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Behavioral Equilibria for a 2x2 "Seller-Buyer" Game Evolutionary Model

Kryazhimskiy, A.V.

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IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 807 □ Fax: +43 2236 71313 □ E-Mail: info@iiasa.ac.at

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Abstract

Equilibric behaviors typical for differential and multi-step games are defined for a 2×2 evolutionary game (two populations of players, two strategies for each player) roughly modeling interactions between sellers and buyers. It is shown that currently optimal behaviors of individuals form long-run equilibric dynamics at both individual and population levels.

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Behavioral Equilibria for a 2×2 "Seller-Buyer" Game-Evolutionary Model

*A. V. Kryazhimskii**

Introduction

We hold the viewpoint that economical evolution is driven by large groups (populations) of individuals interacting in a game style. This viewpoint developed within the framework of the evolutionary game-theoretic approach (see e.g. [Friedman, 1991] and [Young, 1993]) supplements the general theory of economic change (see Nelson and Winter, 1982]) with new analytical tools; these tools allow, in particular, to specify individual behaviors that, being subjectively rational for individuals, guide the whole populations to an equilibrium. Thus, a formal pattern for explaining the driving forces of economical evolution is provided.

In this paper we use another principle to select individual behaviors, which can be called the behavioral equilibrium principle. It extends the Nash equilibrium property – from populations' final states – to individuals' behaviors during the whole gaming process. Thus, within the pool of all potentially admissible collections of individual behaviors, Nash equilibric ones are selected. Nash equilibrium is treated in a usual way: a collection of individual behaviors is equilibric if every individual having any other behavior receives, eventually, a worse payoff.

The principle originates from the theories of multi-step and differential games. Application of the principle to a discrete time evolutionary game model does not reduce it to a standard multi-step game (see [Owen, 1968], [Basar and Olsder, 1982]): in a standard multi-step game payoffs are pre-determined by players' behaviors; in an evolutionary game model it is not so due to uncontrolled mixing of partner pairs (provoking strong uncertainties in dynamics).

In this paper we illustrate the behavioral equilibrium principle at a simplest two-population repeated game with two options (strategies) for each individual. Having in mind the economical background, we treat it as a stylized interaction between "buyers" and "sellers".

Starting with evolutions of individuals (the lower-level model), we, afterwards, convert them into population dynamics (the upper-level model).

Finally, we obtain multi-optimality of equilibric dynamics at both, individual and population levels, with respect to both short- and long-run payoff criteria.

At the individual level, it is reflected in the fact that individuals' instant optimal behaviors intended to increasing current personal benefits (though, due to dynamical uncertainties, not necessarily increasing them actually) satisfy the behavioral equilibrium principle, i.e. entirely fit individuals' long-run interests.

At the population level, it is reflected in the fact that populations' evolutions driven by instant optimal behaviors of individuals, first, are intended to increase current populations' benefits (instant optimality), and, second, satisfy the behavioral equilibrium principle – with the populations regarded as partners.

Moreover, the population behavioral equilibrium principle holds within the extended pool of regulation patterns involving both decentralized and totally centralized ones. Thus for our model, population's decentralized self-regulation (with individuals acting instant optimally with respect to their personal interests) is in the long run equivalent to best centralized regulation.

The subject of the paper adjoins the theory of evolutionary games (see [Maynard Smith, 1982], [Hofbauer and Sigmund, 1988], [Friedman, 1991], [Young, 1993], and the references in the last two papers). Methodologically, we follow in a great deal the spirit of the theory of closed-loop differential games [Krasovskii and Subbotin, 1988].

1 A 2×2 Population Game Model

1.1 The Model

The model we are concerned with is as follows. There are N sellers and N buyers. At every time from the time net $\Delta = \{0, \delta, 2\delta, ...\}$ every buyer visits a seller to buy a unit of good, and every seller deals with a single buyer. Thus, at each time N seller-buyer pairs are formed. The pairs vary arbitrarily.

In a more realistic model a number of sellers should be assumed less than that of buyers. Our approach allows to treat this model too. Wishing to deal with minimum technical details, we focus on the simplest variant.

Every seller controls prices. Following our line to extreme simplification, we suppose that a price of a good unit can be only of two categories: either low or high. Choosing a low or a high price are the two possible strategies of a seller. Let us call them, respectively, *"low price"* strategy and *"high price"* strategy. Every buyer has two strategies: to buy a good unit (*"buy"* strategy) and not to buy it (*"not buy"* strategy).

1.2 Payoff Matrices

Every time a buyer and a seller meet, their srategies match. Four possible outcomes are

"not buy" "low price"	*"buy" "low price"*
"not buy" "high price"	*"buy" "high price"*

Table 1.1

Following the game-theoretical approach, we suppose that every individual (a seller or a buyer) associate each outcome with a certain value serving as a fitness measure of this outcome; these values are called *payoffs*. If one replaces in Table 1.1 the outcomes by the corresponding payoffs for a certain individual, one obtains the *payoff matrix* of this individual. Thus, a seller and a buyer get their 2×2 payoff matrices a and b.

"Low price" and *"high price"* seller's strategies mean, respectively, choosing the first and the second line in a and b; in the standard manner, we will identify these strategies with their line numbers. *"Not buy"* and *"buy"* buyer's strategies mean, respectively, choosing the first and the second column in a and b, and thus will be identified with these column numbers. The payoffs to the seller and the buyer stand respectively in a and b at the intersection of the chosen line and column. This is a standard 2×2 matrix game. The process described in subsection 1.1 is therefore a collection of N 2×2 matrix games played infinitly many times with time step δ by mixing partners.

What payoffs should be associated by a particular seller and a particular buyer with the outcomes indicated in Table 1.1?

The answer is not obvious. The main difficulty is that an individual can hardly nominate a definite payoff value for each outcome. It is much easier for an individual to specify some preference relationships between the outcomes or, equivalently, some inequalities between the elements of her payoff matrix. (Here and in what follows we write "she" instead of "he/she".) If only such specifications are made, an individual actually *does not know* her payoff matrix precisely. What is known to her is a class of matrices containing her payoff matrix (this is the case we will be dealing with). This class is determined by the above inequailities between matrix elements.

We will denote every matrix class of the above kind by several arrows in brackets, every arrow showing the direction from a smaller to a bigger element; sometimes, in order to avoid ambiguity, we use instead of an arrow, the sign $\forall$ meaning no order specification. For instance, the class $(\uparrow\downarrow)$ consists of all matrices c with $c_{11} > c_{21}$ and $c_{12} < c_{22}$; the class $(\leftarrow\rightarrow)$ consists of all c with $c_{11} > c_{12}$ and $c_{21} < c_{22}$, etc. Here and in what follows, c_{ij} stands for the element at the intersection of the i-th line and the j-th column of matrix c.

1.3 Seller's Preferences

Take a particular seller. To sell a good unit for a high price is better for her than to sell it for a low price. Thus, looking at Table 1.1, we conclude that in the second column of the payoff matrix b the lower element is greater than upper.

Compare the outcomes from the first column of Table 1.1. What is better for the seller? To have the good unit not bought for a low price (the upper outcome), or to have it not bought for a high price (the lower outcome) If the good unit has not been bought for a high price, then the seller can think that it could have been bought if the price had been low. This means at last some moral discomfort ("my price strategy was possibly wrong"). If the good unit has not been bought for a low price, the seller does not feel such a moral discomfort ("I did my best, it is definitly not my fault"). The second case is more prefereble for the seller.

Along with the above "moral" justification for the seller's preference of the upper outcome in the first column of Table 1.1, one can imagine an "economical" reason for it. Suppose for instance that the seller pays a tax for profit in advance, i.e. having prices declared and goods not sold. Then, in case the good unit is not bought, the seller looses more money if a price is high. Thus, we assume that in the first column of the seller's payoff matrix a the upper element rather is bigger than lower. Not selling a good unit is worse than selling it, no matter what is the price, and selling a good unit for a high price (the left lower outcome) is worse than selling it for a low price (the right upper outcome).

Summing all up, we conclude that the seller's payoff matrix a belongs to the class $(\uparrow\downarrow\rightarrow\rightarrow\nearrow)$.

1.4 Buyer's Preferences

Take now a particular buyer. Starting with the supposition that the buyer is interested in purchasing a good unit, we turn it as follows: if the price is low, then buying a good unit is better for the buyer than not buying it. Consequently, in the first line of the buyer's payoff matrix b, we have the right element greater than left.

If this is so for the second line too, then *"buy"* is the totally dominating buyer's strategy, and there is no game. We reject it. Thus, we assume that the high price is actually too high for the buyer to buy a desired good unit. In other words, if the price is high, then not buying (saving money) is better than buying. The second line of the matrix b has the bigger element on the left. Finally, buying a good unit for a low price (the upper outcome on the right) is better than buying it for a high price (the lower outcome on the right).

Compare the outcomes in the left column. Not buying a good unit for a low price is a mistake; not buying it for a high price is a right decision. Consequently, the lower outcome on the left is better than the upper. Finally, not buying a good unit for a high price (the lower outcome on the left) is definitely worse than buying it for a low price (the upper outcome on the right).

Thus, b belongs to the class $(\downarrow\uparrow\rightarrow\leftarrow\nearrow)$.

1.5 Identity of Individuals. Individual and Population Goals

For simplicity, we will suppose that all sellers have the same payoff matrix a, and all buyers have the same payoff matrix b. Note once again that individuals do not know their payoff matrices being aware only of their classes. We will also suppose that sellers know the class of the buyers' payoff matrix b.

Each individual desires to increase her income in the long run. Besides, at each time she wants to increase her current income. These two (somewhat different) goals predetermine the indidual's behavior.

Individuals do not care of whole populations, which admittedly have their own goals too. We understand those as reaching maximum total populations' incomes with time going to infinity.

2 Population Dynamics

2.1 Local Events

We suppose that individuals are conservative enough – they do not change their strategies too frequently. Each individual comes to a decision to revise her strategy periodically, at some isolated times. We will call such times *active* for a corresponding individual, or say that an individual is *activated* at such times. All other times it will be called *passive*. We assume that there exists an *activation mechanism* that acivizes every individual by sending her from time to time the message *"decide"*; later we will describe it more definitely.

Besides, we assume that a seller's price change activates a buyer automatically (even without *"decide"*).

Narrate the above rules for individual decision making more accurately. In the sequel, we will use "names" for individuals. Times near which an individual named C receives a *"decide"* will be called *C-active*; all other times will be called *C-passive*. A pair (S, B) of interacting buyer and seller will be called a *partner pair*. Individuals S and B, being activated at time t, select their *actions* $p_S(t)$, $p_B(t)$ in the two-element *action set* $A = \{$ *"change"*, *"keep"* $\}$; *"change"* means a strategy change and *"keep"* means no strategy change.

Using the above terms, specify local sequences of events as follows.

Let t be passive for a buyer B. Take the sellers $S[t]$ and $S[t+\delta]$ playing with B at t and $t+\delta$ respectively. Between t and $t+\delta$, we have the following sequence of events:

1) $S[t]$ and B play their game;

2) B learns the $S[t+\delta]$'s strategy planned for $t+\delta$, and, in case it differs from the strategy of $S[t]$ at t, chooses her action $p_B(t)$.

