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Foreword

In many linear programming problems, we encounter the feature that not all coefficients are known. Frequently, the lack of knowledge of one coefficient is related to the lack of knowledge of other coefficients. This might be modelled by using coefficients which depend on a set of (unknown) parameters.

Such a situation occurs in a global environmental framework model as developed at NIES in Tsukuba, Japan. The question was how to determine for which parameter the lack of knowledge is most detrimental. In order to answer this question, a sensitivity analysis has been developed with respect to the parameters. The present paper explains this sensitivity analysis.

Abstract

When a real world problem is formulated as a linear programming model, we are often faced with difficulties in the parameter specification. We might know the plausible values or the possible ranges of parameters, but there still remains uncertainty. The parameter values could be obtained more exactly by experiments, investigations and/or inspections. However, to make such an experiment, investigation or inspection, expenses would be necessary. Because of capital limitations, we cannot invest in all possible experiments, investigations and inspections. Thus, we have a selection problem, which uncertainty reduction is the most profitable.

In this paper, we discuss an analytic approach to the problem. Because of the difficulty of the global analysis, we make a local analysis around appropriate values of parameters. We focus on giving the decision maker useful information for the selection. First, sensitivity analyses with respect to the uncertain parameters are developed. The sensitivities are available only for the marginal domain without changing the optimal basis. The domain is obtained as an interval. The difficulty of the sensitivity analysis is in the cases of degeneracy and multiplicity of the optimal solutions. A treatment of such difficult cases is proposed. Finally, a numerical example is given for illustrating the proposed approach.

Keywords: Linear programming, sensitivity analysis, simplex method

Contents

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1 Introduction

The linear programming problem is most widely used as model of various real world programming problems because of its computational and analytical tractability. It is not utilized only for making a good decision but also for analyzing the effects of the constraint parameters. However, in many cases, the coefficient parameters are not known exactly even when the structural model can be build as a linear programming problem.

The analysis in this paper has been inspired by such a model. In this case the model is so-called "global environmental framework model" and has been developed by Dr. T. Morita (National Institute for Environmental Studies of Japan, Tsukuba) and Professor Y. Nakamori (Konan University, Kobe). The problem is in this case indeed that several coefficients are not precisely known. If we consider the question of determining the coefficients more precisely, it appears that these coefficients are not independent. A sensible model for describing the imprecision would be to consider the coefficients as known functions of a set of parameters and supposing the parameter values to be imprecisely known. Now the problem becomes: for which parameter would it be most important to obtain a better estimate of its value.

In the present paper, we will develop some tools for treating this problem. In a later paper the use of these tools will be demonstrated.

Sensitivity analysis methods have been proposed for the investigation of effects on the optimal value or solution caused by a change of some parameters. However, in the conventional sensitivity analysis, it is assumed that a parameter changes either objective coefficients or the right-hand side values of constraints (see, for example, Dantzig [1]). Under this assumption, it is not applicable for the above model, since an uncertain parameter changes constraint coefficients as well as objective coefficients in the global environmental framework model.

As a first stage of the investigation of the global environmental framework model, this paper develops an extended sensitivity analysis to a parameter change involved in the constraint and objective coefficients, simultaneously. In Section 2, a linear programming problem with uncertain parameters is described in the mathematical form. Some assumptions are given for the tractability of the mathematical analysis. In Section 3, the sensitivities are represented in the form of matrix calculations. It is emphasized that the sensitivities at the plausible values can be calculated under a weaker assumption than those given in Section 2, i.e., the first order differentiability. The sensitivity is available in a certain domain where the basic variables do not change. The domain is analyzed in Section 4. The most serious difficulty of the sensitivity analysis is in the cases of degeneracy and multiplicity of the optimal solutions. Section 5 is devoted to these difficult cases. A simple numerical example is given to illustrate the proposed approach in Section 6.

2 Linear Programming with Uncertain Parameters

In this paper, the following linear programming problem is treated:

$$\begin{array}{ll} \text{minimize} & \boldsymbol{c}(\boldsymbol{r})\boldsymbol{x}, \\ \text{subject to} & A(\boldsymbol{r})\boldsymbol{x} = \boldsymbol{b}(\boldsymbol{r}), \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{array} \tag{1}$$

where $\boldsymbol{r} = (r_1, r_2, \ldots, r_p)$ is a vector of uncertain parameters. $\boldsymbol{c}(\boldsymbol{r})$, $\boldsymbol{x}$ are n-dimensional vectors and $\boldsymbol{b}(\boldsymbol{r})$ is an m-dimensional vector. $A(\boldsymbol{r})$ is an $m \times n$ matrix. A plausible value of $\boldsymbol{r}$ is supposed to be known as $\boldsymbol{r}^0 = (r_1^0, r_2^0, \ldots, r_p^0)$. Moreover, we introduce the following four assumptions for the sake of analytical tractability:

A1. $\boldsymbol{c}(\boldsymbol{r})$, $A(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$ are linear with respect to each parameter r_i for $i = 1, 2, \ldots, p$. To put it differently, the partial derivatives of $\boldsymbol{c}(\boldsymbol{r})$, $A(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$, i.e., $\partial \boldsymbol{b}(\boldsymbol{r})/\partial r_i$, $\partial A(\boldsymbol{r})/\partial r_i$ and $\partial \boldsymbol{b}(\boldsymbol{r})/\partial r_i$ do not depend on r_i.

A2. Each parameter r_i is included only in either one row or one column of the matrix $A(\boldsymbol{r})$.

A3. For any m-dimensional vector $\boldsymbol{\mu}$, there exists an $\epsilon > 0$ such that $\{\boldsymbol{x} \mid A(\boldsymbol{r}^0)\boldsymbol{x} = \boldsymbol{b}(\boldsymbol{r}^0) + \epsilon\boldsymbol{\mu},\ \boldsymbol{x} \geq \boldsymbol{0}\}$ is non-empty and bounded. As will be seen in Section 5, this assumption is required to obtain all optimal bases in the cases of degeneracy and multiplicity of optimal solutions.

Since we will discuss a sensitivity analysis to each parameter r_i, $i = 1, 2, \ldots, p$ around the plausible value $\boldsymbol{r}^0$, we suppose that substituting $\boldsymbol{r} = \boldsymbol{r}^0$, the linear programming problem (1) has been solved by the simplex method. Then, an optimal solution is obtained together with the associated optimal basis. In what follows, we will use the following notations:

- $B(\boldsymbol{r})$: an optimal basic matrix of the problem (1) for $\boldsymbol{r}$
- $B^{-1}(\boldsymbol{r})$: the inverse matrix of $B(\boldsymbol{r})$
- $A^{\mathrm{N}}(\boldsymbol{r})$: a sub-matrix of $A(\boldsymbol{r})$ which is composed of all columns associated with non-basic variables
- $\boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r})$: a sub-vector of $\boldsymbol{c}(\boldsymbol{r})$ which is composed of all coefficients associated with basic variables
- $\boldsymbol{c}^{\mathrm{N}}(\boldsymbol{r})$: a sub-vector of $\boldsymbol{c}(\boldsymbol{r})$ which is composed of all coefficients associated with non-basic variables
- $\boldsymbol{\pi}(\boldsymbol{r})$: a simplex multiplier for $\boldsymbol{r}$
- $\bar{\boldsymbol{b}}(\boldsymbol{r})$: an optimal basic solution for $\boldsymbol{r}$
- $\bar{z}(\boldsymbol{r})$: the optimal value for $\boldsymbol{r}$.

