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ABSTRACT

A continuous time model is proposed to describe the dynamics of drug consumption in a given country. The model has two state variables, addicts and dealers, and eleven parameters, including the total effort exerted by the State, which is considered as control parameter. The model is highly nonlinear and the analysis shows that it is characterized by a transcritical and a fold bifurcation. This implies that for intermediate values of the State's effort the model has two stable equilibria, one trivial, corresponding to the absence of drugs, and one positive, corresponding to drug consumption. On the contrary, for low and high values of the effort only one of the two equilibria is stable. This suggests a two-step control policy. First, exert a very high effort for a few years, so that the system has the time to approach the trivial equilibrium, and then reduce the effort but maintain it sufficiently high so that drug consumption can not rise anymore. Interesting results on the role played by the price of the drug and the severity of the punishment inflicted to dealers, as well as on the allocation of the effort between therapy and police, have also been obtained.

DYNAMICS OF DRUG CONSUMPTION: A THEORETICAL MODEL

A. Gragnani, G. Feichtinger, and S. Rinaldi

1. INTRODUCTION

Quantitative approaches to the problem of drug use are based on optimization models that try to balance the cost of the effort exerted by the State in different forms (mainly therapies and police) with the various social costs associated with drug consumption. The interested reader can refer to a recent special issue of *Mathematical and Computer Modelling* (J. P. Caulkins, 1993) dedicated to drug markets and drug policy. All these models assume, in one way or another, that the problem is static or, in other words, that the system is at equilibrium and that this equilibrium is unique.

On the contrary, this paper shows that a simple model, encapsulating the main characteristics of the dynamics of drug consumption, has two stable equilibria and that catastrophic transitions from one to another of these equilibria are possible if the effort is varied. This suggests a two-step shock cure to solve the drug problem: first, apply a massive pressure on dealers, thus driving the system to the low consumption equilibrium, and second reduce the enforcement effort but still maintain it at sufficiently high values so that drug consumption can not rise anymore. Although very simple, this analysis seems to be completely new, with one relevant exception (Kleiman, 1993), where the existence of two stable equilibria, associated to two drastically different levels of drug consumption, are conjectured through some game theoretic argument.

In this paper, the enforcement effort is considered a constant parameter. Obviously, the assumption could be relaxed, and a time varying effort could be determined by solving a formal optimal control problem. Alternatively, the effort could be assumed to obey a differential equation interpreting the policy that the State follows reacting to variations of drug consumption. Although both of these approaches deserve attention, the analysis of the constant effort case is already of great interest, particularly in view of the simplicity of the results it implies.

The paper is organized as follows: in Section 2 the model is described; it is a continuous time dynamical model with addicts and dealers as state variables and eleven parameters, among which is the enforcement effort exerted by the State. Then, in Section 3, the stability of the origin (zero drug consumption) is analyzed, and the result is that for reasonable values of the parameters and zero effort the origin is unstable, i.e., the "virus" of drug has the tendency to spread. In Section 4, the analysis of the positive equilibria and their stability is carried out, and the conclusion is that the system can have hysteretic behavior with respect to the control parameter. The influence of the allocation of the effort between therapies and police is then discussed and, finally, in Section 6, the consequences of the analysis are discussed in policy terms.

2. A MODEL WITH CONSTANT EFFORT

In order to describe the dynamics of drug consumption in a given population or community, we use a second order continuous time model where the two variables x, and y represent, respectively, addicts and dealers, i.e.,

$x(t)$ = addicts (fraction of total population) at time t

$y(t)$ = dealers (fraction of total population) at time t

The model has 11 parameters, which are assumed to be constant in time. Among them, there is a control parameter, indicated by z, which represents the enforcement

$$R_s(x,y) = bx(1-x)^2 \frac{y}{h+y} \tag{2}$$

where b is a constant parameter. This can be summarized by writing the *addicts equation*

$$\dot{x} = R_e(x,y) + R_s(x,y) - r\mu zx - dx \tag{3}$$

where r and d are recovery rate and death rate.

