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Preface

One of the basic goals of dynamic models in systems theory is to reflect both the uncertainty in the model and the ability to describe the models' behavior through appropriate decisions (controls). These are generally figured out through the feedback principle on the basis of the on-line position of the system. The aim of such synthesizing control strategies is usually to ensure viability properties and also to achieve some terminal goals despite of the incomplete information about the process.

In this paper a mathematical scheme for solving such problems with the techniques of set-valued calculus is given. The results were mostly achieved within the activity plan of the Dynamic Systems Project at IIASA.

On Viable Tubes Generated by Synthesized Decision Strategies for Uncertain Systems

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Introduction

This paper deals with the description of synthesized trajectories in the target problem for viable systems with input uncertainty. It is shown that the tubes of such trajectories could be described without calculating the synthesizing decision strategies but only through the solution of a system of coupled "funnel equations" with set-valued state space variables. The variety of all possible synthesizing decisions (the set-valued feedback controls) could be then derived from these solutions. It turns out that each of these feedback controls generates one and the same synthesized solution tube – the viable trajectory of the synthesized system [1].

The paper is based on the techniques of set-valued calculus [2].

1 The Problem of Nonlinear Control Synthesis

Let us start with a linear system

$$\dot{x} = A(t)x + u + f(t), \quad t_0 \leq t \leq t_1, \tag{1.1}$$

with restrictions on the control

$$u \in \mathcal{P}(t) \tag{1.2}$$

the uncertain input

$$f(t) \in Q(t) \tag{1.3}$$

and the state space vector

$$x(t) \in Y(t). \tag{1.4}$$

Here $\mathcal{P}(t), Q(t), Y(t)$ are convex compact-valued multifunctions, continuous in t. Given a convex compact terminal set $\mathcal{M}$, the objective is to find a (set-valued) synthesizing control strategy $u = \mathcal{U}(t,x)$, such that all the solutions to the differential inclusion

$$\dot{x} \in \mathcal{U}(t,x) + Q(t), \tag{1.5}$$

that start at a given point $x_\tau = x[\tau]$, $\tau < t_1$, $x_\tau \in W[\tau]$, would satisfy both the state constraint (1.4) and the terminal inclusion

$$x[t_1] \in \mathcal{M}. \tag{1.6}$$

Here $W[\tau] = W(\tau, t_1, \mathcal{M})$ is the *solvability set* for the latter problem, which is the "largest" of sets with respect to inclusion from which the solution does exist at all.

In general, the map $u = \mathcal{U}(t,x)$ is *nonlinear*. The respective class of feasible strategies $\mathcal{U}(t,x)$ will be specified below.

As it was indicated in [4] the multivalued function $W[t]$ satisfies an *evolution equation* with set-valued solutions, which is

$$\lim_{\sigma \to 0} \sigma^{-1} h_+ = (W(t-\sigma) + \sigma Q(t), W(t) \cap \mathcal{Y}(t) - \sigma \mathcal{P}(t)) = 0, \tag{1.7}$$

$$W(t_1) \subseteq \mathcal{M},$$

where $h_+(X', X'')$ is the *Hausdorff semidistance* for $X', X'' \in$ comp $\mathbb{R}^n$, namely $h_+(X', X'') = \inf\{\epsilon : X'' + \epsilon S(0) \supseteq X'\}$, where $S(x^*) = \{x : (x - x^*, x - x^*) \leq 1\}$.

This is true however, under the assumptions that there exists a tube $\varepsilon S(x^*(t))$, such that

$$\varepsilon S(x^*(t)) \subseteq \mathcal{Y}(t)\ ,$$

for some continuous function $x^*(t)$ and some $\varepsilon > 0$. Then the solution to (1.7), namely a solution tube $\mathcal{W}[t]$, does exist, but may not be unique. It was proved that $W[t]$ is the maximal solution to equation (1.7) or namely that

$$\mathcal{W}[t] \subseteq W[t], \quad t_0 \leq t \leq t_1,$$

for any solution $\mathcal{W}[t]$ to equation (1.7).

Clearly, $W[t] \subseteq \mathcal{Y}(t)$ for all $t \in [t_0, t_1]$.

