first strategy for the second.

In reality the system (1.20) can be replaced by the equivalent system of more simple type. Let us consider, for example, the first equation of the system (1.20) and transform it in the following way

$$\dot{x} = -xu_1 + (1-x)u_2 = -x + x(1-u_1) + (1-x)u_2$$

Let us introduce the new control parameter

$$u = x(1-u_1) + (1-x)u_2 \tag{1.21}$$

We determine now the restrictions for the control parameter u. We have

$$u \in P(x), \quad P(x) = P_1(x) + P_2(x)$$

$$P_1(x) = \{x(1-u_1): \quad 0 \le u_1 \le 1, \quad 0 \le x \le 1\} = [0, x]$$
$$P_2(x) = \{(1-x)u_2: \quad 0 \le u_2 \le 1, \quad 0 \le x \le 1\} = [0, 1-x]$$

Hence, the set $P(x) = [0, x] + [0, 1-x]$ is the segment $[0, 1]$ and it does not depend on x. Thus, the first equation of the system (1.20) can be replaced by the equation

$$\dot{x} = -x + u, \quad 0 \le u \le 1$$

Analogously, if we introduce the new control parameter

$$v = y(1-v_1) + (1-y)v_2 \tag{1.22}$$

we obtain the differential equation with respect to the probability y

$$\dot{y} = -y + v, \quad 0 \le v \le 1$$

Thus, we have the following system of differential equations with respect to probabilities x and y

$$\begin{array}{rcl} \dot{x} & = & -x + u, \quad 0 \le u \le 1 \\ \dot{y} & = & -y + v, \quad 0 \le v \le 1 \end{array} \tag{1.23}$$

Finally, let us verify that system (1.23) conserves properties of probabilities x and y.

Proof of Lemma 1.3.

Let $u(s) : [0, +\infty) \mapsto [0, 1]$ be a control function measurable in the sense of Lebesgue. Then according to Cauchy formula we have

$$x(t) = (x_0 + \int_0^t \exp(s)u(s)ds)\exp(-t)$$

If $x(0) = x_0 \ge 0$ then it is obvious that $x(t) \ge 0$. Let $x(0) = x_0 \le 1$. Then

$$\begin{aligned} x(t) \le (1 + \int_0^t \exp(s)u(s)ds)\exp(-t) \le \\ \exp(-t) + \int_{-t}^0 \exp(s)u(s+t)ds \le \\ \exp(-t) + \int_{-t}^0 \exp(s)ds \le 1 \end{aligned} \tag{1.24}$$

Since system (1.23) conserves properties of probabilities then equivalent systems (1.8) and (1.20) also conserve these properties.

2 The Model of "Short-Term" and "Long-Term" Payoffs

2.1 "Short-Term" Payoffs

Let us pass now to the question about evaluation of payoffs of populations. It is naturally to assume that the mathematical expectation connected with the corresponding payoff matrix is the payoff of population at the moment of time t. Namely, quality of the state $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ of dynamical process (1.8) is evaluated for the first population by the mathematical expectation

$$E_A(t) = a_{11}x_1(t) + a_{12}x_2(t) + a_{21}x_3(t) + a_{22}x_4(t) \tag{2.1}$$

and for the second population - by the mathematical expectation

$$E_B(t) = b_{11}x_1(t) + b_{12}x_2(t) + b_{21}x_3(t) + b_{22}x_4(t) \tag{2.2}$$

Taking into account the notations of the equivalent system (1.23) we can rewrite formulas (2.1),(2.2) by means of probabilities $x(t), y(t)$ in the following way

$$\begin{aligned} E_A(t) = a_{11}x(t)y(t) + a_{12}x(t)(1 - y(t)) + \\ a_{21}(1 - x(t))y(t) + a_{22}(1 - x(t))(1 - y(t)) = \\ (a_{11} - a_{12} - a_{21} + a_{22})x(t)y(t) - (a_{22} - a_{12})x(t) - \\ (a_{22} - a_{21})y(t) + a_{22} \end{aligned} \tag{2.3}$$

$$\begin{aligned} E_B(t) = b_{11}x(t)y(t) + b_{12}x(t)(1 - y(t)) + \\ b_{21}(1 - x(t))y(t) + b_{22}(1 - x(t))(1 - y(t)) = \\ (b_{11} - b_{12} - b_{21} + b_{22})x(t)y(t) - (b_{22} - b_{12})x(t) - \\ (b_{22} - b_{21})y(t) + b_{22} \end{aligned} \tag{2.4}$$

Let us note that in the theory of static bimatrix games (see, for example [Vorobjev, 1984]) there are special notations for the coefficients of formulas (2.3),(2.4)

$$\begin{aligned} C_A &= a_{11} - a_{12} - a_{21} + a_{22} \\ \alpha_1 &= a_{22} - a_{12} \\ \alpha_2 &= a_{22} - a_{21} \end{aligned} \tag{2.5}$$

$$\begin{aligned} C_B &= b_{11} - b_{12} - b_{21} + b_{22} \\ \beta_1 &= b_{22} - b_{12} \\ \beta_2 &= b_{22} - b_{21} \end{aligned} \tag{2.6}$$

It is naturally to assume that formulas (2.3)-(2.4) describe the "short-term" (calculated at the fixed moment of time t) payoffs of populations.

2.2 "Long-Term" Payoffs

Let us consider now the dynamical system (1.23) on the infinite interval of time $[0, +\infty)$. Infinity of the interval of time means that we are interested namely in the evolutionary character of behaviour of trajectories generated by the dynamical system.

Let $(x(\cdot), y(\cdot)) = \{(x(t), y(t)) : t \in [0, +\infty)\}$ be an arbitrary trajectory of the system (1.23). We shall estimate the quality of this trajectory by the integral functionals with discount coefficients. For the first population we determine the quality of a trajectory by the functional

$$\begin{aligned} J_1 = J_1(x(\cdot), y(\cdot)) = \\ \int_0^{+\infty} \exp(-\lambda t) E_A(t) dt = \int_0^{+\infty} \exp(-\lambda t) g_1(x(t), y(t)) dt = \\ \int_0^{+\infty} \exp(-\lambda t)(C_A x(t) y(t) - \alpha_1 x(t) - \alpha_2 y(t) + a_{22}) dt \end{aligned} \quad (2.7)$$

and for the second population - by the functional

$$\begin{aligned} J_2 = J_2(x(\cdot), y(\cdot)) = \\ \int_0^{+\infty} \exp(-\lambda t) E_B(t) dt = \int_0^{+\infty} \exp(-\lambda t) g_2(x(t), y(t)) dt = \\ \int_0^{+\infty} \exp(-\lambda t)(C_B x(t) y(t) - \beta_1 x(t) - \beta_2 y(t) + b_{22}) dt \end{aligned} \quad (2.8)$$

Here $\lambda > 0$ is the so-called coefficient of discount (which means discounting of the process with growth of time). Functionals of such type are rather traditional for mathematical models in economics (see, for example, references in [Dolcetta, 1983]). Let us note that integrals (2.7), (2.8) always exist since $0 \leq x(t) \leq 1, \quad 0 \leq y(t) \leq 1$.

Functionals (2.7),(2.8) determine "long-term" (on the infinite interval of time) payoffs of populations in contrast to "short-term" payoffs (2.1)-(2.4).

Integrals (2.7),(2.8) can be interpreted also in terms of "average" mathematical expectations. Indeed, let us, for example, consider integral (2.7). We normalize it by multiplying on the discount coefficient λ. We have

$$\begin{aligned} \lambda \int_0^{+\infty} \exp(-\lambda t)(C_A x(t) y(t) - \alpha_1 x(t) - \alpha_2 y(t) + a_{22}) dt = \\ a_{11} \int_0^{+\infty} \lambda \exp(-\lambda t) x_1(t) dt + a_{12} \int_0^{+\infty} \lambda \exp(-\lambda t) x_2(t) dt + \\ a_{21} \int_0^{+\infty} \lambda \exp(-\lambda t) x_3(t) dt + a_{22} \int_0^{+\infty} \lambda \exp(-\lambda t) x_4(t) dt = \\ \sum_{i,j} a_{ij} \int_0^{+\infty} \lambda \exp(-\lambda t) p_{ij}(t) dt = \\ a_{11} p_{11} + a_{12} p_{12} + a_{21} p_{21} + a_{22} p_{22} = \\ E_A(x(\cdot), y(\cdot)) \end{aligned} \quad (2.9)$$

Here

$$p_{ij} = \int_0^{+\infty} \lambda \exp(-\lambda t) p_{ij}(t) dt, \quad i, j = 1, 2 \quad (2.10)$$

It is easy to verify that $0 \leq p_{ij} \leq 1$ and $\sum_{i,j} p_{ij} = 1$. Hence, one can interpret numbers p_{ij} as some special averaging (2.10) of probabilities $p_{ij}(t), \quad t \in [0, +\infty)$ on the infinite interval of time. Therefore, it is naturally to regard number p_{ij} as "average" probability of the fact that random pairs of individuals play the situation (i, j) on the infinite interval of time. The functional (2.9) is interpreted then as "average" mathematical expectation $E_A(x(\cdot), y(\cdot))$ of payoff for the first population on the infinite interval of time.