The second equation of the model (dealers equation) is also a mass balance equation, taking into account

- recruitment rate (divided into free and forced, as discussed below)
- retirement rate (proportional to y)
- death rate (proportional to y)
- arrest rate (proportional to $(1-\mu)z$ and to the probability $y/(h+y)$ of finding a dealer in a unit time)

The *free recruitment* is the consequence of the attractiveness of a job with high revenue. It is, therefore, proportional to the total consumption of drugs, namely $xy/(h+y)$. On the contrary, the *forced recruitment* is composed of addicted people who, shortly after they start using drugs, are, somehow, obliged to become dealers in order to find their drugs. Therefore, it is natural to assume that this recruitment is a fraction f of the recruitment of addicts. In conclusion, the *dealers equation* is

$$\dot{y} = c\frac{xy}{h+y} + f\,R_e(x,y) + f\,R_s(x,y) - sy - dy - e(1-\mu)z\frac{y}{h+y} \tag{4}$$

where $R_e(x,y)$ and $R_s(x,y)$ are specified by (1) and (2), c is a coefficient increasing with the price of the drug and decreasing with the severity of the punishment inflicted to dealers, s is retirement rate and e efficiency of police.

exerted by the authorities. Such a term is divided into two parts μz and $(1-\mu)z$ corresponding to actions in two different directions: recovery of addicts through suitable therapies and search, arrest, and prosecution of dealers.

The first state equation (addicts equation) is a mass balance equation taking into account

- recruitment rate (divided into endogenous and social, as described below)
- recovery rate (proportional to μzx)
- death rate (proportional to x)

The recruit of an addict is assumed to be possible in two independent ways. The first, called *endogenous*, is due to strictly personal reasons. A non-addict becomes an addict first, by deciding to use drug and then, by being successful in finding a dealer. Since individuals of a given society are in different social strata, with respect to their attitudes toward the use of drugs, it is natural to assume that the probability that a non-addict decides to use drugs for personal reasons is decreasing with the number of addicts. The simplest functional representation of this probability is (1-x). At the same time, the probability of finding a dealer is an increasing and concave function of the number of dealers. The classical Monod function y/(h+y), where h is the half-saturation constant (number of dealers at which the above probability is 0.5), will be used in this paper. Thus, taking into account that the number of potential victims is (1-x), the endogenous recruitment rate $R_e(x,y)$ is

$$R_e(x,y) = a(1-x)^2 \frac{y}{h+y} \qquad (1)$$

where a is a constant parameter characterizing the population.

The second type of recruitment, called *social*, starts with the encounter of a non-addict with an addict (or a group of them) and then follows the same path described for endogenous recruitment. Thus, if the probability of encounters is proportional to the addicted population, the social recruitment rate $R_s(x,y)$ is

Eqs. (3) and (4), taking into account (1) and (2), become

$$\dot{x} = x\left[b(1-x)^2 \frac{y}{h+y} - d - r\mu z \right] + a(1-x)^2 \frac{y}{h+y} \qquad (5)$$

$$\dot{y} = y\left[\frac{1}{h+y}\left(cx + f(a+bx)(1-x)^2 - e(1-\mu)z\right) - d - s \right] \qquad (6)$$

which is a second order dynamical system with eleven parameters. Four of these parameters could be eliminated by rescaling x, y, z and time. This is not done here because the interpretation of the rescaled parameters would be definitely difficult, while the advantage of having seven parameters instead of eleven would be minor. Later in this paper, the role of z and μ which are, somehow, the natural control parameters (they have, in general, to do with the long term planning of the Ministry of Health and Security) will be discussed. The other parameters will be fixed at the following reference values (1 unit of time = 1 year):

$a = 5 \cdot 10^{-4}$	$b = 5 \cdot 10^{-3}$	$c = 5 \cdot 10^{-3}$	$d = 1 \cdot 10^{-2}$	
$e = 4 \cdot 10^{-1}$	$f = 1 \cdot 10^{-1}$	$h = 5 \cdot 10^{-4}$	$r = 1$	$s = 5 \cdot 10^{-2}$

These values are reasonable but somehow naïve estimates of the parameters characterizing European and North American countries. Their determination is not described in the present paper where no specific population is simulated, but, rather, the general consequences of the nonlinearities of the model are explored.