Let t be active for B. Then between t and $t+\delta$, the sequence of events is as follows:

1) $S[t]$ and B play their game;

2) a *"decide"* message comes to B;

3) B learns the $S[t+\delta]$'s strategy planned for $t+\delta$, and chooses her action $p_B(t)$.

Let t be passive for a seller S. Take the buyers $B[t]$ and $B[t+\delta]$ playing with S at t and $t+\delta$, respectively. Between t and $t+\delta$, we have the following:

1) S and $B[t]$ play their game;

2) S keeps her strategy till $t+\delta$ (uses *"keep"*);

3) $B[t+\delta]$ behaves as described above, i.e. learns the S's strategy for $t+\delta$ and works out her action $p_{B[t+\delta]}(t)$ if t is $B[t+\delta]$-active, or t is $B[t+\delta]$-passive, and the S's strategy is new for $B[t+\delta]$ (i.e. differs from the seller's strategy $B[t+\delta]$ played with at t).

Let finally t be active for a seller S. Then:

1) S and $B[t]$ play their game;

2) a *"decide"* message comes to S;

3) S chooses her action $p_S(t)$;

4) $B[t+\delta]$ behaves as above.

2.2 Local Transitions

Take a seller S playing at t and $t+\delta$ strategies $i_S(t)$ and $i_S(t+\delta)$ respectively. Let $p_S(t)$ be a S's action at t. According to the previous description, the following relations are admissible.

t	$p_S(t)$	$i_S(t+\delta)$
S- passive	*"keep"*	$i_S(t)$
S- active	*"keep"*	$i_S(t)$
	"change"	$\neg i_S(t)$

Table 2.1

Here and in what follows, $\neg i$ denotes the individual's strategy complementary to i.

Take a buyer B playing $i_B(t)$ and $i_B(t+\delta)$ at t and $t+\delta$ respectively, and acting $p_B(t)$ at t. Let $S[t]$ and $S[t+\delta]$ be the B's partners at t and $t+\delta$ playing $i_{S[t]}(t)$ and $i_{S[t+\delta]}(t+\delta)$ respectively. Then, the following relations hold.

t	relation between $i_{S[t+\delta]}(t+\delta)$ and $i_{S[t]}(t)$	$p_B(t)$	$i_B(t+\delta)$
B- passive	$=$	*"keep"*	$i_B(t)$
	$\neq$	*"keep"*	$i_B(t)$
		"change"	$\neg i_B(t)$
B- active	$=$	*"keep"*	$i_B(t)$
		"change"	$\neg i_B(t)$
	$\neq$	*"keep"*	$i_B(t)$
		"change"	$\neg i_B(t)$

Table 2.2

2.3 Individual Feedbacks

Individuals S and B choose their control actions $p_S(t)$, $p_B(t)$ based on available information. If B is allowed to make her decision at t, she works out $p_B(t)$ based on her last strategy $i_B(t)$ her partner's strategy $i_{S[t+\delta]}(t+\delta)$ at $t+\delta$ and a strategy $i_{S[t]}(t)$ of her partner $S[t]$ at t. Write it as

$$p_B(t) = P_B(t, i_B(t), i_{S[t]}(t), i_{S[t+\delta]}(t+\delta)) \tag{1}$$

Using the terminology of the control theory, we will call any function P_B associating to every t and every triplet $(i_B(t), i_{S[t]}(t), i_{S[t+\delta]}(t+\delta))$ of strategies a control action (1), a *buyer feedback.*

Introducing, in a similar way, seller feedbacks, one should take into account that a seller S, when forming her action $p_S(t)$, knows a strategy $i_{B[t]}(t)$ played by her partner $B[t]$ at t but does not know $i_{B[t+\delta]}(t+\delta)$:

$$p_S(t) = P_S(t, i_S(t), i_{B[t]}(t)) \tag{2}$$

Thus, a *seller feedback* is identified with an arbitrary function P_S associating to every t and every strategy pair $(i_S(t), i_{B[t]}(t))$ an action (2).

We regard buyer and seller feedbacks as models of admissible individuals' behaviors.

2.4 Control Laws and Motions

Denote by $\mathcal{S}$ and $\mathcal{B}$ the populations of sellers and buyers, respectively. Admissible population motions are driven by collections (families) of seller and buyer feedbacks; such collections will be called control laws. More accurately, define a *$\mathcal{S}$-feedback* to be an arbitrary family (P_S) of seller feedbacks, with S running through $\mathcal{S}$. Similarly, a *$\mathcal{B}$-feedback* as an arbitrary family (P_B) of buyer feedbacks, with B running through $\mathcal{B}$. A pair $((P_S), (P_B))$ composed of a $\mathcal{S}$-feedback (P_S) and a $\mathcal{B}$-feedback (P_B) will be called a *control law.* A control law $((P_S), (P_B))$ generates certain evolutions of strategies $i_S(t)$ of every seller S and strategies $i_B(t)$ of every buyer B. To distinguish such evolutions which are *functions* defined on the time net Δ from particular strategy pairs at time t, we denote them by $i_S(\cdot)$ and $i_B(\cdot)$. A family of all such evolutions describes a population motion.

More accurately, a *population motion* generated by a control law $((P_S), (P_B))$ is identified with a family $\mathcal{M} = ((i_S(\cdot)), (i_B(\cdot)))$, $S \in \mathcal{S}$, $B \in \mathcal{B}$, of maps from Δ into the set of all seller-buyer strategy pairs, such that for every S, B.

(i) the strategies $i_S(t)$, $i_S(t+\delta)$ and the action $p_S(t)$, are subject to Table 2.1, and $p_S(t)$ satisfies (2) ($B[t]$ is the S's partner at t) whenever Table 2.2 does not restrict it to *"keep"*,

(ii) the strategies $i_B(t)$, $i_B(t+\delta)$ and the action $p_B(t)$, are subject to Table 2.1 ($S[t]$ and $S[t+\delta]$ are B's partners at t and $t+\delta$ respectively), and $p_B(t)$ satisfies (1) whenever Table 2.1 does not restrict it to *"keep"*.

Note that, along a population motion, any evolution of partner pairs is admissible.

2.5 Regular Activation

Let us say that an individual C *jumps* at time t (along a population motion $\mathcal{M}$) if the strategy pairs played (along $\mathcal{M}$) by C – together with her partner – at t and $t+\delta$ are different; 0 will formally be regarded as a jump time for every individual.

In the sequel, the following *regularity condition* is assumed: for an arbitrary individual C and an arbitrary time t at which C jumps (along an arbitrary population motion $\mathcal{M}$), there is a time $\xi > t$ at which C either is active or jumps.

Example 2.1 Suppose that each individual C has several critical levels for a current amount of her benefit (i.e. payoffs summed up to a current time); the lower and upper levels could be, respectively, those of survival and prosperity. Whenever one of these levels is reached, C gets an impulse to revise her behavior, or, in our terms, a *"decide"* message. As long as C plays - together with her current partners - a fixed pair of strategies, her current benefit has a constant speed. With this speed, C comes necessarily to one of the critical levels (and thus to an active time) provided she does not jump before. Under this activation mechanism, the regularity condition is satisfied.

3 Population Game

3.1 Game Description

The long-run goal of every individual is to maximize a limit payoff with time going to infinity.

Formulate it more accurately.

Let a population motion $\mathcal{M} = ((i_S(\cdot)), (i_B(\cdot)))$ be performed. The payoff to a particular seller S and a particular buyer B at time t are, respectively, $a_{i_S(t) i_{B[t]}(t)}$ and $b_{i_{S[t]}(t) i_B(t)}$ where $B[t]$ is the S's partner at t, and $S[t]$ is the B's partner at t. Therefore, S's and B's desire is to guide the population motion $\mathcal{M}$ so as to have

$$J^S(\mathcal{M}) = \lim_{t\to\infty} a_{i_S(t) i_{B[t]}(t)}, \quad J^B(\mathcal{M}) = \lim_{t\to\infty} b_{i_{S[t]}(t) i_B(t)}$$

as big as possible. In case these limits do not exist we assume J^S and J^B to be multivalued:

$$J^S(\mathcal{M}) = [\liminf_{t\to\infty} a_{i_S(t) i_{B[t]}(t)}, \limsup_{t\to\infty} a_{i_S(t) i_{B[t]}(t)}]$$

$$J^B(\mathcal{M}) = [\liminf_{t\to\infty} b_{i_{S[t]}(t) i_B(t)}, \limsup_{t\to\infty} b_{i_{S[t]}(t) i_B(t)}]$$

We understand joint maiximzation of these values as reaching an equilibrium in a multi-person game between sellers and buyers playing with their feedbacks. We use the Nash consept of equilibrium. According to the Nash approach, a collection of feedbacks (i.e. a control law) is considered as equilibric if there is no feedback leading a particular individual to a better payoff provided all other individuals play equilibrically. We strengthen it by replacing - in the above equilibrium nonimprovability requirement - a single individual by an arbitrary group of individuals from an equal population. (Multivalidity of J^S and J^B implies several ways to modify the standard definition; we will take the "sharpest" one).

Call a control law $((P_S^0), (P_B^0))$ *(a,b)-equilibric* if

(i) for every set $\mathcal{S}^*$ of sellers, every family (P_S^*), $S \in \mathcal{S}^*$, of seller feedbacks, every population motion $\mathcal{M}^0$ generated by $((P_S^0), (P_B^0))$ and every population motion $\mathcal{M}^{0*}$ generated by $((P_S^{0*}), (P_B^0))$ where $P_S^{0*} = P_S^0$ for $S \notin \mathcal{S}^*$ and $P_S^{0*} = P_S^*$ for $S \in \mathcal{S}_*$, it holds

$$\min J^S(\mathcal{M}^0) \geq \max J^S(\mathcal{M}^{0*})$$

with an arbitrary $S \in \mathcal{S}^*$, and

$$J^S(\mathcal{M}^0) = J^S(\mathcal{M}^{0*})$$

with an arbitrary $S \notin \mathcal{S}^*$;

(ii) for every set $\mathcal{B}^*$ of buyers, every family (P^*_B), $B \in \mathcal{B}^*$, of buyer feedbacks, every population motion $\mathcal{M}^0$ generated by $((P^0_S), (P^0_B))$ and every population motion $\mathcal{M}^{*0}$ generated by $((P^0_S), (P^{*0}_B))$ where $P^{*0}_B = P^0_B$ for $B \notin \mathcal{B}^*$ and $P^{*0}_B = P^*_B$ for $B \in \mathcal{B}^*$, it holds

$$\min J^B(\mathcal{M}^0) \geq \max J^B(\mathcal{M}^{*0})$$

with an arbitrary $B \in \mathcal{B}^*$, and

$$J^B(\mathcal{M}^0) = J^B(\mathcal{M}^{*0})$$

with an arbitrary $B \notin \mathcal{B}^*$.

The long-run goal of an individual can be now expressed as finding her component of an (a, b)-equilibric control law.