Analogously, the normalized functional (2.8) can be considered as "average" mathematical expectation $E_B(x(\cdot), y(\cdot))$ of payoff for the second population in infinite horizon

$$\begin{aligned}
\lambda \int_0^{+\infty} \exp(-\lambda t)(C_B x(t)y(t) - \beta_1 x(t) - \beta_2 y(t) + b_{22})dt = \\
b_{11} \int_0^{+\infty} \lambda \exp(-\lambda t) x_1(t)dt + b_{12} \int_0^{+\infty} \lambda \exp(-\lambda t) x_2(t)dt + \\
b_{21} \int_0^{+\infty} \lambda \exp(-\lambda t) x_3(t)dt + b_{22} \int_0^{+\infty} \lambda \exp(-\lambda t) x_4(t)dt = \\
\sum_{i,j} b_{ij} \int_0^{+\infty} \lambda \exp(-\lambda t) p_{ij}(t)dt = \\
b_{11}p_{11} + b_{12}p_{12} + b_{21}p_{21} + b_{22}p_{22} = \\
E_B(x(\cdot), y(\cdot))
\end{aligned} \tag{2.11}$$

3 Nash Equilibria in Dynamical System

3.1 Feedback Controls, Trajectories of Dynamical System

Let us assume that controls u and v for the first and the second populations in system (1.23) can be formed on the feedback principle. We suppose also that feedback controls (strategies) $U = u(t, x, y, \varepsilon)$ and $V = v(t, x, y, \varepsilon)$ can be discontinuous functions of phase variables (x, y). For definition of motions of the system generated by discontinuous controls $U = u(t, x, y, \varepsilon)$, $V = v(t, x, y, \varepsilon)$ we shall use the approach proposed in [Krasovskii, Subbotin, 1988]. Namely, let $[0, T]$ be an interval of time, $\Delta = \{t_0 = 0 < t_1 < t_2 < ... < t_N = T\}$ be its partition with an instant time-step $\delta = t_{k+1} - t_k$ and $\varepsilon > 0$ be a parameter of accuracy $0 < \delta < \beta(\varepsilon)$ where $\beta(\varepsilon) \downarrow 0$ when $\varepsilon \downarrow 0$. Consider piecewise differentiable trajectory $(x_\Delta(\cdot), y_\Delta(\cdot))$ which is called "Euler spline" and is actually the step-by-step solution of the following differential equation

$$\begin{aligned}
\dot{x}_\Delta(t) &= -x_\Delta(t) + u(t_k, x_\Delta(t_k), y_\Delta(t_k), \varepsilon) \\
\dot{y}_\Delta(t) &= -y_\Delta(t) + v(t_k, x_\Delta(t_k), y_\Delta(t_k), \varepsilon)
\end{aligned} \tag{3.1}$$

$$t \in [t_k, t_{k+1}), \quad k = 0, 1, ..., N-1$$

$$x_\Delta(0) = x_0, \quad y_\Delta(0) = y_0$$

The uniformly continuous limits of "Euler splines" when $N \to \infty, \varepsilon \downarrow 0, \delta \downarrow 0$ are called limit motions or simply motions of the system. The set of all these motions will be denoted by the symbol $X_T(x_0, y_0, U, V)$. This set is a compactum in the space of continuous functions determined on $[0, T]$.

Definition 3.1 *A continuous function $(x(t), y(t)) : [0, +\infty) \to R^2$ is called trajectory of the system (1.23) on the infinite interval of time generated by strategies U and V from the initial position (x_0, y_0) if for any moment $T, 0 < T < \infty$ there exists a trajectory $(x_T(t), y_T(t)) \in X_T(x_0, y_0, U, V)$ which satisfies the condition $(x(t), y(t)) = (x_T(t), y_T(t))$, $t \in [0, T]$. The set of all these trajectories $(x(t), y(t)) : [0, +\infty) \to R^2$ will be denoted by the symbol $X(x_0, y_0, U, V)$.*

3.2 Nash Equilibria

Let us introduce definiton of Nash equilibria for pairs of feedback controls ($U = u(t,x,y,\varepsilon)$, $V = v(t,x,y,\varepsilon)$).

Definition 3.2 *A pair of feedback controls* (U^0, V^0) *is called optimal (equilibrium) in the sense of Nash for the fixed initial position* $(x_0, y_0) \in [0,1] \times [0,1]$ *if for any other feedback controls* U *and* V *the following condition holds. For all trajectories*

$$(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U^0, V^0), \quad \begin{array}{l} (x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V^0) \\ (x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U^0, V) \end{array}$$

inequalities

$$\begin{aligned} J_1(x^0(\cdot), y^0(\cdot)) &\geq J_1(x_1(\cdot), y_1(\cdot)) \\ J_2(x^0(\cdot), y^0(\cdot)) &\geq J_2(x_2(\cdot), y_2(\cdot)) \end{aligned} \tag{3.2}$$

take place.

Remark 3.1 *For construction of Nash equilibria we shall use the scheme proposed in [Kleimenov, 1993]. We shall give the short description of this scheme below in Section 4.*

We need to modify slightly Definition 3.2 because in reality we construct some ε-approximations of optimal feedback controls. Therefore, let us introduce the notion of ε-equilibria.

Definition 3.3 *Let* $\varepsilon > 0$ *and* $(x_0, y_0) \in [0,1] \times [0,1]$. *A pair of feedback controls* $(U_\varepsilon, V_\varepsilon)$ *is called* ε*-optimal (*ε*-equilibrium) in the sense of Nash for the fixed initial position* (x_0, y_0) *if for any other feedback controls* U *and* V *the following condition holds. For all trajectories*

$$(x^0(\cdot), y^0(\cdot)) \in X(x_0, y_0, U_\varepsilon, V_\varepsilon), \quad \begin{array}{l} (x_1(\cdot), y_1(\cdot)) \in X(x_0, y_0, U, V_\varepsilon) \\ (x_2(\cdot), y_2(\cdot)) \in X(x_0, y_0, U_\varepsilon, V) \end{array}$$

inequalities

$$\begin{aligned} J_1(x^0(\cdot), y^0(\cdot)) &\geq J_1(x_1(\cdot), y_1(\cdot)) - \varepsilon \\ J_2(x^0(\cdot), y^0(\cdot)) &\geq J_2(x_2(\cdot), y_2(\cdot)) - \varepsilon \end{aligned} \tag{3.3}$$

take place.

Remark 3.2 *Nash equilibria which will be constructed below possess indeed the more strong properties than the properties indicated in Definitions 3.2, 3.3. These strong properties can be interpreted as dynamical stability of equilibria. Namely, we say that the Nash equilibrium has the property of dynamical stability if it is not profitable for populations to deviate from equilibrium feedback controls even along the whole equilibrium trajectory* $(x^0(\cdot), y^0(\cdot))$, *i.e. the following enequalities hold for all* $t_* > 0$

$$\begin{aligned} \int_{t_*}^{+\infty} \exp(-\lambda t) g_1(x^0(t), y^0(t)) dt &\geq \int_{t_*}^{+\infty} \exp(-\lambda t) g_1(x_1(t), y_1(t)) dt - \varepsilon \\ \int_{t_*}^{+\infty} \exp(-\lambda t) g_2(x^0(t), y^0(t)) dt &\geq \int_{t_*}^{+\infty} \exp(-\lambda t) g_2(x_2(t), y_2(t)) dt - \varepsilon \end{aligned} \tag{3.4}$$

with $\varepsilon = 0$ *for Definition 3.2 and with* $\varepsilon > 0$ *for Definition 3.3.*

3.3 Stability Properties of Dynamical System

Let us formulate the property of stability for dynamical system (1.23).

Lemma 3.1 *Let $u(t) : [0, +\infty) \to [0, 1]$, $\quad v(t) : [0, +\infty) \to [0, 1]$ be measurable controls and $(x_1(\cdot), y_1(\cdot)), (x_2(\cdot), y_2(\cdot))$ be two trajectories of the system (1.23) generated by these controls from different initial positions*

$$(x_1(0), y_1(0)) = (x_1, y_1), \quad (x_2(0), y_2(0)) = (x_2, y_2)$$

Then

$$\begin{aligned} |x_1(t) - x_2(t)| \leq & \ |x_1 - x_2| \exp(-t) \\ |y_1(t) - y_2(t)| \leq & \ |y_1 - y_2| \exp(-t) \end{aligned} \tag{3.5}$$

Moreover, there exists a constant $C > 0$ depending only on coefficients of matrixes A and B such that the following estimates take place

$$\begin{aligned} & |J_k(x_1(\cdot), y_1(\cdot)) - J_k(x_2(\cdot), y_2(\cdot))| \leq \\ & \frac{C}{1 + \lambda} \max\{|x_1 - x_2|, |y_1 - y_2|\}, \quad k = 1, 2 \end{aligned} \tag{3.6}$$

Proof. Consider, for example, differential equations for $x_1(\cdot)$ and $x_2(\cdot)$

$$\begin{aligned} \dot{x}_1(t) &= -x_1(t) + u(t), \quad x_1(0) = x_1 \\ \dot{x}_2(t) &= -x_2(t) + u(t), \quad x_2(0) = x_2 \end{aligned}$$

Subtracting the second equation from the first we obtain

$$\Delta \dot{x}(t) = -\Delta x(t), \quad \Delta x(0) = x_1 - x_2$$
$$\Delta x(t) = x_1(t) - x_2(t)$$

Hence

$$\Delta x(t) = \Delta x(0) \exp(-t)$$

Analogously, we can obtain

$$\Delta y(t) = \Delta y(0) \exp(-t)$$

For the difference of functionals we have the estimate

$$\begin{aligned} & |J_1(x_1(\cdot), y_1(\cdot)) - J_1(x_2(\cdot), y_2(\cdot))| \leq \\ & \int_0^{+\infty} \exp(-\lambda t) |C_A(x_1(t) y_1(t) - x_2(t) y_2(t)) - \alpha_1(x_1(t) - x_2(t)) - \alpha_2(y_1(t) - y_2(t))| dt \leq \\ & C \int_0^{+\infty} \exp(-\lambda t) \max\{|x_1(t) - x_2(t)|, |y_1(t) - y_2(t)|\} dt \leq \\ & C \max\{|x_1 - x_2|, |y_1 - y_2|\} \int_0^{+\infty} \exp(-(1 + \lambda) t) dt = \\ & \frac{C}{1 + \lambda} \max\{|x_1 - x_2|, |y_1 - y_2|\} \end{aligned}$$

Finally we give the following corollary.

Corollary 3.1 *For integral functionals with finite horizon T, $\quad 0 \leq T < +\infty$ the estimate similar to (3.6) takes place*

$$\begin{aligned} & |\int_0^T \exp(-\lambda t) g_k(x_1(t), y_1(t)) dt - \int_0^T \exp(-\lambda t) g_k(x_2(t), y_2(t)) dt| \leq \\ & \frac{C}{1 + \lambda} \max\{|x_1 - x_2|, |y_1 - y_2|\} \end{aligned} \tag{3.7}$$

4 Construction of Nash Equilibria

4.1 Auxiliary Antagonistic (Zero-Sum) Games

In order to construct equilibrium feedback controls we use the approach proposed in the theory of differential games (see, for example, [Kleimenov, 1993]).

Let us consider auxiliary antagonistic (zero-sum) differential games Γ_1 and Γ_2 with the functionals J_1 (2.7) and J_2 (2.8) respectively. In the game Γ_1 the first population tries to maximize the functional $J_1(x(\cdot), y(\cdot))$ using feedback controls $U = u(t, x, y, \varepsilon)$. The second population has the opposite aim, it tries to minimize this functional using feedback controls $V = v(t, x, y, \varepsilon)$. Conversely in the game Γ_2 the first population aims for minimization of the functional $J_2(x(\cdot), y(\cdot))$ and the second population wishes to maximize it.

By the symbols $w_1(x, y)$ and $w_2(x, y)$ we denote value functions of auxiliary antagonistic games Γ_1 and Γ_2. It is known (see [Krasovskii, Subbotin, 1988], [Krasovskii, 1985]) that optimal feedback controls (control synthesis) $U_i = u_i^0(t, x, y, \varepsilon), \quad i = 1, 2$ and $V_j = v_j^0(t, x, y, \varepsilon), \quad j = 1, 2$ of the first and the second population in this antagonistic game can be constructed on the information of value functions $w_k(\cdot), \quad k = 1, 2$.