3. WHY UNCONTROLLED POPULATIONS USE DRUGS

The point x=0, y=0 of the state space is an equilibrium of model (5), (6), corresponding to an ideal population with no addicts and no dealers. If such an equilibrium is unstable, the injection of a few dealers and of a few addicts will give rise to a sharp increase of addicts and dealers and, therefore, of drug consumption. On the contrary, if the equilibrium (0,0) is stable the injection will have no consequence, because the perturbation will be absorbed, and drug consumption will tend to zero.

The discussion of the stability of the origin of model (5), (6) is very simple. In fact, close to the origin, the system is described by

$$\dot{x} = -(d + r\mu z)x + \frac{a}{h}y$$

$$\dot{y} = \left(-d - s - \frac{e(1-\mu)z}{h} + \frac{fa}{h}\right)y$$

which is a system in triangular form, with eigenvalues

$$\lambda_1 = -(d + r\mu z) \qquad\qquad \lambda_2 = -\left(d + s + \frac{e(1-\mu)z}{h}\right) + \frac{fa}{h}$$

The first eigenvalue is always negative, while the second is positive for sufficiently low values of z, provided

$$\frac{fa}{h} > d + s \tag{7}$$

Condition (7) is satisfied by the reference parameter setting ($fa/h=10^{-1}$, $d+s=6\cdot10^{-2}$). This means that in the absence of enforcement effort ($z=0$) the population will start to use drugs as soon as the "virus" of drugs is injected. But λ_2 decreases linearly with z and for

$$z > \frac{fa - (d+s)h}{e(1-\mu)} \tag{8}$$

λ_2 is negative so that the origin is stable. Under these conditions the consumption of drugs will decrease as time goes on, provided addicts and dealers initially present in the population are not too numerous. The minimum value of z that guarantees the stability of the origin is therefore

$$z_{min} = \frac{fa - (d+s)h}{e(1-\mu)} \tag{9}$$

It is interesting to note that z_{min} does not depend upon all parameters and decreases with the efficiency of the police. Moreover, when the control budget is fully allocated to police (i.e., when μ=0), the lowest value of z_{min} is obtained. In this case, $z_{min}=5 \cdot 10^{-5}$ for μ=0, which means that 1 person out of every 20000 should work for the Police Drug Department in order to guarantee that the use of drugs would not have the chance to spread in a country where drugs are not yet used.

This analysis can be summarized by saying that if the parameters of a population are roughly those proposed as a reference setting, the population will start using drugs if it is not controlled (z=0) or if it is too weakly controlled ($z<z_{min}$). On the contrary, if the population is strongly controlled ($z>z_{min}$) the "virus" of drug cannot spread.

4. MULTIPLICITY OF EQUILIBRIA AND HYSTERESIS

When the trivial equilibrium x=y=0 is unstable, all trajectories tend toward a positive stable equilibrium (x^+, y^+), as shown in Fig. 1a. This statement cannot be formally proved, but it is numerically verified for the reference parameter setting and for slightly perturbed values of the parameters. When z increases and reaches the critical value z_{min} given by (9), the trivial equilibrium becomes stable and undergoes a transcritical bifurcation, because one of the two eigenvalues changes sign (Guckenheimer and Holmes, 1983). This means that for a slightly bigger value of z there is a second equilibrium (x^*, y^*) in the positive quadrant. This equilibrium is a saddle and its inset is the boundary of the basins of attraction of the two stable equilibria, namely (0,0) and (x^+, y^+), as shown in Fig. 1b. For increasing values of z the two equilibria (x^*, y^*) and (x^+, y^+) approach and finally collide and disappear at a critical value of z, indicated by z_{max}. In other words, the system has a fold bifurcation at z_{max} and for $z>z_{max}$ there is only one equilibrium, namely the origin, which is globally stable. The conclusion is that for low ($z<z_{min}$) and for high ($z>z_{max}$) values of the effort there is only one stable equilibrium, while for intermediate values of z, i.e., for $z_{min}<z<z_{max}$ there are two stable equilibria. The graph of one of the two components of the equilibrium (or a function of both of them) versus the control parameter is known as hysteresis. In Fig. 2 the hystereses of the addicts, of the dealers y, and of the total consumption of drugs (proportional to xy/(h+y)) are shown for $\mu=0.5$. These graphs suggest the use of a shock cure to eliminate drugs: first apply a high enforcement pressure, $z>z_{max}$, so that addicts and dealers drastically decrease, and then, when drug consumption is at low levels, reduce the effort but still keep it at sufficiently high values, $z>z_{min}$, so that drug consumption can not raise again.