From the theory of the simplex method, we have

$$\boldsymbol{\pi}(\boldsymbol{r}) = \boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r})B^{-1}(\boldsymbol{r}), \tag{2}$$

$$\bar{\boldsymbol{b}}(\boldsymbol{r}) = B^{-1}(\boldsymbol{r})\boldsymbol{b}(\boldsymbol{r}), \tag{3}$$

$$\bar{z}(\boldsymbol{r}) = \boldsymbol{\pi}(\boldsymbol{r})\boldsymbol{b}(\boldsymbol{r}) = \boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r})B^{-1}(\boldsymbol{r})\boldsymbol{b}(\boldsymbol{r}). \tag{4}$$

Moreover, for the sake of simplicity, we define $\boldsymbol{c}^{\mathrm{B}} = \boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r}^0)$, $\boldsymbol{c}^{\mathrm{N}} = \boldsymbol{c}^{\mathrm{N}}(\boldsymbol{r}^0)$, $B = B(\boldsymbol{r}^0)$, $A^{\mathrm{N}} = A^{\mathrm{N}}(\boldsymbol{r}^0)$, $\boldsymbol{b} = \boldsymbol{b}(\boldsymbol{r}^0)$, $\boldsymbol{\pi} = \boldsymbol{\pi}(\boldsymbol{r}^0)$, $\bar{\boldsymbol{b}} = \bar{\boldsymbol{b}}(\boldsymbol{r}^0)$,

$$\Delta\boldsymbol{c}^{\mathrm{B}} = \left.\frac{\partial \boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r})}{\partial r_i}\right|_{\boldsymbol{r}=\boldsymbol{r}^0}, \tag{5}$$

$$\Delta \boldsymbol{c}^{\mathrm{N}} = \left. \frac{\partial \boldsymbol{c}^{\mathrm{N}}(\boldsymbol{r})}{\partial r_i} \right|_{\boldsymbol{r}=\boldsymbol{r}^0}, \tag{6}$$

$$\Delta B = \left. \frac{\partial B(\boldsymbol{r})}{\partial r_i} \right|_{\boldsymbol{r}=\boldsymbol{r}^0}, \tag{7}$$

$$\Delta A^{\mathrm{N}} = \left. \frac{\partial A^{\mathrm{N}}(\boldsymbol{r})}{\partial r_i} \right|_{\boldsymbol{r}=\boldsymbol{r}^0}, \tag{8}$$

$$\Delta \boldsymbol{b} = \left. \frac{\partial \boldsymbol{b}(\boldsymbol{r})}{\partial r_i} \right|_{\boldsymbol{r}=\boldsymbol{r}^0}. \tag{9}$$

3 Sensitivities

Taking the partial derivatives of $\boldsymbol{\pi}(\boldsymbol{r})$ and $\bar{\boldsymbol{b}}(\boldsymbol{r})$ with respect to r_i, we obtain the sensitivities of $\boldsymbol{\pi}(\boldsymbol{r})$ and $\bar{\boldsymbol{b}}(\boldsymbol{r})$ to r_i from (2) and (3) as follows:

$$\frac{\partial \boldsymbol{\pi}(\boldsymbol{r})}{\partial r_i} = \left(\frac{\partial \boldsymbol{c}_{\mathrm{B}}(\boldsymbol{r})}{\partial r_i} - \boldsymbol{\pi}(\boldsymbol{r}) \frac{\partial B(\boldsymbol{r})}{\partial r_i} \right) B^{-1}(\boldsymbol{r}), \tag{10}$$

$$\frac{\partial \bar{\boldsymbol{b}}(\boldsymbol{r})}{\partial r_i} = B^{-1}(\boldsymbol{r}) \left(\frac{\partial \boldsymbol{b}(\boldsymbol{r})}{\partial r_i} - \frac{\partial B(\boldsymbol{r})}{\partial r_i} \bar{\boldsymbol{b}}(\boldsymbol{r}) \right). \tag{11}$$

Further, from (4), we have

$$\frac{\partial \bar{z}(\boldsymbol{r})}{\partial r_i} = \frac{\partial \boldsymbol{c}_{\mathrm{B}}(\boldsymbol{r})}{\partial r_i} \bar{\boldsymbol{b}}(\boldsymbol{r}) - \boldsymbol{\pi}(\boldsymbol{r}) \frac{\partial B(\boldsymbol{r})}{\partial r_i} \bar{\boldsymbol{b}}(\boldsymbol{r}) + \boldsymbol{\pi}(\boldsymbol{r}) \frac{\partial \boldsymbol{b}(\boldsymbol{r})}{\partial r_i}. \tag{12}$$

$\partial \bar{z}(\boldsymbol{r})/\partial r_i$ and $\partial \bar{\boldsymbol{b}}(\boldsymbol{r})/\partial r_i$ show the effects on the optimal value and solution caused by a small change of r_i, respectively. In the global environmental framework model, the effects of a waste control policy are evaluated by the simplex multiplier. Thus, the effects on the simplex multiplier caused by a small change of r_i are also significant and shown by $\partial \boldsymbol{\pi}(\boldsymbol{r})/\partial r_i$.

From (10)-(12), we can easily calculate the sensitivities around $\boldsymbol{r}^0$ by substituting $\boldsymbol{r} = \boldsymbol{r}^0$. As can be seen easily, the sensitivities (10)-(12) can be calculated under a weaker assumption than A1-A3. The required assumption is the first order differentiabilities of $\boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r})$, $B(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$ with respect to r_i. However, the uniqueness of the obtained sensitivities is not guaranteed. If the optimal solution is not degenerate and any other optimal solutions do not exist, then the obtained sensitivities are unique. As will be described in Section 5, in the cases of the degeneracy and the multiplicity of the optimal solutions sensitivities associated with unstable optimal basis[1] are not significant.

Moreover, the sensitivities are effective only in a certain domain where the optimal basis does not change. Analyzing this domain is important not only to know the effective domain itself but also to know the unstableness of the optimal basis. Namely, if the effective domain is empty, the optimal basis is unstable. In the next section, we will discuss the effective domain analysis assuming that the parameters r_j except for r_i are fixed at r_j^0. Under this assumption, the sensitivities (10)-(12) can be regarded as functions of r_i in the effective domain. In the rest of this section, the sensitivities are explicitly represented as functions of a changing parameter. Here we prefer to use q_i instead of r_i,

[1]If an optimal basis is changed for any change in r_i, it is called an unstable optimal basis

where $r_i = r_i^0 + q_i$. Using a p-dimensional vector $\boldsymbol{\delta}_i$ whose component δ_{ij} is a Kronecker's delta, i.e.,

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{13}$$

we have

$$\boldsymbol{r} = \boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i. \tag{14}$$

When $\boldsymbol{r}$ is defined by (14), we have

$$\boldsymbol{c}^{\mathrm{B}}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = \boldsymbol{c}^{\mathrm{B}} + q_i \Delta \boldsymbol{c}^{\mathrm{B}}, \tag{15}$$

$$\boldsymbol{c}^{\mathrm{N}}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = \boldsymbol{c}^{\mathrm{N}} + q_i \Delta \boldsymbol{c}^{\mathrm{N}}, \tag{16}$$

$$B(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = B + q_i \Delta B, \tag{17}$$

$$A^{\mathrm{N}}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = A^{\mathrm{N}} + q_i \Delta A^{\mathrm{N}}, \tag{18}$$

$$\boldsymbol{b}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = \boldsymbol{b} + q_i \Delta \boldsymbol{b}. \tag{19}$$

From assumption A2 and (17), we have the following equations as pointed out by Dobrowolski et al. [2]:

The case where r_i is included in the kth column of B;

$$\det B(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = (1 + q_i v_k) \det B, \tag{20}$$

$$B^{-1}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = B^{-1} - \frac{q_i}{1 + q_i v_k} \boldsymbol{v} B_{k\cdot}^{-1}, \tag{21}$$

where $B_{\cdot k}$ and $B_{k\cdot}^{-1}$ are the kth column of B and the kth row of the inverse of B, respectively. $\boldsymbol{v}$ is defined by

$$\boldsymbol{v} = B^{-1} \Delta B_{\cdot k}. \tag{22}$$

v_k is the kth component of the vector $\boldsymbol{v}$.

The case where r_i is included in the kth row of B;

$$\det B(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = (1 + q_i u_k) \det B, \tag{23}$$

$$B^{-1}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = B^{-1} - \frac{q_i}{1 + q_i u_k} B_{\cdot k}^{-1} \boldsymbol{u}, \tag{24}$$

where $\boldsymbol{u}$ is defined by

$$\boldsymbol{u} = \Delta B_{k\cdot} B^{-1}. \tag{25}$$

u_k is the kth component of $\boldsymbol{u}$.