Definition 1.1 *A feasible control strategy is a set-valued map* $\mathcal{U}(t,x) \neq \emptyset$ *such that* $\mathcal{U}(t,x) \in \mathcal{P}(t)$ *for all* $\{t, x : t \in [t_0, t_1], x \in \mathbb{R}^n\}$ *and such that for any* $x[\tau] = x_\tau \in \mathbb{R}^n$ *the solution* $x[t]$ *to (1.5) does exist and is extendable throughout the interval* $[\tau, t_1]$. *The class of such strategies will be denoted as* $\mathbf{U}$.

Amid the class $\mathbf{U}$ of *feasible control strategies* $U(t,x)$ we will single out the subclass $\mathbf{U}^*$ of synthesizing strategies for the target problem (1.6) with state constraints (1.4).

Definition 1.2 *A strategy* $\mathcal{U}(t,x)$ *will be a* **synthesizing strategy** *for the target problem (1.6) with state constraint (1.4), if for any vector* $x_\tau \in W[\tau]$ *the solution tube* $X[t] = X(t,\tau,x_\tau,\mathcal{U})$ *for the differential inclusion (1.5) will satisfy the finite inclusion* $X[t] \subseteq W[t]$, $\tau \leq t \leq t_1$, *and therefore, the target condition* $X[t_1] \subseteq \mathcal{M}$.

The problem of Control Synthesis is to specify the synthesizing strategy $\mathcal{U}(t,x)$ [3]. The synthesizing strategy thus leads to a nonlinear differential inclusion. The tube of its solutions (the synthesized trajectories) could be therefore also described by evolution equations of the funnel type. Hence the arising question studied in this paper is as follows: would it be possible to present the tube of synthesized trajectories for (1.5) through some kind of solution tubes for the original uncertain linear system (1.1) with constraints (1.2)-(1.4)? Could, for example, the funnel equtions of type (1.7) be used for this purpose?

The other notations used in the sequel are as follows: $2^{\mathbb{R}^n}$ stands for the set of all subsets of $\mathbb{R}^n$, conv $\mathbb{R}^n$ for the set of all nonempty convex compact subsets of $\mathbb{R}^n$, $h_+(X,Y)$ - for the Hausdorff semidistance, as indicated above, provided $X, Y \in$ conv $\mathbb{R}^n$, $h(X,Y) = \max\{h_+(X,Y),\ h_+(Y,X)\}$ for the Hausdorff distance between $X, Y \in$ conv $\mathbb{R}^n$.

We further assume that system (1.1) - (1.4) has undergone a coordinate transformation, so that without loss of generality, we may assume $A(t) \equiv 0$.

Let us begin with the simplest case, when $Q(\cdot) = \{0\}$ and $Y(t) \equiv \mathbb{R}^n$.

2 The Simplest Version

Denote $\mathbf{X}[t] = \mathbf{X}(t,\tau;x_\tau,\mathcal{U})$ to be the cross-section at time t of the tube of all trajectories to system (1.5), $Q(\cdot) = \{0\}$, that start at time τ, from position $x[\tau] = x_\tau$ and are governed by a feasible control $\mathcal{U} \in \mathbf{U}$. This indicates particularly that with initial set $\mathbf{X}^0 = \{x^0\}$, $x^0 = x(t_0)$, given and with the problem of control synthesis being solvable for any $x^0 \in \mathbf{X}^0$, we have

$$\mathbf{X}(t_1, t_0; \mathbf{X}^0, \mathcal{U}) \subseteq \mathcal{M}. \tag{2.1}$$

Consider the evolution equation

$$\lim_{\sigma \to 0} \sigma^{-1} h(W(t-\sigma),\ W(t) - \sigma \mathcal{P}(t)) = 0 \tag{2.2}$$

with boundary condition

$$W(t_1) = \mathcal{M}. \tag{2.3}$$

This equation always has a solution $W(t)$ with values in comp $\mathbb{R}^n$, presented by the set-valued Lebesgue (Aumann) integral, so that

$$W(t) = \mathcal{M} - \int_t^{t_1} \mathcal{P}(\tau) d\tau$$

and its support function

$$\rho(l|W(t)) = \{\max(l, w) | w \in W(t)\}$$

is continuous differentiable in t. Due to [5] this allows to consider the funnel equation

$$\lim_{\sigma \to 0} \sigma^{-1} h(\Gamma(t) + \sigma \mathcal{P}(t)) \cap W(t+\sigma)) = 0 \tag{2.4}$$

with initial condition

$$\Gamma(t_0) = \mathbf{X}^0. \tag{2.5}$$

The solution to (2.4), (2.5) exists and is extendable to the interval $[t_0, t_1]$, particularly if $W[t]$ is the solution to (2.2), (2.3). (see [2,5]). The *solution* to (2.2)-(2.5) is then specified as a pair $\{W(t), \Gamma(t)\}$ of functions defined on $[t_0, t_1]$, with values in comp $\mathbb{R}^n$, that satisfy the equations (2.2), (2.4) for all t and also the boundary conditions (2.3), (2.5).

