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Dosi, G. and Kaniovski, Y.M.

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G. Dosi
Yu. Kaniovski

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IIASA International Institute for Applied Systems Analysis ▫ A-2361 Laxenburg ▫ Austria
Telephone: +43 2236 715210 ▫ Telex: 079 137 iiasa a ▫ Telefax: +43 2236 71313

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Summary

Adaptive (path dependent) processes of growth modeled by urn schemes are important for several fields of applications: biology, physics, chemistry, economics.

In this paper we present a review of studies that have been done in the technological dynamics by means of the urn schemes. Also several new macroeconomic models of technological dynamics are analysed by the same machinery and its new modification allowing to tackle non-homogeneity of the face space. We demonstrate the phenomena of multiple equilibria, different convergence rates for different limit patterns, locally positive and locally negative feedbacks, limit behavior associated with non-homogeneity of economic environment where producers (firms) are operating.

Contents

The Method of Generalized Urn Scheme in the Analysis of Technological and Economic Dynamics

G. Dosi
University "La Sapienza", Rome, Italy

and

Yu. Kaniovski
IIASA, Laxenburg, Austria

1 Introduction

Technical change typically involves diversity amongst the agents who generate or are effected by it; various forms of learning often based on trial - and - error procedures; and mechanisms of selection which reward paricular types of technologies, agents or behaviors at the expenses of others.

These appear to be, indeed, general features of the competitive process driving economic dynamics. "Competition" entails the interaction among heterogeneous firms embodying different technologies, different expectations and, quite often, displaying different behaviours. Moreover, it is often the case that technological and organizational learning is associated with various types of externalities and increasing returns.

Over the last two decades, at last, such dynamic phenomena have drawn an increasing attention within the economic discipline – especially with reference to technological change. A number of conceptual approaches and mathematical tools have been applied, often benefiting from contemporary developments in the analysis of dynamical systems in natural sciences.

In this work, we shall discuss some of these approaches and, in particular, present the basic structure and the interpretation scope of one "formal machinery", namely *generalized urn schemes.* In section 2, we shall outline some phenomena which are central to technological and economic dynamics, and briefly review alternative formal representations of them. Section 3 introduces the basics of urn schemes. In the following sections we illustrate some applications to relatively simple competitive environments (section 4), and further refinements, contemplating local feedback processes (section 5); phenomena of increasing returns deriving from system compatibility (section 6); non-homogeneous environment (section 7). Finally, in the conclusion we shall point out some promising areas of application of this formal apparatus, including the economics of innovation, industrial dynamics, macroeconomics, finance.

2 Processes of Economic Evolution

In very general terms, the impulses driving economic change stem, first, from variations in the knowledge and physical resourses upon which individual agents can draw in order to pursue their activities; second, from the process by which agents learn, adapt, invent – on the grounds of whatever they perceive to be the available knowledge and resourses, and, third, from the interactions amongst the agents themselves. Of course, these sourses of change are by no means independent: for example, learning activities obviously affect the available knowledge and the efficiency by which resourses are used; interactions might trigger learning and entail externalities; learning itself may be associated with particular forms of economic activity, such as learning – by – doing. The variety of sourses and mechanisms of economic change highlighted by economic history, most likely, in our view, precludes the identification of some unique or archetypical dynamic form which could apply across industries, phases of development, historical context. Still, it might be possible (and indeed is a challenging area of research) to identify few relatively invariant characteristics of the process of change and, with them, also the "formal machineries" most apt to represent them.

Some basic features of economic evolution are the following: (i) imperfect and time-consuming microeconomic learning; (ii) microheterogeneity; (iii) most often, various form of increasing returns – especially in the accumulation of knowledge – and non-linearities; (iv) aggregate dynamics driven by both individual learning and collective selection mechanisms; (v) "orderly" structural properties resulting from non-equilibrium fluctuations.

Correspondingly, let us examine the formal representations which can account for at least some of these features of evolutionary dynamics. As a general reference, let us start from "order – through – fluctuation" dynamics (cf. Nicolis and Prigogine (1971) and (1989), Prigogine and Stengers (1984)): it is a quite broad paradigm for the interpretation of complex non-linear processes, initially developed with reference to physical chemistry and molecular biology, but more generally emphasizing the properties of self-reinforcing mechanisms and out – of – equilibrium self-organization. Such systems turn out to be sensitive to (however small) early perturbations and display multiplicity of patterns in their long-term behaviour. The cumulation of small early disturbances (or small disturbances around unstable states) "pushes" the system toward one of these patterns and thus "select" the structure toward which the system will eventually tend. These properties apply to a very wide class of dynamical systems, highlighting, loosely speaking, some general "evolutionary" features well beyond the domain of social sciences and biology.

Further specifications of evolutionary dynamics come from mathematical biology (see Eigen and Schuster (1979)). Evolution in many of such models occurs in a way that some integral characteristics (mean fitness for biological systems or mean "competitiveness" in the economic analogy) "improves" along the tragectory. In the simplest case of Fisher's selection model, "improvements" straightforwardly imply that the mean fitness increases along the path. However, even in biology this equivalence does not hold in general (due, for example, to phenomena of hyperselection, co-evolution, symmetry-breaking: see Allen (1988) and Silverberg (1988) for discussions directly linked to economic applications). Even more so, this *non*-equivalence between "evolution" and "increasing fitness", however defined, is likely to emerge whenever there is no identifiable "fundamental law of nature" or conservation principle. Putting it another way: evolutionary dynamics – in biology as well as in economics – involves some kind of selection process grounded on the relevant distributions of agents' characteristics, on the one hand, and on some environmental criterion of "adaptiveness", on the other. (Until recently, most economic models have avoided the issue simply by *assuming* that all the agents were perfectly "adapted", either via some unspecified selection process that occured just before the economist started looking at the world or via some optimization process that occured in the head of the agents themselves.) Replicator dynamics is a common formal tool to represent such selection-driven adaptation (for applications to economics, see Silverberg *et al.* (1988) and with reference to game-theoretical problems, Banerjee and Weibull (1992), Cabrales (1992), Kandori *et al.* (1990)). However, at least the simplest replicator process impose quite strigent conditions on the ways selection occurs. In essence, these restrictions turn out to be negative feedbacks, i.e. diminishing returns, deriving from some underlying "conservation principle"[1]. On the contrary, positive feedbacks lead to multiple limit states and generate a much richer variety of trajectories which the system may follow. For example, it is increasingly acknowledged that technological innovations are likely to involve some forms of dynamic increasing returns – hence, positive feedbacks – along their development and diffusion (cf. Freeman (1982), Dosi *et al.* (1988), Anderson *et al.* (1988), David (1993), and for an interpretation of the empirical evidence, Dosi (1988)). Relatedly, these is no guarantee that the particular economic outcome which happens to be historically selected amongst many notional alternatives will be the "best" one, irrespectively of the "fitness" or welfare yardstick.

[1]Conventionally, in economics, profit (or utility) maximization under a constraint of given and scarse resourses clearly performs this role.