The case where r_i is not included in any column nor in any row of B;

$$B^{-1}(\boldsymbol{r}^0 + q_i \boldsymbol{\delta}_i) = B^{-1}. \tag{26}$$

For the above three cases, we have the following representations of the sensitivities as functions of q_i:

The case where r_i is included in the kth column of B; Since

$$\boldsymbol{\pi}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) = \left(\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}}\right)\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right) \tag{27}$$

holds, we have

$$\frac{\partial\boldsymbol{\pi}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \left\{\Delta\boldsymbol{c}^{\mathrm{B}} - \left(\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}}\right)\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right)\Delta B\right\} \times\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right), \tag{28}$$

$$\frac{\partial\bar{\boldsymbol{b}}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right) \times\left\{\Delta\boldsymbol{b} - \Delta B\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right)(\boldsymbol{b} + q_i\Delta\boldsymbol{b})\right\}, \tag{29}$$

$$\frac{\partial\bar{z}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \left(\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}}\right)\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right) \times\left\{\Delta\boldsymbol{b} - \Delta B\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right)(\boldsymbol{b} + q_i\Delta\boldsymbol{b})\right\} + \Delta\boldsymbol{c}^{\mathrm{B}}\left(B^{-1} - \frac{q_i}{1+q_iv_k}\boldsymbol{v}B^{-1}_{k\cdot}\right)(\boldsymbol{b} + q_i\Delta\boldsymbol{b}). \tag{30}$$

The case where r_i is included in the kth row of B; Since

$$\boldsymbol{\pi}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) = \left(\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}}\right)\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right) \tag{31}$$

holds, we have

$$\frac{\partial\boldsymbol{\pi}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \left\{\Delta\boldsymbol{c}^{\mathrm{B}} - \left(\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}}\right)\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right)\Delta B\right\} \times\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right), \tag{32}$$

$$\frac{\partial\bar{\boldsymbol{b}}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right) \times\left\{\Delta\boldsymbol{b} - \Delta B\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right)(\boldsymbol{b} + q_i\Delta\boldsymbol{b})\right\}, \tag{33}$$

$$\frac{\partial\bar{z}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \left(\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}}\right)\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right) \times\left\{\Delta\boldsymbol{b} - \Delta B\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right)(\boldsymbol{b} + q_i\Delta\boldsymbol{b})\right\} + \Delta\boldsymbol{c}^{\mathrm{B}}\left(B^{-1} - \frac{q_i}{1+q_iu_k}B^{-1}_{\cdot k}\boldsymbol{u}\right)(\boldsymbol{b} + q_i\Delta\boldsymbol{b}). \tag{34}$$

The case where r_i is not included in any column nor in any row of B; Since the inverse of the basic matrix does not change, we have

$$\frac{\partial\boldsymbol{\pi}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \Delta\boldsymbol{c}^{\mathrm{B}}B^{-1}, \tag{35}$$

$$\frac{\partial \bar{z}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = B^{-1}\Delta\boldsymbol{b}, \tag{36}$$

$$\frac{\partial \bar{\boldsymbol{b}}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)}{\partial q_i} = \Delta\boldsymbol{c}^{\mathrm{B}}B^{-1}(\boldsymbol{b} + q_i\Delta\boldsymbol{b}) + (\boldsymbol{c}^{\mathrm{B}} + q_i\Delta\boldsymbol{c}^{\mathrm{B}})B^{-1}\Delta\boldsymbol{b}. \tag{37}$$

4 Effective Domain Analysis

As stated in the previous section, the sensitivities are available only in a certain domain where the optimal basis does not change. In this section, we discuss the domain. To this end, we consider the following three conditions, as discussed by Dobrowolski et al. [2]:

1. Non-singularity of the basic matrix,
2. Non-negativity of the basic variables,
3. Necessary condition of optimality.

In what follows, the domain where these three conditions are satisfied is obtained in the classification of the previous section:

The case where r_i is included in the kth column of B;

1. Non-singularity of the basic matrix.

   The condition can be written as det $B(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) \neq 0$. Since our interest is in the behavior around $\boldsymbol{r} = \boldsymbol{r}^0$, i.e., $q_i = 0$, the condition becomes that det $B(\boldsymbol{r}^0+q_i\boldsymbol{\delta}_i) \neq 0$ has the same sign as det B, i.e., det $B(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) \times$ det $B > 0$. From (20), the condition is equivalent to

   $$1 + q_iv_k > 0. \tag{38}$$

   Hence, from the non-singularity condition, we obtain the following domain Q_1:

   $$Q_1 = \begin{cases} \left(-\infty, -\dfrac{1}{v_k}\right) & \text{if } v_k < 0, \\ (-\infty, +\infty) & \text{if } v_k = 0, \\ \left(-\dfrac{1}{v_k}, +\infty\right) & \text{if } v_k > 0. \end{cases} \tag{39}$$

2. Non-negativity of the basic variables.

   The non-negativity can be checked by $\bar{\boldsymbol{b}}(\boldsymbol{r}^0+q_i\boldsymbol{\delta}_i) = B^{-1}(\boldsymbol{r}^0+q_i\boldsymbol{\delta}_i)\boldsymbol{b}(\boldsymbol{r}^0+q_i\boldsymbol{\delta}_i) \geq \boldsymbol{0}$. From (19), (21) and (38), we have

   $$(v_k\Delta\bar{b}_j - v_j\Delta\bar{b}_k)q_i^2 + (v_k\bar{b}_j - v_j\bar{b}_k + \Delta\bar{b}_j)q_i + \bar{b}_j \geq 0,\ j = 1, 2, \ldots, m, \tag{40}$$

   where $\bar{b}_j$ and $\Delta\bar{b}_j$ are the jth component of $\bar{\boldsymbol{b}}$ and $\Delta\bar{\boldsymbol{b}}$, respectively. $\Delta\bar{\boldsymbol{b}}$ is defined by

   $$\Delta\bar{\boldsymbol{b}} = B^{-1}\Delta\boldsymbol{b}. \tag{41}$$

   For the sake of simplicity, we define

   $$\alpha_1 = v_k\Delta\bar{b}_j - v_j\Delta\bar{b}_k, \tag{42}$$

   $$\beta_1 = v_k\bar{b}_j - v_j\bar{b}_k + \Delta\bar{b}_j, \tag{43}$$

$$\gamma_1 = \bar{b}_j. \tag{44}$$

Here let us consider the following quadratic equation:

$$F_1^j(q_i) = \alpha_1 q_i^2 + \beta_1 q_i + \gamma_1 = 0. \tag{45}$$

The discriminant D_1 can be defined by

$$D_1 = \beta_1^2 - 4\alpha_1\gamma_1. \tag{46}$$

We have the following seven cases by taking into account the fact that $F_1^j(0) \geq 0$:

$D_1 \geq 0$ and $\alpha_1 < 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = \left[-\frac{\beta_1 - \sqrt{D_1}}{2\alpha_1}, -\frac{\beta_1 + \sqrt{D_1}}{2\alpha_1}\right]. \tag{47}$$

Especially, if $D_1 = 0$, then $Q_2^j = \{0\}$.

$D_1 \leq 0$ and $\alpha_1 > 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = (-\infty, \infty). \tag{48}$$

$D_1 > 0$, $\alpha_1 > 0$ and $-\beta_1 - \sqrt{D_1} \geq 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = \left[-\frac{\beta_1 - \sqrt{D_1}}{2\alpha_1}, \infty\right). \tag{49}$$

$D_1 > 0$, $\alpha_1 > 0$ and $-\beta_1 + \sqrt{D_1} \leq 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = \left(-\infty, -\frac{\beta_1 + \sqrt{D_1}}{2\alpha_1}\right]. \tag{50}$$

$\alpha_1 = 0$ and $\beta_1 > 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = \left[-\frac{\gamma_1}{\beta_1}, \infty\right). \tag{51}$$

$\alpha_1 = 0$ and $\beta_1 < 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = \left(-\infty, -\frac{\gamma_1}{\beta_1}\right]. \tag{52}$$

$\alpha_1 = 0$, $\beta_1 = 0$ and $\gamma_1 \geq 0$. We obtain a domain Q_2^j where $F_1^j(q_i) \geq 0$ as

$$Q_2^j = (-\infty, \infty). \tag{53}$$

Calculating Q_2^j for all $j = 1, 2, \ldots, m$, we obtain the domain in which (40) is fulfilled as

$$Q_2 = \bigcap_{j=1}^{m} Q_2^j. \tag{54}$$

3. The necessary condition of optimality.

   From the theory of the simplex method, the necessary condition of optimality can be written as

$$\boldsymbol{\pi}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i)A^{\mathrm{N}}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) - \boldsymbol{c}^{\mathrm{N}}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) \leq \mathbf{0}. \tag{55}$$

