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Jean Pierre Aubin

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IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

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FOREWORD

In cooperative game theory as well as in some domaine of economic regulation by shortages (queues or unempoyment), one is confronted to the problem of optimizing coalitions of players or economic agents. Since coalitions are subsets and cannot be represented by vectors — except if we embed subsets in the family of fuzzy sets, which are functions — the need to adapt the theory of optimization under constraint for coalitions or subsets instead of vectors did emerge.

The "power spaces" in which coalitions, images, shapes, etc. have to be chosen are metric spaces without a linear structure. However, one can extend the differential calculus to a mutational calculus *for maps from one metric space to another, as we shall explain in this paper. The simple idea is to replace half-lines allowing to define difference quotients of maps and their various limits in the case of vector space by "transitions" with which we can also define differential quotients of a map. Their various limits are called "mutations" of a map.*

Many results of differential calculus and set-valued analysis, including the Inverse Function Theorem, do not really rely on the linear structure and can be adapted to the nonlinear case of metric spaces and exploited. This is the purpose of this paper.

Optimization of Coalitions

The Mutational Approach

Jean-Pierre Aubin

Introduction

The topic of this paper is to build a "differential calculus" in metric spaces in order to study optimization under constraints in metric spaces.

This study was motivated by problems arising in cooperative game theory and economics where coalitions of players or economic agents play an important role which was neglected by lack of adequate mathematical tools.

It was also motivated by "visual servoing", where one needs to find feedback controls feeding back on subsets (shapes) instead of vectors (see [17, 21, Doyen] for further results, applications and references). Mathematical morphology, introduced in [26, Matheron] is also another field of motivations (see [30, Mattioli]).

But a differential calculus unabling to derive necessary conditions for optimization did exist in "shape optimization".

The suggestion we propose here is inspired by the concept of **shape derivatives** of **shape maps** V, which are in some sense "set-defined maps", mapping subsets $K \subset E$ to vectors $V(K) \in Y$ in a finite dimensional vector space Y. (See [12, Céa], [13,14,15,16, Delfour & Zolésio], [18, Doyen], [39, Zolésio], etc.). Their idea was to replace the usual differential quotients $\frac{U(x+hv)-U(x)}{h}$ measuring the variation of a function U on **half-lines** $x+hv$ by differential quotients $\frac{V(\vartheta_{\varphi}(h,K))-V(K)}{h}$ where $\varphi : E \mapsto E$ is a Lipschitz map, $\vartheta_{\varphi}(h,x) := x(h)$ denotes the value at time h of the solution to the differential equation $x' = \varphi(x)$ starting at x at time 0 and $\vartheta_{\varphi}(h,K) := \{\vartheta_{\varphi}(h,x)\}_{x \in K}$ the **reachable set** from K at time h of φ.

In other words, the "curve" $h \mapsto \vartheta_{\varphi}(h,K)$ plays the role of the half lines $h \mapsto x+hv$ for defining differential quotients measuring the variations of the function V along it. Since the set $\mathcal{K}(E)$ of nonempty compact subsets of E is only a metric space, without linear structure, replacing half-lines by curves to measure variations is indeed a very reasonable strategy. For this special metric space, these "curves" ϑ_{φ}, which are examples of "**transitions**" defined below, are in one to one correspondence with the space $\text{Lip}(E,E)$ of Lipschitz maps φ. They play the role of directions when one defines directional derivatives of usual functions. Hence, if the limit

$$\overset{\circ}{V}(K)\varphi := \lim_{h\to 0+} \frac{V(\vartheta_\varphi(h,K)) - V(K)}{h}$$

exists, it is called the **directional shape derivative** of V at K in the "direction" φ. With such a concept, an inverse function theorem allowing to inverse locally a shape map V whenever its shape derivative $\text{Lip}(E,E) \mapsto Y$ is surjective is proved in [18, Doyen] and many applications to shape optimization under constraints are derived in Doyen's paper.

Since this strategy works well for shape maps, it should work as well for set-valued maps, and indeed, it does for solving certain classes of problems.

For instance, in the case of tubes $t \rightsquigarrow P(t)$ with **nonempty compact values**, we suggest to look for differential quotients of the form

$$\frac{d(\vartheta_\varphi(h, P(t)), P(t+h))}{h}$$

which compare the variation $P(t+h)$ and the variation $\vartheta_\varphi(h, P(t))$ produced by a transition ϑ_φ applied to $P(t)$.