Concerning the mathematical tools that have been proposed within and outside economics for the analysis of the competitive process, ordinary differential equations have a paramount importance (not surprisingly, since they are also the most common language of modern science and especially physics). They are applied to most analyses of economic and technological dynamics (for our purposes here, cf. Nelson and Winter (1982), Polterovich and Henkin (1988), Day (1992), and the works surveyed in Boldrin (1988); in general cf. Brock and Malliaris (1989)). In particular ordinary differential equations with trajectories on the unit simplex – i.e. of the replicator type – borrow, as already mentioned an idea of selection-driven evolution from biology (cf. Silverberg *et al.* (1988))[2]. For stochastic (Markov) perturbations of these equations see Nicolis and Prigogine (1971) – for general equations – and Forster and Young (1990), – for equations of the replicator type. However, while these continuous-time formulations work well, they involve a not so harmless approximation for events that are by nature discrete (the main example being a phase space which is discrete and changes by discrete increments). More intuitively, the continuous-time approximation is bound to take very literally the old saying that *natura non facet saltum.*

Moreover, from a technical point of view, the approximation carries unnecessary hypotheses of mathematical nature (a classical example is the Lipschitz condition on the coefficients of the differential equation describing the system) and specific difficulties (such as the requirement of rigorously defining the stochastic perturbations of replicator equations). In this respect, it might be worth mentioning here some recent results from so-called "evolutionary games" showing convergence to conventional Nash-type equilibria in the continuous approximation but not in the discrete formulation (Banerjee and Weibull (1992), Dekel and Scotchmer (1991)). Moreover, formal representations of selection processes in economics often rely on replicator dynamics satisfying the monotonicity condition (Friedman (1991), Samuelson and Zhang (1991), Banerjee and Weibull (1992)) (loosely speaking, the condition guarantees that, given an environment, there is no reversal in the "forces of selection" along the trajectory). However, even in simple cases the results on limit properties obtained under replicator dynamics night not hold under more general selection processes (see, for example, Cabrales (1992)).

To summarize this brief overwiev of the formalisms applied to economic dynamics and evolution: ideally, one would like some machinery able to capture as adequately as possible (a) increasing-returns phenomena, i.e. positive feedbacks; (b) "ugly" and badly behaved selection dynamics, involving also "jumps" and discontinuities, co-evolutionary effects, *etc.*; and (c) a large

[2]Of course, this does not bear any implication for the sources of "mutation" upon which environmental selection operates. For example, Silverberg *et al.* (1988) assume an exogenous drift in innovative opportunities with learning – by – using and diffusion-related externalities.

variety of individual processes of adaptation and innovation (and, thus, being quite agnostic on the processes driving the perturbations).

In the following, we shall assess to what extend an alternative class of models, namely *generalized urn schemes*, can fulfill these tasks. These schemes, sometimes called *non-linear Polya processes* or *adaptive processes of growth*, generate stochastic discrete-time dynamic systems with trajectories on the set of points with rational coordinates from the unit simplex (cf. Arthur (1988), Arthur *et al.* (1983) and (1987c), Glaziev and Kaniovski (1991), Dosi *et al.* (1991), Arthur and Ruszczinski (1992)). The mathematical background comes from Hill *et al.* (1980) and Arthur *et al.* (1983), (1987a) and (1988). This formal apparatus enables one to handle positive and/or negative feedbacks (possibly coexisting in the same process): see Arthur (1988) and Arthur *et al.* (1987c). In particular these feedbacks may have a "local" nature – in the sense that they may occur only under paricular states on the trajectories (Dosi *et al.* (1991)). This approach allows also to treat complementarities and network externalities in the adoption of competing technologies (Arthur *et al.* (1987b)), whereby individual commodities – say, computers or telecomunication equipment – operate within networks requiring compatibility[3]. It must be also emphasized that in this work we generally suggest examples of application of this formalism drawn from the economics of innovation, but similar properties can easily be found in many other economic domains: rather than technologies, one could also consider e.g. organizational forms or strategies in business economics; cognitive models and decision rules in finance; *etc.* (see the final section). Using the generalized urn schemes one can analyse the emergence of random market structure with more than one limit state occuring with positive probability (cf. Arthur *et al.* (1983) and also Glaziev and Kaniovski (1991)). Moreover, one may determine the different convergence rates to the various limit states attainable with positive probability (Arthur *et al.* (1988)).

In this work we shall analyse some of the patterns of system evolution which can be discovered by means of generalized urn schemes. In order to do this we shall use some known models of technological dynamics and also introduce some novel modification highlighting the complex limit structures that these models generate.

Let us start with the simplest definition of a generalized urn scheme.

[3]System compatibility implies that one ought to consider combinations amongst individual technologies. In turn, this can hardly be done by adding to the "technological space" where choices are made all possible combinations of technologies existing at any one time. At the very least, this procedure would lead to an enormous growth in the dimension of the phase space. For example, if N new technologies come to the market, considering all their possible combinations would imply the "explosion" of the dimension of the phase space up to $2^N - 1$.

3 The Basic Elements of the Theory of Generalized Urn Schemes

To simplify the presentation let us restrict ourselves to the case of two competing technologies which correspond to urn schemes with balls of two colors (Hill *et al.* (1980) and Arthur *et al.* (1983)). Think of an urn of infinite capacity with black and white balls. Starting with $n_w \geq 1$ white balls and $n_b \geq 1$ black balls into the urn, a new ball is added into the urn at time instants $t = 1, 2 \ldots$. It will be white with probability $f_t(X_t)$ and black with probability $1 - f_t(X_t)$. Here $f_t(\cdot)$ is a function[4], which maps $R(0,1)$ in [0,1] ($R(0,1)$ stands for the set of rational numbers from $(0,1)$). By X_t we designate the proportion of white balls into the urn at time t. The dynamics of X_t is given by the relation

$$X_{t+1} = X_t + (t + n_w + n_b)^{-1}[\xi_t(X_t) - X_t], \quad t \geq 1, \quad X_1 = n_w(n_w + n_b)^{-1}.$$

Here $\xi_t(x)$, $t \geq 1$, are random variables independent in t, such that

$$\xi_t(x) = \begin{cases} 1 & \text{with probability} \quad f_t(x)\,, \\ 0 & \text{with probability} \quad 1 - f_t(x)\,. \end{cases}$$

Designate $\xi_t(x) - \mathrm{E}\xi_t(x) = \xi_t(x) - f_t(x)$ by $\zeta_t(x)$. Then we have

$$X_{t+1} = X_t + (t + n_w + n_b)^{-1}\{[f_t(X_t) - X_t] + \zeta_t(X_t)\}, \quad t \geq 1, \quad X_1 = n_w(n_w + n_b)^{-1}. \tag{1}$$

Due to $\mathrm{E}\zeta(x) = 0$, the system (1) shifts on average at time $t \geq 1$ from a point x onto the value $(t + n_w + n_b)^{-1}[f_t(x) - x]$. Consequently limit points of the sequence $\{X_t\}$ have to belong to the "set of zeros" of the function $f_t(x) - x$ (for $x \in [0,1]$). It will really be the set of zeros if $f_t(\cdot)$ does not depend on t, i.e. $f_t(\cdot) = f(\cdot)$, $t \geq 1$, for $f(\cdot)$ being a continuous function.

For the general case one needs a specific mathematical machinery to describe this "set of zeros" (see Hill *et al.* (1980) for the case when the probabilities are discontinuous and do not depend on t; and Arthur *et al.* (1987b) for the case when the probabilities are discontinuous functions and depend on t).