The support function $\rho(l|\Gamma(t))$ for $\Gamma(t)$ is absolutely continuous in t.

It is not difficult to prove the following

Lemma 2.1 *The solution $\{W(t), \Gamma(t)\}$ to the set-valued boundary problem (2.2) - (2.5) does exist if there exists a function $x[t]$ that satisfies both the differential inclusion*

$$\dot{x} \in \mathcal{P}(t) \tag{2.6}$$

and the finite inclusions

$$x[t_0] \in \mathbf{X}^0;\ x(t) \in W(t) \tag{2.7}$$

$$t_0 \le t \le t_1.$$

A stronger requirement is given by

Assumption A. There exists a trajectory $x^*[t]$ of system (2.6) such that $x^*[t_0] \in X^0$ and

$$x^*[t] \in \text{ int } W(t), \quad t_0 \leq t \leq t_1.$$

Theorem 2.1 *Let Assumption A be fulfilled. Then the following assertions are true:*

(i) In order that there would exist a feasible strategy $\mathcal{U} \in \mathbf{U}$ that ensures (2.1), it is necessary and sufficient that the system (2.2) - (2.5) would have a solution $\{W(\cdot), \Gamma(\cdot)\}$.

(ii) A strategy $\mathcal{U} \subseteq \mathbf{U}$ ensures the relation (2.1) if and only if the inclusion

$$\mathbf{X}(t, t_0; \mathbf{X}^0, \mathcal{U}) \subseteq \Gamma(t)$$

is true for all $t \in [t_0, t_1]$.

(iii) The variety of set-valued maps $\{\mathbf{X}(\cdot) = \mathbf{X}(\cdot, t_0; X^0, \mathcal{U}) | \mathcal{U} \in \mathbf{U}, \mathbf{X}[t_1] \subseteq \mathcal{M}\}$ has a unique maximal element $\mathbf{X}^*(\cdot)$ with respect to the inclusion and

$$\mathbf{X}^*(t) \equiv \Gamma(t). \tag{2.8}$$

Theorem 2.1 indicates that the tube $\mathbf{X}^*(t)$ of solutions to a synthesized system (1.5), $(Q = \{0\})$ may be calculated without the knowledge of the control strategy $\mathcal{U}$ itself, but just through calculating the tube $\Gamma(t)$. It is therefore natural to pose the question: how to find the strategy $u = \mathcal{U}(t, x)$ that generates $\Gamma(t)$ provided $\Gamma(t)$ is known?

Let $\Pi_\Gamma x$ denote the *metric projection* of $x \in \mathbb{R}^n$ on set $\Gamma \in$ conv $\mathbb{R}^n$, namely $\Pi_\Gamma x = \{x^* : \|x - x^*\| = d(x, \Gamma)\}$ where $d(x, \Gamma) = \min\{\|x - z\| | z \in \Gamma\}$ and

$$T_\Gamma(t, x) = \{u : \lim_{\sigma \to 0} \sigma^{-1} h_+(\Pi_{\Gamma(t)} x + \sigma u, \Gamma(t + \sigma)) = 0\}.$$

It is obvious that $\Pi_\Gamma(t, x) = x$ if $x \in \Gamma(t)$ and that $T_\Gamma(t, x)$ is the *tangent cone* to $\Gamma(t)$ at point x if $x \in \Gamma(t)$.

Denote

$$\mathcal{U}^*(t, x) = T_\Gamma(t, x) \cap \mathcal{P}(t).$$

Lemma 2.2 *The set-valued map $\mathcal{U}^*(t, x)$ is a feasible synthesizing control strategy for system (1.5), $Q = \{0\}$.*

The proof of Lemma 2.2 follows the standard techniques of set-valued calculus.