Strategies $u_1^0(t, x, y, \varepsilon)$ and $v_2^0(t, x, y, \varepsilon)$ can be interpreted as strategies which have positive nature (we shall call them "positive" strategies) because they are aimed for maximization of their own quality functional. Let us mention that these strategies are cautious (guaranteed) feedback controls. Strategies $u_2^0(t, x, y, \varepsilon)$ and $v_1^0(t, x, y, \varepsilon)$ can be considered as strategies of "punishment" because they minimize the payoff functional of another population.

4.2 Equilibrium Feedback Controls

Let us construct now the pair of feedback strategies which forms Nash equilibrium by pasting together "positive" and "punishment" strategies $u_i^0(t, x, y, \varepsilon)$ and $v_j^0(t, x, y, \varepsilon), \quad i, j = 1, 2$ of two populations.

Let $(x_0, y_0) \in [0, 1] \times [0, 1]$ be an arbitrary initial position, $\varepsilon > 0$ be an accuracy parameter and $(x(\cdot), y(\cdot)) \in X(x_0, y_0, u_1(\cdot), v_2(\cdot))$ be a trajectory generated by "positive" strategies $u_1^0(t, x, y, \varepsilon)$ and $v_2^0(t, x, y, \varepsilon)$. Let $T_\varepsilon > 0$ be such a moment of time that

$$\int_{T_\varepsilon}^{+\infty} \exp(-\lambda t) |g_i(x(t), y(t))| dt < \varepsilon$$

By the symbols $u_\varepsilon(t) : [0, T_\varepsilon) \to [0, 1], \quad v_\varepsilon(t) : [0, T_\varepsilon) \to [0, 1]$ we denote step-by-step realizations of strategies $u_1^0(t, x, y, \varepsilon), \quad v_2^0(t, x, y, \varepsilon)$ such that the corresponding step-by-step motion $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ satisfies the condition

$$\max_{t \in [0, T_\varepsilon]} \|(x(t), y(t)) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon$$

It is affirmed that the following pair of feedback controls $U^0 = u^0(t, x, y, \varepsilon), V^0 = v^0(t, x, y, \varepsilon)$ pasted with the help of "positive" strategies $u_1^0(t, x, y, \varepsilon), v_2^0(t, x, y, \varepsilon)$ and "punishment" strategies $u_2^0(t, x, y, \varepsilon), v_1^0(t, x, y, \varepsilon)$ forms an ε-equilibrium situation in the sense of Nash

$$U^0 = u^0(t, x, y, \varepsilon) = \begin{cases} u_\varepsilon(t) & \text{if } \|(x, y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon \\ u_2^0(t, x, y, \varepsilon) & \text{otherwise} \end{cases} \tag{4.1}$$

$$V^0 = v^0(t,x,y,\varepsilon) = \begin{cases} v_\varepsilon(t) & \text{if } \|(x,y) - (x_\varepsilon(t), y_\varepsilon(t))\| < \varepsilon \\ v_1^0(t,x,y,\varepsilon) & \text{otherwise} \end{cases} \tag{4.2}$$

Let us remind that programming controls $u_\varepsilon(t)$, $v_\varepsilon(t)$ are realizations of "positive" strategies $u_1^0(t,x,y,\varepsilon), v_2^0(t,x,y,\varepsilon)$. In other words the "acceptable" trajectory $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ is generated by "positive interests" of populations. The number ε can be interpreted as a parameter of "reliance" of populations to each other or a level of "risk" which populations allow in the game.

5 Value Functions of Differential Games

5.1 Value Functions for Games with Infinite Horizon

Let us consider now the auxiliary antagonistic (zero-sum) game Γ_1 the dynamics of which is described by equations

$$\begin{aligned} \dot{x} &= -x + u \\ \dot{y} &= -y + v \end{aligned} \tag{5.1}$$

and the payoff functional is determined by relation

$$J_1(x(\cdot), y(\cdot)) = \int_0^{+\infty} \exp(-\lambda t) g_1(x(t), y(t)) dt =$$

$$\int_0^{+\infty} \exp(-\lambda t)(C_A x(t) y(t) - \alpha_1 x(t) - \alpha_2 y(t) + a_{22}) dt \tag{5.2}$$

The aim of the first population is to maximize the functional J_1 (5.2) on trajectories $(x(\cdot), y(\cdot))$ of the system (5.1) by disposing of control parameter $U = u(t,x,y,\varepsilon)$. The aim of the second population is opposite: to minimize the functional J_1 (5.2) on trajectories $(x(\cdot), y(\cdot))$ of the system (5.1) by disposing of control parameter $V = v(t,x,y,\varepsilon)$.

According to formalization proposed in [Krasovskii, Subbotin, 1988] the antagonistic game (5.1),(5.2) has the value function $(x_0, y_0) \to w_1(x_0, y_0)$

$$w_1(x_0, y_0) = \sup_U \inf_{(x(\cdot),y(\cdot)) \in X(x_0,y_0,U)} J_1(x(\cdot), y(\cdot)) =$$

$$\inf_V \sup_{(x(\cdot),y(\cdot)) \in X(x_0,y_0,V)} J_1(x(\cdot), y(\cdot)) \tag{5.3}$$

Here symbols $X(x_0, y_0, U), X(x_0, y_0, V)$ denote trajectories of dynamical system (5.1) generated by feedback controls $U = u(t,x,y,\varepsilon)$ and $V = v(t,x,y,\varepsilon)$.

5.2 Properties of Value Functions

Let us indicate some properties of value function $w_1 : [0,1] \times [0,1] \to R$.

Property 5.1 *Let $w_1(t,x,y)$ be the value function for the differential game with dynamics (5.1) and payoff functional*

$$J_1(t, x(\cdot), y(\cdot)) = \int_t^{+\infty} \exp(-\lambda s) g_1(x(s), y(s)) ds \tag{5.4}$$

$$x(t) = x, \quad y(t) = y$$

Then value functions $w_1(x,y)$ and $w_1(t,x,y)$ are connected by relation

$$w_1(t,x,y) = \exp(-\lambda t) w_1(x,y) \tag{5.5}$$

Property 5.2 *Let $w_1(t,T,x,y)$ be the value function for the differential game with dynamics (5.1) and payoff functional*

$$J_1(t,T,x(\cdot),y(\cdot)) = \int_t^T \exp(-\lambda s) g_1(x(s),y(s))ds \tag{5.6}$$

$$x(t) = x, \quad y(t) = y$$

Then value functions $w_1(t,x,y)$ and $w_1(t,T,x,y)$ are connected by inequality

$$\max_{(t,x,y)\in[0,T]\times[0,1]\times[0,1]} |w_1(t,x,y) - w_1(t,T,x,y)| \leq \frac{K}{\lambda}\exp(-\lambda T) \tag{5.7}$$

Here parameter K depends only on coefficients a_{ij} of matrix $A = \{a_{ij}\}$.

Property 5.3 *Value function w_1 is bounded*

$$\max_{(x,y)\in[0,1]\times[0,1]} |w_1(x,y)| \leq \frac{K}{\lambda} \tag{5.8}$$

Property 5.4 *Value function w_1 satisfies the Lipschitz condition*

$$\begin{aligned}|w_1(x_1,y_1) - w_1(x_2,y_2)| \leq \frac{K}{1+\lambda}(|x_1 - x_2| + |y_1 - y_2|) \leq \\ \frac{\sqrt{2}K}{1+\lambda}((x_1-x_2)^2 + (y_1-y_2)^2)^{\frac{1}{2}}\end{aligned} \tag{5.9}$$

Proof. Let us choose arbitrarily $\varepsilon > 0$. Determine a moment of time T, $0 \leq T < +\infty$ from the relation

$$\exp(-\lambda T)K < \varepsilon$$

i.e.

$$T > \frac{1}{\lambda}\ln\frac{K}{\varepsilon}$$

Consider value function $w_1(0,T,x,y)$. According to Property 5.2 we have

$$|w_1(x,y) - w_1(0,T,x,y)| \leq \frac{K}{\lambda}\exp(-\lambda T) \leq \frac{K\varepsilon}{\lambda}$$

Let U^0 and V^0 be feedback controls realizing external extremum in relations which determine value function $w_1(0,T,x,y)$, i.e.

$$\begin{aligned} & w_1(0,T,x,y) = \\ \max_U \min_{(x(\cdot),y(\cdot))\in X(x,y,U)} & J_1(0,T,x(\cdot),y(\cdot)) = \\ \min_{(x(\cdot),y(\cdot))\in X(x,y,U^0)} & J_1(0,T,x(\cdot),y(\cdot)) = \\ \min_V \max_{(x(\cdot),y(\cdot))\in X(x,y,V)} & J_1(0,T,x(\cdot),y(\cdot)) = \\ \max_{(x(\cdot),y(\cdot))\in X(x,y,V^0)} & J_1(0,T,x(\cdot),y(\cdot))\end{aligned}$$

Consider the difference

$$
\begin{aligned}
&w_1(0,T,x_1,y_1) - w_1(0,T,x_2,y_2) = \\
\min_{(x(\cdot),y(\cdot))\in X(x,y,U^0)} J_1(0,T,x(\cdot),y(\cdot)) - &\max_{(x(\cdot),y(\cdot))\in X(x,y,V^0)} J_1(0,T,x(\cdot),y(\cdot)) = \\
&J_1(0,T,x_0(\cdot),y_0(\cdot)) - J_1(0,T,x^0(\cdot),y^0(\cdot)) \leq \\
&J_1(0,T,x_1(\cdot),y_1(\cdot)) - J_1(0,T,x_2(\cdot),y_2(\cdot)) + \varepsilon
\end{aligned}
$$

Here $(x_1(\cdot),y_1(\cdot))$ and $(x_2(\cdot),y_2(\cdot))$ are "Euler splines" which are close enough to trajectories $(x_0(\cdot),y_0(\cdot))$ and $(x^0(\cdot),y^0(\cdot))$ realizing corresponding extremum.

According to the stability property of dynamical system (see Corollary 3.1) we have the following estimate

$$
\begin{aligned}
J_1(0,T,x_1(\cdot),y_1(\cdot)) - J_1(0,T,x_2(\cdot),y_2(\cdot)) \leq \\
\frac{K}{1+\lambda}(|x_1-x_2|+|y_1-y_2|)
\end{aligned}
$$

Let us note that the last estimate does not depend on T.