To summarize the properties of the above urn scheme that are important for our purposes recall the following:

1) the process X_t develops on the one-dimensional unit simplex $[0,1]$ taking (discrete) values from the set $R(0,1)$ of rational points from $(0,1)$ (more precisely, at time $t+1$ it can take the values $i(t + n_w + n_b)^{-1}$, $n_w \leq i \leq n_w + t$);

2) since in general we do not require any regularity of $f_t(\cdot)$, $t \geq 1$, the process can display a very complicated behavior (for example, its trajectories can "sweep off" an interval with probability 1 (see Arthur *et al.* (1987b));

[4]When it does not depend on t it is called (Hill *et al.* (1980)) *urn function*.

3) if for a function $f(\cdot)$ one has $f_t(\cdot) = f(\cdot) + \delta_t(\cdot)$ and $\sup_{x \in R(0,1)} |\delta_t(x)| \to 0$ sufficiently fast as $t \to \infty$, then, for an isolated root θ of $f(x) - x$, one can have convergence of X_t to θ with positive or zero probability (we call such points *attainable* or *unattainable*, correspondingly) depending upon

$$(f(x) - x)(x - \theta) \leq 0 \tag{2}$$

or

$$(f(x) - x)(x - \theta) \geq 0 \tag{3}$$

in a neighborhood of θ (see Hill *et al.* (1980), Arthur *et al.* (1988) and Dosi *et al.* (1991));

4) the convergence rate to those θ, to which the process X_t converges with positive probability, depends upon the smoothness of $f(\cdot)$ at θ. In particular, if the smoothness decrease from differentiability, i.e.

$$f(x) = f'(\theta)(x - \theta) + o(|x - \theta|) \text{ as } x \to \theta,$$

to the Hölder differentiability of the order $\gamma > 1/2$, i.e.

$$f(x) = f'_H(\theta)\text{sgn}(x - \theta)|x - \theta|^\gamma + o(|x - \theta|^\gamma) \text{ as } x \to \theta,$$

the order of convergence of X_t to θ increases from $t^{-1/2}$ to $t^{-1/(1+\gamma)}$ (see Kaniovski and Pflug (1992)).

Now, thinking of the urn as a market, a white ball as a unit of A technology, a black ball as a unit of B technology, we can analyse the process of diffusion of A and B on the market (of infinite capacity) by means of the forgoing urn scheme. Let us consider first some conceptual examples of technological dynamics in homogeneous economic environments, where competing firms, producing either one of the technologies, are operating.

4 Some Examples of Competition Under Global Feedbacks in an Homogeneous Economic Environment

We start with the simplest model which displays (global) positive feedback and, as a consequence, multiple (two in this case) patterns of limit behavior.

Suppose that we have two competing technologies, say, A and B, and a market with imperfectly informed and risk-averse adopters[5]. The two technologies have already been introduced

[5]Note that some general system properties – such as the multiplicity of limit states under positive feedbacks – are independent from the exact characterization of microeconomic decision rules, although the latter influence both the processes and the nature of limit structures themselves.

to the market, say $n_A \geq 1$ units of A and $n_B \geq 1$ units of B. Let us study their diffusion on the market. At time instants $t = 1, 2, \ldots$ one new consumer adopts a unit of either technology. Since the adopters, in the example here, are incompletely informed and risk-averse, they use some "boundedly rational" decision rule to make their choice[6]. For example, in Arthur *et al.* (1983) and Glaziev and Kaniovski (1991) the following rule was considered:

$R1$. Ask an odd number $r > 1$ of the users of alternative technologies. If the majority of them use A, choose A. Otherwise choose B.

According to this rule, the technologies are symmetric. Alternatively suppose that they are not. For example, A comes from a wellknown firm with a lot of "goodwill" and B from a new and unknown one. Hence, potential users perceive a different risk in this choice and require different evidence. Assume that this correspond to the following rule:

$R2$. Fix $\alpha \in [1/2, 1)$. Ask $q \geq 3$ users of the technologies. If more than αq of them use A, choose A. Otherwise choose B.

Here α measures the relative uncertainty of the adopters concerning the two technologies. If $\alpha = 1/2$ and q is an odd number, then $R2$ converts into $R1$.

Another interpretation, to the same effect, of the choice process described by $R1$ and $R2$ is in terms of increasing returns to the technologies rather than risk-aversion of the adopters: the later know that the greater the number of past adopters the bigger are also the improvements which a technology has undergone (although the improvements themselves are not directly observable). Hence, in this case, sampling provides an indirect measure of unobservable technological characteristics.

Rule $R1$ generates the probability to choose A as a function of its current proportion on the market, of the following form

$$f_t(x) = p_{R1}(x) + \delta_t(x). \tag{4}$$

Here

$$p_{R1}(x) = \sum_{i=\frac{r+1}{2}}^{r} C_r^i x^i (1-x)^{r-i},$$

$$\sup_{x \in R(0,1)} |\delta_t(x)| = O(t^{-1})$$

and C_r^i stands for the number of combinations from r to i. The function $p_{R1}(x) - x$ has three roots 0, 1/2 and 1 on $[0, 1]$. The root 1/2, satisfying (3), proves to be unattainable, i.e. there

[6] In any case, fascinating issue, which cannot be pusued here, regards the meaning of "rationality" in environments driven by positive feedbacks and showing multiple limit states. For example, even if the agents knew the "true" urn model, what use can they make of this cognitive representation? How could they be more than "boundedly rational"?

is no feasible asymptotic market structure corresponding to it or, speaking in mathematical terms, X_t converges to this root with zero probability as $t \to \infty$ (see Glaziev and Kaniovski (1991)). The roots 0 and 1, satisfying (2), are attainable, i.e. X_t converges to each of them with positive probability for any ratio between $n_A \geq 1$ and $n_B \geq 1$. Also the probability for A (B) to dominate in the limit (i.e. that $X_t \to 1$ ($X_t \to 0$) as $t \to \infty$) will be greater than 1/2 if the initial number of units n_A (n_B) of the technology is greater than the initial number of units of the alternative technology (for details see Glaziev and Kaniovski (1991)).

Consequently we observe here a mechanism of "selection" which is "history-dependent": the past shapes, in probability, the future, and this effect self-reinforces along the diffusion trajectory.

Quite similarly, rule $R2$ generates

$$f_t(x) = p_{R2}(x) + \delta_t(x), \tag{5}$$

where

$$p_{R2}(x) = \sum_{i=[\alpha q]}^{q} C_q^i x^i (1-x)^{q-i},$$

$$\sup_{x \in R(0,1)} |\delta_t(x)| = O(t^{-1}).$$

Here we designate by $[a]$ the largest integer in a. The function $p_{R2}(x) - x$ has three roots 0, θ and 1 on $[0, 1]$, where $\theta \geq 1/2$ and shifts to the right as α increases. It can be shown, that similarly to the previous case, also this rule generates a mechanism for establishing of dominance of one of the competing technologies (and both have a positive probability to dominate). However, one cannot explicitly trace here the influence of the initial frequencies of the technologies on the probabilities to dominate.

The two forgoing examples display (global) positive feedbacks. Examples of (global) negative feedbacks can be similarly derived.

Consider the following rules:

*R*3. Ask an odd number r of the users of alternative technologies. If the majority of them use A, choose B. Otherwise choose A.

*R*4. Fix $\alpha \in [1/2, 1)$. Ask $q \geq 3$ users of the technologies. If more than αq of them use A, choose B. Otherwise choose A.