Calculating the full derivative in time t

$$\frac{d}{dt}d(x,\Gamma(t))$$

due to the system

$$\dot{x}=u, \quad u\in\mathcal{P}(t), \tag{2.9}$$

define

$$\mathcal{U}^{**}(t,x)=\{u:\frac{d}{dt}d(x,\Gamma(t))|\leq 0\}. \tag{2.10}$$

It is not difficult to observe the following assertion

Lemma 2.3 *The set-valued map $\mathcal{U}^{**}(t,x)$ is a feasible synthesizing control strategy for system (1.5), $Q=\{0\}$.*

A particular version of $\mathcal{U}^{**}(t,x)$ is the "extremal aiming" map

$$\mathcal{U}^{l}(t,x)=\partial_l\rho(-l^0(t,x)|\mathcal{P}(t)),$$

where $\partial_l f(l,t)$ is the subdifferential of $f(t,l)$ in the variable l, and$l^0=l^0(t,x)$ is the unique maximizer for

$$d[t,x]=\max\{(l,x)-\rho(l|W[t])|||l||\leq 1\}$$

$$(l^0=\{0\} \text{ if } d[t,x]=0).$$

Theorem 2.2 *Assume system (2.2)-(2.5) to have a solution $\{W(\cdot),\Gamma(\cdot)\}$. Then, whatever is the set $X^0\in W(t_0)$, each of the strategies $\mathcal{U}(t,x)=\mathcal{U}^{*}(t,x)$, $\mathcal{U}(t,x)=\mathcal{U}^{*}(t,x)$, $\mathcal{U}(t,x)=U^{e}(t,x)$ generates one and the same solution tube $\mathbf{X}^0[t]$ to the inclusion*

$$\dot{x}\in\mathcal{U}(t,x), \quad \mathbf{X}[t_0]=\mathbf{X}^0, \tag{2.11}$$

and

$$\mathbf{X}^0[t]=\Gamma(t), \quad t_0\leq t\leq t_1.$$

Therefore, from the point of view of the trajectory tube $\Gamma(t)\equiv X^0[t]$ for the synthesized system (2.11), the strategies $\mathcal{U}^{*}(t,x)$, $\mathcal{U}^{*}(t,x)$, $\mathcal{U}^{l}(t,x)$ cannot be distinguished. What follows from Theorem 2.2 is that there exists a variety $\mathbf{V}^0=\{\mathcal{U}(t,x)\}\subseteq\mathbf{U}^*$ of strategies, each of which generates one and the same tube $\Gamma(t)\equiv\mathbf{X}^0[t]$.

Theorem 2.3 *Suppose that* $\mathcal{U} = \mathcal{U}^0(t,x) \subseteq \mathbf{V}^0$ *and that* $x[t]$ *is a solution to the system*

$$\dot{x} \in \mathcal{U}^0(t,x), \; x[t_0] \in \mathbf{X}^0, \; t_0 \leq t \leq t_1.$$

Then the function $\dot{x}[t] \equiv u[t]$ *satisfies the relation*

$$u[t] \in \mathcal{U}^*(t,x).$$

The given theorem asserts that the tube of all the solutions to the differential inclusion (2.11) with $\mathcal{U} = \mathcal{U}(t,x) \subseteq \mathbf{V}^0$ consists only of those trajectories that satisfy (2.11) with $\mathcal{U} = \mathcal{U}^*(t,x)$.

Theorem 2.4 *Let* $\mathcal{X}[t]$ *be the solution to the "forward" funnel equation*

$$\lim_{\sigma \to 0} \sigma^{-1} h(\mathcal{X}[t+\sigma], \; x[t] + \sigma \mathcal{P}(t)) = 0.$$

Then

$$\Gamma(t) \equiv \mathcal{X}[t] \cap W[t]. \tag{2.12}$$

Remark 2.1. Since $\Gamma(t)$ is also the attainability domain for system (2.6) with state constraints (2.7), the relation (2.12) indicates that in the particular problem considered here the state constraint for (2.6) is given precisely by $W(t)$, the set $\Gamma(t)$ turns to be the intersection of the attainability domain $\mathcal{X}[t]$ for the system (2.6) without state constraints and the state constraint $W[t]$ itself. This property is not true in general (see for example [6]).

3 The Synthesized Tube of Viable Trajectories

Assume now that the state constraint (1.6) holds and that the condition $Q = \{0\}$ is still true. Here the synthesizing strategy $\mathcal{U} = \mathcal{U}(t,x)$ should ensure both (2.1) and the inclusion

$$\mathbf{X}(t, t_0; \mathbf{X}^0, \mathcal{U}) \subseteq Y(t), \quad t_0 \leq t \leq t_1 \tag{3.1}$$

(provided $\mathbf{X}^0 \subseteq W[t_0]$).