   Developing the right-hand side, we have

$$\begin{aligned}&(v_k\Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + \Delta\boldsymbol{\pi}\Delta A^{\mathrm{N}}_{\cdot j} - v_k\Delta \boldsymbol{c}^{\mathrm{N}}_j)q_i^2\\ &\quad + (\Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + v_k\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + \boldsymbol{\pi}\Delta A^{\mathrm{N}}_{\cdot j} - \Delta\boldsymbol{c}^{\mathrm{N}}_j - v_k\boldsymbol{c}^{\mathrm{N}}_j)q_i\\ &\quad + (\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \boldsymbol{c}^{\mathrm{N}}_j) \leq 0,\ j = 1, 2, \ldots, n-m,\end{aligned} \tag{56}$$

   where $\boldsymbol{c}^{\mathrm{N}}_j$ and $\Delta\boldsymbol{c}^{\mathrm{N}}_j$ are jth components of $\boldsymbol{c}^{\mathrm{N}}$ and $\Delta\boldsymbol{c}^{\mathrm{N}}$, respectively. $\Delta\boldsymbol{\pi}$ is defined by

$$\Delta\boldsymbol{\pi} = \Delta\boldsymbol{c}^{\mathrm{B}}B^{-1}. \tag{57}$$

   For the sake of simplicity, we define

$$\alpha_2 = v_k\Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + \Delta\boldsymbol{\pi}\Delta A^{\mathrm{N}}_{\cdot j} - v_k\Delta\boldsymbol{c}^{\mathrm{N}}_j, \tag{58}$$

$$\beta_2 = \Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + v_k\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + \boldsymbol{\pi}\Delta A^{\mathrm{N}}_{\cdot j} - \Delta\boldsymbol{c}^{\mathrm{N}}_j - v_k\boldsymbol{c}^{\mathrm{N}}_j, \tag{59}$$

$$\gamma_2 = \boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \boldsymbol{c}^{\mathrm{N}}_j, \tag{60}$$

$$F_2^j(q_i) = \alpha_2 q_i^2 + \beta_2 q_i + \gamma_2, \tag{61}$$

$$D_2 = \beta_2^2 - 4\alpha_2\gamma_2. \tag{62}$$

   By the same discussion as with Q_2^j, we obtain the domain within which $F_2^j(q_i) \leq 0$ as

$$Q_3^j = \begin{cases} \left[-\dfrac{\beta_2+\sqrt{D_2}}{2\alpha_2}, -\dfrac{\beta_2-\sqrt{D_2}}{2\alpha_2}\right], & \text{if } D_2 \geq 0 \text{ and } \alpha_2 > 0,\\ (-\infty,\infty), & \text{if } D_2 \leq 0 \text{ and } \alpha_2 \leq 0,\\ \left[-\dfrac{\beta_2+\sqrt{D_2}}{2\alpha_2}, -\infty\right), & \text{if } D_2 > 0,\ \alpha_2 < 0 \text{ and } -\beta_2-\sqrt{D_2} \geq 0,\\ \left(-\infty, -\dfrac{\beta_2-\sqrt{D_2}}{2\alpha_2}\right], & \text{if } D_2 > 0,\ \alpha_2 < 0 \text{ and } -\beta_2+\sqrt{D_2} \leq 0,\\ \left[-\dfrac{\gamma_2}{\beta_2}, \infty\right), & \text{if } \alpha_2 = 0 \text{ and } \beta_2 < 0,\\ \left(-\infty, -\dfrac{\gamma_2}{\beta_2}\right], & \text{if } \alpha_2 = 0 \text{ and } \beta_2 > 0. \end{cases} \tag{63}$$

   Calculating Q_3^j for all $j = 1, 2, \ldots, n-m$, we obtain the domain in which (56) is fulfilled as

$$Q_3 = \bigcap_{j=1}^{n-m} Q_3^j. \tag{64}$$

4. Conjunction of three conditions.

   Taking the intersection of Q_1, Q_2 and Q_3, we obtain the domain Q within which three conditions, the non-singularity, the non-negativity and the optimality are satisfied as

$$Q = Q_1 \cap Q_2 \cap Q_3. \tag{65}$$

The case where r_i is included in the kth row of B;

The way of the derivation is the same as that in the case where r_i is included in the kth column of B. Here we describe it briefly.

1. Non-singularity of the basic matrix.

   From (23), the non-singularity condition is equivalent to

$$1 + q_i u_k > 0. \tag{66}$$

   Hence, from the non-singularity condition, we obtain the following domain Q_1:

$$Q_1 = \begin{cases} \left(-\infty, -\dfrac{1}{u_k}\right) & \text{if } u_k < 0, \\ (-\infty, +\infty) & \text{if } u_k = 0, \\ \left(-\dfrac{1}{u_k}, +\infty\right) & \text{if } u_k > 0. \end{cases} \tag{67}$$

2. Non-negativity of the basic variables.

   From (19), (24) and (66), the non-negativity condition can be written as

$$(u_k \Delta \bar{b}_j - B_{jk}^{-1} \boldsymbol{u} \Delta \boldsymbol{b}) q_i^2 + (u_k \bar{b}_j - B_{jk}^{-1} \boldsymbol{u}\boldsymbol{b} + \Delta \bar{b}_j) q_i + \bar{b}_j \geq 0, \quad j = 1, 2, \ldots, m, \tag{68}$$

   where B_{jk}^{-1} is the (j,k)-component of B^{-1}.

   For the sake of simplicity, we define

$$\alpha_3 = u_k \Delta \bar{b}_j - B_{jk}^{-1} \boldsymbol{u} \Delta \boldsymbol{b}, \tag{69}$$

$$\beta_3 = u_k \bar{b}_j - B_{jk}^{-1} \boldsymbol{u}\boldsymbol{b} + \Delta \bar{b}_j, \tag{70}$$

$$\gamma_3 = \bar{b}_j, \tag{71}$$

$$F_3^j(q_i) = \alpha_3 q_i^3 + \beta_3 q_i + \gamma_3, \tag{72}$$

$$D_3 = \beta_3^2 - 4\alpha_3\gamma_3. \tag{73}$$

   Thus, we obtain the domain Q_2^j within which $F_3^j(q_i) \leq 0$ as

$$Q_2^j = \begin{cases} \left[-\dfrac{\beta_3 + \sqrt{D_3}}{2\alpha_3}, -\dfrac{\beta_3 - \sqrt{D_3}}{2\alpha_3}\right], & \text{if } D_3 \geq 0 \text{ and } \alpha_3 < 0, \\ (-\infty, \infty), & \text{if } D_3 \leq 0 \text{ and } \alpha_3 \geq 0, \\ \left[-\dfrac{\beta_3 + \sqrt{D_3}}{2\alpha_3}, \infty\right), & \text{if } D_3 > 0,\ \alpha_3 > 0 \text{ and } -\beta_3 - \sqrt{D_3} \geq 0, \\ \left(-\infty, -\dfrac{\beta_3 - \sqrt{D_3}}{2\alpha_3}\right], & \text{if } D_3 > 0,\ \alpha_3 > 0 \text{ and } -\beta_3 + \sqrt{D_3} \leq 0, \\ \left[-\dfrac{\gamma_3}{\beta_3}, \infty\right), & \text{if } \alpha_3 = 0 \text{ and } \beta_3 > 0, \\ \left(-\infty, -\dfrac{\gamma_3}{\beta_3}\right], & \text{if } \alpha_3 = 0 \text{ and } \beta_3 < 0. \end{cases} \tag{74}$$

   Calculating Q_2^j for all $j = 1, 2, \ldots, m$, we obtain the domain in which (68) is fulfilled as in (54).