If $\alpha = 1/2$ and q is an odd number, then $R4$ converts into $R3$.

These rules may accommodate behaviors such as the search for diversity in consumption or implicitly capture the outcomes of strategic behaviors on the side of the producers of the technologies aimed at the exploitation of "market power" (cf. Dosi *et al.* (1991) and Glaziev and Kaniovski (1991)). We have relations here similar to (4) and (5) with

$$p_{R3}(x) = \sum_{i=0}^{\frac{r-1}{2}} C_r^i x^i (1-x)^{r-i},$$

and

$$p_{R4}(x) = \sum_{i=0}^{[\alpha q]} C_q^i x^i (1-x)^{q-i}.$$

In both cases there is unique solution of the corresponding equations $p_{R3}(x) - x = 0$ and $p_{R4}(x) - x = 0$. For $R3$ it is 1/2, and for $R4$ the root θ is greater than 1/2 and increases as α increases. The negative feedback determines a limit market structure where by both technologies are represented in the market with equal shares ($R3$) or they share the market in the proportion $\theta : (1-\theta)$ (the limit for the ratio of the number of units of A to the number of units of B).

For both rules, we know the rates of convergence of X_t to the root, i.e. $\sqrt{t}(X_t - 1/2)$ for $R3$ or $\sqrt{t}(X_t - \theta)$ for $R4$ are asymptotically normal as $t \to \infty$. The means of the limit normal distributions equal zero for both cases and one can also specify the corresponding variances (see Arthur *et al.* (1983) for the case of $R3$). Consequently we can characterize the rate of emergence of the limit market structures[7].

More complicated $f(\cdot)$ functions appear if we introduce additional hypotheses concerning the characteristics and/or dynamics of the pool of adopters. If we assume that adopters who use some decision rule R_i happen with frequency (probability) $\alpha_i > 0$, $i = 1, 2, \ldots, k$, $(\sum_{i=1}^k \alpha_i = 1)$, then the function $f_t(\cdot)$, corresponding to the behaviour of the whole pool, is a a randomization with weights α_i of functions $f_t^i(\cdot)$ generated by the rules R_i, i.e.

$$f_t(x) = \sum_{i=1}^{k} \alpha_i f_t^i(x), \quad x \in R(0,1),\ t \geq 1.$$

The simplest example, whenever adopters who use $R1$ come up with probability $\alpha > 0$, while those who use $R3$ come up with probability $1 - \alpha > 0$, has been considered in Dosi *et al.* (1991).

Beyond these properties of general positive and negative feedbacks, let us now consider those more complicated situations with *locally* positive and/or *locally* negative feedbacks.

5 Examples of Technological Dynamics Under Local Feedbacks in Homogeneous Economic Environments

Let us introduce a price dynamics for the two technologies. As in Dosi *et al.* (1991), assume that two firms (producers of A and B, respectively) use the following strategy: up to a certain

[7]For this particular rule one can determine an even sharper asymptotic characterization – the law of iterated logarithm (see Arthur *et al.* (1983)).

market share, defined by the proportion of the product of the firm among all products which have been sold until the current time (usually greater than 1/2) reduce the price, above that level increase it. Let us consider the simplest (linear) case of this policy which is graphically represented in figure 1. Here $Pr_A(x_A)$ designates the dependence of the price of technology A

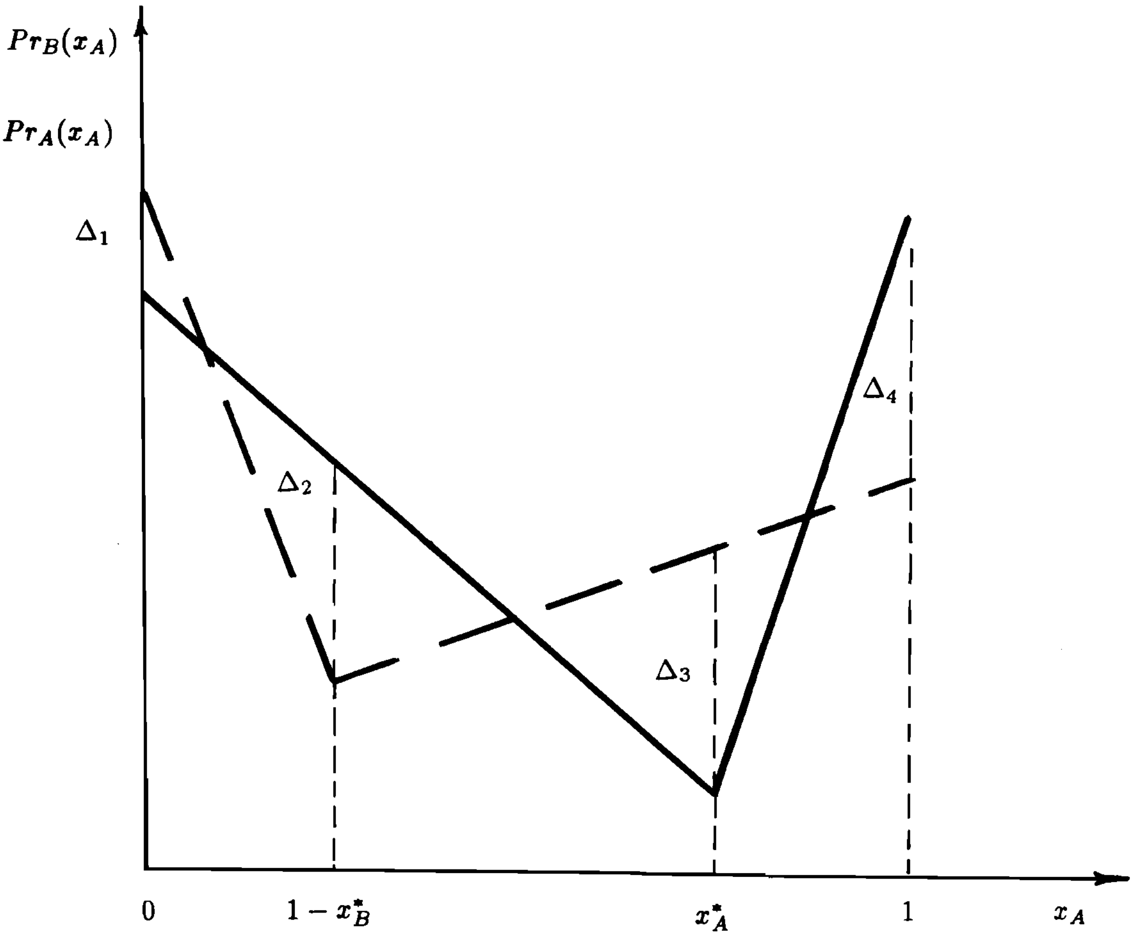

Figure 1: Dependence of prices of A and B on the market share of A.

as a function of its proportion x_A among adopters who are using either technology. $Pr_B(x_A)$ designates the dependence of the price of the technology B as a function of x_A. (Note, that the proportions of the technologies A and B are related by: $x_A + x_B = 1$.) Define x_A^* and x_B^* as the "critical" market shares which switch from falling- to rising-price rules. Hence the dependence of the price of the A (B) technology on its proportion on the market $x_A(x_B)$ is given by four parameters: $Pr_A(0)$, x_A^*, $Pr_A(x_A^*)$, $Pr_A(1)(Pr_B(1)$, x_B^*, $Pr_B(1 - x_B^*)$, $Pr_B(0))$[8].