Combining all estimates together we obtain one-sided inequality for the Lipschitz condition (5.9). Changing places of equal "maxmin" and "minmax" in the previous arguments we come to the complementary estimate of the Lipschitz condition. Thus, condition (5.9) is proven.

Remark 5.1 *Properties 5.1 - 5.4 are valid also for the value function w_2*

$$
\begin{aligned}
w_2(x_0,y_0) = \sup_V \inf_{(x(\cdot),y(\cdot))\in X(x_0,y_0,V)} J_2(x(\cdot),y(\cdot)) = \\
\inf_U \sup_{(x(\cdot),y(\cdot))\in X(x_0,y_0,U)} J_2(x(\cdot),y(\cdot))
\end{aligned}
\tag{5.10}
$$

of the second auxiliary differential game.

6 Value Functions and Minimax (Viscosity) Solutions of Hamilton-Jacobi Equations

6.1 Hamilton-Jacobi Equations

The most principal properties of value functions are so-called properties of stability (u and v stability [Krasovskii, Subbotin, 1988]) which express the principle of optimality (suboptimality, superoptimality) of dynamical programming. At points where value function $w_1(x,y)$ is differentiable these properties convert to the first order partial differential equation of Hamilton-Jacobi type which is called Bellman-Isaacs equation or the basic equation for optimal control problems. For our optimal guaranteed control problem (differential game) (5.1),(5.2) the corresponding Bellman-Isaacs equation has the following form

$$
\begin{aligned}
-\lambda w(x,y) - \frac{\partial w}{\partial x}x - \frac{\partial w}{\partial y}y + \\
\max_{0\leq u\leq 1} \frac{\partial w}{\partial x}u + \min_{0\leq v\leq 1} \frac{\partial w}{\partial y}v = 0
\end{aligned}
\tag{6.1}
$$

Remark 6.1 *Equation (6.1) does not depend on time t. It is an equation of stationary type.*

It is easy to see that

$$\max_{0\le u\le 1}\frac{\partial w}{\partial x}u = \max\left\{0,\frac{\partial w}{\partial x}\right\}$$
$$\min_{0\le v\le 1}\frac{\partial w}{\partial y}v = \min\left\{0,\frac{\partial w}{\partial y}\right\} \tag{6.2}$$

Therefore, equation (6.1) can be rewritten in the form

$$-\lambda w(x,y) - \frac{\partial w}{\partial x}x - \frac{\partial w}{\partial y}y +$$
$$\max\left\{0,\frac{\partial w}{\partial x}\right\} + \min\left\{0,\frac{\partial w}{\partial y}\right\} = 0 \tag{6.3}$$

6.2 Generalized Derivatives, Differential Inequalities

Usually value function $w_1(x,y)$ is not differentiable everywhere. It satisfies only the Lipschitz condition (5.9), i.e. it is only almost everywhere differentiable according to Rademaher theorem.

It is shown in the theory of minimax (viscosity) solutions [Subbotin, 1980, 1991], [Crandall, Lions, 1983, 1984] that value function w_1 must satisfy generalized differential inequalities at points where it is not differentiable (measure of this set is equal to zero). These inequalities generalize Bellman-Isaacs equation and express the optimality principle of dynamical programming in infinitesimal form.

In order to write the principle of dynamical programming in infintesimal form let us introduce the notions of directional derivatives and conjugate derivatives for functions which satisfy the Lipschitz condition.

Let function $w(x,y) : [0,1]\times[0,1] \to R$ satisfy the Lipschitz condition.

Definition 6.1 *Lower and upper derivatives of function w at a point $(x,y)\in(0,1)\times(0,1)$ in a direction $h=(h_1,h_2)\in R^2$ are determined by relations*

$$\partial_- w(x,y)|(h) = \liminf_{\delta\downarrow 0}\frac{w(x+\delta h_1, y+\delta h_2) - w(x,y)}{\delta}$$
$$\partial_+ w(x,y)|(h) = \limsup_{\delta\downarrow 0}\frac{w(x+\delta h_1, y+\delta h_2) - w(x,y)}{\delta} \tag{6.4}$$

Definition 6.2 *Lower and upper conjugate derivatives of function w at a point $(x,y)\in(0,1)\times(0,1)$ are determined by equalities*

$$D^* w(x,y)|(s) = \sup_{h\in R^2}(\langle s,h\rangle - \partial_- w(x,y)|(h))$$
$$D_* w(x,y)|(s) = \inf_{h\in R^2}(\langle s,h\rangle - \partial_+ w(x,y)|(h)) \tag{6.5}$$

It is proven (see, for example, [Subbotin, 1980, 1991], [Crandall, Lions, 1983, 1984], [Subbotin, Tarasyev, 1985], [Dolcetta, 1983], [Adiatulina, Tarasyev, 1987]) in the theory of minimax (viscosity) solution that value function $w_1(x, y)$ is uniquely determined by the pair of differential inequalities which connect conjugate derivatives with the Hamiltonian of dynamical system. Let us give this result for differential game (5.1),(5.2).

Theorem 6.1 *For a Lipschitz continuous function $w : [0, 1] \times [0, 1] \to R$ to be the value function of differential game (5.1),(5.2) it is necessary and sufficient that the following differential inequalities hold for all $(x, y, s) \in (0, 1) \times (0, 1) \times R^2$*

$$D^* w(x, y)|(s) \geq -\lambda w(x, y) + H(x, y, s) \tag{6.6}$$

$$D_* w(x, y)|(s) \leq -\lambda w(x, y) + H(x, y, s) \tag{6.7}$$

Here the symbol $H(x, y, s)$ denotes the Hamiltonian of dynamical system (5.1)

$$H(x, y, s) = -s_1 x - s_2 y + \max\{0, s_1\} + \min\{0, s_2\} + g_1(x, y) \tag{6.8}$$

$$s = (s_1, s_2) \in R^2$$

$$g_1(x, y) = C_A xy - \alpha_1 x - \alpha_2 y + a_{22}$$

Remark 6.2 *Inequalities (6.6),(6.7) turn into Bellman-Isaacs equation (6.3) at points where function w is differentiable.*

Remark 6.3 *Differential inequality (6.7) expresses the so-called property of u-stability of the value function w which implies the existence of directions of nondecrease. Similarly, differential inequality (6.6) expresses property of v-stability which means that there exist directions of nonincrease for the value function w. Thus, relations (6.6),(6.7) can be interpreted as infinitesimal form of the dynamical programming principle.*

6.3 Piecewise Smooth Value Function

The prevalent situation is the piecewise smooth construction for the value function w. In this case smooth components of the value function must satisfy Bellman-Isaacs (Hamilton-Jacobi) equation (6.3) and on surfaces of continuous contraction of these smooth components differential inequalities (6.6),(6.7) must hold. Realization of differential inequalities (6.6),(6.7) on surfaces of contraction is essential. There exist numerous examples demonstrating that there exist piecewise smooth functions which satisfy Hamilton-Jacobi equation at points of their differentiability but these functions are not the value function because they don't satisfy relations (6.6),(6.7).

For piecewise smooth functions directional derivatives and conjugate derivatives can be calculated in the framework of nonsmooth and convex analysis. Let us give corresponding formulas. Assume that for function w the following equalities are valid in some neighborhood $O_\varepsilon(x_*, y_*)$ of point $(x_*, y_*) \in (0, 1) \times (0, 1)$

$$w(x, y) = \min_{i \in I} \max_{j \in J} \varphi_{ij}(x, y) = \max_{j \in J} \min_{i \in I} \varphi_{ij}(x, y) \tag{6.9}$$

$$w(x_*, y_*) = \varphi_{ij}(x_*, y_*), \quad i \in I, \quad j \in J$$

Then directional derivatives are determined by relations

$$\partial_- w(x_*, y_*)|(h) = \partial_+ w(x_*, y_*)|(h) = \partial w(x_*, y_*)|(h) = \\ \min_{i \in I} \max_{j \in J} \langle b_{ij}, h \rangle = \max_{j \in J} \min_{i \in I} \langle b_{ij}, h \rangle \quad (6.10)$$

$$b_{ij} = \left(\frac{\partial \varphi_{ij}}{\partial x}, \frac{\partial \varphi_{ij}}{\partial y} \right), \quad h = (h_1, h_2)$$

Let us introduce the following notations

$$C = \bigcap_{i \in I} B_i, \quad B_i = co\{b_{ij} : j \in J\}$$
$$D = \bigcap_{j \in J} B_j, \quad B_j = co\{b_{ij} : i \in I\}$$

Then conjugate derivatives are determined by relations

$$D^* w(x_*, y_*)|(s) = \begin{cases} 0 & \text{if } s \in C \\ +\infty & \text{otherwise} \end{cases} \quad (6.11)$$

$$D_* w(x_*, y_*)|(s) = \begin{cases} 0 & \text{if } s \in D \\ -\infty & \text{otherwise} \end{cases} \quad (6.12)$$

Remark 6.4 *The sets C and D may be empty. In this case the corresponding conjugate derivatives have infinite values and differential inequalities (6.6),(6.7) are obviously fulfilled.*

6.4 Example

Let us consider an example in which the value function is differentiable and can be found by the method of indetermined coefficients.

We rewrite Hamilton-Jacobi equation in the following way

$$-\lambda w(x,y) + (C_A xy - \alpha_1 x - \alpha_2 y + a_{22}) - \\ \frac{\partial w}{\partial x} x - \frac{\partial w}{\partial y} y + \max\left\{0, \frac{\partial w}{\partial x}\right\} + \min\left\{0, \frac{\partial w}{\partial y}\right\} = 0 \quad (6.13)$$

Without loss of generality we will assume that $C_A > 0$. Let us find solution w in the form

$$w(x,y) = Dxy - \gamma_1 x - \gamma_2 y + d \quad (6.14)$$

Here D, γ_1, γ_2, d are indetermined coefficients. In additition we will suppose that partial derivatives conserve their signs

$$\frac{\partial w}{\partial x} = Dy - \gamma_1 \leq 0 \quad (6.15)$$

$$\frac{\partial w}{\partial y} = Dx - \gamma_2 \geq 0 \quad (6.16)$$

and, hence, the following relations hold

$$\max\left\{0, \frac{\partial w}{\partial x}\right\} = 0$$

$$\min\left\{0, \frac{\partial w}{\partial y}\right\} = 0$$

We substitute function w (6.14) to (6.13) and obtain the equation

$$-\lambda(Dxy - \gamma_1 x - \gamma_2 y + d) + (C_A xy - \alpha_1 x - \alpha_2 y + a_{22}) -$$
$$(Dy - \gamma_1)x - (Dx - \gamma_2)y = 0$$

Setting equal coefficients at similar terms we come to equations with respect to indetermined coefficients D, γ_1, γ_2, d. Solving these equations we obtain the following solution

$$\begin{aligned} D &= \frac{C_A}{2+\lambda} \\ \gamma_1 &= \frac{\alpha_1}{1+\lambda} \\ \gamma_2 &= \frac{\alpha_2}{1+\lambda} \\ d &= \frac{a_{22}}{\lambda} \end{aligned} \tag{6.17}$$

Coefficients D, γ_1, γ_2 must satisfy conditions (6.15),(6.16). Substituting (6.17) to (6.15),(6.16) we find conditions for the initial matrix A which ensure that the value function w is differentiable and has the form (6.14),(6.17). We have from (6.15),(6.17)

$$y \leq \frac{\alpha_1(2+\lambda)}{C_A(1+\lambda)}, \quad y \in [0,1]$$

The last inequality holds if and only if coefficients of matrix A satisfy condition

$$\frac{\alpha_1}{C_A} \geq \frac{1+\lambda}{2+\lambda} \tag{6.18}$$

From (6.16),(6.17) we obtain

$$x \geq \frac{\alpha_2(2+\lambda)}{C_A(1+\lambda)}, \quad x \in [0,1]$$

The last inequality is equivalent to the following condition for coefficients of matrix A

$$\alpha_2 \geq 0 \tag{6.19}$$

Thus, if relations (6.18),(6.19) hold then function w (6.14),(6.17) is the differentiable solution of Hamilton-Jacobi equation (6.13).