3.2 Instant Optimal Feedbacks

The instant goal of an individual is maximizing a payoff increment at a current time step. Let us model a possible way of thinking of a particular buyer B and a particular seller S at a current time t; we will end up with some heuristic decision making patterns which will be called instant optimal.

The B's speculations could be as follows. "The $S[t+\delta]$'s strategy $i_{S[t+\delta]}(t+\delta)$ is known to me. My action $p^0_B(t)$ should bring me to a $i_B(t+\delta)$ such that

$$b_{i_{S[t+\delta]}(t+\delta)\ i_B(t+\delta)} \geq b_{i_{S[t+\delta]}(t+\delta)\ \neg i_B(t+\delta)}$$

The fact that my payoff matrix b belongs to the class $(\rightarrow\leftarrow)$ (subsection 1.4) lead me to the following table."

$i_{S[t+\delta]}(t+\delta)$	$i_B(t)$	$p^0_B(t)$
1	1	*"change"*
1	2	*"keep"*
2	1	*"keep"*
2	2	*"change"*

Table 3.1

We consider Table 3.1 as a buyer feedback; call it *instant optimal.* If the activated buyer B holding this feedback finds herself at time t not buying a low-price good unit (i.e. playing the strategy pair $(1,1)$) and before $t+\delta$ learns that the price will not be high at $t+\delta$, she buys a good unit at $t+\delta$. In a similar way, other values of $p^*_B(t)$ can be commented. One easily sees that this buyer's behavior corresponds to the best-reply principle.

Now turn to the seller S deciding what $p^0_S(t)$ is instant optimal for her. Let, first, $(i_S(t), i_B(t)) = (1,1)$. The S's way of thinking could be like that. "Suppose that at $B[t+\delta]$ played the same strategy as $B[t]$. Suppose that I act *"change"*. Then $B[t+\delta]$ is automatically activated and reacts optimally with $p^0_{B[t+\delta]}(t) =$ *"keep"* (I refer to Table 3.1). In this case we move to $(2,1)$. Suppose that I act *"keep"*. If $B[t+\delta]$ is not activated with a *"decide"*, then we stay at $(1,1)$. If $B[t+\delta]$ is activated, then, with her optimal reaction $p^0_{B[t+\delta]}(t) =$ *"change"*, we move to $(1,2)$. Each of the last two strategy pairs is better for me than the first one, since my payoff matrix a is in $(\uparrow\downarrow\nearrow)$. Consequently, my instant optimal action $p^0_S(t)$ is *"keep"*." Represent the above pattern schematically:

(*"keep"*, *"keep"*) (1,1)	(*"keep"*, $p^0_{B[t+\delta]}(t)$) (1,2)
↑	↑
(*"change"*, $p^0_{B[t+\delta]}(t)$) (2,1)	

Table 3.2

Here arrows mark S's preferences. They show that at $(1,1)$ the instant optimal S's action is *"keep"*. Rolling, similarly, S's speculations under the assumption that $B[t+\delta]$ played at t the strategy different from that played by $B[t]$, we come to the simpler variant of Table 3.2 where the left option on the top is removed (due to the fact that $B[t+\delta]$ is definitely activated at t).

For other strategy pairs the decision making patterns are

(*"keep"*, *"keep"*) (1,2)	(*"keep"*, $p^0_{B[t+\delta]}(t)$) (1,2)
↑	↑
(*"change"*, $p^0_{B[t+\delta]}(t)$) (2,1)	

Table 3.3

(*"keep"*, *"keep"*) (2,1)	(*"keep"*, $p^0_{B[t+\delta]}(t)$) (2,1)
↓	↓
(*"change"*, $p^0_{B[t+\delta]}(t)$) (1,2)	

Table 3.4

(*"keep"*, *"keep"*) (2,2)	(*"keep"*, $p^0_{B[t+\delta]}(t)$) (2,1)
↑	↓
(*"change"*, $p^0_{B[t+\delta]}(t)$) (1,2)	

Table 3.5

(We omit the simplified variants of the Tables where the left options on the top are removed; they occure, as above, under the supposition that at t the strategies of $B[t+\delta]$ and $B[t]$ were different).

Tables 3.3 and 3.4 show that at $(1,2)$ and $(2,1)$ the instant optimal S's actions are, respectively, *"keep"* and *"change"*. At $(2,2)$ (*"buy"*, *"high price"*) there is no definite preference for S (Table 3.5): *"keep"* is better if B is not activated at t (the uparrow in Table 3.5); otherwise, *"change"* is better (the downarrow in Table 2.6). In other words, if the seller expects that the buyer that buyes a good unit for the high price at t will not do it at $t+\delta$ (being activated), then the seller should lower down the price; if the seller expects that a good unit will be bought at $t+\delta$ for the same (i.e. high) price (i.e. the buyer is not activated), she should keep the price high. Thus, if S believes that B will be activated before the next play, she uses *"change"*; if her belief is opposite, she uses *"keep"*. Each belief is possible and therefore each of the two actions can be assumed as instant optimal. Summing up, we come to the following table for $p^0_S(t)$:

	(1,1)	(1,2)	(2,1)	(2,2)
$p_S^0(t)$	*"keep"*	*"keep"*	*"change"*	*"keep"* or *"change"*

Table 3.6

We consider Table 3.6 as a seller feedback; call it *instant optimal.*

3.3 Equilibria

We will call $\mathcal{S}$- and $\mathcal{B}$-feedbacks composed, respectively, of instant optimal seller and buyer feedbacks *instant optimal.* A control law composed of instant optimal $\mathcal{S}$- and $\mathcal{B}$-feedbacks will be called *instant optimal.*

Our main conjecture is

Proposition 3.1 *An instant optimal control law is (a, b)-equilibric.*

Proposition 3.2 *Let $\mathcal{M}^0 = ((i_S^0(\cdot)), (i_B^0(\cdot)))$ be a population motion generated by an instant optimal control law. Then for every seller S and every buyer B, $(i_S^0(t), i_B^0(t)) = (1, 2) =$ ("buy", "low price") for all t sufficiently large.*

Prove Propositions 3.1 and 3.2. Fix an instant optimal control law $((P_S^0), (P_B^0))$, and arbitrary control law $((P_S^*), (P_B^*))$, arbitrary groups $\mathcal{S}^*$ and $\mathcal{B}^*$ of sellers and buyers, respectively. Build the control law $((P_S^{0*}), (P_B^0))$ where $P_S^{0*} = P_S^0$ for $S \notin \mathcal{S}^*$ and $P_S^{0*} = P_S^*$ for $S \in \mathcal{S}^*$. Take an arbitrary population motion $\mathcal{M}^{0*}$ generated by this control law. Similarly, build the control law $((P_S^0), (P_B^{*0}))$ where $P_B^{*0} = P_B^0$ for $B \notin \mathcal{B}^*$ and $P_B^{*0} = P_B^*$ for $S \in \mathcal{S}^*$, and take an arbitrary population motion $\mathcal{M}^{*0}$ generated by this control law.

We base on the following lemmas.

Lemma 3.1 *Let B be an arbitrary buyer and $S[t]$ be the B's partner at t along the population motion $\mathcal{M}^{0*}$. Then $(i_{S[t]}(t), i_B(t)) \in \{(1, 2), (2, 1)\}$ for all t sufficiently large.*

Lemma 3.2 *Let a seller S not belong to $\mathcal{S}^*$ and $B[t]$ be the S's partner at t along the population motion $\mathcal{M}^{0*}$. Then $(i_S(t), i_{B[t]}(t)) = (1, 2)$ for all t sufficiently large.*

Lemma 3.3 *Let a buyer B not belong to $\mathcal{B}^*$ and $S[t]$ be the B's partner at t along the population motion $\mathcal{M}^{*0}$. Then $(i_{S[t]}(t), i_B(t)) = (1, 2)$ for all t sufficiently large.*

Proposition 3.2. follows immediately from Lemmas 3.2 and 3.3 by putting $P_S^* = P_S^0$, $P_B^* = P_B^0$.

Prove Proposition 3.1. By Proposition 3.2 for an arbitrary seller S, it holds $\min J^S(\mathcal{M}^0) = a_{12}$. By Lemmas 3.1 and 3.2 we have, respectively, $\max J^S(\mathcal{M}^{0*}) \le \max\{a_{12}, a_{21}\} = a_{12}$ if $S \in \mathcal{S}^*$, and $\min J^S(\mathcal{M}^{0*}) = a_{12}$ if $S \notin \mathcal{S}^*$. The first property from the definition of an equilibric control law is verified. The second one is verified similarly. Namely, by Proposition 3.2 for an arbitrary buyer B, it holds $\min J^B(\mathcal{M}^0) = b_{12}$. Obviously $\max J^B(\mathcal{M}^{*0}) \le \max\{b_{11}, b_{12}, b_{21}, b_{22}\} = b_{12}$ if $B \in \mathcal{B}^*$, and by Lemma 3.3 $\min J^B(\mathcal{M}^{*0}) = b_{12}$ if $B \notin \mathcal{B}^*$. Proposition 3.1 is proved.

Prove Lemma 3.1. Take an arbitrary buyer B. If at a certain t $S[t]$ and B play (along $\mathcal{M}^{0*}$) (1,2) or (2,1), the same is at $t + \delta$, i.e. $S[t + \delta]$ and B play (1,2) or (2,1). Indeed, if $S[t + \delta]$ plays the same strategy as $S[t]$, then, no matter if B is activated at t or not, she acts *"keep"* (see Table 3.1), and plays with S the same as with $S[t]$. If S plays the other strategy as $S[t]$, then B acts *"change"* (see Table 3.1), and therefore plays with S (2,1) or (1,2) provided she plays with $S[t]$ (1,2) or (2,1), respectively. Hence for all times greater than t, B plays with her partners (1,2) or (2,1). Let at a certain time ξ B play – with her partner – (1,1) or (2,2). By the regularity condition, there is a minimum $t \geq \xi$ at wich B either jumps or is activated. If B jumps at t, then either she changes her strategy at t, and thus t is necessarily B-active, or her partner's strategy at $t + \delta$ differs from that at t; the latter also implies that t is B-active. Therefore, we conclude that B is necessarily activated at t. Hence, B acts at t in accordance with Table 3.1 (where S is the B's partner at $t + \delta$). Her action $p_B^0(t)$ moves (S, B) to (1,2) or (2,1). We get the previous situation. Lemma 3.1 is proved.

Prove Lemma 3.2. Take a seller S not belonging to $\mathcal{S}^*$. By Lemma 3.1 for all sufficiently large t every buyer B plays with her partner (along $\mathcal{M}^{0*}$) either (1,2) or (2,1) (we take into account that the number of buyers is finite). Since for every such t, S plays with a certain buyer, we conclude that for all sufficiently large times S and her partner play either (1,2) or (2,1). Let us consider only such large times. At (1,2) the S's action is *"keep"* (see Table 3.6); therefore if S arrives at (1,2), she never leaves it. If S plays (2,1) at some t, then, by the regularity condition, there exists a $\xi \geq t$ at which S either jumps or is active. A jump at ξ without S being active at this time means that at $\xi + \delta$ S – together with her partner – play (2,2). This is impossible. Hence S is active at ξ. According to Table 3.6, the S's action at ξ is *"change"*. Thus, at $\xi + \delta$ S – with her partner – move to (1,1) or (1,2). The first case is impossible. Consequently, starting from $\xi + \delta$, S plays with her current partners only (1,2).