Let $B(K,\varepsilon)$ denote the closed ball of radius ε around K. If

$$\lim_{h\to 0+} \frac{d(\vartheta_\varphi(h, P(t)), P(t+h))}{h} = 0 \tag{0.1}$$

or, equivalently, if there exists $\beta(h) \to 0$ with h such that, for all $h \in]0,1]$,

$$\vartheta_\varphi(h, P(t)) \subset B(P(t+h), \beta(h)h) \ \& \ P(t+h) \subset B(\vartheta_\varphi(h, P(t)), \beta(h)h)$$

it is tempting to say that the transition ϑ_φ, or, equivalently, that the associated Lipschitz map $\varphi \in \text{Lip}(E,E)$, plays the role of the directional derivative of the tube P at t in the forward direction 1.

This is what we shall do: we propose to call **mutation** $\overset{\circ}{P}(t)$ **of the tube** P at t the set of Lipschitz maps φ satisfying the property (0.1). We do have to coin a new name, because many concepts of derivatives of a set-valued map — **graphical derivatives**[1], such as contingent derivatives[2], circatangent

[1] according to a term coined by R.T. Rockafellar. See [33,?, Rockafellar], [35, Rockafellar & Wets], SET-VALUED ANALYSIS, [9, Aubin & Frankowska] and VIABILITY THEORY, [6, Aubin], among other authors for an exposition of their properties.

[2] introduced in [2, Aubin].

derivatives[3] or adjacent derivatives[4], as well as other **pointwise concepts**[5] — have been used extensively.

The need to extend concepts of derivatives in metric spaces is not new. As early as 1946, T. Ważewski introduced in [37,38, Ważewski] the concept of **allongements contingentiels supérieur et inférieur** (upper and lower contingent elongations) of a map $X \mapsto Y$[6] to prove implicit function theorems in metric spaces. More recently, H. Frankowska used first order and higher order "variations" in [23,24, Frankowska] to prove sophisticated inverse function theorems in metric spaces and L. Doyen to shape maps in [18, Doyen]. But we follow here another track motivated by the evolution of tubes, shape analysis and mathematical morphology.

The main concepts of set-valued analysis shall then be transferred to set-valued maps $F : X \rightsquigarrow Y$ from a metric space X to a metric space Y, by defining **contingent mutations** of a set-valued map at a point of its graph and other concepts of tangent mutations.

The main concepts of nonsmooth analysis shall also be extended to functions defined on metric spaces. By using **epimutations**, we will adapt to optimization of functions on metric spaces the Fermat and Ekeland rules.

1 Transitions on Metric spaces

Transitions adapt to metric spaces the concept of half line $x + hv$ starting from x in the direction v by replacing it by "curved" half-lines $\vartheta(h, x)$. Indeed, the "linear" structure of half lines in vector spaces is not really needed to build a differential calculus.

[3] introduced in [5, Aubin].

[4] introduced in [?,?,?, Frankowska].

[5] See [10, Banks & Jakobs], [11, De Blasi], [25, Martelli & Vignoli] among many other authors.

[6] Namely,

$$\overline{\text{all}}f(x) := \limsup_{x' \to x} \frac{d(f(x'), f(x))}{d(x', x)} \;\;\&\;\; \underline{\text{all}}f(x) := \liminf_{x' \to x} \frac{d(f(x'), f(x))}{d(x', x)}$$

Definition 1.1 *Let X be a metric space for a distance $\mathbf{d}$. A map $\vartheta : [0,1] \times X \mapsto X$ satisfying*

$$\begin{cases} i) & \vartheta(0,x) = x \\ ii) & \|\vartheta(x)\| := \sup_{h \neq k} \frac{\mathbf{d}(\vartheta(h,x), \vartheta(k,x))}{|h-k|} < +\infty \\ iii) & \|\vartheta\|_\Lambda := \sup_{h\in[0,1],\, x\neq y} \frac{\mathbf{d}(\vartheta(h,x), \vartheta(h,y))}{\mathbf{d}(x,y)} < +\infty \\ iv) & \lim_{h\to 0+} \frac{\mathbf{d}(\vartheta(t+h,x), \vartheta(h, \vartheta(t,x)))}{h} = 0 \end{cases}$$

is called a transition. *When $\|\vartheta\|_\Lambda \leq 1$ in the above inequality, we say that ϑ is a* nonexpansive transition.

We denote by $\overline{\Theta}(X)$ the vector space of all transitions on X[7].