Equation (2.2) will now be substituted by

$$\lim_{\sigma \to 0} \sigma^{-1} h(W(t-\sigma), (W(t) - \sigma P(t)) \cap Y(t-\sigma)) = 0 \tag{3.2}$$

$$W(t_1) = M, \tag{3.3}$$

while equation (2.4) remains formally unchanged.

We further introduce

Assumption B. The support function $\rho(l|Y(t))$ is Lipschitz in t.

Assumption C. The Assumption A is fulfilled for equation (3.1), (3.2).

Theorem 3.1 *Suppose Assumptions B and C are fulfilled. Then a control strategy $\mathcal{U} \subseteq \mathbf{U}^*$ ensures the inclusions (2.1), (3.1) if and only if there exists a solution $\{W(t), \Gamma(t)\}$ to the system (3.1), (3.2), (2.4), (2.5). In the latter case $\Gamma(\cdot) = \mathbf{X}^0(\cdot)$ is the unique maximal solution (with respect to inclusion) among the solution tubes $\mathbf{X}(\cdot) = \mathbf{X}(\cdot, t_0, X^0, \mathcal{U})$ that satisfy the relations (2.3), (3.1).*

The synthesizing control strategies for the target problem (1.4), (1.6) may be again described as in the previous section and the analogies of theorems 2.2 - 2.1 are still true with the solution $\{W(\cdot), \Gamma(\cdot)\}$ now taken from (3.2), (3.3), (2.4), (2.5). The class $\mathbf{V}^*$ is defined accordingly.

A relatively simple way of calculating $\Gamma(t)$ is with the aid of a solution $\mathcal{X}[t]$ to the funnel equation

$$\lim_{\sigma \to 0} \sigma^{-1} h(\mathcal{X}(t+\sigma),\ (\mathcal{X}(t) + \sigma P(t)) \cap Y(t)) = 0 \tag{3.4}$$

$$\mathcal{X}(t_0) = X^0. \tag{3.5}$$

Theorem 3.2 *Let $\{W(\cdot), \Gamma(\cdot)\}$ be the solution to system (3.2), (3.3), (2.4), (2.5). Then the strategies $\mathcal{U}^*, \mathcal{U}^{**}, \mathcal{U}^l$ generate one and the same tube $\Gamma(\cdot) = \mathbf{X}^0[\cdot]$ the tube $\Gamma(\cdot)$ may be represented as*

$$\Gamma(t) \equiv \mathcal{X}[t] \cap W(t),$$

where $W(t)$ and $\mathcal{X}[t]$ are the solutions to (3.2), (3.3) and (3.4), (3.5) respectively.

4 Synthesizing Viable Tubes Under Counteraction

The general case that incorporates both the state constraints (1.6) and the input uncertainty ($Q(\cdot) \neq \emptyset$) does not allow a direct propagation of the results of Sections 2 and 3. To treat the problem we will use the notion of Hausdorff semidistance introduced in Section 1.

Definition 4.1 *A multivalued map $Z(\cdot) : [t_0, t_1] \to$ conv $\mathbb{R}^n$ is h_+ - absolutely continuous from the left (the right) if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\sum_i (t_i'' - t_i') < \delta$$

implies

$$\sum_i h_+(Z(t_i')Z(t_i'')) < \varepsilon$$

$$(\sum_i h_+(Z(t_i''), Z(t_i')) < \varepsilon).$$

Let us now introduce the following pair of evolution equations

$$\lim_{\sigma \to 0} \sigma^{-1} h_+(W(t-\sigma) + \sigma Q(t),\ W(t) \cap Y(t) - \sigma P(t)) = 0 \tag{4.1}$$

$$W(t_1) \subseteq \mathcal{M} \tag{4.2}$$

$$\lim_{\sigma \to 0} \sigma^{-1} h_+(\Gamma(t+\sigma), \Gamma(t) \cap W(t) + \sigma P(t) + \sigma Q(t)) = 0 \tag{4.3}$$

$$\Gamma(t_0) = \mathbf{X}^0. \tag{4.4}$$

Definition 4.2 *The solution $\{W(\cdot), \Gamma(\cdot)\}$ to the system (4.1) - (4.4) will be defined as a pair of h_+ - absolutely continuous set-valued functions $W(t), \Gamma(t), t \in [t_0, t_1]$, (with the first of these possessing this property from the right and the second from the left), that satisfy equations (4.1), (4.3) almost everywhere and also the boundary conditions (4.2), (4.4).*

Let $\mathbf{X}_Q(t, t_0, \mathbf{X}^0, \mathcal{U})$ denote the tube of all solutions to (1.5), $x^0 \in \mathbf{X}^0$.