Remark 6.5 *As a pattern of matrix which satisfy conditions (6.18),(6.19) one can take, for example, the matrix*

$$A = \begin{pmatrix} 3 & 0 \\ 2 & 2 \end{pmatrix} \tag{6.20}$$

Parameter λ is constrained here by the inequality

$$0 < \lambda \leq 1$$

Let us note that matrix A (6.20) does not have a dominating line but has the dominating first column.

Remark 6.6 *One can find analogously the differentiable solution of Hamilton-Jacobi equation (6.13) for other combinations of inequalities of the type (6.15),(6.16).*

Remark 6.7 *In the general case inequalities of the type (6.15),(6.16) are not valid for all* $(x, y) \in [0, 1] \times [0, 1]$. *In such situations value functions are not everywhere differentiable. The structure of the value function is rather complex in this case and it does not have analytical description. Therefore, below we propose the numerical method for construction of value functions.*

7 Approximation Operators and Method of Contraction Mappings for Construction of Generalized Solutions of Hamilton-Jacobi Equations

7.1 Discrete Approximation of Hamilton-Jacobi Equations

Let us consider the discrete approximation of Hamilton-Jacobi equation (6.1). We fix parameter $h \in (0, \frac{1}{\lambda})$. Note that parameter h can be interpreted as quantization step of time intervals.

Definition 7.1 *Algebraic equation of the type*

$$\begin{aligned} -\underline{w}_{1,h}(x, y) + hg_1(x, y) + \\ (1 - \lambda h) \max_{0 \leq u \leq 1} \min_{0 \leq v \leq 1} \underline{w}_{1,h}(x + h(-x + u), y + h(-y + v)) = 0 \end{aligned} \tag{7.1}$$

$$(x, y) \in [0, 1] \times [0, 1]$$

$$g_1(x, y) = C_A xy - \alpha_1 x - \alpha_2 y + a_{22}$$

is called discrete approximation of Hamilton-Jacobi equation (6.1).

Remark 7.1 *The solution* $\underline{w}_{1,h}(x, y)$ *of equation (7.1) is an approximation for generalized solution* $w_1(x, y)$ *of Hamilton-Jacobi equation (6.1). It is known (see, for example, [Dolcetta, 1983], [Adiatulina, Tarasyev, 1987]) that when* $h \downarrow 0$ *functions* $\underline{w}_{1,h}(x, y)$ *tend to function* $w_1(x, y)$ *in the space of Lipschitz continuous functions and order of approximation estimate is* $h^{\frac{1}{2}}$

$$\max_{(x,y) \in [0,1] \times [0,1]} |\underline{w}_{1,h}(x, y) - w_1(x, y)| \leq Eh^{\frac{1}{2}} \tag{7.2}$$

Remark 7.2 *Parallel with equation (7.1) one can consider the discrete approximation of Hamilton-Jacobi equation (6.1) in which the sequence of operations "maxmin" is replaced by the sequence "minmax"*

$$\begin{aligned} -\overline{w}_{1,h}(x, y) + hg_1(x, y) + \\ (1 - \lambda h) \min_{0 \leq v \leq 1} \max_{0 \leq u \leq 1} \overline{w}_{1,h}(x + h(-x + u), y + h(-y + v)) = 0 \end{aligned} \tag{7.3}$$

The solution $\overline{w}_{1,h}(x, y)$ *of equation (7.3) also converges to generalized solution* $w_1(x, y)$ *with the estimate*

$$\max_{(x,y) \in [0,1] \times [0,1]} |\overline{w}_{1,h}(x, y) - w_1(x, y)| \leq Eh^{\frac{1}{2}} \tag{7.4}$$

7.2 Method of Successive Approximations

Let us pass now to the question of finding solutions $\underline{w}_{1,h}(x,y), \overline{w}_{1,h}(x,y)$ of equations (7.1),(7.3). One can prove (see, for example, [Dolcetta, 1983], [Adiatulina, Tarasyev, 1987]) that equations (7.1),(7.3) contain contraction operators and, therefore, can be solved by the method of successive approximations. Let us indicate these contraction operators Π_* and Π^*

$$\Pi_* w(x,y) = hg_1(x,y) +$$
$$(1-\lambda h)\max_{0\le u\le 1}\min_{0\le v\le 1} w(x+h(-x+u), y+h(-y+v)) \tag{7.5}$$

$$\Pi^* w(x,y) = hg_1(x,y) +$$
$$(1-\lambda h)\min_{0\le v\le 1}\max_{0\le u\le 1} w(x+h(-x+u), y+h(-y+v)) \tag{7.6}$$

Contraction coefficients of operators Π_* and Π^* are equal to $(1-\lambda h)$. According to the principle of contraction mappings we can formulate the following statement.

Theorem 7.1 *Equations (7.1),(7.3)*

$$\underline{w}_{1,h}(x,y) = \Pi_* \underline{w}_{1,h}(x,y)$$
$$\overline{w}_{1,h}(x,y) = \Pi^* \overline{w}_{1,h}(x,y)$$

have unique solutions in the class of bounded, Lipschitz continuous functions. Moreover, iterative procedures

$$\underline{w}^n_{1,h}(x,y) = \Pi_* \underline{w}^{n-1}_{1,h}(x,y) = hg_1(x,y) +$$
$$(1-\lambda h)\max_{0\le u\le 1}\min_{0\le v\le 1} \underline{w}^{n-1}_{1,h}(x+h(-x+u), y+h(-y+v)) \tag{7.7}$$

$$\overline{w}^n_{1,h}(x,y) = \Pi^* \overline{w}^{n-1}_{1,h}(x,y) = hg_1(x,y) +$$
$$(1-\lambda h)\min_{0\le v\le 1}\max_{0\le u\le 1} \overline{w}^{n-1}_{1,h}(x+h(-x+u), y+h(-y+v)) \tag{7.8}$$

converge uniformly to solutions of equations (7.1),(7.3) for any initial approximations $\underline{w}^0_{1,h}(x,y)$, $\overline{w}^0_{1,h}(x,y)$ *which are bounded and Lipschitz continuous.*

As patterns for initial approximations $\underline{w}^0_{1,h}(x,y)$, $\overline{w}^0_{1,h}(x,y)$ one can take function $g_1(x,y)$ or zero function.

Remark 7.3 *According to the principle of contraction mappings the following estimates are also valid*

$$\max_{(x,y)\in[0,1]\times[0,1]} |\underline{w}_{1,h}(x,y) - \underline{w}^n_{1,h}(x,y)| \le \frac{K}{\lambda}(1-\lambda h)^n \tag{7.9}$$

$$\max_{(x,y)\in[0,1]\times[0,1]} |\overline{w}_{1,h}(x,y) - \overline{w}^n_{1,h}(x,y)| \le \frac{K}{\lambda}(1-\lambda h)^n \tag{7.10}$$

$$n = 0, 1, 2, ...$$

Taking into account estimates (7.2),(7.4),(7.9),(7.10) for functions w_1, $\underline{w}_{1,h}$, $\overline{w}_{1,h}$, $\underline{w}^n_{1,h}$, $\overline{w}^n_{1,h}$, $n = 0,1,2,...$ one can obtain estimates of convergence of functions $\underline{w}^n_{1,h}$, $\overline{w}^n_{1,h}$, to the function w_1. More precisely, the following statement takes place.

Theorem 7.2 *Estimates (7.11),(7.12) are valid*

$$\max_{(x,y)\in[0,1]\times[0,1]} |w_1(x,y) - \underline{w}^m_{1,h}(x,y)| \le Gh^{\frac{1}{2}} \tag{7.11}$$

$$\max_{(x,y)\in[0,1]\times[0,1]} |w_1(x,y) - \overline{w}^m_{1,h}(x,y)| \le Gh^{\frac{1}{2}} \tag{7.12}$$

Here $h \in (0, \min\left\{\frac{1}{\lambda}, 1\right\})$*,* G *is a constant which does not depend on* h*, a number* m *depends on* h *and is determined by condition*

$$(1-\lambda h)^m \le h^{\frac{1}{2}} \tag{7.13}$$

7.3 Discrete Approximations for the Second Differential Game

Analogous results are valid for the differential game with the second payoff functional J_2. Namely, let us consider discrete approximations of Hamilton-Jacobi equation for the value function $w_2(x,y)$

$$\begin{gathered} -\underline{w}_{2,h}(x,y) + hg_2(x,y) + \\ (1-\lambda h)\max_{0\le v\le 1}\min_{0\le u\le 1} \underline{w}_{2,h}(x+h(-x+u), y+h(-y+v)) = 0 \end{gathered} \tag{7.14}$$

$$\begin{gathered} -\overline{w}_{2,h}(x,y) + hg_2(x,y) + \\ (1-\lambda h)\min_{0\le u\le 1}\max_{0\le v\le 1} \overline{w}_{2,h}(x+h(-x+u), y+h(-y+v)) = 0 \end{gathered} \tag{7.15}$$

$$(x,y) \in [0,1]\times[0,1]$$
$$g_2(x,y) = C_B xy - \beta_1 x - \beta_2 y + b_{22}$$