Lemma 3.3 is proved similarly. Namely, modifyting unessentially the proof of Lemma 3.1, we establish that for all t suffiviently large, it holds $(i_{S[t]}(t), i_B(t)) \in \{(1,2), (2,1)\}$. The rest of the proof is similar to that of Lemma 3.2, the above property playing the role of Lemma 3.1.

3.4 Stationarity of Equilibric Motions

The property of population motions indicated in Proposition 3.2 will be called $(1,2)$-*stationarity.* Proposition 3.2 admits the following generalization.

Proposition 3.3 *If $((P_S^*), (P_B^*))$ is (a, b)-equilibric, then every population motion generated by $((P_S^*), (P_B^*))$ is $(1,2)$-stationary.*

Indeed, suppose that there is a population motion $\mathcal{M}^* = ((i_S^*(\cdot), (i_B^*(\cdot)))$ generated by $((P_S^*), (P_B^*))$ which is not $(1,2)$-stationary. Then one of the two cases takes place: (i) there exists a seller S such that for infinitely many t, $(i_S^*(t), i_{B[t]}^*(t)) = (k, l) \neq (1, 2)$ where $B[t]$ is the S's partner at t along the population motion $\mathcal{M}^*$, or (ii) there exists a buyer B such that for infinitely many t, $(i_{S[t]}^*(t), i_B^*(t)) = (k, l) \neq (1, 2)$ where $S[t]$ is the B's partner at t along $\mathcal{M}^*$. If (i) takes place, then among buyers $B[t]$ a one buyer B is repeated infinitely many times, and we have (ii) with $S[t] = S$. Similarly, (ii) implies (i) with $B[t] = B$. Therefore assume without loss of generality that there exist a seller S and a buyer B such that along the population motion $\mathcal{M}^*$, S and B are partners at infinitely many times t every time playing (k, l). The minimum of J^S at $\mathcal{M}^*$ – denote it J_*^S – is no bigger than a_{kl}, and the minimum of J^B at $\mathcal{M}^*$ – denote it J_*^B – is no bigger than b_{kl}. Let $(k, l) = (2, 2)$. Then $J_*^B \leq b_{22} < \min\{b_{12}, b_{21}\}$. The last value is no bigger than the minimum of J^B at an arbitrary population motion generated by $((P_S^*), (P_B^0))$ where (P_B^0) is the instant optimal $\mathcal{B}$-feedback; this follows from Lemma 3.2 (where $\mathcal{S}^* = \mathcal{S}$ and $\mathcal{B}^*$ coincides with $\mathcal{B}$ minus B). Consequently, $((P_S^*), (P_B^*))$ is not (a, b)-equilibric. The same is obtained similarly in the case $(k, l) = (1, 1)$. Let $(k, l) = (2, 1)$. Then $J_*^S \leq a_{21} < \min\{a_{11}, a_{12}\}$. The last value is no bigger than the minimum of J^S at an arbitrary population motion generated by $((P_S), (P_B^*))$ where P_S prescribes *"change"* at the lower line and *"keep"* at the upper line. Consequently, (P_S^*, P_B^*) is not (a, b)-equilibric. Proposition 3.3 is proved.

Proposition 3.4 *Proposition 3.3 is irreversible. Namely, if a control law $((P_S^*), (P_B^*))$ is such that an arbitrary population motion generated by it is $(1,2)$-stationary, then $((P_S^*), (P_B^*))$ is not necessarily (a, b)-equilibric.*

Indeed, let P_S^* prescribe *"keep"* at the upper line and *"change"* at the lower line, and P_B^* prescribe *"keep"* at the right column and *"change"* at the left column. The $(1,2)$-stationarity property takes place for any population motion generated by $((P_S^*), (P_B^*))$. Hence for an arbitrary seller S, J^S takes value a_{12} at any such population motion. On the other hand, J^S takes the greater value a_{22} at a population motion generated by the control law $((P_S), (P_B^*))$ where P_S prescribes *"keep"* in any position; the above (constant) population motion occurs if all sellers and buyers play 2 at $t = 0$. Consequently, $((P_S^*), (P_B^*))$ is not (a, b)-equilibric.

4 Evolutions of Cluster Numbers

4.1 Clusters of Partner Pairs

In this section we look at the population evolutions from a "far away" point where a single individual is not seen. Only the numbers of the groups (clusters) of partner pairs playing different strategies are distinguished. In other words, we project population motions onto the space of the numbers of clusters and track their evolutions. The goal is to describe macro-level transitions resulting into individual efforts and driving the populations along equilibric trajectories.

We understand a *cluster* as a group of partner pairs playing the same strategies. For a cluster of partner pairs playing at time t a strategy pair (k, l) we will use the notation $G_{kl}(t)$. The sets of all sellers and buyers playing within a cluster $G_{kl}(t)$ will be denoted, respectively, $G_{kl}^{\mathcal{S}}(t)$ and $G_{kl}^{\mathcal{B}}(t)$. By $n_{kl}(t)$ we will denote the number of the partner pairs belonging to a cluster $G_{kl}(t)$, or, equivalently, the number of individuals in each of the groups $G_{kl}^{\mathcal{S}}(t)$ and $G_{kl}^{\mathcal{B}}(t)$. Note that collections $n(t)$ of numbers $n_{kl}(t)$ $(k, l = 1, 2)$ determine the total payoffs $K^{\mathcal{S}}(t)$ and $K^{\mathcal{B}}(t)$ to the populations $\mathcal{S}$ and $\mathcal{B}$ at time t; namely,

$$K^{\mathcal{S}}(n(t)) = \sum_{k,l=1,2} a_{kl} n_{kl}(t), \quad K^{\mathcal{B}}(n(t)) = \sum_{k,l=1,2} b_{kl} n_{kl}(t). \tag{1}$$

In this sense, a collection of numbers $n_{kl}(t)$ provides minimal characterization of a population state at time t. We will be interested in modeling evolutions of these numbers.

Each population motion $\mathcal{M}$ (speaking of a population motion we always imply that it is generated by a certain control law) determines an evolution of clusters: at every time t $G_{kl}(t)$ is the collection of all partner pairs (S, B) occuring at t along $\mathcal{M}$ and playing (k, l). This evolution identified with the function $G(\cdot) = (G_{11}(\cdot), (G_{12}(\cdot), (G_{21}(\cdot), (G_{22}(\cdot))$ will be called the *cluster image* of the population motion $\mathcal{M}$. Taking cluster numbers $n_{kl}(t)$ occuring along the cluster image $G(\cdot)$, we come to the function $n(\cdot) = (n_{11}(\cdot), n_{12}(\cdot), n_{21}(\cdot), n_{22}(\cdot))$; call it the *number image* of the population motion $\mathcal{M}$. Number images of population motions are now in the focus of our study.

4.2 Local Transitions and Number Evolutions

Specify the mechanism driving number images of population motions.

Take the number image $n(\cdot) = (n_{11}(\cdot), n_{12}(\cdot), n_{21}(\cdot), n_{22}(\cdot))$ of an arbitrary population motion $\mathcal{M}$ and consider the transition from $n(t)$ to $n(t+\delta)$. We represent it as the following three-step procedure.

Step 1 (sellers' decision making). The individual games at t are over, a part of sellers are activated, and some of them act *"change"*. The output at this step is the numbers $X_1(t+\delta)$ and $X_2(t+\delta)$ of sellers who will play 1 and 2 respectively at $t+\delta$. The transition is expressed as follows. Let $u_{kl}(t)$ be the number of all sellers from $G^{\mathcal{S}}_{kl}(t)$ acting *"change"*. Note that

$$u_{kl}(t) \leq n_{kl}(t) \quad (k, l = 1, 2) \tag{2}$$

Put

$$u(t) = -u_{11}(t) - u_{12}(t) + u_{21}(t) + u_{22}(t) \tag{3}$$

Then

$$X_1(t+\delta) = X_1(t) + u(t), \quad X_2(t+\delta) = X_2(t) - u(t) = N - X_1(t+\delta) \tag{4}$$

where

$$X_1(t) = n_{11}(t) + n_{12}(t), \quad X_2(t) = n_{21}(t) + n_{22}(t) = N - X_1(t) \tag{5}$$

are, respectively, the numbers of sellers playing 1 and 2 at t. Note that $u(t)$ is the increment of the number of sellers playing 1 which one gets after the t-to $(t+\delta)$ transition ($u(t)$ could be negative).

Step 2 (partners mixing). Buyers come to sellers; partner pairs for $t+\delta$ are formed. At the output we have the numbers $m_{kl}(t)$ of the *pre-clusters* $E_{kl}(t+\delta)$. The pre-cluster $E_{kl}(t+\delta)$. is the set of those partner pairs where the seller is ready to play k and the buyer (having yet no strategy for $t+\delta$) played l at t. Let $q_{il,k}(t)$ be the number of all buyers who emerge from $G^{\mathcal{B}}_{il}(t)$ and meet sellers ready to play k at $t+\delta$. Note that $q_{il,k}(t)$ is subject to

$$q_{il,1}(t) + q_{il,2}(t) = n_{il}(t) \quad (i, l = 1, 2) \tag{6}$$

$$\sum_{i,l=1,2} q_{il,k}(t) = X_k(t+\delta) \quad (k = 1, 2) \tag{7}$$

Then for the numbers $m_{kl}(t)$ of the pre-clusters we have

$$m_{kl}(t) = q_{1l,k}(t) + q_{2l,k}(t) \tag{8}$$

Step 3 (buyers' decision making). A part of buyers (those who are sent a message *"decide"* or find their partners playing strategies different from those at t) are activated and some of them act *"change"*. Pre-clusters $E_{kl}(t+\delta)$ turn into clusters $G_{kl}(t+\delta)$. Note that a buyer's l-to-$\neg l$ strategy change moves a partner pair from a pre-cluster indexed kl to the cluster indexed $k\neg l$. The corresponding transition is expressed as follows. Let $v_{kl}(t)$ be the number of buyers who, after finding themselves – together with their partners – in the pre-cluster $E_{kl}(t+\delta)$, change their strategies before $t+\delta$. Clearly,

$$v_{kl}(t) \leq m_{kl}(t) \quad (k, l = 1, 2) \tag{9}$$

For the numbers $n_{kl}(t+\delta)$ of clusters $G_{kl}(t+\delta)$ we have

$$n_{kl}(t+\delta) = m_{kl}(t) - v_{kl}(t) + v_{k\neg l}(t) \quad (k, l = 1, 2) \tag{10}$$

Thus we conclude that if $n(\cdot) = (n_{11}(\cdot), n_{12}(\cdot), n_{21}(\cdot), n_{22}(\cdot))$ is a number image of a certain population motion, then at every time t there exist nonnegative integers $u_{kl}(t)$, satisfying (2), $X_k(t+\delta)$ satisfying (4), (3), $q_{il,k}(t)$ satisfying (6), (7), and $v_{kl}(t)$ satisfying (9), such that the transition from $n(t)$ to $n(t+\delta)$ is given by (8), (10). Every function $n(\cdot) = (n_{11}(\cdot), n_{12}(\cdot), n_{21}(\cdot), n_{22}(\cdot))$ having the above property will be called a *number evolution*. The brief summary is

Proposition 4.1 *The number image of an arbitrary population motion is a number evolution.*

Proposition 4.1 is inverted as follows.