3. The necessary condition of optimality.

   The necessary condition of optimality can be written as

$$
\begin{aligned}
&(u_k\Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \Delta\pi_k\boldsymbol{u}A^{\mathrm{N}}_{\cdot j} + u_k\boldsymbol{u}A^{\mathrm{N}}_{\cdot j} - u_k\Delta c^{\mathrm{N}}_j)q_i^2 \\
&\quad + (\boldsymbol{\pi}\Delta A^{\mathrm{N}}_{\cdot j} - \pi_k\boldsymbol{u}A^{\mathrm{N}}_{\cdot j} + \Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + u_k\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \Delta c^{\mathrm{N}}_j - u_k c^{\mathrm{N}}_j)q_i \\
&\quad + (\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \boldsymbol{c}^{\mathrm{N}}_j) \le 0,\ j = 1,2,\ldots n-m,
\end{aligned} \tag{75}
$$

   where π_k and c^{N}_j are the kth component of $\boldsymbol{\pi}$ and $\boldsymbol{c}^{\mathrm{N}}$, respectively.

   For the sake of simplicity, we define

$$\alpha_4 = u_k\Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \Delta\pi_k\boldsymbol{u}A^{\mathrm{N}}_{\cdot j} + u_k\boldsymbol{u}A^{\mathrm{N}}_{\cdot j} - u_k\Delta c^{\mathrm{N}}_j, \tag{76}$$

$$\beta_4 = \boldsymbol{\pi}\Delta A^{\mathrm{N}}_{\cdot j} - \pi_k\boldsymbol{u}A^{\mathrm{N}}_{\cdot j} + \Delta\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} + u_k\boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \Delta c^{\mathrm{N}}_j - u_k c^{\mathrm{N}}_j, \tag{77}$$

$$\gamma_4 = \boldsymbol{\pi}A^{\mathrm{N}}_{\cdot j} - \boldsymbol{c}^{\mathrm{N}}_j, \tag{78}$$

$$F^j_4(q_i) = \alpha_4 q_i^4 + \beta_4 q_i + \gamma_4, \tag{79}$$

$$D_4 = \beta_4^2 - 4\alpha_4\gamma_4. \tag{80}$$

   Thus, we obtain the domain Q^j_3 within which $F^j_3(q_i) \le 0$ as

$$
Q^j_3 = \begin{cases}
\left[-\dfrac{\beta_4+\sqrt{D_4}}{2\alpha_4}, -\dfrac{\beta_4-\sqrt{D_4}}{2\alpha_4}\right], & \text{if } D_4 \ge 0 \text{ and } \alpha_4 > 0,\\
(-\infty,\infty), & \text{if } D_4 \le 0 \text{ and } \alpha_4 \le 0,\\
\left[-\dfrac{\beta_4+\sqrt{D_4}}{2\alpha_4}, \infty\right), & \text{if } D_4 > 0,\ \alpha_4 < 0 \text{ and } -\beta_4-\sqrt{D_4} \ge 0,\\
\left(-\infty, -\dfrac{\beta_4-\sqrt{D_4}}{2\alpha_4}\right], & \text{if } D_4 > 0,\ \alpha_4 < 0 \text{ and } -\beta_4+\sqrt{D_4} \le 0,\\
\left[-\dfrac{\gamma_4}{\beta_4}, \infty\right), & \text{if } \alpha_4 = 0 \text{ and } \beta_4 < 0,\\
\left(-\infty, -\dfrac{\gamma_4}{\beta_4}\right], & \text{if } \alpha_4 = 0 \text{ and } \beta_4 > 0.
\end{cases} \tag{81}
$$

   Calculating Q^j_3 for all $j = 1,2,\ldots,n-m$, we obtain the domain in which (75) is fulfilled as in (64).

4. Conjunction of three conditions.

   Taking the intersection of Q_1, Q_2 and Q_3, we obtain the domain Q within which three conditions, the non-singularity, the non-negativity and the optimality are satisfied as in (65).

The case where r_i is not included in any column nor in any row of B;

1. Non-singularity of the basic matrix.

   Since the basic matrix never changes in this case, we do not care about the condition of non-singularity. Thus, we have $Q_1 = (-\infty,\infty)$.

2. Non-negativity of the basic variables.

   From the condition $\bar{\boldsymbol{b}}(\boldsymbol{r}^0 + q_i\boldsymbol{\delta}_i) \geq 0$, we have

$$\bar{b}_j + q_i B_{j\cdot}^{-1}\Delta\boldsymbol{b} \geq 0,\ j = 1,2,\ldots,m. \tag{82}$$

   Thus, the domain Q_2^j where $\bar{b}_j + q_i B_{j\cdot}^{-1}\Delta\boldsymbol{b} \geq 0$ is fulfilled is obtained as

$$Q_2^j = \begin{cases} \left(-\infty, -\dfrac{\bar{b}_j}{B_{j\cdot}^{-1}\Delta\boldsymbol{b}}\right], & \text{if } B_{j\cdot}^{-1}\Delta\boldsymbol{b} < 0, \\ (-\infty,\infty), & \text{if } B_{j\cdot}^{-1}\Delta\boldsymbol{b} = 0, \\ \left[-\dfrac{\bar{b}_j}{B_{j\cdot}^{-1}\Delta\boldsymbol{b}}, \infty\right), & \text{if } B_{j\cdot}^{-1}\Delta\boldsymbol{b} > 0. \end{cases} \tag{83}$$

   Calculating Q_2^j for all $j = 1,2,\ldots,m$, we obtain the domain in which (82) is fulfilled as in (54).

3. The necessary condition of optimality.

   From (55), we have

$$(\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - c_j^{\mathrm{N}}) + (\Delta\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - \Delta c_j^{\mathrm{N}})q_i \leq 0,\ j = 1,2,,\ldots,n-m. \tag{84}$$

   Thus, the domain Q_3^j where the jth condition of (84) is satisfied is given as

$$Q_3^j = \begin{cases} \left(-\infty, -\dfrac{\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - c_j^{\mathrm{N}}}{\Delta\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - \Delta c_j^{\mathrm{N}}}\right], & \text{if } \Delta\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - \Delta c_j^{\mathrm{N}} > 0, \\ (-\infty,\infty), & \text{if } \Delta\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - \Delta c_j^{\mathrm{N}} = 0, \\ \left[-\dfrac{\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - c_j^{\mathrm{N}}}{\Delta\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - \Delta c_j^{\mathrm{N}}}, \infty\right), & \text{if } \Delta\boldsymbol{\pi}A_{\cdot j}^{\mathrm{N}} - \Delta c_j^{\mathrm{N}} < 0. \end{cases} \tag{85}$$

   Calculating Q_3^j for all $j = 1,2,\ldots,m$, we obtain the domain in which (84) is fulfilled as in (64).

4. Conjunction of three conditions

   Taking the intersection of Q_1, Q_2 and Q_3, we obtain the domain as (65).

5 The Cases of Degeneracy and Multiplicity of the Optimal Solutions

The most serious difficulty of the sensitivity analysis is met in the cases of degeneracy and multiplicity of the optimal solutions. Akgül [3] has developed an excellent approach to such difficult cases. Since the approach is proposed only for calculating the simplex multipliers, unfortunately, it is not available for our extended sensitivity analysis. Here we chose a primitive approach for the difficult cases. Namely, all optimal bases are generated and for each optimal basis, the above analyses are applied. All optimal bases can be continuously generated from an optimal basis by pivot operations.

$\bar{A} = B^{-1}(\boldsymbol{r}^0)A(\boldsymbol{r}^0)$ and $\boldsymbol{w} = \boldsymbol{\pi}(\boldsymbol{r}^0)A(\boldsymbol{r}^0) - \boldsymbol{c}(\boldsymbol{r}^0)$. When an optimal basis is obtained, we have $\bar{\boldsymbol{b}} \geq \boldsymbol{0}$ and $\boldsymbol{w} \leq \boldsymbol{0}$. If the optimal basis is degenerate, there exists at least one zero component of $\bar{\boldsymbol{b}}$, i.e., $\bar{b}_j = 0$. If there exist more than one optimal solutions, at least one component of $\boldsymbol{w}$ is zero, i.e., $w_j = 0$. In those cases, we can calculate all optimal bases by the following algorithm:

Algorithm for generating all optimal bases

Step 1. Set S^{u} as a singleton with the initial optimal basis and $S^{\mathrm{e}} = \emptyset$, where S^{u} and S^{e} are unexplored and explored sets of bases.

Step 2. If S^{u} is empty, terminate the algorithm. Otherwise, take one basis from S^{u}.

Step 3. Calculate the $\bar{A}$, $\bar{\boldsymbol{b}}$ and $\boldsymbol{w}$ corresponding to the selected basis.