We define an equivalence relation $\sim_x$ between transitions by

$$\vartheta_1 \sim_x \vartheta_2 \text{ if and only if } \lim_{h\to 0+} \frac{d(\vartheta_1(h,x), \vartheta_2(h,x))}{h} = 0$$

We say that $(X, \Theta(X))$ is a (complete) mutational space *if X is a (complete) metric space and $\Theta(X) \subset \overline{\Theta}(X)$ is a nontrivial vector subspace of transitions, closed in $C([0,1] \times X, X)$ supplied with the pointwise convergence.*

Remark — We could have introduced the factor space of equivalence classes of transitions, by identifying at each point equivalent transitions. But this may be too cumbersome. □

[7] One may sometimes need more regular transitions: A transition is strict if

$$\limsup_{y\to x} \sup_{h\neq k} \frac{\mathbf{d}(\vartheta(h,y), \vartheta(k,y))}{|h-k|} < +\infty$$

and

$$\liminf_{h\to 0+,\, y\to x} \frac{\mathbf{d}(\vartheta(t+h,y), \vartheta(h, \vartheta(t,y)))}{h} = 0$$

We shall say that ϑ_1 and ϑ_2 are strictly equivalent if

$$\vartheta_1 \sim_x \vartheta_2 \text{ if and only if } \lim_{h\to 0+,\, y\to x} \frac{\mathbf{d}(\vartheta_1(h,x), \vartheta_2(h,y))}{h} = 0$$

One observes that the transitions $\vartheta(h,\cdot)$ are Lipschitz uniformly with respect to $h \in [0,1]$ and that for every $x \in X$, the maps $\vartheta(\cdot,x)$ are Lipschitz. The **unit transition** defined by $\mathbf{1}(h,x) = x$ is denoted by $\mathbf{1}$.

Example: Transitions on Normed Spaces Let E be a finite dimensional vector space. We can associate with any $v \in E$ the transition $\vartheta_v \in \Theta(E)$ defined by

$$\vartheta_v(h,x) := x + hv$$

for which we have $\|\vartheta_v(x)\| = \|v\|$ and $\|v\|_\Lambda = 1$ (it is nonexpansive).

Therefore, we shall identify a normed space E with the mutational space (E,E) by taking for space of transitions the space $\Theta(E) = E$ of vectors regarded as "directions".

We can enlarge the space of transitions by using the Cauchy-Lipschitz Theorem. We associate with any Lipschitz map $\varphi : X \mapsto X$ the transition $\vartheta_\varphi \in \Theta(E)$ defined by

$$\vartheta_\varphi(h,x) := x(h)$$

where $x(h)$ is the unique solution to the differential equation $x'(t) = \varphi(x(t))$ starting from x.

Indeed, we deduce from the Cauchy-Lipschitz Theorem that

$$\|\vartheta_\varphi(x)\| \leq e^{\|\varphi\|_\Lambda} \frac{e^{\|\varphi\|_\Lambda} - 1}{\|\varphi\|_\Lambda} \|\varphi(x)\|$$

and that $\|\vartheta_\varphi\|_\Lambda \leq e^{\|\varphi\|_\Lambda}$ because

$$d(\vartheta_\varphi(h,x), \vartheta_\varphi(h,y)) \leq e^{\|\varphi\|_\Lambda} d(x,y)$$

They satisfy $\vartheta_\varphi(h+t,x) = \vartheta_\varphi(h, \vartheta_\varphi(t,x))$.

We also deduce that

$$d_\Lambda(\vartheta_\varphi, \vartheta_\psi) \leq \frac{e^{\|\varphi\|_\Lambda} - 1}{\|\varphi\|_\Lambda} \|\varphi - \psi\|_\infty$$

because

$$d(\vartheta_\varphi(h,x), \vartheta_\psi(h,x)) \leq \frac{e^{\|\varphi\|_\Lambda h} - 1}{\|\varphi\|_\Lambda h} \|\varphi - \psi\|_\infty$$

Then the space of Lipschitz maps $\varphi : E \mapsto E$ can be embedded in the space $\overline{\Theta}(E)$ of all transitions:

$$E \subset \text{Lip}(E,E) \subset \overline{\Theta}(E)$$

We observe that for any $x \in E$, φ is equivalent to the vector $\varphi(x)$ at x: $\varphi \sim_x \varphi(x)$.

Example: Transitions on a subset of a vector space

Let $M \subset E$ be a closed subset of a finite dimensional vector space E. We denote by $T_M(x)$ its **contingent cone** and by $N_M(x) := (T_M(x))^-$ the **subnormal cone.**

Nagumo's Theorem for differential equations (see [31, Nagumo], VIABILITY THEORY, [6, Aubin]) states that M is invariant under $\varphi \in \text{Lip}(E, E)$ if and only if

$$\forall\, x \in M,\ \ \varphi(x) \in -T_M(x) \cap T_M(x)$$

and, actually[8], if and only if

$$\forall\, x \in M,\ \ \forall\, p \in N_M(x),\ \ \langle p, \varphi(x)\rangle \;=\; 0 \qquad (1.1)$$

We shall set

$$\text{Lip}_0(M, E) \;:=\; \{\varphi \in \text{Lip}(E, E) \mid \text{satisfying (1.1)}\}$$

When φ is Lipschitz, we denote by

$$\|\varphi\|_\Lambda \;:=\; \sup_{x \neq y} \frac{\|\varphi(x) - \varphi(y)\|}{\|x - y\|}$$

the Lipschitz semi-norm of φ.