Proposition 4.2 *Every number evolution is the number image of a certain population motion.*

We omit the proof of Proposition 4.2.

Propositions 4.1 and 4.2 show that number evolutions and only they are number images of population motions.

4.3 Number Images under Instant Optimal $\mathcal{B}$-Feedback

Let us focus on number images of "partially optimal" population motions, i.e. those generated by control laws where one of the feedbacks is instant optimal (recall that instant optimal control laws are equilibric, see subsection 3.3). Our goal is to provide a rational interpretation for local transitions of cluster numbers along these motions. Namely, we will show that these transitions follow the principle of local increasing of a total payoff; current total payoffs have thus the sense of fitness functions ([see Friedman, 1992]) guiding the populations equilibrically.

In this and the next subsections we deal with instant optimal feedbacks of the population of buyers (a $\mathcal{B}$-feedback).

Fix the instant optimal $\mathcal{B}$-feedback (P_B^0) and take a population motion $\mathcal{M}$ generated by the control law $((P_S), (P_B^0))$ where (P_S) is an arbitrary $\mathcal{S}$-feedback. Consider the number image $n(\cdot)$ of $\mathcal{M}$. The transition from $n(t)$ to $n(t+\delta)$ passes through the three-steps procedure described in the previous subsection. The numbers introduced in this procedure, satisfy the relationships (2) – (10). The $\mathcal{B}$-feedback (P_B^0) determines the values $v_{kl}(t)$. Recall that $v_{kl}(t)$ is the number of buyers who find themselves – together with their partners – in the pre-cluster $E_{kl}(t+\delta)$, and change their strategies. We put

$$v_{kl}(t) = v_{kl}^{(aut)}(t) + v_{kl}^{(dec)}(t) \qquad (11)$$

where $v_{kl}^{(aut)}(t)$ and $v_{kl}^{(dec)}(t)$ are the numbers of the above buyers who are activated, respectively, automatically (by a strategy change of their partners) and through a message *"decide"*.

Consider the first of the above groups of buyers.

Recall that the pre-cluster $E_{kl}(t+\delta)$ is the set of those partner pairs (at $t+\delta$) where the seller is ready to play k and the buyer played l at t. The fact that a buyer B coming from $G_{il}^{\mathcal{B}}(t)$ to $E_{kl}(t+\delta)$ notices that her partner plays at $t+\delta$ a strategy differing from that played by her partner at t is reflected by $i \neq k$. Every such buyer B is automatically activated and implements her action $p_B(t)$ in accordance with her instant optimal feedback P_B^0. This action is subject to Table 3.1 where $i_S(t+\delta) = k$ and $i_B(t) = l$. From Table 3.1 we see that $p_B(t) =$ *"change"* if and only if $k = l$. Thus we conclude that the buyers coming from $G^{\mathcal{B}}_{\neg kk}(t)$ to $E_{kk}(t+\delta)$ and only they are activated automatically and change their strategies. The number of such buyers is $q_{\neg kk,k}(t)$. Each of them moves the corresponding partner pair from the pre-cluster $E_{kk}(t+\delta)$ to the cluster $G_{k\neg k}(t+\delta)$. Consequently

$$v_{11}^{(aut)}(t) = q_{21,1}(t), \quad v_{12}^{(aut)}(t) = 0$$

$$v_{21}^{(aut)}(t) = 0, \quad v_{22}^{(aut)}(t) = q_{12,2}(t)$$

A buyer B activated by a *"decide"* acts *"change"* if and only if she finds herself – together with her partner – in $E_{kk}(t+\delta)$. Hence

$$v_{12}^{(dec)}(t) = 0, \quad v_{21}^{(dec)}(t) = 0$$

All buyers coming to $E_{kk}(t+\delta)$ from $G^{\mathcal{B}}_{\neg kk}(t)$ are activated automatically; they were already taken into account. All other buyers activated by a *"decide"* and acting *"change"* come to $E_{kk}(t+\delta)$ from $G^{\mathcal{B}}_{kk}(t)$; their number is no bigger than $q_{kk,k}(t)$. Hence

$$v_{11}^{(dec)}(t) \le q_{11,1}(t), \quad v_{22}^{(dec)}(t) \le q_{22,2}(t) \tag{12}$$

Note that if a buyer B travels – together with her partner – from $G^{\mathcal{B}}_{kk}(t)$ to $E_{kk}(t+\delta)$, and does not change her strategy, she plays – together with partner – at $t+\delta$ the same strategy pair as at t. Due to the regularity condition this can be repeated only a finite number of times; finally, at a certain t B is activated (a B's jump implies that B is activated as well). At this t B acts *"change"* yielding $v_{kk}^{(dec)}(t) > 0$. Thus we conclude that every sequence of adjoining times ξ where $q_{kk,k}(\xi) > 0$, is either empty or stops at a t where $v_{kk}^{(dec)}(t) > 0$ $(k = 1,2)$. This property of functions $v_{kk}^{(dec)}(\cdot)$ will be called *nondegeneracy.*

Substituting the obtained values for $v_{kl}(t)$ in (8), (10), we get

$$n_{11}(t+\delta) = q_{11,1}(t) - v_{11}^{(dec)}(t), \quad n_{12}(t+\delta) = q_{12,1}(t) + q_{22,1}(t) + q_{21,1}(t) + v_{11}^{(dec)}(t) \tag{13}$$

$$n_{21}(t+\delta) = q_{21,2}(t) + q_{11,2}(t) + q_{12,2}(t) + v_{22}^{(dec)}(t), \quad n_{22}(t+\delta) = q_{22,2}(t) - v_{22}^{(dec)}(t) \tag{14}$$

We summarize this as follows.

Proposition 4.3 *Let $(P^0_{\mathcal{B}})$ be the instant optimal $\mathcal{B}$-feedback, $(P_{\mathcal{S}})$ be an arbitrary $\mathcal{S}$-feedback, $\mathcal{M}$ be a population motion generated by the control law $((P_{\mathcal{S}}),(P^0_{\mathcal{B}}))$, and $n(\cdot)$ be the number image of $\mathcal{M}$. Then the equalities (17), (18) hold where functions $v_{11}^{(dec)}(\cdot)$, $v_{22}^{(dec)}(\cdot)$ are nondegenerate and satisfy (16).*

Backward speculations lead to the reverse conjecture (we omit the proof):

Proposition 4.4 *Let $n(\cdot)$ be a number evolution such that the equalities (17), (18) hold where functions $v_{11}^{(dec)}(\cdot)$, $v_{22}^{(dec)}(\cdot)$ are nondegenerate and satisfy (16). Then there exist an $\mathcal{S}$-feedback $(P_{\mathcal{S}})$ and a population motion $\mathcal{M}$ generated by the control law $((P_{\mathcal{S}}),(P^0_{\mathcal{B}}))$ such that $n(\cdot)$ is the number image of $\mathcal{M}$.*

4.4 Number Images under Instant Optimal $\mathcal{B}$-Feedback

Let us focus on number images of "partially optimal" population motions, i.e. those generated by control laws where one of the feedbacks is instant optimal (recall that instant optimal control laws are equilibric, see subsection 3.3). Our goal is to provide a rational interpretation for local transitions of cluster numbers along these motions. Namely, we will show that these transitions follow the principle of local increasing of a total payoff; current total payoffs have thus the sense of fitness functions ([see Friedman, 1992]) guiding the populations equilibrically.

In this and the next subsections we deal with instant optimal feedbacks of the population of buyers (a $\mathcal{B}$-feedback).

Fix the instant optimal $\mathcal{B}$-feedback (P_B^0) and take a population motion $\mathcal{M}$ generated by the control law $((P_S),(P_B^0))$ where (P_S) is an arbitrary $\mathcal{S}$-feedback. Consider the number image $n(\cdot)$ of $\mathcal{M}$. The transition from $n(t)$ to $n(t+\delta)$ passes through the three-steps procedure described in the previous subsection. The numbers introduced in this procedure, satisfy the relationships (2) - (10). The $\mathcal{B}$-feedback (P_B^0) determines the values $v_{kl}(t)$. Recall that $v_{kl}(t)$ is the number of buyers who find themselves – together with their partners – in the pre-cluster $E_{kl}(t+\delta)$, and change their strategies. We put

$$v_{kl}(t) = v_{kl}^{(aut)}(t) + v_{kl}^{(dec)}(t) \tag{15}$$

where $v_{kl}^{(aut)}(t)$ and $v_{kl}^{(dec)}(t)$ are the numbers of the above buyers who are activated, respectively, automatically (by a strategy change of their partners) and through a message *"decide"*.

Consider the first of the above groups of buyers.

Recall that the pre-cluster $E_{kl}(t+\delta)$ is the set of those partner pairs (at $t+\delta$) where the seller is ready to play k and the buyer played l at t. The fact that a buyer B coming from $G_{il}^{\mathcal{B}}(t)$ to $E_{kl}(t+\delta)$ notices that her partner plays at $t+\delta$ a strategy differing from that played by her partner at t is reflected by $i \neq k$. Every such buyer B is automatically activated and implements her action $p_B(t)$ in accordance with her instant optimal feedback P_B^0. This action is subject to Table 3.1 where $i_S(t+\delta) = k$ and $i_B(t) = l$. From Table 3.1 we see that $p_B(t) =$ *"change"* if and only if $k = l$. Thus we conclude that the buyers coming from $G_{\neg kk}^{\mathcal{B}}(t)$ to $E_{kk}(t+\delta)$ and only they are activated automatically and change their strategies. The number of such buyers is $q_{\neg kk,k}(t)$. Each of them moves the corresponding partner pair from the pre-cluster $E_{kk}(t+\delta)$ to the cluster $G_{k\neg k}(t+\delta)$. Consequently

$$v_{11}^{(aut)}(t) = q_{21,1}(t), \quad v_{12}^{(aut)}(t) = 0$$

$$v_{21}^{(aut)}(t) = 0, \quad v_{22}^{(aut)}(t) = q_{12,2}(t)$$

A buyer B activated by a *"decide"* acts *"change"* if and only if she finds herself – together with her partner – in $E_{kk}(t+\delta)$. Hence

$$v_{12}^{(dec)}(t) = 0, \quad v_{21}^{(dec)}(t) = 0$$

All buyers coming to $E_{kk}(t+\delta)$ from $G_{\neg kk}^{\mathcal{B}}(t)$ are activated automatically; they were already taken into account. All other buyers activated by a *"decide"* and acting *"change"* come to $E_{kk}(t+\delta)$ from $G_{kk}^{\mathcal{B}}(t)$; their number is no bigger than $q_{kk,k}(t)$. Hence

$$v_{11}^{(dec)}(t) \leq q_{11,1}(t), \quad v_{22}^{(dec)}(t) \leq q_{22,2}(t) \tag{16}$$

Note that if a buyer B travels – together with her partner – from $G_{kk}^{\mathcal{B}}(t)$ to $E_{kk}(t+\delta)$, and does not change her strategy, she plays – together with partner – at $t+\delta$ the same strategy pair as at t. Due to the regularity condition this can be repeated only a finite number of times; finally, at a certain t B is activated (a B's jump implies that B is activated as well). At this t B acts *"change"* yielding $v_{kk}^{(dec)}(t) > 0$. Thus we conclude that every sequence of ajoining times ξ where $q_{kk,k}(\xi) > 0$, is either empty or stops at a t where $v_{kk}^{(dec)}(t) > 0$ $(k = 1, 2)$. This property of functions $v_{kk}^{(dec)}(\cdot)$ will be called *nondegeneracy.*

Substituting the obtained values for $v_{kl}(t)$ in (8), (10), we get

$$n_{11}(t+\delta) = q_{11,1}(t) - v_{11}^{(dec)}(t), \quad n_{12}(t+\delta) = q_{12,1}(t) + q_{22,1}(t) + q_{21,1}(t) + v_{11}^{(dec)}(t) \quad (17)$$

$$n_{21}(t+\delta) = q_{21,2}(t) + q_{11,2}(t) + q_{12,2}(t) + v_{22}^{(dec)}(t), \quad n_{22}(t+\delta) = q_{22,2}(t) - v_{22}^{(dec)}(t) \quad (18)$$

We summarize this as follows.