This price dynamics embodies both positive and negative feedback mechanisms of diffusion. Within the domain of positive feedback the price falls with increasing market share possibly due to learning economies, dynamic increasing returns, *etc.*, and/or, on the behavioural side, to market-penetration strategies. Then, above a certain market share, the price, driven by

[8]Note that one accounts also for the circumstances when $Pr_A(1) \leq Pr_A(x_A^*)$ ($Pr_B(0) \leq Pr_B(1 - x_B^*)$), such as when $x_A^* = 1(x_B^* = 1)$: in this case, firm $A(B)$ still reduces the price on its product as its proportion on the market increases.

negative feedbacks, starts to rise, possibly due to monopolistic behaviours of the firm or to the progressive exhaustion of technological opportunities to lower production costs. Note that the model accounts also for those particular cases when firms follow different "non-symmetric" policies – e.g. one increases the price and another lowers it, or both increase (lower) them[9], or one increases (lowers) price and the other follows the above general strategy. These special cases can be obtained from the general one by simply changing the relations between $Pr_A(0)$, $Pr_A(x_A^*)$, $Pr_A(1)(Pr_B(1)$, $Pr_B(1-x_B^*)$, $Pr_B(0))$.

It is natural to suppose that, in the case when the performance of the technologies is approximately the same and potential adopters know about it, the technology which is cheaper has more chances to be sold, i.e. the A technology is bought if $Pr_A(x_A) - Pr_B(x_A) < 0$. However if the prices only slightly differ or consumers have some specific preferences (which can be characterized only statistically or on average), that may sometimes lead to the adoption of the more expensive technology. This case mathematically can be formalized in the following way (see also Hanson (1985)). The A technology is bought if $Pr_A(x_A) - Pr_B(x_A) + \xi < 0$, where ξ is a random variable. (Consequently B technology is bought if $Pr_A(x_A) - Pr_B(x_A) + \xi > 0$.) To preserve the symmetry of the decision rule we should avoid the situation when the event "$Pr_B(x_A) - Pr_A(x_A) = \xi$" has nonzero probability. This is definitely not the case when the distribution of ξ possesses a density with respect to the Lebesgue measure on the set of real numbers. Consequently we will assume that the distribution of ξ has a density in R^1. The probability $f(x_A)$ to choose the A technology, as a function of x_A, equals to $\mathcal{P}\{\xi < Pr_B(x_A) - Pr_A(x_A)\}$. To avoid unnecessary sophistications of the model, we shall assume that ξ has a uniform distribution on $[-\alpha, \alpha]$. The probability to choose A as a function of x_A in this case has the form

$$f(x_A) = \begin{cases} 1 & \text{if } Pr_B(x_A) - Pr_A(x_A) \geq \alpha \ , \\ 0 & \text{if } Pr_B(x_A) - Pr_A(x_A) \leq -\alpha \ , \\ [Pr_B(x_A) - Pr_A(x_A) + \alpha]/2\alpha & \text{if } -\alpha < Pr_B(x_A) - Pr_A(x_A) < \alpha \ . \end{cases}$$

For $\alpha > \min_{i=1,2,3,4} \Delta_i$ this is graphically represented in figure 2. Here we have three roots – θ_1, θ_2 and θ_3 – of the function $f(x) - x$ on $[0,1]$. Satisfying (3), the root θ_2 proves to be unattainable, while θ_1 and θ_3, satisfying (2), are attainable, i.e. the process X_t converges to each of them with positive probability for any initial proportions of the technologies on the market. Using results of Arthur *et al.* (1988) we find the rates of convergence to the attainable roots

$$\theta_1 = \frac{(\alpha + \Delta_1)(1 - x_B^*)}{2\alpha(1 - x_B^*) + \Delta_1 + \Delta_2},$$

[9]For the case when both lower prices, see Glaziev and Kaniovski (1991) where formally the same situation is interpreted somewhat differently.

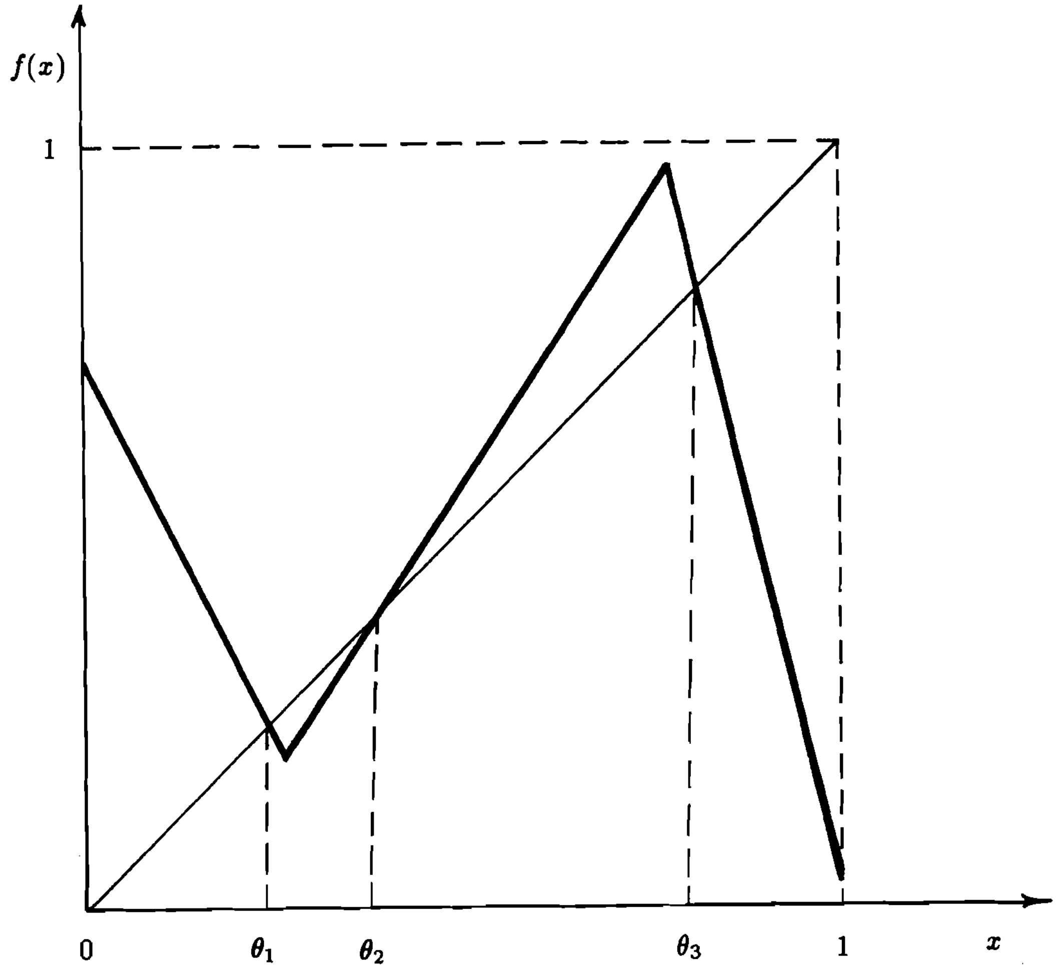

Figure 2: Probability to choose A depending on its market-share.