We supply it with the distance $\|\varphi_1 - \varphi_2\|_\infty \;:=\; \sup_{x \in M} \|\varphi_1(x) - \varphi_2(x)\|$ of uniform convergence.

We thus infer that

$$\text{Lip}_0(M, E) \;\subset\; \overline{\Theta}(M)$$

is a space of transitions of the metric subset M.

Example: Transitions on Power Sets This is our main example. Let $M \subset E$ be a closed subset of a finite dimensional vector space E and $X := \mathcal{K}(M)$ be the family of nonempty compact subsets $K \subset M$.

We can also associate with any Lipschitz map $\varphi : E \mapsto E$ a transition $\vartheta_\varphi \in \Theta(X)$ defined by

$$\vartheta_\varphi(h, K) \;:=\; \{\vartheta_\varphi(h, x)\}_{x \in K}$$

[8] See VIABILITY THEORY, [6, Aubin], Theorem 3.2.4.

Indeed, we deduce that

$$\|\vartheta_\varphi(K)\| \leq e^{\|\varphi\|_\Lambda} \frac{e^{\|\varphi\|_\Lambda} - 1}{\|\varphi\|_\Lambda} \|\varphi(K)\|$$

and that

$$\|\vartheta_\varphi\|_\Lambda \leq e^{\|\varphi\|_\Lambda}$$

because

$$d(\vartheta_\varphi(h,K), \vartheta_\varphi(h,L)) \leq e^{\|\varphi\|_\Lambda} d(K,L)$$

We also observe that

$$d_\Lambda(\vartheta_\varphi, \vartheta_\psi) \leq \frac{e^{\|\varphi\|_\Lambda} - 1}{\|\varphi\|_\Lambda} \|\varphi - \psi\|_\infty$$

Therefore,

$$\mathrm{Lip}_0(M,E) \subset \overline{\Theta}(\mathcal{K}(M))$$

is a space of transitions of $\mathcal{K}(M)$ and $(\mathcal{K}(M), \mathrm{Lip}_0(M,E))$ is a mutational space, the one we presented in the introduction.

Actually, there are other transitions on the metric space $\mathcal{K}(M)$.

2 Mutations of Smooth Single-Valued Maps

2.1 Definitions

We adapt first some classical definitions of differential calculus and notations to single-valued maps from a metric space to another.

Definition 2.1 *Consider two mutational spaces $(X, \Theta(X))$, $(Y, \Theta(Y))$ and a single-valued map $f : X \mapsto Y$ from X to Y.*

We shall say that the mutation $\overset{\circ}{f}(x)$ of f at x *is the set-valued map from $\Theta(X)$ to $\Theta(Y)$ defined by*

$$\tau \in \overset{\circ}{f}(x)\vartheta \text{ if and only if } \lim_{h \to 0+} \frac{d(f(\vartheta(h,x)), \tau(h, f(x)))}{h} = 0$$

We shall say that f is mutable *at x in the directions $\vartheta \in \Theta(X)$ if $\overset{\circ}{f}(x)\vartheta$ is nonempty for every $\vartheta \in \Theta(X)$ and that f is* strictly mutable *if*

$$\tau \in \overset{\circ}{f}(x)\vartheta \text{ if and only if } \lim_{h \to 0+,\, x' \to x} \frac{d(f(\vartheta(h,x')), \tau(h, f(x')))}{h} = 0$$

Proposition 2.2 *Consider two metric spaces X, Y and a single-valued map $f : X \mapsto Y$ from X to Y. If f is mutable at x, then two transitions $\tau_1 \in \overset{\circ}{f}(x)\vartheta$ and $\tau_2 \in \overset{\circ}{f}(x)\vartheta$ are equivalent at $f(x)$: $\tau_1 \sim_{f(x)} \tau_2$.*

If f is Lipschitz and if $\vartheta_1 \sim_x \vartheta_2$ are equivalent at x, then transitions $\tau_1 \in \overset{\circ}{f}(x)\vartheta_1$ and $\tau_2 \in \overset{\circ}{f}(x)\vartheta_2$ are also equivalent at $f(x)$.