Proposition 4.5 *Let (P_B^0) be the instant optimal $\mathcal{B}$-feedback, (P_S) be an arbitrary $\mathcal{S}$-feedback, $\mathcal{M}$ be a population motion generated by the control law $((P_S), (P_B^0))$, and $n(\cdot)$ be the number image of $\mathcal{M}$. Then the equalities (17), (18) hold where functions $v_{11}^{(dec)}(\cdot)$, $v_{22}^{(dec)}(\cdot)$ are nondegenerate and satisfy (16).*

Backward speculations lead to the reverse conjecture (we omit the proof):

Proposition 4.6 *Let $n(\cdot)$ be a number evolution such that the equalities (17), (18) hold where functions $v_{11}^{(dec)}(\cdot)$, $v_{22}^{(dec)}(\cdot)$ are nondegenerate and satisfy (16). Then there exist an $\mathcal{S}$-feedback (P_S) and a population motion $\mathcal{M}$ generated by the control law $((P_S), (P_B^0))$ such that $n(\cdot)$ is the number image of $\mathcal{M}$.*

4.5 $\mathcal{B}$-Fitting

Consider a number evolution $n(\cdot)$ and a population motion $\mathcal{M}$ satisfying the conditions of Proposition 4.5 (4.6). Compare the numbers $n_{kl}(t+\delta)$ of clusters $G_{kl}(t+\delta)$ with the numbers $m_{kl}(t)$ of pre-clusters $E_{kl}(t+\delta)$. Due to (17), (18) and (8) we have

$$n_{11}(t+\delta) \le m_{11}(t), \quad n_{12}(t+\delta) \ge m_{12}(t) \quad (19)$$

$$n_{21}(t+\delta) \ge m_{21}(t), \quad n_{22}(t+\delta) \le m_{22}(t) \quad (20)$$

Moreover, the first two inequalities are strict if either $q_{21}(t) > 0$ or $v_{11}^{(dec)}(t) > 0$, and the second two inequalities are strict if either $q_{12}(t) > 0$ or $v_{22}^{(dec)}(t) > 0$. Hence (recall the class of the matrix b)

$$K^{\mathcal{B}}(n(t+\delta)) \ge K^{\mathcal{B}}(m(t)) \quad (21)$$

with the strict inequality holding provided one of the above mentioned conditions takes place. Here (see (1)) $K^{\mathcal{B}}(m(t))$ $m(t)$ is the collection of numbers $m_{kl}(t)$) is the payoff the population $\mathcal{B}$ would get at $t+\delta$ if all buyers following $\mathcal{M}$ up to t would not change their strategies between t and $t+\delta$, and $K^{\mathcal{B}}(n(t+\delta))$ is the actual payoff to $\mathcal{B}$ along $\mathcal{M}$ at $t+\delta$.

Thus we conclude that along $\mathcal{M}$, at every current state where new partner pairs are formed, the resulting action of the population $\mathcal{B}$ is such that it improves a current total payoff to $\mathcal{B}$.

A number evolution $n(\cdot)$ possessing the above property will be called *$\mathcal{B}$-fitting.* Summarize:

Proposition 4.7 *Let (P_B^0) be the instant optimal $\mathcal{B}$-feedback, (P_S) be an arbitrary $\mathcal{S}$-feedback, $\mathcal{M}$ be a population motion generated by the control law $((P_S), (P_B^0))$, and $n(\cdot)$ be the number image of $\mathcal{M}$. Then $n(\cdot)$ is $\mathcal{B}$-fitting.*

Proposition 4.8 *Proposition 4.7 is irreversible. Namely, if a number evolution $n(\cdot)$ is $\mathcal{B}$-fitting, then for every population motion $\mathcal{M}$ whose number image is $n(\cdot)$ there may not exist a $\mathcal{S}$-feedback (P_S) such that $\mathcal{M}$ is generated by $((P_S),(P_B^0))$ where (P_B^0) is the instant optimal $\mathcal{B}$-feedback.*

Indeed, let a number evolution $n(\cdot)$ be such that $v_{12}(t) = 0$, $v_{22}(t) = 0$, and $v_{11}(t)$ and $v_{21}(t)$ are positive whenever, respectively, $m_{11}(t)$ and $m_{21}(t)$ are positive (we use the notations of subsection 4.2). Note that due to (1) and (10)

$$K^{\mathcal{B}}(n(t+\delta)) - K^{\mathcal{B}}(m(t)) = (b_{12} - b_{11})v_{11}(t) + (b_{22} - b_{21})v_{21}(t)$$

where the first and the second brackets are, respectively, positive and negative; hence, if

$$v_{11}(t)/v_{21}(t) > (b_{21} - b_{22})/(b_{12} - b_{11}) \tag{22}$$

the inequality (21) is satisfied strictly. Assume (22) if both $m_{11}(t)$ and $m_{21}(t)$ are positive, and $v_{11}(t) = v_{21}(t) = 0$ otherwise. Then (21) holds, and consequently $n(\cdot)$ is $\mathcal{B}$-fitting.

Put $u_{kl}(t) = 0$ (all sellers act *"keep"* everywhere), and let $n_{11}(0)$ and $n_{21}(0)$ be positive. It can easily be shown there are t such that $m_{11}(t)$ and $m_{21}(t)$ are positive. At such a t, by definition $v_{11}(t)$ and $v_{21}(t)$ are positive, yielding (see (10))

$$n_{21}(t+\delta) < m_{21}(t), \quad n_{22}(t+\delta) > m_{22}(t) \tag{23}$$

Hence $n(\cdot)$ is not the number image of a population motion generated by a control law $((P_S),(P_B^0))$; indeed, otherwise the inequalities (20) opposite to (23) would hold.

4.6 Number Images under Instant $\mathcal{S}$-Optimal Feedback

Fix an instant optimal $\mathcal{S}$-feedback (P_S^0) and take a population motion $\mathcal{M}$ generated by the control law $((P_S^0),(P_B))$ where (P_B) is an arbitrary $\mathcal{B}$-feedback. Consider the number image $n(\cdot)$ of $\mathcal{M}$. In the three-step procedure of the transition from $n(t)$ to $n(t+\delta)$, the $\mathcal{B}$-feedback (P_S^0) determines the numbers $u_{kl}(t)$ of sellers from $G_{kl}^{\mathcal{S}}(t)$ acting *"change"*. According to the Table 3.6 P_S^0 prescribes

$$u_{11}(t) = 0, \quad u_{12}(t) = 0 \tag{24}$$

The fact that P_S^0 prescribes *"change"* at (2,1) and the regularity condition imply (like in the previous subsection) that every sequence of ajoining times ξ where $n_{21}(\xi) > 0$, is either empty or stops at a t where $u_{21}(t) > 0$. This property of $u_{21}(\cdot)$ will be called *nondegeneracy*. For the number $u(t)$ (3) of the sellers changing 1 to 2 we have

$$u(t) = u_{21}(t) + u_{22}(t) \geq 0 \tag{25}$$

Referring to (2) we get

$$u_{21}(t) \leq n_{21}(t) \quad u_{22}(t) \leq n_{22}(t) \tag{26}$$

Thus we have

Proposition 4.9 *Let (P_S^0) be an instant optimal $\mathcal{S}$-feedback, (P_B) be an arbitrary $\mathcal{B}$-feedback, $\mathcal{M}$ be a population motion generated by the control law $((P_S^0),(P_B))$, and $n(\cdot)$ be the number image of $\mathcal{M}$. Then (24), (26) are satisfied and the function $u_{21}(\cdot)$ is nondegenerate.*

Backward speculations lead to the reverse conjecture:

Proposition 4.10 *Let $n(\cdot)$ be a number evolution such that the function $u(\cdot)$ is nondegenerate and satisfies n21. Then there exist a $\mathcal{B}$-feedback (P_B) and a population motion $\mathcal{M}$ generated by the control law $((P_S^0),(P_B))$ such that $n(\cdot)$ is the number image of $\mathcal{M}$.*

4.7 $\mathcal{S}$-Fitting

Consider a number evolution $n(\cdot)$ defined in Proposition 4.9.

Fix a t and do the following (imaginary) operations.

Replace the collection $U(t)$ of values $u_{kl}(t)$ actually formed at the first step of the t-to-$(t+\delta)$ transition procedure (subsection 4.3) by an arbitrary admissible collection $U^*(t) = (u^*_{kl}(t))$, i.e. that satisfying

$$u^*_{kl}(t) \leq n_{kl}(t) \quad (k, l = 1, 2)$$

At step 1 of the t-to-$(t+\delta)$ transition procedure use $U^*(t)$ (instead of $U(t)$).