Step 4. If there does not exist a j such that $\bar{b}_j = 0$, then go to Step 6. Otherwise, for every j such that $\bar{b}_j = 0$, obtain all $s \in \{1, 2, \ldots, n\}$ such that

$$\frac{w_s}{\bar{A}_{jt}} = \min\{\frac{w_t}{\bar{A}_{jt}} \mid \bar{A}_{jt} < 0\} ,$$

and $\bar{A}_{js} < 0$.

Step 5. For all pairs (j, s) obtained at Step 4, if the basis where the jth basic variable is replaced with x_s is not a member of $S^{\mathrm{e}} \cup S^{\mathrm{u}}$, put it in S^{u}.

Step 6. If there does not exist an s such that $w_s = 0$, then return to Step 2. Otherwise, for every s such that $w_s = 0$, obtain all $j \in \{1, 2, \ldots, m\}$ such that

$$\frac{\hat{b}_j}{\bar{A}_{js}} = \min\{\frac{\hat{b}_t}{\bar{A}_{ts}} \mid \bar{A}_{ts} > 0\} ,$$

and $\bar{A}_{js} < 0$.

Step 7. For all pairs (j, s) obtained at Step 6, if the basis where the jth basic variable is replaced with x_s is not a member of $S^{\mathrm{e}} \cup S^{\mathrm{u}}$, put it in S^{u}. Return to Step 2.

In this algorithm, $\bar{A}$, $\bar{\boldsymbol{b}}$ and $\boldsymbol{w}$ at Step 3, can be calculated easily by a pivot operation. As shown below, the existence of s at Step 4 and j at Step 6 are guaranteed by the assumption A3. The existence of j at Step 6 follows directly from the assumption A3, since the feasible area of the problem (1) with $\boldsymbol{r} = \boldsymbol{r}^0$ is bounded. The right-hand side values of constraints of the problem (1) are the objective coefficients of the dual problem. A change of the right-hand side values of the problem (1) is a change of the objective function of the dual problem. By the assumption A3, the problem (1) with $\boldsymbol{r} = \boldsymbol{r}^0$ has at least one optimal solution even if the right-hand side values slightly change. Applying the duality theorem in the linear programming problem, the dual problem with $\boldsymbol{r} = \boldsymbol{r}^0$ has also at least one optimal solution even if the objective coefficients slightly change. This means that the dual problem with $\boldsymbol{r} = \boldsymbol{r}^0$ has a bounded optimal solution set. Hence, s at Step 4 exists under the assumption A3.

Table 1. The optimal bases

No.	basis	value	x_1	x_2	s_1	s_2	s_3	s_4
1	x_1	4	1	0	2	−1	0	0
	x_2	9	0	1	−1	1	0	0
	s_3	5	0	0	−2	1	1	0
	s_4	0	0	0	1	−1	0	1
	w_j	−13	0	0	−1	0	0	0
2	x_1	9	1	0	0	0	1	0
	x_2	4	0	1	1	0	−1	0
	s_2	5	0	0	−2	1	1	0
	s_4	5	0	0	−1	0	1	1
	w_j	−13	0	0	−1	0	0	0
3	x_1	4	1	0	1	0	0	−1
	x_2	9	0	1	0	0	0	1
	s_2	0	0	0	−1	1	0	−1
	s_3	5	0	0	−1	0	1	1
	w_j	−13	0	0	−1	0	0	0

6 A Numerical Example

Let us consider the following linear programming problem with uncertain parameters:

$$\begin{array}{rl} \text{maximize} & (0.5r_1+0.5r_2)x_1+x_2, \\ \text{subject to} & x_1+r_2x_2 \le 13, \\ & r_1x_1+2r_1r_2x_2 \le 22, \\ & 0 \le x_1 \le 9, \\ & 0 \le x_2 \le 9, \end{array} \tag{86}$$

where (r_1, r_2) are uncertain parameters with the plausible values $(r_1^0, r_2^0) = (1,1)$. Introducing slack variables s_1, s_2, s_3 and s_4, (86) can be transformed to the following standard form:

$$\begin{array}{rl} \text{minimize} & -(0.5r_1+0.5r_2)x_1-x_2, \\ \text{subject to} & x_1+r_2x_2+s_1=13, \\ & r_1x_1+2r_1r_2x_2+s_2=22, \\ & x_1+s_3=9, \\ & x_2+s_4=9, \\ & (x_1,x_2,s_1,s_2,s_3,s_4) \ge \mathbf{0}. \end{array} \tag{86'}$$

As can be checked easily, the assumptions A1-A3 are satisfied.

Substituting $(r_1, r_2) = (1,1)$ in the problem (86), we have two optimal extreme points $(x_1, x_2) = (4,9)$ and $(x_1, x_2) = (9,4)$ as shown in Figure 1. The solutions on the line segment between these points are all optimal. Thus we have multiple optimal solutions. Three border lines, $x_1 + x_2 = 13$, $x_1 + 2x_2 = 22$ and $x_2 = 9$ cross at one point $(x_1, x_2) = (4,9)$. Thus, the optimal solution $(x_1, x_2) = (4,9)$ is degenerate. Applying the algorithm for generating all optimal bases, we obtain three optimal bases. The corresponding simplex tableaus are shown in Table 1.

For each optimal basis, let us calculate the effective domain with respect to $q_1 = r_1 - r_1^0$ and $q_2 = r_2 - r_2^0$ and if it is not equal to $\{0\}$, let us calculate the sensitivities.

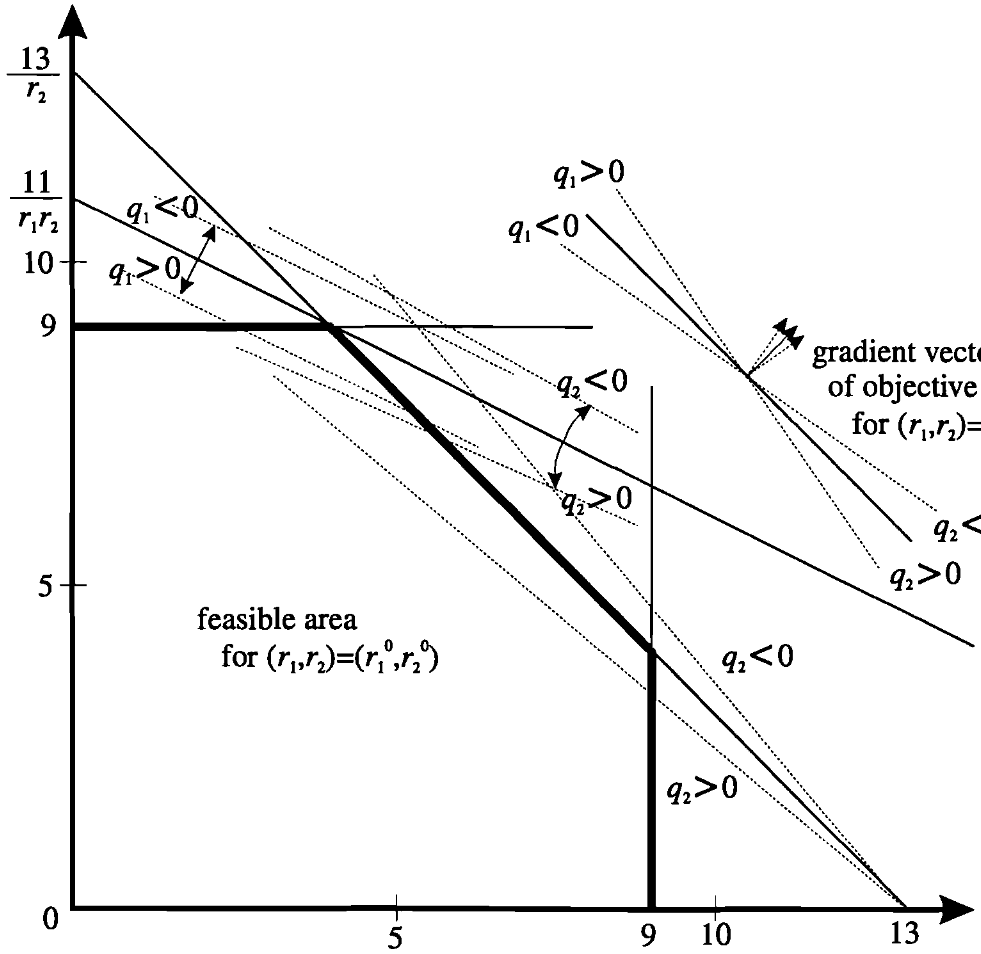

Figure 1: An illustration of the problem

Analysis with respect to r_1

Tableau 1. In this case, we have

$$B^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{c}^{\mathrm{B}} = (-1,-1,0,0), \quad \Delta\boldsymbol{c}^{\mathrm{B}} = (-0.5,0,0,0),$$