Remark — When the context allows it, we may identify the transitions $\tau \in \overset{\circ}{f}(x)\vartheta$ since they are equivalent at $f(x)$ and make the mutation $\overset{\circ}{f}(x)$ single-valued by taking the factor space of $\Theta(Y)$. □

For maps defined from a mutational space $(X, \Theta(X))$ to a vector space F, we restrict naturally the transitions to be vectors $u \in F$ by taking $\Theta(F) = F$, so that mutations $\overset{\circ}{f}(x)$ induce maps from $\Theta(X)$ to F defined by

$$\overset{\circ}{f}(x)\vartheta = \lim_{h \to 0+} \frac{f(\vartheta(h,x)) - f(x)}{h}$$

Let X and E be finite dimensional vector space s and $Y := \mathcal{K}(E)$. We regard a set-valued map $P : X \rightsquigarrow E$ with nonempty compact images as a single valued map $P : X \mapsto \mathcal{K}(E)$. We associate the mutational spaces (X,X) and $(\mathcal{K}(E), \text{Lip}(E,E))$.

We thus restrict the transitions $\vartheta \in \Theta(X)$ to be just vectors $u \in E$ and the transitions $\tau \in \Theta(\mathcal{K}(E))$ to be Lipschitz maps $\varphi \in \text{Lip}(E,E)$, so that mutations $\overset{\circ}{P}(x)$ are set-valued maps from X to $\text{Lip}(E,E)$ defined by

$$\varphi \in \overset{\circ}{P}(x)u \text{ if and only if } \lim_{h \to 0+} \frac{d(P(x+hu), \vartheta_\varphi(h, P(x)))}{h} = 0$$

In other words, the mutation $\overset{\circ}{P}(x)(u)$ is a set of Lipschitz maps $\varphi : E \mapsto E$ such that

$$\vartheta_\varphi(h, P(x)) \subset B(P(x+hu), \beta(h)h) \;\&\; P(x+hu) \subset B(\vartheta_\varphi(h, P(x)), \beta(h)h) \;\square$$

Remark — The contingent derivative of a set-valued map $P : X \rightsquigarrow E$ at a point (x,y) of its graph has no relations with the concept of mutation of this set-valued map regarded as a single-valued map from X to the power space $Y := \mathcal{K}(E)$.

In the first instance, the contingent derivative is a set-valued map $DP(x,y)$ from X to E depending upon a point $(x,y) \in \text{Graph}(P)$ whereas in the second point of view, the mutation $\overset{\circ}{P}(x)$ is a set-valued map from X to $\text{Lip}(E,E)$ depending only upon x and not on the choice of $y \in P(x)$.

This is the reason why we had to coin the word **mutation** instead of derivative to avoid this confusion. □

Let $M \subset E$ be a closed subset of a finite dimensional vector space, $X := \mathcal{K}(M)$ be the metric space of nonempty compact subsets of M and Y be a normed space. We associate with them the mutational spaces $(\mathcal{K}(M), \text{Lip}_0(M,E))$ and (Y,Y).

A map $f : \mathcal{K}(M) \mapsto Y$ is often called a **shape map**, since they have been extensively used in shape design and shape optimization (see [12, Céa], [39, Zolésio], [13,14,15,16, Delfour & Zolésio], [18, Doyen], etc.).

Then, by restricting transitions on $\mathcal{K}(M)$ to $\text{Lip}_0(M,E)$ and the transitions on Y to be directions $v \in Y$, we see that a mutation $\overset{\circ}{f}(K)$ is a set-valued map from the vector space $\text{Lip}_0(M,E)$ to Y associating with a Lipschitz map φ the direction v defined by

$$v = \overset{\circ}{f}(K)\varphi := \lim_{h \to 0+} \frac{f(\vartheta_\varphi(h,K)) - f(K)}{h}$$

Assume that the interior Ω of M is not empty. Denote by $\mathcal{D}(\Omega, E)$ the space of indefinitely differentiable maps with compact support from Ω to Y. Let $f : \mathcal{K}(M) \mapsto \mathbf{R}$ be a shape function. If

$$\varphi \in \mathcal{D}(\Omega, E) \cap \text{Lip}(E,E) \mapsto \overset{\circ}{f}(K)\varphi \text{ is linear and continuous}$$

then $\overset{\circ}{f}(K)$ is a **vector distribution** called the **shape gradient** of f at $K \subset M$.

2.2 Shape Derivatives

The main example of mutation of a map is the shape derivative of shape maps associating with a subset K the average of a a given function.