Equations (7.14),(7.15) contain contraction operators Φ_* and Φ^* with contraction coefficients equal to $(1-\lambda h)$

$$\begin{gathered} \Phi_* w(x,y) = hg_2(x,y) + \\ (1-\lambda h)\max_{0\le v\le 1}\min_{0\le u\le 1} w(x+h(-x+u), y+h(-y+v)) \end{gathered}$$

$$\begin{gathered} \Phi^* w(x,y) = hg_2(x,y) + \\ (1-\lambda h)\min_{0\le u\le 1}\max_{0\le v\le 1} w(x+h(-x+u), y+h(-y+v)) \end{gathered}$$

Equations (7.14),(7.15) can be rewritten in the form

$$\underline{w}_{2,h}(x,y) = \Phi_* \underline{w}_{2,h}(x,y)$$
$$\overline{w}_{2,h}(x,y) = \Phi^* \overline{w}_{2,h}(x,y)$$

The method of successive approximations for these equations provides convergence to the unique solution. Namely, the following iterative procedures converge

$$\begin{gathered} \underline{w}^n_{2,h}(x,y) = \Phi_* \underline{w}^{n-1}_{2,h}(x,y) = hg_2(x,y) + \\ (1-\lambda h)\max_{0\le v\le 1}\min_{0\le u\le 1} \underline{w}^{n-1}_{2,h}(x+h(-x+u), y+h(-y+v)) \end{gathered} \tag{7.16}$$

$$\overline{w}_{2,h}^n(x,y) = \Phi^* \overline{w}_{2,h}^{n-1}(x,y) = hg_2(x,y) +$$

$$(1-\lambda h) \min_{0 \le u \le 1} \max_{0 \le v \le 1} \overline{w}_{2,h}^{n-1}(x + h(-x+u), y + h(-y+v)) \tag{7.17}$$

Moreover, approximations $\underline{w}_{2,h}^n$ and $\overline{w}_{2,h}^n$ converge also to the value function w_2 when $n \to \infty$, $h \downarrow 0$ and the following estimates of convergence take place

$$\max_{(x,y)\in[0,1]\times[0,1]} |w_2(x,y) - \underline{w}_{2,h}^m(x,y)| \le Gh^{\frac{1}{2}} \tag{7.18}$$

$$\max_{(x,y)\in[0,1]\times[0,1]} |w_2(x,y) - \overline{w}_{2,h}^m(x,y)| \le Gh^{\frac{1}{2}} \tag{7.19}$$

for number m satisfying condition (7.13).

Thus, we need to construct two approximations $\underline{w}_{1,h}^m$, $\overline{w}_{1,h}^m$ for the value function w_1 and two approximations $\underline{w}_{2,h}^m$, $\overline{w}_{2,h}^m$ for the value function w_2.

7.4 Optimal Feedback Controls

Very important detail of the considered construction consists in the fact that functions $\underline{w}_{1,h}^m$, $\overline{w}_{1,h}^m$, $\underline{w}_{2,h}^m$, $\overline{w}_{2,h}^m$ keep information not only about approximation values of generalized solutions w_1 and w_2 but simultaneously they allow to determine approximations of optimal feedback controls: "positive feedback controls" $u_{1,h}^0(x,y)$, $v_{2,h}^0(x,y)$ and "punishment feedback controls" $u_{2,h}^0(x,y)$, $v_{1,h}^0(x,y)$. Namely, "positive feedback controls" $u_{1,h}^0(x,y)$, $v_{2,h}^0(x,y)$ can be determined as arguments u, v which realize external maximum in formulas (7.7),(7.16) when $n = m$

$$u_{1,h}^0(x,y) = \arg \max_{0 \le u \le 1} \min_{0 \le v \le 1} \underline{w}_{1,h}^{m-1}(x + h(-x+u), y + h(-y+v)) \tag{7.20}$$

$$v_{2,h}^0(x,y) = \arg \max_{0 \le v \le 1} \min_{0 \le u \le 1} \underline{w}_{2,h}^{m-1}(x + h(-x+u), y + h(-y+v)) \tag{7.21}$$

"Punishment feedback controls" $u_{2,h}^0(x,y)$, $v_{1,h}^0(x,y)$ are determined analogously as arguments u, v which realize external minimum in formulas (7.17),(7.8) when $n = m$

$$u_{2,h}^0(x,y) = \arg \min_{0 \le u \le 1} \max_{0 \le v \le 1} \overline{w}_{2,h}^{m-1}(x + h(-x+u), y + h(-y+v)) \tag{7.22}$$

$$v_{1,h}^0(x,y) = \arg \min_{0 \le v \le 1} \max_{0 \le u \le 1} \overline{w}_{1,h}^{m-1}(x + h(-x+u), y + h(-y+v)) \tag{7.23}$$

Let us remind that quadruple $u_{i,h}^0(x,y)$, $v_{j,h}^0(x,y)$ $i,j = 1,2$ forms one of Nash equilibria. Namely, "positive feedback controls" $u_{1,h}^0(x,y)$, $v_{2,h}^0(x,y)$ generate the trajectory "acceptable" for both populations. And "punishment feedback controls" $u_{2,h}^0(x,y)$, $v_{1,h}^0(x,y)$ are the instrument which forces populations to follow this trajectory.

7.5 Conjecture on the Structure of Optimal Synthesis

Taking into account that payoff functions $g_1(x,y)$, $g_2(x,y)$ are bilinear, their gradients $\partial g_k/\partial x$, $\partial g_k/\partial y$, $k = 1,2$ are linear and control parameters u, v are linearly presented in the dynamical system we can propose the following conjecture about the structure of optimal feedback controls $u_{i,h}^0(x,y)$, $v_{j,h}^0(x,y)$ $i,j = 1,2$.

Proposition 7.1 (Conjecture on the structure of optimal synthesis).
There is a curve in the square $[0,1]\times[0,1]$ *of phase states which passes through the point* $(\alpha_2/C_A, \alpha_1/C_A)$ *and divides the square into two parts. In one part the optimal feedback control is equal to zero and in another part - to unit.*

Moreover, in some neighborhood of such curve for "positive" control synthesis $u^0_{1,h}(x,y)$, $v^0_{2,h}(x,y)$ *corresponding approximation functions* $\underline{w}^m_{1,h}$, $\underline{w}^m_{2,h}$ *are concave and for "punishment" control synthesis* $u^0_{2,h}(x,y)$, $v^0_{1,h}(x,y)$ *corresponding approximation functions* $\overline{w}^m_{2,h}$, $\overline{w}^m_{1,h}$ *are convex.*

For the optimal control $u^0_{1,h}(x,y)$, *for example, such curve must be disposed in the domain* G

$$G = \{(x,y) \in [0,1]\times[0,1] : g_1(x,y) \geq V_A\}$$

and passes through the point $(\alpha_2/C_A, \alpha_1/C_A)$. *Here* $V_A = (a_4 C_A - \alpha_1\alpha_2)/C_A$ *is the value of the corresponding matrix game.*

8 Numerical Realization of Iterative Procedure for Construction of Value Functions and Synthesis of Controls

8.1 Grid Schemes for Construction of Value Functions

For numerical realization of iterative procedures (7.7),(7.8) and (7.16),(7.17) we use grid approximation for corresponding iterative functions which leads in fact to the grid scheme for solving Hamilton-Jacobi equation. Formally it is necessary to calculate formulas (7.7),(7.8) and (7.16),(7.17) at all points (x,y) of the square of phase states. In order to make this procedure finite we will fulfill these calculations only at nodes of the fixed grid given on the square. Let us assume that values of iterative functions $\underline{w}^n_{1,h}$, $\overline{w}^n_{1,h}$, $\underline{w}^n_{2,h}$, $\overline{w}^n_{2,h}$ determined at nodes of the grid are interpolated linearly to the whole square for the given triangulation Ω.

Let us give the description of the proposed numerical procedure. Let the following quantization steps be given:

h	be a quantization step of the time interval
Δx	be a quantization step of the square by variable x
Δy	be a quantization step of the square by variable y
Δp	be a quantization step of the segment $[0,1]$ of constraints for control parameter u
Δq	be a quantization step of the segment $[0,1]$ of constraints for control parameter v

We shall assume that there is linear relation between steps h, Δx, Δy, Δp, Δq, i.e. the following relations hold

$$\begin{aligned} \Delta x = K_x h, \quad \Delta y = K_y h \\ \Delta p = K_p h, \quad \Delta q = K_q h \end{aligned} \tag{8.1}$$

Next, let us suppose that values of iterative functions $w^{n-1} = \underline{w}^{n-1}_{i,h}$ or $w^{n-1} = \overline{w}^{n-1}_{i,h}$, $i = 1,2$ have been calculated already at nodes of the grid GR

$$GR = \{(x_i, y_j) \in [0,1]\times[0,1] : x_i = i\Delta x, \quad y_j = j\Delta y\} \tag{8.2}$$

which is given on the square of phase states and determined by quantization steps Δx, Δy.

Assume for definiteness that functions w^{n-1} are interpolated in the square of phase states according to the following triangulation. Consider triangulation Ω of the square $[0,1] \times [0,1]$ by simplexes of types S_+ and S_-

$$S_+ = co\{(x_i, y_j), (x_i + \Delta x, y_j), (x_i, y_j + \Delta y)\} \tag{8.3}$$

$$S_- = co\{(x_i, y_j), (x_i - \Delta x, y_j), (x_i, y_j - \Delta y)\} \tag{8.4}$$

It is clear that for any point $(x, y) \in [0,1] \times [0,1]$ there exists simplex T of type S_+ or S_- such that $(x, y) \in T$. It can be determined by the following relations. Let $(x, y) \in [0,1] \times [0,1]$ and

$$i = int(\frac{x}{\Delta x}), \quad j = int(\frac{y}{\Delta y})$$
$$x_i = i\Delta x, \quad y_j = j\Delta y$$

Two cases can appear.