$$\theta_3 = 1 - \frac{(\alpha + \Delta_4)(1 - x_A^*)}{2\alpha(1 - x_A^*) + \Delta_3 + \Delta_4}.$$

In particular,

$$\lim_{t\to\infty} \mathcal{P}\{\sqrt{t}(X_t - \theta_i) < y,\ X_s \to \theta_i\} = \mathcal{P}\{X_s \to \theta_i\}\mathcal{P}\{\mathcal{N}(0, \sigma_i^2) < y\}. \tag{6}$$

Also $\mathcal{N}(0, \sigma_i^2)$ stands for a Gaussian distribution with zero mean and variance

$$\sigma_i^2 = \frac{\theta_i(1 - \theta_i)}{1 - 2f'(\theta_i)}, \tag{7}$$

where $f'(\cdot)$ designates the derivative of $f(\cdot)$. It can be shown that

$$f'(\theta_1) = -\frac{\Delta_1 + \Delta_2}{2\alpha(1 - x_B^*)} \tag{8}$$

and

$$f'(\theta_3) = -\frac{\Delta_3 + \Delta_4}{2\alpha(1 - x_A^*)}. \tag{9}$$

One sees from (6) – (9) that convergence to both θ_1 and θ_3 occurs with the rate $t^{-1/2}$ but the random fluctuations, which are determined by the variances of the corresponding limit distributions, around this dominant can be different.

In this example, the above dynamics of prices together with the described behaviour of adopters generate multiple limit patterns with slightly different rates of emergence. Under the same price dynamics and marginally more sophisticated assumptions concerning the behaviour of adopters, one can have even more complicated limit market structure where the initial proportions of the technologies on the market influence those structures (see Dosi *et al.* (1991)). Similar the considerations concerning convergence rates also apply (with corresponding modifications).

The analytical procedure is to introduce further specifications on the statistical frequences (probabilities) of the producers for $A(B)$ to follow a particular shape of the above price dynamics and/or hypotheses concerning statistical frequences of the adopters who use variants of the above decision rules: thus, one can construct much more complicated functions $f_t(\cdot)$.

Next let us discuss one important generalization of the urn scheme presented so far.

6 Urn Schemes with Multiple Additions – a Tool for Analysis of System Compatibilities

As mentioned in section 2, quite a few modern high-technology products require compatibility. We have also hinted earlier that considering all notional combinations of new technologies as a sort of "higher level" new technologies, although formally possible, does not look too attractive. An alternative method for handling inter-technological compatibilities has been introduced by Arthur *et al.* (1987a). For the case of two (A and B) competing technologies it looks like the following.

Consider Z_+^2, the set of two dimensional vectors with non-negative integer coordinates. Introduce $\vec{\xi}_t(x)$, $t \geq 1$, $x \in R(0,1)$, independent in t, random vectors with values in Z_+^2. If $\vec{\xi}_t(x)$ takes value $\vec{i} = (i_1, i_2)$ we can interpret this both as additions $i_1 \geq 0$ white and $i_2 \geq 0$ black balls into an urn of infinite capacity with black and white balls or as adoptions on a market of infinite capacity of i_1 units of A and i_2 units of B.

Mathematical results similar to those presented in section 3 are obtained (see Arthur *et al.* (1987a), (1987b) and (1988)). An important property of this generalization is that $\vec{\xi}_t(x)$ can take the value $\vec{0} = (0,0)$ with nonzero probability. Consequently no adoption might happen at time t. Hence, taking into account that the scheme allows multiple adoptions, one sees that sequential instances of adoption do not coincide with physical time "periods". Hence, loosely speaking, history may "accelerate" by discrete jumps of variable length.

Further, let us introduce the urn model corresponding to the case when competition occurs in non-homogeneous economic environments.

7 Generalized Urn Schemes with Non-Homogeneous Economic Environments

Think of m urns of infinite capacity with black and white balls. Starting with $n_i^w \geq 1$ white balls and and $n_i^b \geq 1$ black balls into the i-th urn a ball is added in one of the urns at time instants $t = 1, 2 \ldots$. It will be added with probability $f_i(\vec{X}(t))$ to the i-th urn. Also it will be white with probability $f_i^w(\vec{X}(t))$ and black with probability $f_i^b(\vec{X}(t))$. Here $\vec{f}(\cdot)$, $\vec{f}^w(\cdot)$, $\vec{f}^b(\cdot)$, are vector functions which map $R(\vec{0},\vec{1})$ in S_m and $\vec{f}^w(\cdot) + \vec{f}^b(\cdot) = \vec{f}(\cdot)$. By $R(\vec{0},\vec{1})$ we designate the Cartesian product of m copies of $R(0,1)$ and

$$S_m = \{\vec{x} \in R^m : \quad x_i \geq 0, \sum_{i=1}^{m} x_i = 1\}.$$

Also $\vec{X}(t)$ stands for the vector whose i-th coordinate $X_i(t)$ represents the proportion of white balls in i-th urn at time t. To introduce the dynamics of $\vec{X}(t)$ consider $\xi^t(\vec{x})$, $t \geq 1$, $\vec{x} \in R(\vec{0},\vec{1})$, independent in t, random $m \times 2$ matrices with the elements $\xi_{i,j}^t(\vec{x})$, $\quad i = 1, 2, \ldots, m$, $j = 1, 2$, such that $\mathcal{P}\left\{\xi_{i,1}^t(\vec{x}) = 1\right\} = f_i^w(\vec{x})$ and $\mathcal{P}\left\{\xi_{i,2}^t(\vec{x}) = 1\right\} = f_i^b(\vec{x})$. Then the total number γ_i^t of balls in i-th urn at time $t \geq 1$ follows the dynamics

$$\gamma_i^{t+1} = \gamma_i^t + \xi_{i,1}^t(\vec{X}(t)) + \xi_{i,2}^t(\vec{X}(t)),\ t \geq 1,\ \gamma_i^1 = n_i^w + n_i^b\,. \tag{10}$$

Since

$$\mathbf{E}[\xi_{i,1}^t(\vec{x}) + \xi_{i,2}^t(\vec{x})] = f_i(\vec{x}), \tag{11}$$

then, requiring that

$$f_i(\vec{x}) \geq f_i^0 > 0, \tag{12}$$

one has

$$f_i^0 \leq \liminf_{t\to\infty} \frac{\gamma_i^t}{t} \leq \limsup_{t\to\infty} \frac{\gamma_i^t}{t} \leq 1. \tag{13}$$

The number w_i^t of white balls and the number b_i^t of black balls in the urn follow the dynamics

$$w_i^{t+1} = w_i^t + \xi_{i,1}^t(\vec{X}(t)),\ t \geq 1,\ w_i^1 = n_i^w\,, \tag{14}$$

$$b_i^{t+1} = b_i^t + \xi_{i,2}^t(\vec{X}(t)),\ t \geq 1,\ b_i^1 = n_i^b\,.$$

Dividing (14) on (10) one has the following dynamics for for the proportion of white balls in the i-th urn

$$X_i(t+1) = X_i(t) + \frac{1}{\gamma_i^t} \frac{\xi_{i,1}^t(\vec{X}(t)) - X_i(t)[\xi_{i,1}^t(\vec{X}(t)) + \xi_{i,2}^t(\vec{X}(t))]}{1 + (\gamma_i^t)^{-1}[\xi_{i,1}^t(\vec{X}(t)) + \xi_{i,2}^t(\vec{X}(t))]},\ t \geq 1,\ X_i(1) = \frac{n_i^w}{\gamma_i^1}. \tag{15}$$