Proposition 2.3 *Let us consider a shape function W defined by*

$$W(K) := \int_K \alpha(x)dx$$

where α is C^1. It is shape differentiable:

$$\overset{\circ}{W}(K)(\varphi) = \int_K \text{div}(\alpha(x)\varphi(x))dx$$

2.3 Contingent Transition Sets

Definition 2.4 (Contingent Transition Sets) *Let $(X, \Theta(X))$ be a mutational space, $K \subset X$ be a subset of X and $x \in K$ belong to K. The* contingent[9] transition set *$T_K(x)$ is defined by*

$$T_K(x) := \left\{ \vartheta \in \Theta(X) \mid \liminf_{h \to 0+} \frac{d_K(\vartheta(h,x))}{h} = 0 \right\}$$

It is very convenient to have the following characterization of this transition set in terms of sequences:

$$\begin{cases} \vartheta \in T_K(x) \text{ if and only if } \exists\, h_n \to 0+,\ \exists\, \varepsilon_n \to 0+ \\ \text{and } \exists\, x_n \in K \to x \text{ such that } \forall\, n,\ d(\vartheta(h_n,x),x_n) \le \varepsilon_n h_n \end{cases}$$

Naturally, if $\vartheta_1 \sim_x \vartheta_2$ are equivalent at $x \in K$ and if ϑ_1 belongs to $T_K(x)$, then ϑ_2 is also a contingent transition to K at x.

Example: Normed Spaces Let E be a normed vector space. We can associate with any $v \in E$ the transition $\vartheta_v \in \overline{\Theta}(E)$ defined by

$$\vartheta_v(h,x) := x + hv$$

Then the vector $v \in E$ is contingent to K at $x \in K$ (in the usual sense of contingent cones to subsets in normed spaces) if and only if the associated transition ϑ_v is contingent to K at x.

Let us associate with any Lipschitz map $\varphi : X \mapsto X$ the transition $\vartheta_\varphi \in \overline{\Theta}(E)$ defined by

$$\vartheta_\varphi(h,x) := x(h)$$

where $x(\cdot)$ is the unique solution to the differential equation $x'(t) = \varphi(x(t))$ starting from x.

[9]This termed has been coined by G. Bouligand in the 30's. Since this is a concept consistent with the concept of contingent direction as we shall see below, we adopted the same terminology.

Then the associated transition is contingent to K at x if and only if the vector $\varphi(x)$ is contingent to K at x.

Example: Contingent Transition Sets on Power Sets

Let $M \subset E$ be a closed subset of a finite dimensional vector space and consider the mutational space $(\mathcal{K}(M), \mathrm{Lip}_0(M, E))$. Let $\mathcal{M} \subset \mathcal{K}(M)$ be the a family of nonempty compact subsets of M.

We shall say that a Lipschitz map $\varphi \in \mathrm{Lip}_0(M, E)$ is contingent to $\mathcal{M}$ at $K \in \mathcal{M}$ if and only if the associated transition ϑ_φ is contingent to $\mathcal{M}$ at K, i.e.,

$$\liminf_{h \to 0+} \frac{d_{\mathcal{M}}(\vartheta_\varphi(h, K))}{h} = 0$$

or again, if and only if there exist sequences h_n and ε_n converging to 0 and a sequence of subsets $K_n \in \mathcal{M}$ such that

$$\vartheta_\varphi(h_n, K) \subset K_n + \varepsilon_n h_n B \ \& \ K_n \subset \vartheta_\varphi(h_n, K) + \varepsilon_n h_n B$$

This contingent cone has been introduced and studied in [18, Doyen] under the name of **velocity cone**.

Constrained Inverse Function Theorems, a calculus of contingent cones and Lagrange multipliers for shape optimization under constraints, which use such concepts of tangent cones, have been obtained in [18, Doyen].

3 Inverse Function Theorem

Let us consider now a complete mutational space $(X, \Theta(X))$, a normed space Y, a closed subset $K \subset X$ and a continuous (single-valued) map $f : K \mapsto Y$.

We shall say that a set-valued map $F : X \rightsquigarrow Y$ is said to be **pseudo-Lipschitz** around $(x, y) \in \mathrm{Graph}(F)$ if there exist a positive constant λ and neighborhoods $\mathcal{U} \subset \mathrm{Dom}(F)$ of x and $\mathcal{V}$ of y such that

$$\forall\, x_1, x_2 \in \mathcal{U}, \ \ F(x_1) \cap \mathcal{V} \subset F(x_2) + \lambda d(x_1 - x_2) B_Y$$

Inverse Function Theorems provide criteria for the inverse of a map to be pseudo-Lipschitz around a point of its graph.

Sophisticated Inverse Function Theorems in metric spaces have already been provided in [23,24, Frankowska], using first order and higher order "variations". We extend here the Inverse Function Theorem proved in SET-VALUED ANALYSIS, [9, Aubin & Frankowska] to the case of metric spaces.