Case 1. If

$$(x - x_i)\Delta y + (y - y_j)\Delta x \leq \Delta x \Delta y$$

then

$$(x, y) \in co\{(x_i, y_j), (x_i + \Delta x, y_j), (x_i, y_j + \Delta y)\} = T_1$$

$$(x, y) = \lambda_1(x_i, y_j) + \lambda_2(x_i + \Delta x, y_j) + \lambda_3(x_i, y_j + \Delta y)$$

$$\lambda_2 = \frac{x - x_i}{\Delta x} \geq 0$$

$$\lambda_3 = \frac{y - y_j}{\Delta y} \geq 0$$

$$\lambda_1 = 1 - \lambda_2 - \lambda_3 \geq 0$$

Here simplex T_1 is of the type S_+. The value of function w^{n-1} at point (x, y) is interpolated linearly

$$w^{n-1}(x, y) = \lambda_1 w^{n-1}(x_i, y_j) + \lambda_2 w^{n-1}(x_i + \Delta x, y_j) + \lambda_3 w^{n-1}(x_i, y_j + \Delta y) \tag{8.5}$$

Case 2. If

$$(x - x_i)\Delta y + (y - y_j)\Delta x > \Delta x \Delta y$$

then

$$(x, y) \in co\{(x_i + \Delta x, y_j + \Delta y), (x_i + \Delta x, y_j), (x_i, y_j + \Delta y)\} = T_2$$

$$(x, y) = \lambda_1(x_i + \Delta x, y_j + \Delta y) + \lambda_2(x_i + \Delta x, y_j) + \lambda_3(x_i, y_j + \Delta y)$$

$$\lambda_3 = \frac{(x_i + \Delta x) - x}{\Delta x} \geq 0$$

$$\lambda_2 = \frac{(y_j + \Delta y) - y}{\Delta y} \geq 0$$

$$\lambda_1 = 1 - \lambda_2 - \lambda_3 \geq 0$$

Here simplex T_2 is of the type S_-. The value of function w^{n-1} at point (x, y) is interpolated linearly

$$w^{n-1}(x,y) = \lambda_1 w^{n-1}(x_i + \Delta x, y_j + \Delta y) + \\ \lambda_2 w^{n-1}(x_i + \Delta x, y_j) + \lambda_3 w^{n-1}(x_i, y_j + \Delta y) \quad (8.6)$$

Let us give now formulas for calculating the following iterations w^n based on values of previous iterations w^{n-1} at nodes of the grid GR. Let $x_i = i\Delta x$, $y_j = j\Delta y$. For values of functions $\underline{w}^n_{1,h}$, $\overline{w}^n_{1,h}$, $\underline{w}^n_{2,h}$, $\overline{w}^n_{2,h}$ at nodes x_i, y_j we have the following relations

$$\underline{w}^n_{1,h}(x_i, y_j) = hg_1(x_i, y_j) + (1 - \lambda h) \max_k \min_l \underline{w}^{n-1}_{1,h}(x_k, y_l) \quad (8.7)$$

$$\overline{w}^n_{1,h}(x_i, y_j) = hg_1(x_i, y_j) + (1 - \lambda h) \min_l \max_k \overline{w}^{n-1}_{1,h}(x_k, y_l) \quad (8.8)$$

$$\underline{w}^n_{2,h}(x_i, y_j) = hg_2(x_i, y_j) + (1 - \lambda h) \max_l \min_k \underline{w}^{n-1}_{2,h}(x_k, y_l) \quad (8.9)$$

$$\overline{w}^n_{2,h}(x_i, y_j) = hg_2(x_i, y_j) + (1 - \lambda h) \min_k \max_l \overline{w}^{n-1}_{2,h}(x_k, y_l) \quad (8.10)$$

$$x_k = x_i + h(-x_i + k\Delta p)$$

$$y_l = y_j + h(-y_j + l\Delta q)$$

Without loss of generality of arguments we use here the same notations as we do before for corresponding approximations. Let us note that values of functions w^{n-1} in formulas (8.7)-(8.10) are calculated according to (8.5),(8.6). In addition, operations *max* and *min* in (8.7)-(8.10) are determined on finite sets of indexes k, l and, therefore, can be easily calculated.

As to convergence of numerical procedures (8.7)-(8.10) then we can prove here the following statement using results of papers [Souganidis, 1985], [Subbotin, Tarasyev, Ushakov, 1993], [Tarasyev, 1994], [Bardi, Osher, 1991].

Theorem 8.1 *Approximation grid schemes (8.7)-(8.10) converge to corresponding solutions w_1, w_2 of Hamilton-Jacobi equations (value functions of correspondig differential games) when $n \to \infty$, $h \downarrow 0$. The estimate of convergence has the order $h^{\frac{1}{2}}$.*

8.2 Grid Approximation of Control Synthesis

Let us remind that together with values of functions $\underline{w}^n_{1,h}$, $\overline{w}^n_{1,h}$, $\underline{w}^n_{2,h}$, $\overline{w}^n_{2,h}$ we calculate also approximations $u^n_1(x_i, y_j)$, $v^n_2(x_i, y_j)$ of "positive" feedback controls $u^0_1(x_i, y_j)$, $v^0_2(x_i, y_j)$ and aproximations $u^n_2(x_i, y_j)$, $v^n_1(x_i, y_j)$ of "punishment" feedback controls $u^0_2(x_i, y_j)$, $v^0_1(x_i, y_j)$ at nodes (x_i, y_j) of the grid GR. Namely, approximations $u^n_1(x_i, y_j)$, $v^n_2(x_i, y_j)$ are determined as arguments which realize external maximum in (8.7),(8.9)

$$u^n_1(x_i, y_j) = k^* \Delta p \\ k^* = k^*(n) = \arg\max_k \min_l \underline{w}^{n-1}_{1,h}(x_k, y_l) \quad (8.11)$$

$$v^n_2(x_i, y_j) = l^* \Delta q \\ l^* = l^*(n) = \arg\max_l \min_k \underline{w}^{n-1}_{2,h}(x_k, y_l) \quad (8.12)$$

Similarly, approximations $u_2^n(x_i, y_j)$, $v_1^n(x_i, y_j)$ are determined as arguments which realize external minimum in (8.10),(8.8)

$$u_2^n(x_i, y_j) = k_* \Delta p$$
$$k_* = k_*(n) = \arg \min_k \max_l \overline{w}_{2,h}^{n-1}(x_k, y_l) \tag{8.13}$$

$$v_1^n(x_i, y_j) = l_* \Delta q$$
$$l_* = l_*(n) = \arg \min_l \max_k \overline{w}_{1,h}^{n-1}(x_k, y_l) \tag{8.14}$$

$$x_k = x_i + h(-x_i + \Delta pk)$$
$$y_l = y_j + h(-y_j + \Delta ql)$$

Let us note that feedback controls (8.11)-(8.14) are determined only at nodes (x_i, y_j) of the grid GR. If we accept the conjecture on structure of optimal synthesis (see Proposition (7.1)) then we can interpolate linearly values of controls (8.11)-(8.14) to the whole square $[0, 1] \times [0, 1]$ of phase states. More precisely, the following proposition is valid.

Proposition 8.1 *Linear interpolations $u_1^n(x, y)$, $u_2^n(x, y)$, $v_1^n(x, y)$, $v_2^n(x, y)$ of values $u_1^n(x_i, y_j)$, $u_2^n(x_i, y_j)$, $v_1^n(x_i, y_j)$, $v_2^n(x_i, y_j)$ according to triangulation Ω (8.3),(8.4) of the square $[0, 1] \times [0, 1]$ can guarantee result on generated trajectories $(x(t), y(t)), t \in [0, +\infty)$ which is arbitrarily close to the value $w_1(x(0), y(0))$ or $w_2(x(0), y(0))$ of corresponding differential games.*

Remark 8.1 *If conjecture on structure of optimal synthesis postulated in Proposition (7.1) does not fulfill then we need high order quantization of phase variables in comparison with time step h*

$$\Delta x = K_x h^2, \quad \Delta y = K_y h^2$$

In this case piecewise constant interpolations $u_1^n(x, y)$, $u_2^n(x, y)$, $v_1^n(x, y)$, $v_2^n(x, y)$ of values $u_1^n(x_i, y_j)$, $u_2^n(x_i, y_j)$, $v_1^n(x_i, y_j)$, $v_2^n(x_i, y_j)$ ensure result on generated trajectories $(x(t), y(t)), t \in [0, +\infty)$ which is arbitrarily close to the value $w_1(x(0), y(0))$ or $w_2(x(0), y(0))$ of corresponding differential games.

9 Alliance of "Long-Term" and "Short-Term" Interests of Populations and Individuals

9.1 "Short-Term" Interests of Individuals and Constraints on Control Parameters in the Game Problem for "Long-Term" Interests of Populations

Let us remind that in the considered nonantagonistic (nonzero sum) game of two populations with dynamics (5.1) and payoff functionals J_1 (2.7) for the first population and J_2 (2.8) for the second population control parameters u, v are constrained by the segment $[0, 1]$. As it was mentioned above, extreme values of these parameters can be interpreted as control signals for populations: "to change" one behavioral action for another. In the general case these signals can contradict to "short-term" interests of individuals. Contradiction for the first population is interpreted here in the sense that the following relations take place simultaneously

$$u_1^0(x, y) = 0 \tag{9.1}$$

and

$$u_1^*(x,y) = \arg\max_{0\le u\le 1} \frac{\partial g_1}{\partial x} u = 1 \tag{9.2}$$

or vice versa

$$u_1^0(x,y) = 1 \tag{9.3}$$

and

$$u_1^*(x,y) = \arg\max_{0\le u\le 1} \frac{\partial g_1}{\partial x} u = 0 \tag{9.4}$$

Relations (9.1),(9.3) correspond to "long-term" interests of the first population and relations (9.2),(9.4) correspond to "short-term" interests of individuals. In other words the payoff function g_1 and the value function w_1 can have "gradients" (subgradients) of different signs at a point (x,y). "Gradient" for the value function is understood here in the generalized sense.

Contradiction for the second population is understood analogously.

The indicated contradiction between "long-term" interests determined by the value function and "short-term" interests determined by the payoff function can be overcome in the statement of the problem if we insert information about "short-term" interests of individuals into constraints on control parameters u, v. Of course, these new constraints have peculiarity: they depend on a phase state (x,y) of dynamical system (5.1). More precisely, contradictions between "long-term" and "short-term" interests of the first population is eliminated if control parameter u satisfies the following constraints

$$\begin{aligned} &x \le \phi_1(x,y) \le u \le \phi_2(x,y) \le 1 \quad \text{if} \quad \frac{\partial g_1}{\partial x} = C_A y - \alpha_1 \ge 0 \\ &0 \le \phi_3(x,y) \le u \le \phi_4(x,y) \le x \quad \text{if} \quad \frac{\partial g_1}{\partial x} = C_A y - \alpha_1 < 0 \end{aligned} \tag{9.5}$$

Restrictions on control parameter v can be written analogously

$$\begin{aligned} &y \le \psi_1(x,y) \le v \le \psi_2(x,y) \le 1 \quad \text{if} \quad \frac{\partial g_2}{\partial y} = C_B x - \beta_2 \ge 0 \\ &0 \le \psi_3(x,y) \le v \le \psi_4(x,y) \le 1 \quad \text{if} \quad \frac{\partial g_2}{\partial y} = C_B x - \beta_2 < 0 \end{aligned} \tag{9.6}$$

For example, one can take the following functions $\phi_i, \psi_i, i = 1,2,3,4$

$$\phi_1 = \phi_4 = x, \quad \phi_2 = 1, \quad \phi_3 = 0 \tag{9.7}$$

$$\psi_1 = \psi_4 = y, \quad \psi_2 = 1, \quad \psi_3 = 0 \tag{9.8}$$

9.2 Replicator Dynamics and Constraints on Control Parameters

The most interesting cases connected with classical models of evolutionary dynamics (see, for example, [Hofbauer, Sigmund, 1988]) are constraints in which the so-called replicator dynamics is presented

$$\phi_1 = \phi_4 = x + x(x-1)(C_A y - \alpha_1) \tag{9.9}$$

$$\psi_1 = \psi_4 = y + y(y-1)(C_B x - \beta_2) \tag{9.10}$$

For such constraints control parameters u, v switch dynamical system (5.1) from the optimal modes $u_1^0(x,y) = 0$ or $u_1^0(x,y) = 1$ and $v_2^0(x,y) = 0$ or $v_2^0(x,y) = 1$ (which don't cotradict to the "short-term" principle of optimality) to the replicator dynamics.