Since

$$\mathbf{E}\left\{\frac{1}{\gamma_i^t}\frac{\xi_{i,1}^t(\vec{X}(t)) - X_i(t)[\xi_{i,1}^t(\vec{X}(t)) + \xi_{i,2}^t(\vec{X}(t))]}{1+(\gamma_i^t)^{-1}[\xi_{i,1}^t(\vec{X}(t)) + \xi_{i,2}^t(\vec{X}(t))]}|\vec{X}(t) = \vec{x},\ \vec{\gamma}^t = \vec{\gamma}\right\} = \frac{1}{\gamma_i}\frac{f_i^w(\vec{x}) - x_i f_i(\vec{x})}{1+(\gamma_i)^{-1}f_i(\vec{x})},$$

relations (13) and (15) allow to show that $\vec{X}(t)$ converges with probability 1 as $t \to \infty$ to the set of zeros (defined properly) on $R(\vec{0},\vec{1})$ of the m-dimensional vector-function $\vec{F}(\cdot)$ whose i-th coordinate is $f_i^w(\vec{x}) - x_i f_i(\vec{x})$. Assume that both $\vec{f^w}(\cdot)$ and $\vec{f^b}(\cdot)$ are continuous and there is a limit $\vec{X}^0$ for $\vec{X}(t)$. Then from equality (11) one can conclude that $\vec{\gamma}^t$ converges with probability 1 as $t \to \infty$ and the limit $\vec{\gamma}^0$ has the form

$$\gamma_i^0 = f_i(\vec{X}^0), \quad i = 1,2,\ldots,m. \tag{16}$$

Using above relations we can obtain analogs of the results listed in section 3 for the basic generalized urn scheme.

Consider a particular model of technological dynamics in a non-homogeneous economic environment which can be treated by means of the modification given here.

Suppose that we have two possible locations - 1 and 2, for the producers of two competing technologies - A and B. At each of the locations there are one firm producing A and one firm producing B. Producers use the strategy described in section 5 (with their own sets of parameters). Then for each of the locations there exists a minimal price of the technologies as a function of the current concentration of, say, A, i.e. $M(x_A) = \min(Pr_A(x_A), Pr_B(x_A))$. For the case represented by figure 1, the function is given on figure 3. Note that at points λ_i technologies reverse their order as the cheaper ones. Designate the proportion of A for the first and the second locations by x_1 and x_2 correspondingly. Also designate by λ_j^i, $j = 1,2,3$, $i = 1,2$, the points where the minimal prices switch from one technology to other. (Consequently we consider the case when the minimal prices for the both locations have a shape similar to that presented in figure 3.) Suppose that at time instants $t = 1,2,\ldots$ a consumer buys a unit of either technology. He adopts the cheapest among the technologies, but, as before (section 5), because of some specific preferences or other reasons which can be taken into account statistically, he measures the difference between $M_1(x_1)$ and $M_2(x_2)$ with a random error. Here $M_i(\cdot)$ stands for the minimal price for the i-th location as a function of the market-share of A at this location. A unit of the technologies from the first location is bought if $M_1(x_1) - M_2(x_2) + \zeta < 0$; otherwise, i.e. when $M_1(x_1) - M_2(x_2) + \zeta > 0$, a unit from the second location is bought. As before (section 5) to preserve the symmetry of the decision rule we should avoid the situation when the event "$M_2(x_2) - M_1(x_1) = \zeta$" has nonzero probability. Consequently we should again assume that the distribution of ζ possesses a density with respect to the Lebesgue measure on the set of real numbers. The probability to choose the first location is $f_1(x_1,x_2) = \mathcal{P}\{\zeta < M_2(x_2) - M_1(x_1)\}$.

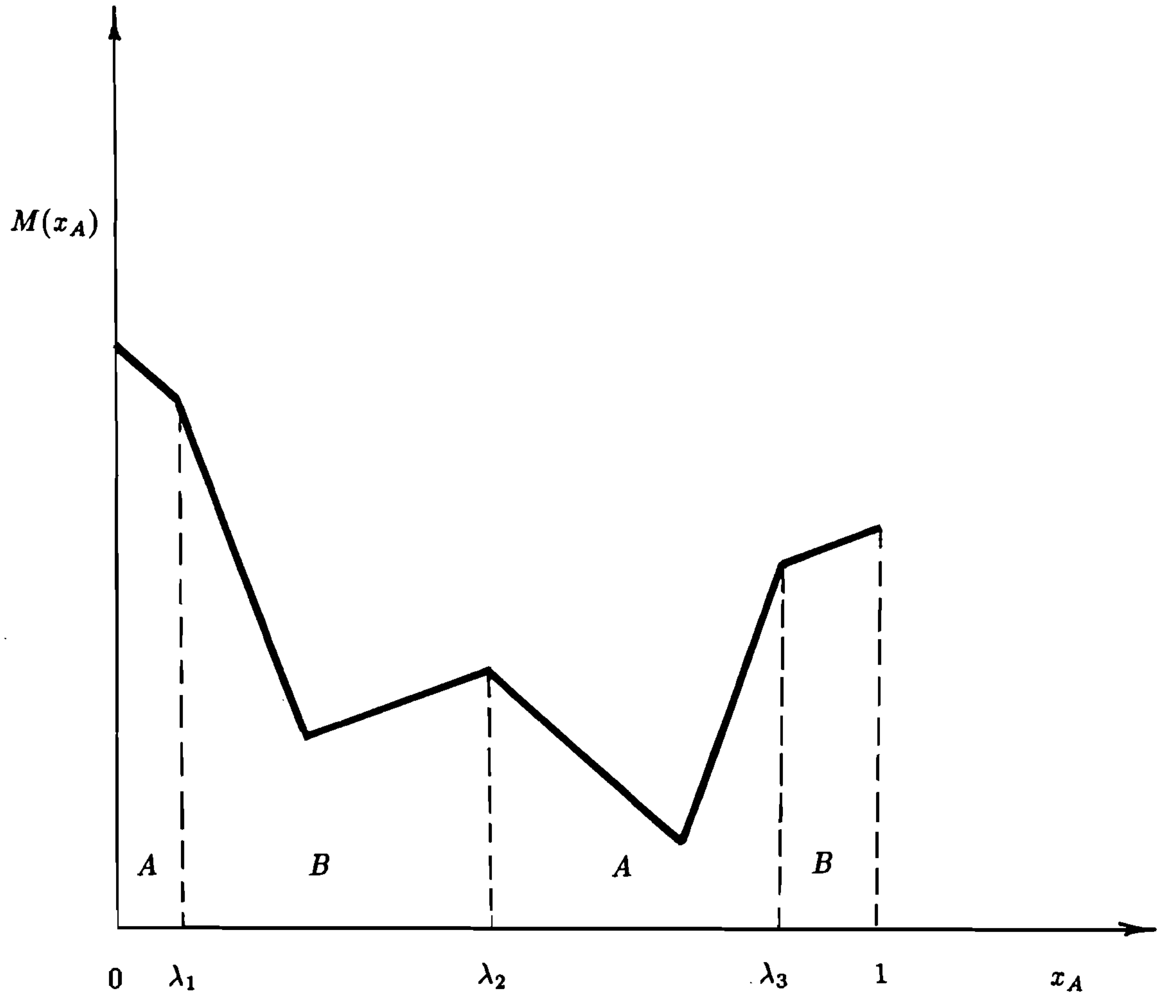

Figure 3: The price of the cheapest among A and B technologies as a function of x_A.