Of course, the hysteresis that has been detected depends upon many, if not all, parameters of the model. Eq. (9) shows the dependence of z_{min} upon parameters. A similar expression for z_{max} cannot be found, because the fold bifurcation characterizing the upper catastrophe of the hysteresis can not be discussed analytically. Nevertheless, with suitable packages for bifurcation analysis of dynamical systems, it is possible to produce graphs showing the variation of z_{max} with respect to any parameter. Fig. 3 shows one of these graphs obtained with LOCBIF, a program implementing a powerful continuation technique for bifurcation analysis (Khibnik et al., 1993). The figure shows that z_{max} increases with c, namely with the price of the drug. On the contrary, z_{min} is not influenced by c (see (9)). This means that the cost of the shock cure, obviously determined by the maximum effort, would be lower if the application of the high enforcement pressure could be combined with a relevant reduction of the price of drugs and with a strengthening of the punishment inflicted to dealers.

5. ALLOCATION OF THE EFFORT

The parameter μ, specifying the allocation of the effort between therapy and police, is also a control parameter. Therefore, it is of interest to know if there is a rational way of selecting this parameter. It has already been shown (see (9)) that in order to have the lowest possible value of z_{min}, μ must be fixed at zero. In other words, in order to minimize the cost of enforcement, while keeping drug consumption at low levels, the budget should be fully allocated to police. This is quite obvious, since there is no need for therapists in a population with no addicts. On the contrary, the dependence of z_{max} upon μ is more complex, as shown in Fig. 4 for two different parameter settings. In Fig. 4a the value of μ that minimizes z_{max} is μ=0 (100% of the budget to police) while in Fig. 4b, corresponding to a higher efficiency r of the therapies and to a lower efficiency e of the police, the minimum value is μ=0.43 (57% of the budget to police). This means that the tuning of the allocation of the effort between therapy and police can be an important problem in order to minimize the cost of the shock cure.

6. CONCLUSIONS AND SUGGESTIONS

A second order continuous time model of drug consumption has been proposed in this paper. It is a highly nonlinear model with two stable equilibria in a whole range $[z_{min}, z_{max}]$ of the enforcement effort z. One equilibrium is associated with high level drug consumption, while the other to low level (virtually none). This is very similar to what has been recently argued by Kleiman (1993) on the basis of totally different arguments and without any formal stability analysis.

The most interesting consequence of the analysis is the identification of a shock cure policy (see Kleiman, 1993) which could, in principle, solve the problem of drug consumption. First, at time t', the pressure on the dealers should be increased and kept very high, i.e. $z=z'>z_{max}$ for a sufficiently long time. During this period, the system has only one stable equilibrium, the low level one, so that dealers, addicts and drug consumption gradually decrease (see Fig. 5). Then, at time t'' when the system is sufficiently close to the low equilibrium, the enforcement effort (and therefore the cost) can be reduced to a value $z>z_{min}$ and then kept constant. Of course, the switch should be performed at the right time, namely, when the state of the system is already in the basin of attraction of the low level equilibrium. Otherwise, if the switch is performed too early, the system tends toward the high level equilibrium $x^{+}(z)$ as shown in Fig. 5. The analysis carried out in the paper also indicates that, in order to have more chances of success, the application of the high enforcement pressure should be combined with a policy of reinforcement of the punishment inflicted to dealers and with a policy of free drugs. In fact, these are the ideal conditions for quickly driving the system into the basin of attraction of the low level equilibrium, without using paramount efforts.