Theorem 4.1 *In order that there would exist a synthesizing strategy for the target problem*

$$\mathbf{X}_Q(t, t_0, \mathbf{X}^0, \mathcal{U}) \subseteq Y(t), \quad t \in [t_0, t_1], \tag{4.5}$$

$$\mathbf{X}_Q(t, t_0, \mathbf{X}^0, \mathcal{U}) \subseteq \mathcal{M} \tag{4.6}$$

due to the system (1.5), it is necessary and sufficient that there would exist a solution $\{W(\cdot), \Gamma(\cdot)\}$ to system (4.1) - (4.4).

The introduction of funnel equations involving semidistances h_+ rather than distances h does not require to impose any additional regularity assumptions.

We could have also done this in the previous sections. However in this case the unicity of the solution $\{W(\cdot), \Gamma(\cdot)\}$ would be lost. The absence of unicity could be compensated through the notion of maximal solution.

Once the variety of solutions to (4.1) - (4.4) is nonvoid, there exists a unique maximal solution $\{W^0(\cdot), \Gamma^0(\cdot)\}$ to this system (with respect to inclusion)

Theorem 4.2 *Let the system (4.1) - (4.4) be resolvable by a strategy $\mathcal{U}(t,x)$ and let $\{W^0(\cdot), \Gamma^0(\cdot)\}$ be the maximal solution to this system. Then the multivalued map $\Gamma^0(\cdot)$ coincides with the maximal tube $\mathbf{X}_Q^0(\cdot)$ of the variety of all tubes $\mathbf{X}_Q(\cdot) = \mathbf{X}_Q(\cdot, t_0, \mathbf{X}^0, \mathcal{U})$ that satisfy (4.5) , (4.6).*

Presuming $\Gamma^0(t)$ to be known, let us now use this knowledge to specify a strategy $\mathcal{U}(t,x)$ that resolves the problem (1.5), (4.5), (4.6).

Consider the distance $d(x, \Gamma^0(t)) = h_+(x, \Gamma^0(t))$, and further, the set

$$\mathcal{U}(t,x,v) = \{u \in P(t) : \frac{d\, d(x, \Gamma^0(t))}{dt}\Big|_{(4.7),f} \leq 0\}$$

Here $d\, d(t,x)/dt$ is the derivative in time $t+0$ of the distance $d(x, \Gamma^0(t))$ due to the system

$$\dot{x} = u + f. \tag{4.7}$$

Denote

$$\mathcal{U}_+(t,x) = \cap_{v \in Q(t)} \mathcal{U}(t,x,v).$$

(We further assume $\mathcal{U}_+(t,x) \neq \emptyset$.)

Also define the strategy $\mathcal{U}_+^l(t,x)$ as $\mathcal{U}^l(t,x)$ in Section 2, where $\Gamma(t)$ is now substituted by $\Gamma^0(t)$.

Lemma 4.1 *Each of the strategies $\mathcal{U}_+(t,x)$, $\mathcal{U}_+^l(t,x)$ is a feasible synthesizing strategy for the problem (1.5), (4.5), (4.6)*

Theorem 4.3 *If $\Gamma^0(t)$ is the maximal solution to (4.1)-(4.4), then*

$$\mathbf{X}_Q(t, t_0, \mathbf{X}^0, \mathcal{U}_+^l) \equiv \mathbf{X}_Q(t, t_0, \mathbf{X}^0, \mathcal{U}_+) \equiv \Gamma^0(t).$$

Conclusion

We have demonstrated here that the tubes of viable trajectories to an uncertain system that also satisfy a target inclusion could be specified through the knowledge of only the system itself and the constraints on the uncertainty, as the solution tubes to a certain system of coupled "funnel equations". The respective synthesizing decision strategies (the feedback controls) could be then determined from these tubes through techniques of set-valued calculus.

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