To simplify our considerations let us suppose that ζ has a uniform distribution on $[-\beta, \beta]$. Then the probability to choose the first location is

$$f_1(\vec{x}) = \begin{cases} 1 & \text{if } M_2(x_2) - M_1(x_1) \geq \beta \,, \\ 0 & \text{if } M_2(x_2) - M_1(x_1) \leq -\beta \,, \\ [M_2(x_2) - M_1(x_1) + \beta]/2\beta & \text{if } -\beta < M_2(x_2) - M_1(x_1) < \beta \,. \end{cases}$$

Suppose that $\beta > \max_{0 \leq x_i \leq 1, i=1,2} |M_2(x_2) - M_1(x_1)|$. Then (11) holds with

$$f_1^0 = \{\min_{0 \leq x_i \leq 1, i=1,2} [M_2(x_2) - M_1(x_1)] + \beta\}/2\beta$$

and

$$f_2^0 = 1/2 - \{\max_{0 \leq x_i \leq 1, i=1,2} [M_2(x_2) - M_1(x_1)]\}/2\beta.$$

The simplest decision rule for choosing a specific technology when a location has been chosen is the following: a unit of $A(B)$ is adopted at the i-th location if $x_i \in I_i^A$ $x_i \in I_i^B$. Here $I_i^A = (0, \lambda_1^i) \bigcup (\lambda_2^i, \lambda_3^i)$ and $I_i^B = [\lambda_1^i, \lambda_2^i] \bigcup [\lambda_3^i, 1)$. The corresponding vector-function $\vec{F}(\cdot)$ has the form

$$F_i(\vec{x}) = \begin{cases} (1 - x_i) f_i(\vec{x}) & \text{for } x_i \in I_i^A \,, \\ -x_i f_i(\vec{x}) & \text{for } x_i \in I_i^B \,. \end{cases}$$

We can show that $\vec{X}(t)$ converges (for any initial number of A and B at the both locations) with probability 1 as $t \to \infty$ to a random vector $\vec{X}$. The limit takes with positive probability four values: $(\lambda_1^1, \lambda_1^2)$, $(\lambda_1^1, \lambda_3^2)$, $(\lambda_3^1, \lambda_1^2)$, $(\lambda_3^1, \lambda_3^2)$. Finally note that one may easily refine these examples by introducing more complicated decision rules (e.g. mixed strategies randomizing the choice among technologies after having chosen the location, etc.).

8 Conclusions

Innovation and technology diffusion generally involve competition among different technologies, and, most often, endogenous changes in the costs/prices of technologies themselves and in adopters' choices. In the economic domain (as well as in other disciplines) the formal representation of such processes involves some dynamics of competing "populations" (i.e., technologies, firms, or even behavioral traits and "models" of expectation formation). A growing literature on such dynamics has begun studying the properties of those (generally non-linear) processes that innovation and diffusion entails. As by now robustly established, multiple equilibria are normally to be expected and "history matters", also in the sense that out-of-equilibrium fluctuations may bear system-level consequences on notional asymptotic outcomes. Developing on previous results showing – under dynamic increasing returns – the likely "lock-in" of diffusion trajectories onto particular technologies, we have presented a formal modeling apparatus aimed at handling the interaction between diffusion patterns, on the one hand, and endogenous preferences formation and/or endogenous price formation, on the other. As examples, we presented three classes of stochastic models of shares dynamics on a market of infinite capacity by two competing new technologies. In the first of them, we assumed that the adoption dynamics is essentially driven by endogenous changes in the choices of risk-averse, imperfectly informed adopters (or, in a formally equivalent analogy, by some positive or negative externality imperfectly estimated by would-be users of alternative technologies). In the second example, we considered an endogenous price dynamics of two alternative technologies, driven by e.g., changes in their costs of production and/or by the intertemporal behaviours of their producers. In the third example we dealt with the same economic set-up as in the second one, but with an explicit "spatial" representation of the location of producers.

In all of the cases, the diffusion process is allowed to embody some stochasticity, due to e.g., "imperfect" learning from other people's choices, marginal and formally undetectable differences in users' preferences, or some inertia in adjusting between differently priced but identical-return technologies.

The formal apparatus presented here, based on the idea of the generalized urn scheme, allows quite general analytical accounts of the relationships between some system-parameters (e.g., proxies for information "imperfection" by adopters; dynamic increasing returns and monopolistic exploitation of new technologies by their producers) and limit market shares. While path-dependency (i.e., "history matters") applies throughout, the foregoing analytical techniques appear to be able, at the very least, to discriminate those which turn out to be feasible limit equilibria (i.e., those which are attainable with positive probabilities) and, also, to discover the different rates of emergence of the limit patterns.

The apparatus can also be used for numerical simulation. In this case it proves to be as general as ordinary differential equations and as easy to implement. By means of numerical simulation one can also study much more complicated and "inductively rich" models. Still, the developed mathematical machinery serves in such numerical studies as a means of prediction and verification, showing the general kind of behaviour one ought to expect. Yet another complementarity between the analytical exploration of these models and their numerical simulation concern the study of their non-limit properties, e.g. the "transient" structures that might emerge along the trajectories and their degrees of persistence.

As the foregoing modeling illustrations show, "market imperfections" and "informational imperfections" often tend to foster technological variety, i.e., the equilibrium co-existence of different technologies and firms. Moreover, stochasticity in the choice process may well bifurcate limit market-shares outcomes. Finally, it is shown, corporate pricing strategies-possibly based on rationally-bounded procedures, imperfect informational and systematically "wrong" expectation-formation mechanisms – are generally bound to influence long-term outcomes. Under all these circumstances, the foregoing modeling techniques allow, at the very least, a "qualitative" analytical assessment of diffusion/competition processes by no means restricted to those circumstances whereby microeconomic expectations, on average, represent unbiased estimations of the future.

If all this analytical representation is empirically adequate, there seem to no *a priori* reasons to restrict it to technological dynamics. In fact, under suitable modifications, it may apply as well to interdependent expectations, decisions and returns in many other economic domains. Just to give few examples: the evolution of strategies and organizational forms in industrial dynamics; the dynamics of location in economic geography (Arthur (1990)); adaptive processes and the emergence of social norms; "mimetic" effects and speculation on financial markets; macroeconomic coordination[10]. The list is likely to be indeed very long. Ultimately, what we have tried to implement is a relatively general analytical apparatus able to handle at least some qualitative

[10]For some works these different domains that link at least in spirit with the approach to economic dynamics suggested here, see among others, Kirman (1991), Kuran (1991), Boyer and Orléan (1992), Durlauf (1991).

properties of dynamic stochastic processes characterized by both positive, and, possibly negative, feedbacks of a functional form as "badly-behaved" as possible. Indeed, we believe, quite a few of the processes of economic change fall into this category, related to technological change but also to interdependent (possibly "disequilibrium") changes in e.g., industrial structures, but also financial or product-market expectations and behaviours.

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