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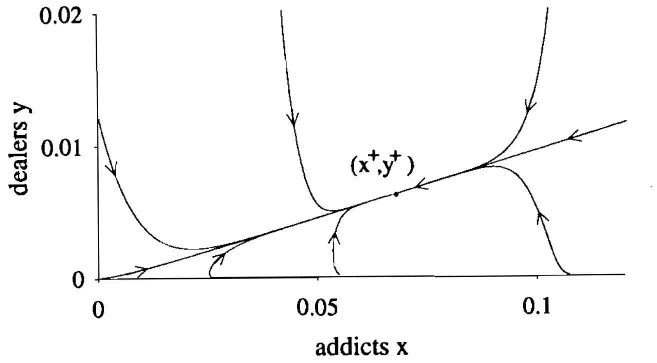
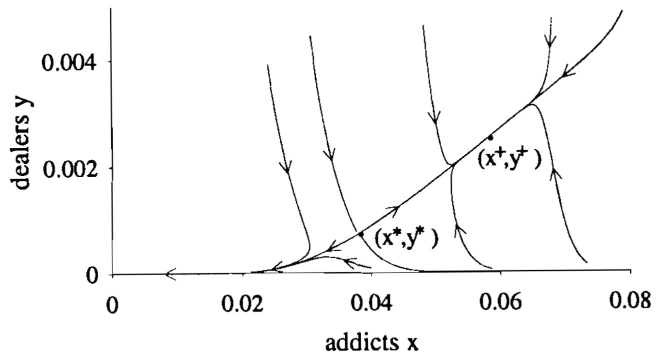


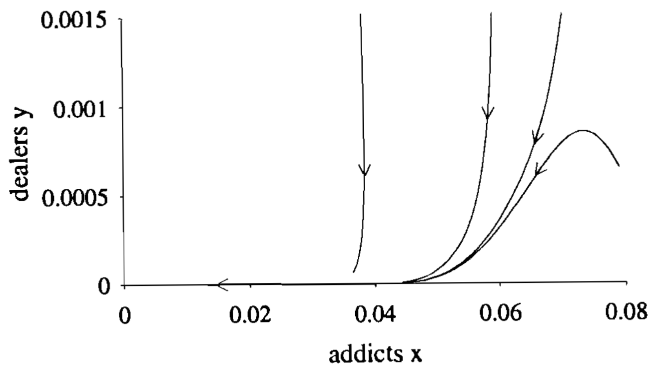
$z < z_{min}$

(a)



$z_{min} < z < z_{max}$

(b)



$z > z_{max}$

(c)

Fig. 1 Phase portraits of system (5), (6) for the reference parameter values, μ=0.1 and for three different values of the control parameter z: (a) $z=10^{-5}$; (b) $z = 5 \cdot 10^{-4}$; (c) $z=10^{-3}$.