At step 2 take an arbitrary mixture of partner pairs that could, potentially, happen at $t+\delta$ (after using $U^*(t)$); this mixture is determined by certain values $q_{il,k}(t)$ satisfying (6) and

$$\sum_{i,l=1,2} q^*_{il,k}(t) = X^*_k(t+\delta) \quad (k = 1, 2) \tag{27}$$

where $X^*_1(t+\delta)$, $X^*_2(t+\delta)$ are given by

$$X^*_1(t+\delta) = X_1(t) + u^*(t), \quad X^*_2(t+\delta) = N - X^*_1(t+\delta) \tag{28}$$

$$u^*(t) = -u^*_{11}(t) - u^*_{12}(t) + u^*_{21}(t) + u^*_{22}(t) \tag{29}$$

Take arbitrary increments $v_{kl}(t)$ (15) that might occur at step 3 provided (i) a part of buyers would be activated at t, and (ii) all activated buyers would act in accordance with their instant optimal feedbacks. Consider the corresponding collection $n^*(t+\delta) = (n^*_{kl}(t+\delta))$ of cluster numbers (resulting in step 3); as it was shown in subsection 4.3, (see (17), (18)) we have

$$n^*_{11}(t+\delta) = q_{11,1}(t) - v^{(dec)}_{11}(t), \quad n^*_{12}(t+\delta) = q_{12,1}(t) + q_{22,1}(t) + q_{21,1}(t) + v^{(dec)}_{11}(t) \tag{30}$$

$$n^*_{21}(t+\delta) = q_{21,2}(t) + q_{11,2}(t) + q_{12,2}(t) + v^{(dec)}_{22}(t), \quad n^*_{22}(t+\delta) = q_{22,2}(t) - v^{(dec)}_{22}(t) \tag{31}$$

Obtained is a collection of cluster numbers that might occur at $t+\delta$ if $U^*(t)$ would act at step 1, a certain mixture of partner pairs would be performed at step 2, and all buyers would act instant optimally at step 3. Take the corresponding payoff $K^{\mathcal{S}}(n_*(t+\delta))$. Minimize it with respect to all admissible (above described) second- and third-step transition indexes $q_{il,k}(t)$ and $v_{kl}(t)$; denote the obtained infimum by $K^{\mathcal{S}}(n(t), U^*(t))$. This value has the sense of the worst payoff that could be provided to the population $\mathcal{S}$ at $t+\delta$ if at step 1 all sellers' actions would result in $U^*(t)$ and at step 3 all buyers would respond instant optimally. Call $K^{\mathcal{S}}(n(t), U^*(t))$ the *next worst payoff* (at $n(t)$). Observing steps 2 and 3 we easily see that the next worst payoff $K^{\mathcal{S}}(n(t), U^*(t))$ depends only on the resulting number $X^*_1(t+\delta)$ of sellers playing 1 at $t+\delta$ (determined by $U^*(t)$ through (28), (29) at step 1). Taking this into account, use for the next worst payoff the notation $K^{\mathcal{S}}(n(t), X^*_1(t+\delta))$.

Consider the *passive* next worst payoff $K^{\mathcal{S}}_0(n(t))$ corresponding to the case where the numbers of sellers playing 1 and 2 are not changed while passing from t to $t+\delta$ (say, all sellers play their old strategies). In this case $u^*(t) = 0$, $X^*_1(t+\delta) = X_1(t)$ (see (28)); consequently

$$K^{\mathcal{S}}_0(n(t)) = K^{\mathcal{S}}(n(t), X_1(t)) \tag{32}$$

We will show that the actual increment $u(t)$ (3) of the number of 1-playing sellers resulting step 1 of the t-to-$(t+\delta)$ transition procedure along $n(\cdot)$, is such that the corresponding next worst payoff is no smaller (from time to time strictly greater) than the passive one. More accurately, for $X_1(t+\delta)$ given by (4), first, it holds

$$K^S(n(t), X_1(t+\delta)) \geq K_0^S(n(t)) \tag{33}$$

and, second, every sequence of adjoining times ξ where $n_{21}(\xi) > 0$, is either empty or stops at a t where the inequality (33) is strict. A number evolution $n(\cdot)$ posessing this property will be called *$\mathcal{S}$-fitting.*

Our resulting statement is

Proposition 4.11 *Let (P_S^0) be an instant optimal $\mathcal{S}$-feedback, (P_B) be an arbitrary $\mathcal{B}$-feedback, $\mathcal{M}$ be a population motion generated by the control law $((P_S^0),(P_B))$, and $n(\cdot)$ be the number image of $\mathcal{M}$. Then $n(\cdot)$ is $\mathcal{S}$-fitting.*

The proposition follows immediately from (32), the inequality $u(t) \geq 0$ (see (25)), nondegeneracy of $u_{21}(\cdot)$ and the following lemma.

Lemma 4.1 *The next worst payoff $K^S(n(t), X_1^*(t+\delta))$ is strictly increasing in $X_1^*(t+\delta)$.*

Let us prove Lemma 4.1. Due to (30), (31) we have

$$\begin{aligned}
K^S(n_*(t+\delta)) &= a_{11}(q_{11,1}(t) - v_{11}^{(dec)}(t)) + \\
&\quad a_{12}(q_{12,1}(t) + q_{22,1}(t) + q_{21,1}(t) + v_{11}^{(dec)}(t)) + \\
&\quad a_{21}(q_{21,2}(t) + q_{11,2}(t) + q_{12,2}(t) + v_{22}^{(dec)}(t)) + \\
&\quad a_{22}(q_{22,2}(t) - v_{22}^{(dec)}(t))
\end{aligned}$$

In order to obtain $K^S(n(t), X_1^*(t+\delta))$, we must minimize the above expression first, over $v_{11}^{(dec)}(t)$, $v_{22}^{(dec)}(t)$, and then over $q_{il,k}(t)$. Perform the first minimization. Recall that $v_{11}^{(dec)}(t)$, $v_{22}^{(dec)}(t)$ are nonnegative and subject to the constraints (16). The inequalities $a_{11} < a_{12}$ and $a_{21} < a_{22}$ (recall the class of the sellers' payoff matrix a) yield that $K^S(n_*(t+\delta))$ is minimized by $v_{11}^{(dec)}(t)) = 0$, $v_{22}^{(dec)}(t)) = q_{22,2}(t)$. With these $v_{11}^{(dec)}(t)$, $v_{22}^{(dec)}(t)$, we get

$$\begin{aligned}
K^S(n_*(t+\delta)) &= a_{11}q_{11,1}(t) + \\
&\quad a_{12}(q_{12,1}(t) + q_{22,1}(t) + q_{21,1}(t)) + \\
&\quad a_{21}(q_{21,2}(t) + q_{11,2}(t) + q_{12,2}(t) + q_{22,2}(t))
\end{aligned}$$

Taking into account (27) and (28), continue as follows

$$\begin{aligned}
K^S(n_*(t+\delta)) &= a_{11}q_{11,1}(t) + a_{12}(X_1^*(t+\delta) - q_{11,1}(t)) + a_{21}(N - X_1^*(t+\delta)) \\
&= (a_{11} - a_{12})q_{11,1}(t) + a_{21}N + (a_{12} - a_{21})X_1^*(t+\delta)
\end{aligned} \tag{34}$$

Minimize this value with respect to $q_{11,1}(t)$. From (6), (27) we get the constraint

$$q_{11,1}(t) \leq \min\{n_{11}(t), X_1^*(t+\delta)\} \tag{35}$$

Let us show that the equality is admissible. Suppose that the minimum in (35) equals $X_1^*(t+\delta)$. Then the numbers

$$\begin{aligned} q_{11,1}(t) &= X_1^*(t+\delta) \\ q_{il,1}(t) &= 0, \quad ((i,l) \neq (1,1)) \\ q_{il,2}(t) &= n_{il}(t) - q_{il,1}(t) \end{aligned}$$

satisfy the equalities (6), (27); consequently the equality in (35) is admissible. If the minimum in (35) equals $n_{11}(t)$, then (6), (27) are satisfied by

$$\begin{aligned} q_{11,1}(t) &= n_{11}(t) \\ q_{12,1}(t) &= n_{12}(t) + u^*(t) \\ q_{il,1}(t) &= 0, \quad ((i,l) \neq (1,1),(1,2)) \\ q_{il,2}(t) &= n_{il}(t) - q_{il,1}(t) \end{aligned}$$

for $u^*(t) \leq 0$, and by

$$\begin{aligned} q_{11,1}(t) &= n_{11}(t) \\ q_{12,1}(t) &= n_{12}(t) + u^*(t) \\ q_{21,1}(t) &= \min\{n_{21}(t), u^*(t)\} \\ q_{22,1}(t) &= \max\{u^*(t) - q_{21,1}(t), 0\} \\ q_{il,2}(t) &= n_{il}(t) - q_{il,1}(t) \end{aligned}$$

for $u^*(t) > 0$. Thus, indeed, the equality in (35) is admissible. Coming back to (34) and noticing that $a_{11} - a_{12} < 0$ we see that (34) is minimized by the maximal $q_{11,1}(t)$; the latter is, as it was stated, the right hand side of (35). If $q_{11,1}(t) = n_{11}(t)$, then

$$K^S(n_*(t+\delta)) = (a_{11} - a_{12})n_{11}(t) + a_{21}N + (a_{12} - a_{21})X_1^*(t+\delta)$$

If $q_{11,1}(t) = X_1^*(t+\delta)$, then

$$\begin{aligned} K^S(n_*(t+\delta)) &= (a_{11} - a_{12})X_1^*(t+\delta) + a_{21}N + (a_{12} - a_{21})X_1^*(t+\delta) \\ &= a_{21}N + (a_{11} - a_{21})X_1^*(t+\delta) \end{aligned}$$

In both cases the obtained expression providing $K^S(n(t), X_*(t+\delta))$ has positive coefficients by $X_*(t+\delta)$. By this, the lemma is proved.

Backward speculations (we omit them) allow to establish

Proposition 4.12 *Proposition 4.11 is reversible. Namely, if a number evolution $n(\cdot)$ is $\mathcal{S}$-fitting, then for every population motion $\mathcal{M}$ whose number image is $n(\cdot)$ there exists a $\mathcal{B}$-feedback (P_B) such that $\mathcal{M}$ is generated by $((P_S^0),(P_B))$ where (P_S^0) is an instant optimal $\mathcal{S}$-feedback.*

5 Upper-Level Population Game

5.1 Centralized Feedbacks

In this section we compare our previous model based on individual decision making with a centralized one. The latter assumes that the individuals have no personal interests;

their goal is to maximize the limit payoff to a whole population. In reaching this goal, the centralized decision making pattern seems to be the best. We assume it. Thus, we suppose that the sellers (buyers) delegate the right to working out current individuals' actions to a Center. The individuals agree beforehand that they perform all commands of their Center. Our purpose is to show that the centrailzed control pattern is unable to provide a better payoff to a population than the decentralized one.

In this subsection we describe admissible regulation rules in the centralized pattern (centralized feedbacks).

It is assumed that at every time step the Centers work out their commands to their individuals having complete information of a current populations' history. A *populations' history* up to time t is a collection

$$H(t) = ((i_S(\cdot) \mid t)), (i_B(\cdot) \mid t)), \mathcal{P}(\cdot \mid t))$$

where families $(i_S(\cdot) \mid t))$ $(S \in \mathcal{S})$ and $(i_B(\cdot) \mid t))$ $(B \in \mathcal{B})$ are histories of strategies played by sellers and, respectively, buyers up to, and including, t (the dot stands for the past time argument τ running from 0 to t), and $\mathcal{P}(\cdot \mid t)$ is the history of partner pairs: for every past time τ $\mathcal{P}(\tau \mid t)$ is the collection of all partner pairs (S, B) occured at τ.

A *centralized feedback* of the population $\mathcal{S}$ (a *centralized $\mathcal{S}$-feedback*) is a rule to prescribe a particular action to every seller at step 1 of every t-to-$(t+\delta)$ transition procedure (subsection 4.2), on the basis of a current populations' history $H(t)$. Working out a centralized $\mathcal{S}$-feedback and implementing it is, in accordance with the centralization principle, the task of the sellers' Center. Given a centralized $\mathcal{S}$-feedback $\Pi^{\mathcal{S}}$ and a populations' history $H(t)$, every action preassigned by $\Pi^{\mathcal{S}}$ at $H(t)$ to every seller is performed at step 1.