Theorem 3.1 (Constrained Inverse Function Theorem) *Let $(X, \Theta(X))$ be a complete mutational space and Y be a normed space. We consider a (single-valued) continuous map $f : X \mapsto Y$, a closed subset $K \subset X$ and an element x_0 of K.*

We assume that f is strictly mutable at x_0 and we posit the following **transversality assumption***:*

there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that

$$\forall x \in K \cap B(x_0, \eta), \ B_Y \subset \overset{\circ}{f}(x)(T_K(x) \cap cB_X) + \alpha B_Y$$

Then $f(x_0)$ belongs to the interior of $f(K)$ and the set-valued map $y \rightsquigarrow f^{-1}(y) \cap K$ is **pseudo-Lipschitz** *around $(f(x_0), x_0)$.*

This theorem is a consequence of the still more general Theorem 3.2 below:

Indeed, not only are we interested in knowing whether a solution to the constrained problem does exist, but we wish to approximate it by solutions to the approximate problems

$$\text{find } x_n \in L_n \text{ and } y_n \in M_n \text{ such that } f_n(x_n) = y_n$$

where x_n converges to x_0, y_n converges to $y_0 = f(x_0)$ and f_n converges to f in some sense.

Theorem 3.2 *Let $(X, \Theta(X))$ be a complete mutational space and Y a Banach space. Consider a sequence of continuous single-valued maps f_n from X to Y, a sequence of closed subsets $L_n \subset X$ and $M_n \subset Y$ and elements x_0, y_0 in the lower limits of the subsets L_n and M_n respectively.*

We assume that f_n are strictly mutable on a neighborhood of x_0 and verify the following **stability assumption***:*

there exist constants $c > 0$, $\alpha \in [0, 1[$ and $\eta > 0$ such that

$$\begin{cases} \forall x \in L_n \cap B(x_0, \eta), \ \forall y \in M_n \cap B(y_0, \eta) \\ B_Y \subset \overset{\circ}{f}_n(x)(T_{L_n}(x) \cap cB_X)) - T^{\flat}_{M_n}(y) \cap cB_X) + \alpha B_Y \end{cases} \tag{3.1}$$

Then there exist $l > 0$ and $\gamma > 0$ such that for any $x_{0n} \in B_{L_n}(x_0, \gamma)$ and any $y_{0n} \in B_{M_n}(y_0, \gamma)$ satisfying $\|y_{0n} - f_n(x_{0n})\| \leq \gamma$, we have

$$\max(d(\widehat{x}_n - x_{0n}), \|\widehat{y}_n - y_{0n}\|) \leq l\|y_{0n} - f_n(x_{0n})\|$$

Proof of Theorem 3.2 — We choose $\rho > 0$, $\varepsilon > 0$ such that

$$\frac{3\rho}{\eta} < \varepsilon < \frac{1-\alpha}{c}$$

and consider elements x_{0n} and y_{0n} satisfying

$$d(x_{0n}, x_0) \leq \eta/3 \quad \& \quad \|x_{0n} - x_0\| \leq \eta/3$$

and

$$\|f_n(x_{0n}) - y_{0n}\| \leq \rho$$

By Ekeland's Variational Principle applied to the function

$$V(x,y) := \|y_n - f_n(x)\|$$

on the complete metric space $L_n \times M_n$, we know that there exists a solution $(\widehat{x}_n, \widehat{y}_n) \in L_n \times M_n$ to

$$\left\{\begin{array}{ll} i) & \|\widehat{y}_n - f_n(\widehat{x}_n)\| + \varepsilon \max(d(\widehat{x}_n, x_{0n}), \|\widehat{y}_n - y_{0n}\|) \leq \|y_{0n} - f_n(x_{0n})\| \\ ii) & \forall\, x_n \in L_n,\ y_n \in M_n, \\ & \|\widehat{y}_n - f_n(\widehat{x}_n)\| \leq \|y_n - f_n(x_n)\| + \varepsilon \max(d(x_n, \widehat{x}_n), \|y_n - \widehat{y}_n\|) \end{array}\right. \tag{3.2}$$

We deduce from inequality (3.2) i) that

$$\max(d(\widehat{x}_n, x_{0n}), \|\widehat{y}_n - y_{0n}\|) \leq \frac{1}{\varepsilon}\|y_{0n} - f_n(x_{0n})\| \leq \frac{\rho}{\varepsilon} < \frac{\eta}{3}$$

so that

$$d(\widehat{x}_n, x_0) \leq \eta/3 + d(x_{0n}, x_0) \leq 2\eta/3$$

In the same way, we show that $\|\widehat{y}_n - y_0\| \leq 2\eta/3$.