References

Krasovskii, N.N., Control of a Dynamical System, Moscow, Nauka, 1985 (in Russian)

Krasovskii, N.N., Subbotin A.I., Game-Theoretical Control Problems, NY, Berlin, Springer, 1988

Kleimenov A.F., Nonantagonistic Positional Differential Games, Ekaterinburg, Nauka, 1993 (in Russian)

Subbotin A.I., Generalization of the Basic Equation of the Differential Games Theory, Dokl. Akad. Nauk SSSR, 1980, Vol. 254, No. 2, PP. 293-297 (in Russian)

Subbotin A.I., Minimax Inequalities and Hamilton-Jacobi Equations, Moscow, Nauka, 1991 (in Russian)

Crandall M.G., Lions P.-L., Viscosity Solutions of Hamilton-Jacobi Equations, Trans. Amer. Math. Soc., 1983, Vol. 277, No. 4, PP. 1-42

Crandall M.G., Lions P.-L., Two Approximations of Solutions of Hamilton-Jacobi Equations, Math. Comput., 1984, Vol. 43, No. 167, PP. 1-19

Isaacs R., Differential games, John Wiley & Sons, Inc., New York, 1965

Vorobyev N.N., Games Theory, Moscow, Nauka, 1985 (in Russian)

Basar, T., Olsder, G.J., Dynamic Noncooperative Game Theory, London, NY, Acad. Press, 1982

Friedman, D., Evolutionary Games in Economics, Econometrica, 1991, Vol. 59, No. 3, PP. 637-666

Hofbauer, J., Sigmund K., The Theory of Evolution and Dynamic Systems, Cambridge, Cambridge Univ. Press, 1988

Young, P., The Evolution of Conventions, Econometrica, 1993, Vol. 61, No. 1, PP.57-84

Nelson, R., Winter, S., An Evolutionary of Economic Change, The Belknap Press of Harvard University Press, Cambridge, Massachusetts and London, England, 1982

Kryazhimskii, A.V., Behavioral Equilibria for a 2×2 "Seller-Buyer" Game-Evolutionary Model, IIASA Working Paper, Laxenburg, Austria, 1994, (to appear)

Subbotin, A.I., Tarasyev, A.M., Conjugate Derivatives of the Value Function of a Differential Game, Soviet Math. Dokl., 1985, Vol. 32, No. 2, PP. 162-166

Subbotin, A.I., Tarasyev, A.M., Ushakov, V.N., Generalized Characteristics of Hamilton-Jacobi Equations, Izv. Russian Acad. Sci, Techn. Cybernet., 1993, No. 1, PP. 190-197 (in Russian)

Souganidis, P.E., Approximation Schemes for Viscosity Solutions of Hamilton-Jacobi Equations, J. Different. Equat., 1985, Vol. 59, No. 1, PP. 1-43

Dolcetta, I.C., On a Discrete Approximation of the Hamilton-Jacobi Equation of Dynamic Programming, Appl. Math. Optimiz., 1983, Vol. 10, No. 4, PP. 367-377

Bardi, M., Osher, S., The Nonconvex Multi-Dimensional Riemann Problem for Hamilton-Jacobi Equations, SIAM J. Math. Annal., 1991, Vol. 22, No. 2, PP. 344-351

Adiatulina, R.A., Tarasyev, A.M., A Differential Game of Unlimited Duration, PMM USSR, 1987, Vol. 51, No. 4, PP. 415-420

Tarasyev A.M., Approximation construction of minimax solutions of Hamilton-Jacobi Equations, Prikl. Math. and Mech., 1994, Vol.58, No.2, PP. 22-36 (in Russian)

Appendix. Results of Numerical Experiments.

The algorithms of numerical construction of value functions w_1, w_2 and feedback controls u_i^0, v_j^0, $i,j = 1,2$ described in Sections 7, 8 were realized in computer programs (PASCAL programs) and corresponding illustrative results were produced by N.Mel'nikova. I would like to thank her for this great work.

For numerical simulations two basic combinations of matrixes A and B generating three or one static Nash equilibria in bimatrix games were considered. Let us remind that three Nash equilibria appear in the case of "one-type" interests of populations which are characterized by matrixes A and B. Broken lines (zigzags) of acceptable situations are differently oriented (right and left zigzagz) and have three points of intersection - Nash equilibria. The situation of one Nash equilibrium arises for "almost antagonistic" interests of populations. Broken lines (zigzags) of acceptable situations generated by matrixes A and B have the same orientation (both zigzags are right or left).

For the first case with three Nash equilibria the following payoff matrixes A and B of "one-type" interests of populations were be taken

$$A = C1 = \begin{pmatrix} 11 & 2 \\ 3 & 6 \end{pmatrix}$$

$$C_A = a_{11} - a_{12} - a_{21} + a_{22} = 12$$

$$\alpha_1 = a_{22} - a_{12} = 4$$

$$\alpha_2 = a_{22} - a_{21} = 3$$

$$B = C2 = \begin{pmatrix} 3 & 1 \\ 0 & 4 \end{pmatrix}$$

$$C_B = b_{11} - b_{12} - b_{21} + b_{22} = 6$$

$$\beta_1 = b_{22} - b_{12} = 3$$

$$\beta_2 = b_{22} - b_{21} = 4$$

There exist the following saddle points $SP1 = (1/4, 1/3)$ (for matrix A), $SP2 = (2/3, 1/2)$ (for matrix B) in corresponding matrix games. In the bimatrix game with matrixes $A = C1$ and $B = C2$ there are three Nash equilibria $NE1 = (0,0)$, $NE2 = (1,1)$, $NE3 = (2/3, 1/3)$.

As a result of numerical realization of algorithms described in Sections 7, 8 the following value functions and feedback controls for the dynamical game with matrixes $A = C1$ and $B = C2$ were constructed. "Positive" feedback controls u_1^0, v_2^0 have the form represented on Figures 1, 3. "Punishment" feedback controls u_2^0, v_1^0 are depicted on Figures 2, 4. In the shaded domains of these Figures controls u and v are equal to unit and in the unshaded domains they are equal to zero. On Figures 14, 15 graphs of the corresponding value functions w_1, w_2 are given.

The "acceptable" trajectories (TR) $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ generated by "positive" feedback controls u_1^0, v_2^0 from different initial positions (IP) (x_0, y_0) are shown on Figures 7-10. These trajectories consist of the pieces of characteristics corresponding to Hamilton-Jacobi equations. In this problem characteristics are straigt lines directed to different corners of the unit square. Switching from one characteristic to another takes place when trajectory crosses switch lines (curves) $SW1$, $SW2$ generated by "positive" feedback controls u_1^0, v_2^0. For the considered initial positions (x_0, y_0) "acceptable" trajectories $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ come to the corner Nash equilibria $NE1$ or $NE2$.

In the second case with one Nash equilibrium the following payoff matrixes A and B realizing "almost antagonistic" interests of populations were considered

$$A = C1 = \begin{pmatrix} 11 & 2 \\ 3 & 6 \end{pmatrix}$$

$$B = C3 = \begin{pmatrix} 2 & 4 \\ 5 & 1 \end{pmatrix}$$

$$C_B = b_{11} - b_{12} - b_{21} + b_{22} = -6$$

$$\beta_1 = b_{22} - b_{12} = -3$$

$$\beta_2 = b_{22} - b_{21} = -4$$

The antagonistic (zero-sum) games have the same saddle points as in the previous case $SP1 = (1/4, 1/3)$, $SP2 = (2/3, 1/2)$. But there is only one Nash equilibrium $NE = (2/3, 1/3)$ in the corresponding bimatrix game.

The following numerical results were obtained for the differential game of populations with payoff matrixes $A = C1$, $B = C3$. The structure of "positive" feedback controls u_1^0, v_2^0 is represented on Figures 1, 5. "Punishment" feedback controls u_2^0, v_1^0 are given on Figures 2, 6. Controls u, v are equal to unit in the shaded domains and they have zero values otherwise. On Figures 14, 16 graphs of the value functions w_1, w_2 corresponding to payoff matrixes $A = C1$, $B = C3$ are depicted. The "acceptable" trajectories $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ generated by "positive" feedback controls u_1^0, v_2^0 from different initial positions (IP) (x_0, y_0) are shown on Figures 11-13. These trajectories as in the previous case are piecewise linear and consist of pieces of characteristics directed to the corners of the square. Switching from one characteristic to another happens at points of intersection of the "acceptable" trajectory with switch lines $SW1$, $SW2$ of feedback controls u_1^0, v_2^0.

Let us note the very remarkable result which appears for "acceptable" trajectories in the considered examples depicted on Figures 11-13. The "acceptable" trajectories don't converge to the static Nash equilibrium NE (or don't circulate in a neighborhood of this point NE) but tend to the point of intersection of switch lines $SW1$, $SW2$. Moreover, the values of "long-term" payoffs J_1, J_2 calculated on the "acceptable" trajectory $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ are better than the values of J_1, J_2 calculated on trajectories $(x(\cdot), y(\cdot))$ which start from the same initial position (x_0, y_0) but converge to the static Nash equilibrium NE.

In conclusion let us consider the question on dependence of obtained solutions on a discount coefficient λ (remind that this coefficient is one of basic parameters for the examined problem). The last Figure (Figure 17) illustrates weak dependence of switch lines on variation of the discount coefficient λ. Namely, on Figure 17 we give switch lines of "positive" feedback controls u_1^0 for the payoff matrix $A = C1$ calculated for discount coefficients: $\lambda = DC1 = 1$, $\lambda = DC2 = 0.1$. One can see that these curves differ very slightly from each other although the ratio of $DC1$ to $DC2$ is comparetively large $DC1/DC2 = 10$.