A *centralized $\mathcal{B}$-feedback* is a rule to prescribe an action to every buyer at step 3 of every t-to-$(t+\delta)$ transition procedure, on the basis of a current populations' history $H(t)$ and a collection $(E_{kl}(t+\delta))$ of pre-clusters resulting step 2; pairs $(H(t), (E_{kl}(t+\delta)))$ will be called *extended population histories.* Given a centralized $\mathcal{B}$-feedback $\Pi^{\mathcal{B}}$ and an extended populations' history $(H(t), (E_{kl}(t+\delta)))$, the actions worked out by the Central Regulation Office of the population $\mathcal{B}$ in accordance with $\Pi^{\mathcal{B}}$ are performed by all buyers at step 3.

In such a way, a *centralized control law* being a pair $(\Pi^{\mathcal{S}}, \Pi^{\mathcal{B}})$ of a centralized $\mathcal{S}$-feedback and a centralized $\mathcal{B}$-feedback, generates a population motion $\mathcal{M}$.

5.2 Coupling Centralized and Decentralized Feedbacks

In order to compare centralized feedbacks with decentralized ones, we will consider population motions guided simultaneously by centralized and decentralized feedbacks. Along such a motion, individuals of one population are regulated from the Center, and those of the other one behave independently, being activated in accordance with the decision making pattern described in subsection 2.1. Recall that this pattern implies that individuals are activated by messages *"decide"* (sellers' strategy changes activate buyers automatically). Thus, in a natural way we define a population motion generated by a *$\mathcal{S}$-centralized control law* identified with a pair $(\Pi^{\mathcal{S}}, (P_B))$ where $\Pi^{\mathcal{S}}$ is a centralized $\mathcal{S}$-feedback, and (P_B) is a (decentralized) $\mathcal{B}$-feedback, and a population motion generated by a *$\mathcal{B}$-centralized control low* identified with a pair $((P_S), \Pi^{\mathcal{B}})$ where $\Pi^{\mathcal{B}}$ is a centralized $\mathcal{B}$-feedback, and (P_S) is a (decentralized) $\mathcal{S}$-feedback.

Recall that previously the decentralized decision making pattern implied that individuals are activated under the regularity condition (subsection 2.5). We keep the regularity condition for a decentralized population playing against a centralized one; this will eventually result in the equivalency of both regulation patterns.

Thus, we assume the following *$\mathcal{S}$-regularity condition*: along a population motion generated by an arbitrary $\mathcal{B}$-centralized control law, it holds: for every seller S and every time t at which S jumps, there is a time $\xi > t$ at which S is either active or jumps.

Symmetrically, the *$\mathcal{B}$-regularity condition* is assumed: along a population motion generated by an arbitrary $\mathcal{S}$-centralized control law, it holds: for every buyer B and every time t at which B jumps, there is a time $\xi > t$ at which B is either active or jumps.

Note that assumptions outlined in Example 2.1 imply the $\mathcal{S}$- and $\mathcal{B}$-regularity conditions.

5.3 Upper-Level Population Game

Now, keeping in mind that all individuals pursue only the long run aims of their populations, we allow every population choose between the centralized and decentralized control patterns. Extension of the number of control patterns – from 1 (centralization) to 2 (centralization or decentralization) – by no means decreases expected populations' benefits.

According to which control patterns are chosen, four types of feedback pairs may occur. These are control laws occuring if both $\mathcal{S}$ and $\mathcal{B}$ are decentralized (subsection 2.4), $\mathcal{S}$-centralized control laws corresponding to $\mathcal{S}$ centralized and $\mathcal{B}$ decentralized, $\mathcal{B}$-centralized control laws corresponding to the opposite case (subsection 5.2), and centralized control laws occuring in case both populations are centralized. All these types of control procedures will be called *extended control laws.* Thus, writing an extended control law as a pair $(\Pi^{\mathcal{S}}, \Pi^{\mathcal{B}})$, we imply that each component of it is either a centralized or a decentralized feedback of the corresponding population. Population motions generated by extended control laws of each of the above classes were defined earlier. We will operate with them as well as with their number images defined as in subsection 4.1. Note that in the course of a population motion, the Center of a centralized population can not detect (not being informed of that directly) whether the other population is centralized or not.

The long run goal of the population $\mathcal{S}$ is maximization of the limit payoff, i.e. the limit of $K^{\mathcal{S}}(n(t))$ where $n(t) = (n_{kl}(t))$ is the collection of cluster numbers at time t, with t going to infinity. If there is no limit, we, like in section 3, deal with the interval between the lower and upper limits. The interest of the population $\mathcal{B}$ is symmetric. Thus, we define on the population motions $\mathcal{M}$ the multivalued objective functionals $J^{\mathcal{S}}$, $J^{\mathcal{B}}$ of the populations $\mathcal{S}$ and $\mathcal{B}$ by

$$J^{\mathcal{S}}(\mathcal{M}) = [\liminf_{t\to\infty} K^{\mathcal{S}}(n(t)),\ \limsup_{t\to\infty} K^{\mathcal{S}}(n(t))]$$

$$J^{\mathcal{B}}(\mathcal{M}) = [\liminf_{t\to\infty} K^{\mathcal{B}}(n(t)),\ \limsup_{t\to\infty} K^{\mathcal{B}}(n(t))]$$

where $n(\cdot)$ is the number image of $\mathcal{M}$. The upper-level population game is played between the populations $\mathcal{S}$ and $\mathcal{B}$ with respect to these functionals. A solution of the game is understood as a Nash equilibrium in the class of all extended control laws.

Call an extended control law $(\Pi_0^{\mathcal{S}}, \Pi_0^{\mathcal{B}})$ *upper-level (a,b)-equilibric* if for every extended control law $(\Pi_*^{\mathcal{S}}, \Pi_*^{\mathcal{B}})$, and every population motion $\mathcal{M}^0$ generated by $(\Pi_0^{\mathcal{S}}, \Pi_0^{\mathcal{B}})$, it holds $\min J^{\mathcal{S}}(\mathcal{M}^0) \geq \max J^{\mathcal{S}}(\mathcal{M}^*)$ where $\mathcal{M}^{0*}$ is an arbitrary population motion generated by $(\Pi_*^{\mathcal{S}}, \Pi_0^{\mathcal{B}})$, and $\min J^{\mathcal{B}}(\mathcal{M}^0) \geq \max J^{\mathcal{B}}(\mathcal{M}^{*0})$ where $\mathcal{M}^{*0}$ is an arbitrary population motion generated by $(\Pi_0^{\mathcal{S}}, \Pi_*^{\mathcal{B}})$.

We understand now the long-run goal of a population as finding its component of an upper-level (a,b)-equilibric extended control law.

5.4 Decentralized Upper-Level Equilibria

Our main observation is that an upper-level equilibrium is reached in the class of decentralized control laws; moreover, it is provided by instant optimal control laws being equilibric in the decentralized game between individuals (Proposition 3.1). This means that decentralized regulation where all individuals make their decisions independently and instant optimally with respect to their personal interests, is equivalent to best centralized regulation.

The accurate formulation is as follows.

Proposition 5.1 *An instant optimal control law is upper-level (a,b)-equilibric.*

Proposition 5.2 *Let $n^0(\cdot) = (n^0_{kl}(\cdot))$ be the number image of a population motion generated by an instant optimal control law. Then $n^0_{12}(t) = N$ for all t sufficiently large.*

Proposition 5.2 follows immediately from Proposition 3.2.

To justify Proposition 5.1, we refer to subsections 4.3 and 4.5 where number images of population motions guided by instant $\mathcal{S}$- and $\mathcal{B}$-optimal feedbacks were considered. Let us take a population motion $\mathcal{M}^0 = ((i^0_S(\cdot)),(i^0_B(\cdot))$ generated by an instant optimal control law $((P^0_S),(P^0_B))$ and a population motion $\mathcal{M}^{0*}$ generated by the extended control law $(\Pi^S_*,(P^0_B))$ where Π^S_* is an arbitrary centralized or decentralized $\mathcal{S}$-feedback. Let $n(\cdot)$ be the number image of $\mathcal{M}^{0*}$. Now recall the three-step t-to-$(t+\delta)$ transition procedure for the number image $n(\cdot)$ (subsection 4.2) and consider variables $v_{kl}(t)$ (15) and $q_{kl}(t)$ appearing at steps 2 and 3. Repeating the speculations which led us to Proposition 4.5, we come to the same statement for $n(\cdot)$ (the fact that (P^0_B) is coupled with, possibly, a centralized $\mathcal{S}$-feedback does not matter). The statement claims that the equalities (17), (18) hold and functions $v^{(dec)}_{11}(\cdot)$, $v^{(dec)}_{22}(\cdot)$ are nondegenerate and satisfy the equalities (16). Now using Proposition 4.5, we conclude that there exist an $\mathcal{S}$-feedback (P_S) and a population motion $\mathcal{M} = ((i_S(\cdot)),(i_B(\cdot))$ generated by the control law $((P_S),(P^0_B))$ such that $n(\cdot)$ is the number image of $\mathcal{M}$. By Proposition 3.1 the control law $((P^0_S),(P^0_B))$ is (a,b)-equilibric. Hence (see the definition in subsection 3.1) for every seller S, it holds

$$\min J^S(\mathcal{M}^0) \geq \max J^S(\mathcal{M})$$

Referring to the form of J^S (subsection 3.1) and taking in account that the number of sellers is finite, we obtain

$$a_{i^0_S(t) i^0_{B^0[t]}(t)} \geq a_{i_S(t) i_{B[t]}(t)} - \epsilon$$

for an arbitrary $\epsilon > 0$ and all sufficiently large t; here $B^0[t]$ and $B[t]$ are the S's partners at time t along the motions $\mathcal{M}^0$ and $\mathcal{M}$ respectively. Consequently, at every large t, the total payoffs $K^S(n^0(t))$, $K^S(n(t))$ to the population $\mathcal{S}$ along $\mathcal{M}^0$ and $\mathcal{M}$ satisfy

$$K^S(n^0(t)) \geq K^S(n(t)) - N\epsilon$$

Passing to the lower limit on the left and the upper limit on the right provides

$$\min J^S(\mathcal{M}^0) \geq \max J^S(\mathcal{M}) - N\epsilon$$

and, due to arbitrariness of ϵ,

$$\min J^S(\mathcal{M}^0) \geq \max J^S(\mathcal{M})$$

But $\mathcal{M}$ has the same number image $n(\cdot)$ as $\mathcal{M}^{0*}$ yielding $J^S(\mathcal{M}) = J^S(\mathcal{M}^{0*})$. Consequently,

$$\min J^S(\mathcal{M}^0) \geq \max J^S(\mathcal{M}^{0*})$$

Thus, for $((P_S^0), (P_B^0))$ the first property of an an upper-level (a, b)-equilibric extended control law is established.

Similarly, referring to Propositions 4.9, 4.10, we establish the second (symmetrical) property.

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