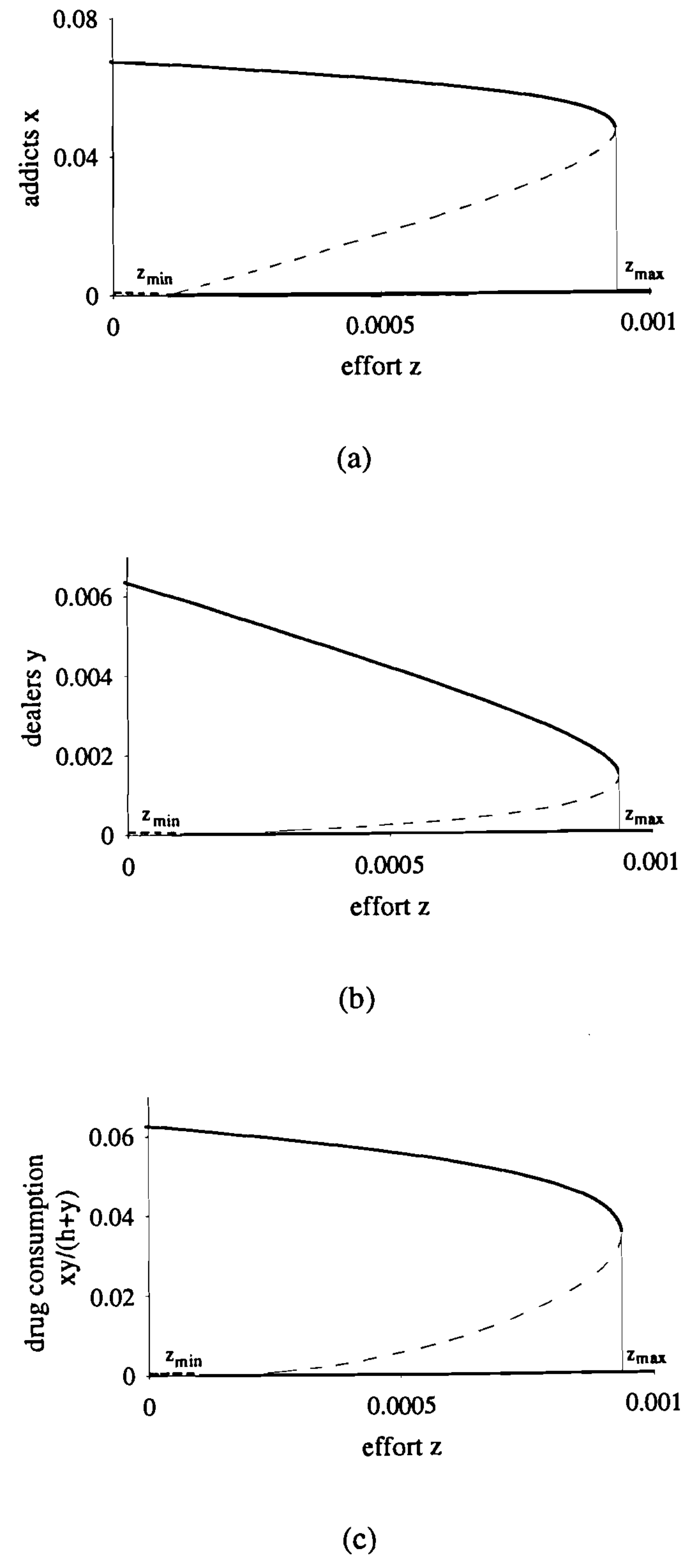


Fig. 2 Hysteresis of addicts (a), dealers (b), and drug consumption (c) for μ=0.5. Solid and dashed lines represent, respectively, stable and unstable equilibria.

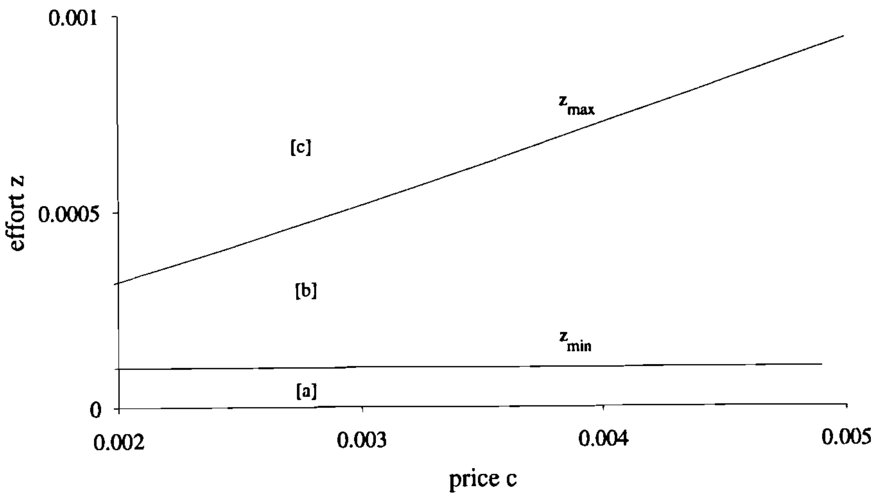
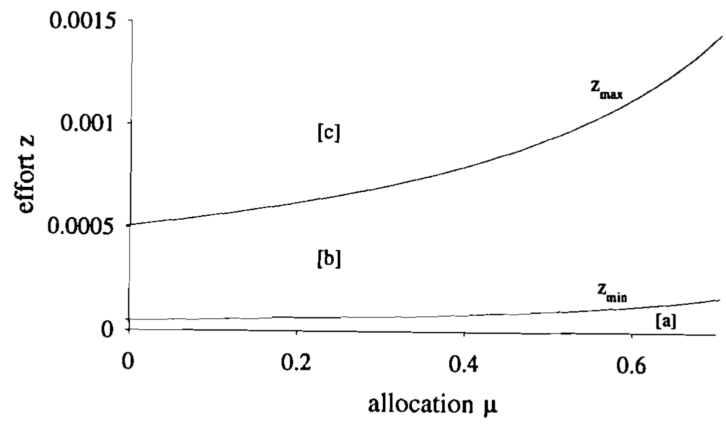
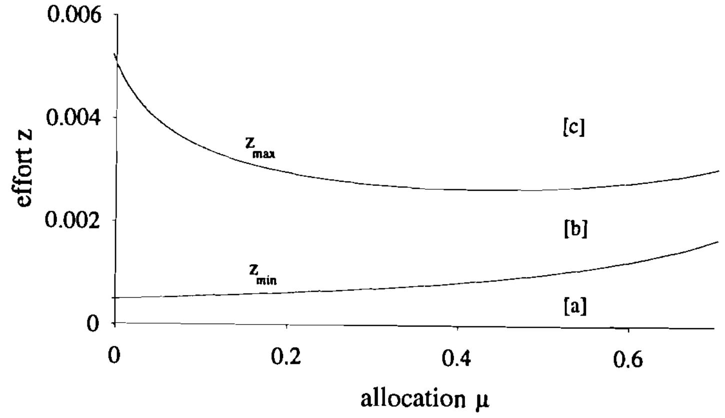