$$\boldsymbol{c}^{\mathrm{N}} = (0,0), \quad \Delta\boldsymbol{c}^{\mathrm{N}} = (0,0),$$

$$A^{\mathrm{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta A^{\mathrm{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{b} = (13,22,9,9)^{\mathrm{T}}, \quad \Delta\boldsymbol{b} = (0,0,0,0)^{\mathrm{T}},$$

$$\bar{\boldsymbol{b}} = (4,9,5,0)^{\mathrm{T}}, \quad \Delta\bar{\boldsymbol{b}} = (0,0,0,0)^{\mathrm{T}},$$

$$\boldsymbol{\pi} = (-1,0,0,0), \quad \Delta\boldsymbol{\pi} = (-1,0.5,0,0)$$

and $k = 2$.

$\boldsymbol{u}$ defined by (25) is calculated as

$$\boldsymbol{u} = \Delta B_2.B^{-1} = (0,1,0,0).$$

We obtain $Q_1 = (-1,\infty)$. By the calculation of Q_2^j, $j = 1,2,\ldots,4$, we have

$$\begin{aligned} Q_2^1 &= \left[-\tfrac{2}{13},\infty\right), \\ Q_2^2 &= \left(-\infty,\tfrac{9}{13}\right], \\ Q_2^3 &= \left(-\infty,\tfrac{5}{17}\right], \\ Q_2^4 &= [0,\infty). \end{aligned}$$

Hence, $Q_2 = [0,\frac{5}{17}]$. Similarly, calculating Q_3^j, $j = 1,2$, we obtain

$$\begin{aligned} Q_3^1 &= \left[-\tfrac{1}{2},\infty\right), \\ Q_3^2 &= (-\infty,0]. \end{aligned}$$

Hence, $Q_3 = [-\frac{1}{2},0]$. Taking the intersection of Q_1, Q_2 and Q_3, we have $Q = [0,0]$.

Since $Q = \{0\}$, this optimal basis is unstable and we do not calculate the sensitivities with respect to this optimal basis.

Tableau 2. In this case, we have

$$B^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{c}^{\mathrm{B}} = (-1,-1,0,0), \quad \Delta\boldsymbol{c}^{\mathrm{B}} = (-0.5,0,0,0),$$

$$\boldsymbol{c}^{\mathrm{N}} = (0,0), \quad \Delta\boldsymbol{c}^{\mathrm{N}} = (0,0),$$

$$A^{\mathrm{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Delta A^{\mathrm{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{b} = (13,22,9,9)^{\mathrm{T}}, \quad \Delta\boldsymbol{b} = (0,0,0,0)^{\mathrm{T}},$$

$$\bar{\boldsymbol{b}} = (9,4,5,5)^{\mathrm{T}}, \quad \Delta\bar{\boldsymbol{b}} = (0,0,0,0)^{\mathrm{T}},$$

$$\boldsymbol{\pi} = (-1,0,0,0), \quad \Delta\boldsymbol{\pi} = (0,0,-0.5,0)$$

and $k = 2$.

$\boldsymbol{u}$ defined by (25) is calculated as

$$\boldsymbol{u} = \Delta B_{2\cdot} B^{-1} = (2, 0, -1, 0).$$

We obtain $Q_1 = (-\infty, \infty)$. By the calculation of Q_2^j, $j = 1, 2, \ldots, 4$, we have

$$\begin{aligned} Q_2^1 &= (-\infty, \infty), \\ Q_2^2 &= (-\infty, \infty), \\ Q_2^3 &= \left(-\infty, \tfrac{5}{17}\right], \\ Q_2^4 &= (-\infty, \infty). \end{aligned}$$

Hence, $Q_2 = (-\infty, \frac{5}{17}]$. Similarly, calculating Q_3^j, $j = 1, 2$, we obtain

$$\begin{aligned} Q_3^1 &= (-\infty, \infty), \\ Q_3^2 &= [0, \infty). \end{aligned}$$

Hence, $Q_3 = [0, \infty)$. Taking the intersection of Q_1, Q_2 and Q_3, we have $Q = [0, \frac{5}{17}]$.

Since $Q \neq \{0\}$, let us proceed to the calculation of sensitivities. From (32) to (34), we have

$$\frac{\partial \boldsymbol{\pi}((1+q_1, 1))}{\partial q_1} = (0, -0.5, 0, 0),$$

$$\frac{\partial \bar{\boldsymbol{b}}((1+q_1, 1))}{\partial q_1} = (0, 0, -17, 0)^{\mathrm{T}},$$

$$\frac{\partial \bar{z}((1+q_1, 1))}{\partial q_1} = -4.5,$$

for $q_1 \in [0, \frac{5}{17}]$

Tableau 3. In this case, we have

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{c}^{\mathrm{B}} = (-1, -1, 0, 0), \qquad \Delta \boldsymbol{c}^{\mathrm{B}} = (-0.5, 0, 0, 0),$$

$$\boldsymbol{c}^{\mathrm{N}} = (0, 0), \qquad \Delta \boldsymbol{c}^{\mathrm{N}} = (0, 0),$$

$$A^{\mathrm{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Delta A^{\mathrm{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{b} = (13, 22, 9, 9)^{\mathrm{T}}, \qquad \Delta \boldsymbol{b} = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\bar{\boldsymbol{b}} = (4, 9, 0, 5)^{\mathrm{T}}, \qquad \Delta \bar{\boldsymbol{b}} = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\boldsymbol{\pi} = (-1, 0, 0, 0), \qquad \Delta \boldsymbol{\pi} = (-0.5, 0, 0, 0.5)$$

and $k = 2$.

$\boldsymbol{u}$ defined by (25) is calculated as

$$\boldsymbol{u} = \Delta B_{2\cdot} B^{-1} = (1, 0, 0, 1).$$

We obtain $Q_1 = (-\infty, \infty)$. By the calculation of Q_2^j, $j = 1, 2, \ldots, 4$, we have

$$\begin{aligned} Q_2^1 &= (-\infty, \infty), \\ Q_2^2 &= (-\infty, \infty), \\ Q_2^3 &= (-\infty, 0], \\ Q_2^4 &= (-\infty, \infty). \end{aligned}$$

Hence, $Q_2 = (-\infty, 0]$. Similarly, calculating Q_3^j, $j = 1, 2$, we obtain

$$\begin{aligned} Q_3^1 &= [-2, \infty), \\ Q_3^2 &= (-\infty, 0]. \end{aligned}$$

Hence, $Q_3 = [-2, 0]$. Taking the intersection of Q_1, Q_2 and Q_3, we have $Q = [-2, 0]$.

Since $Q \neq \{0\}$, let us proceed to the calculation of sensitivities. From (32) to (34), we have

$$\begin{aligned} \frac{\partial \boldsymbol{\pi}((1+q_1, 1))}{\partial q_1} &= (-0.5, 0, 0, 0.5), \\ \frac{\partial \bar{\boldsymbol{b}}((1+q_1, 1))}{\partial q_1} &= (0, -22, 0, 0)^{\mathrm{T}}, \\ \frac{\partial \bar{z}((1+q_1, 1))}{\partial q_1} &= -2, \end{aligned}$$

for $q_1 \in [-2, 0]$.

Analysis with respect to r_2

Tableau 1. In this case, we have

$$B^{-1} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\begin{aligned} &\boldsymbol{c}^{\mathrm{B}} = (-1, -1, 0, 0), && \Delta \boldsymbol{c}^{\mathrm{B}} = (-0.5, 0, 0, 0), \\ &\boldsymbol{c}^{\mathrm{N}} = (0, 0), && \Delta \boldsymbol{c}^{\mathrm{N}} = (0, 0), \end{aligned}$$

$$A^{\mathrm{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Delta A^{\mathrm{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\begin{aligned} &\boldsymbol{b} = (13, 22, 9, 9)^{\mathrm{T}}, && \Delta \boldsymbol{b} = (0, 0, 0, 0)^{\mathrm{T}}, \\ &\bar{\boldsymbol{b}} = (4, 9, 5, 0)^{\mathrm{T}}, && \Delta \bar{\boldsymbol{b}} = (0, 0, 0, 0)^{\mathrm{T}}, \\ &\boldsymbol{\pi} = (-1, 0, 0, 0), && \Delta \boldsymbol{\pi} = (-1, 0.5, 0, 0) \end{aligned}$$

and $k = 2$.

$\boldsymbol{v}$ defined by (22) is calculated as

$$\boldsymbol{v} = B^{-1} \Delta B_{\cdot 2} = (0, 1, 0, -1)^{\mathrm{T}}.$$

We obtain $Q_1 = (-1, \infty)$. By the calculation of Q_2^j, $j = 1, 2, \ldots, 4$, we have

$$\begin{aligned} Q_2^1 &= [-1, \infty), \\ Q_2^2 &= (-\infty, \infty), \\ Q_2^3 &= [-1, \infty), \\ Q_2^4 &= [0, \infty). \end{aligned}$$

Hence, $Q_2 = [0, \infty]$. Similarly, calculating Q_3^j, $j = 1, 2$, we obtain

$$\begin{aligned} Q_3^1 &= (-\infty, \infty), \\ Q_3^2 &= [-1, 0]. \end{aligned}$$

Hence, $Q_3 = [-1, 0]$. Taking the intersection of Q_1, Q_2 and Q_3, we have $Q = [0, 0]$.

Since $Q = \{0\}$, this optimal basis is unstable and we do not calculate the sensitivities with respect to this basis.

Tableau 2. In this case, we have

$$B^{-1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ -2 & 1 & 1 & 0 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{c}^{\mathrm{B}} = (-1, -1, 0, 0), \qquad \Delta \boldsymbol{c}^{\mathrm{B}} = (-0.5, 0, 0, 0),$$

$$\boldsymbol{c}^{\mathrm{N}} = (0, 0), \qquad \Delta \boldsymbol{c}^{\mathrm{N}} = (0, 0),$$

$$A^{\mathrm{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \Delta A^{\mathrm{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{b} = (13, 22, 9, 9)^{\mathrm{T}}, \qquad \Delta \boldsymbol{b} = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\bar{\boldsymbol{b}} = (9, 4, 5, 5)^{\mathrm{T}}, \qquad \Delta \bar{\boldsymbol{b}} = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\boldsymbol{\pi} = (-1, 0, 0, 0), \qquad \Delta \boldsymbol{\pi} = (0, 0, -0.5, 0)$$

and $k = 2$.

$\boldsymbol{v}$ defined by (22) is calculated as

$$\boldsymbol{v} = B^{-1} \Delta B_{\cdot 2} = (0, 1, 0, -1)^{\mathrm{T}}.$$

We obtain $Q_1 = (-1, \infty)$. By the calculation of Q_2^j, $j = 1, 2, \ldots, 4$, we have

$$\begin{aligned} Q_2^1 &= [-1, \infty), \\ Q_2^2 &= (-\infty, \infty), \\ Q_2^3 &= [-1, \infty), \\ Q_2^4 &= \left[-\tfrac{5}{9}, \infty\right). \end{aligned}$$

Hence, $Q_2 = [-\frac{5}{9}, \infty)$. Similarly, calculating Q_3^j, $j = 1, 2$, we obtain

$$\begin{aligned} Q_3^1 &= [-1, \infty), \\ Q_3^2 &= [0, \infty). \end{aligned}$$

Hence, $Q_3 = [0, \infty)$. Taking the intersection of Q_1, Q_2 and Q_3, we have $Q = [0, \infty)$.

Since $Q \neq \{0\}$, let us proceed to the calculation of sensitivities. From (28) to (30), we have

$$\frac{\partial \boldsymbol{\pi}((1, 1 + q_2))}{\partial q_2} = \frac{1}{2(1 + q_2)^2}(2, 0, -3 - 2q_2 - q_2^2, 0),$$

$$\frac{\partial \bar{\boldsymbol{b}}((1, 1 + q_2))}{\partial q_2} = \frac{4}{(1 + q_2)^2}(0, -1, 0, 1)^{\mathrm{T}},$$

$$\frac{\partial \bar{z}((1, 1 + q_2))}{\partial q_2} = \frac{1}{2(1 + q_2)^2}(-1 - 18q_2 - 9q_2^2),$$

for $q_2 \in [0, \infty)$.

Tableau 3. In this case, we have

$$B^{-1} = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & 0 & 1 & 1 \end{pmatrix}, \quad \Delta B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{c}^{\mathrm{B}} = (-1, -1, 0, 0), \quad \Delta \boldsymbol{c}^{\mathrm{B}} = (-0.5, 0, 0, 0),$$

$$\boldsymbol{c}^{\mathrm{N}} = (0, 0), \quad \Delta \boldsymbol{c}^{\mathrm{N}} = (0, 0),$$

$$A^{\mathrm{N}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Delta A^{\mathrm{N}} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix},$$

$$\boldsymbol{b} = (13, 22, 9, 9)^{\mathrm{T}}, \quad \Delta \boldsymbol{b} = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\bar{\boldsymbol{b}} = (4, 9, 0, 5)^{\mathrm{T}}, \quad \Delta \bar{\boldsymbol{b}} = (0, 0, 0, 0)^{\mathrm{T}},$$

$$\boldsymbol{\pi} = (-1, 0, 0, 0), \quad \Delta \boldsymbol{\pi} = (-0.5, 0, 0, 0.5)$$

and $k = 2$.

$\boldsymbol{v}$ defined by (22) is calculated as

$$\boldsymbol{v} = B^{-1} \Delta B_{\cdot 2} = (1, 0, 1, -1)^{\mathrm{T}}.$$

We obtain $Q_1 = (-\infty, \infty)$. By the calculation of Q_2^j, $j = 1, 2, \ldots, 4$, we have

$$\begin{aligned} Q_2^1 &= \left(-\infty, \tfrac{4}{9}\right], \\ Q_2^2 &= (-\infty, \infty), \\ Q_2^3 &= (-\infty, 0], \\ Q_2^4 &= \left[-\tfrac{5}{9}, \infty\right). \end{aligned}$$

Hence, $Q_2 = [-\frac{5}{9}, 0]$. Similarly, calculating Q_3^j, $j = 1, 2$, we obtain

$$\begin{aligned} Q_3^1 &= [-2, \infty), \\ Q_3^2 &= (-\infty, 0]. \end{aligned}$$

Hence, $Q_3 = [-2, 0]$. Taking the intersection of Q_1, Q_2 and Q_3, we have $Q = [-\frac{5}{9}, 0]$.

Since $Q \neq \{0\}$, let us proceed to the calculation of sensitivities. From (28) to (30), we have

$$\frac{\partial \boldsymbol{\pi}((1, 1 + q_2))}{\partial q_2} = (-0.5, 0, 0, 1.5 + q_2),$$

$$\frac{\partial \bar{\boldsymbol{b}}((1, 1 + q_2))}{\partial q_2} = (-9, 0, -9, 9)^{\mathrm{T}},$$

$$\frac{\partial \bar{z}((1, 1 + q_2))}{\partial q_2} = 7 + 9q_2,$$

for $q_2 \in [-\frac{5}{9}, 0]$.

7 Conclusions

In the present paper we consider sensitivity analysis with respect to a set of parameters which determine the objective coefficients, the right-hand side values and the matrix coefficients. It is assumed that each coefficient depends linearly on each parameter and each parameter occurs at most in one row or column of the matrix. The sensitivities of simplex multipliers, an optimal solution and the optimal value are explicitly represented. The effective domain where the sensitivities are available is analyzed and obtained as an interval. Moreover, an approach for treating the cases of degeneracy and multiplicity of the optimal solutions is proposed.

The proposed analysis is applicable under certain assumptions. From a theoretical point of view, making a sensitivity analysis under weaker assumptions will be a future topic along the line of this paper. On the other hand, application to the global environment model will be our next step.

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