Fig. 3 Dependence of z_{min} and z_{max} upon the cost c of the drug. In the three regions [a], [b], and [c] the phase portraits of the system are like the corresponding plots of Fig. 1. The figure has been obtained for the reference parameter values and for μ=0.5.



(a)



(b)

Fig. 4 Dependence of z_{min} and z_{max} upon the allocation parameter μ. Case (a) corresponds to the reference parameter setting, while case (b) corresponds to $r=10$ and $e=4 \cdot 10^{-2}$ while all other parameters are at the reference values. In the regions [a], [b], and [c] the phase portraits of the system are like those in the corresponding plots of Fig. 1.

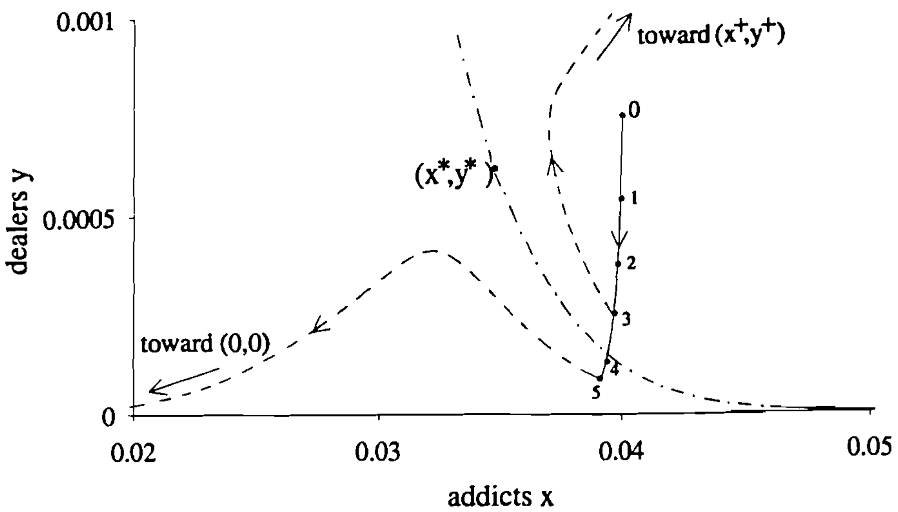


Fig. 5 Evolution of addicts (x) and dealers (y) corresponding to a shock cure policy. The initial point 0 is in the basin of attraction of (x^{+},y^{+}) for $z=6\cdot10^{-4}$ (the boundary of such a basin (dashed-dotted line) is the stable manifold of the saddle (x^{*},y^{*})). The solid line is a trajectory with constant effort $z'=2\cdot10^{-3}$. The dashed lines are trajectories with constant effort $z=6\cdot10^{-4}$. Figures along trajectories indicate time (in years). Parameters are at their reference values and $\mu=0.3$.