Stability assumption (5.1) implies that there exist $\vartheta_n \in T_{L_n}(\widehat{x}_n)$, $u_n \in T^{\flat}_{M_n}(\widehat{y}_n)$ and $w_n \in Y$ satisfying

$$\left\{\begin{array}{ll} i) & \widehat{y}_n - f_n(\widehat{x}_n) = \overset{\circ}{f}_n(\widehat{x}_n)\vartheta_n + u_n + w_n \\ ii) & \|\vartheta_n(\widehat{x}_n)\| \leq c\|\widehat{y}_n - f_n(\widehat{x}_n)\| \quad \& \quad \|u_n\| \leq c\|\widehat{y}_n - f_n(\widehat{x}_n)\| \\ & \|w_n\| \leq \alpha\|\widehat{y}_n - f_n(\widehat{x}_n)\| \end{array}\right.$$

By definition of the contingent transition set, there exist elements $h_p > 0$ converging to $0+$ and $x_p \in L_n$ such that,

$$d(\vartheta_n(h_p, \widehat{x}_n), x_p) \leq h_p \varepsilon_p$$

where ε_p converges to 0 with h_p, and by definition of the adjacent transition set, there exist elements $y_p \in M_n$ such that,

$$\|\widehat{y}_n - h_p u_n - y_p\| \ \leq \ h_p \varepsilon_p$$

To say that f_n is strictly mutable at $\widehat{x}_n$ means that

$$\|f_n(\widehat{x}_n) + h_p(y_n - f_n(\widehat{x}_n) - u_n - w_n) - f_n(x_p)\| \ \leq \ \beta_p h_p$$

Hence

$$\|y_p - f_n(x_p)\| \ \leq \ (1 - h_p)\|\widehat{y}_n - f_n(\widehat{x}_n)\| + h_p\|w_n\| + \beta_p h_p$$

By taking in inequality (3.2) *ii*) such an $(x_p, y_p) \in L_n \times M_n$, we deduce that

$$h_p\|\widehat{y}_n - f_n(\widehat{x}_n)\| \ \leq \ h_p\|w_n\| + \beta_p h_p + \varepsilon \max\left(d(x_p, \widehat{x}_n), \|y_p - \widehat{y}_n\|\right)$$

We note that

$$\begin{cases} +d(x_p, \widehat{x}_n) \ \leq \ h_p \varepsilon_p + d(\vartheta_n(h_p, \widehat{x}_n), x_p) \ \leq \ h_p(\varepsilon_p + \|\vartheta_n(\widehat{x}_n)\|) \\ \|y_p - \widehat{y}_n\| \ \leq \ h_p(\varepsilon_p + \|u_n\|) \end{cases}$$

Dividing by $h_p > 0$ and letting $p \to +\infty$, we get:

$$\|y_n - f_n(\widehat{x}_n)\| \ \leq \ \|w_n\| + \varepsilon \max(\|\vartheta_n(\widehat{x}_n)\|, \|u_n\|) \ \leq \ (\alpha + \varepsilon c)\|\widehat{y}_n - f_n(\widehat{x}_n)\|$$

Since we have chosen ε such that $\alpha + \varepsilon c < 1$, we infer that $(\widehat{y}_n, \widehat{x}_n)$ is a solution to

$$\widehat{x}_n \ \in \ L_n, \ \ \widehat{y}_n \ \in \ M_n \ \ \& \ \ f_n(\widehat{x}_n) \ = \ \widehat{y}_n$$

satisfying

$$\max\left(d(\widehat{x}_n - x_{0n}), \|\widehat{y}_n - y_{0n}\|\right) \ \leq \ \frac{1}{\varepsilon}\|y_{0n} - f_n(x_{0n})\|$$

from which the error estimate follows.

Observe also that setting $K_n := L_n \cap f^{-1}(M_n)$, the above estimate implies that

$$d(x_{0n}, K_n) \leq \frac{1}{\varepsilon}\|y_{0n} - f_n(x_{0n})\|$$

since $\widehat{x}_n = f^{-1}(\widehat{y}_n)$ belongs to K_n. By letting ε converge to $c/(1-\alpha)$, we obtain the estimate

$$d(x_{0n}, K_n) \leq \frac{c}{1-\alpha}\|y_{0n} - f_n(x_{0n})\| \quad \square$$

4 Tangent transition sets to subsets defined by equality and inequality constraints

Consider a metric space X and two strictly mutable maps

$$g := (g_1, \ldots, g_p) : X \mapsto \mathbf{R}^p \ \& \ h := (h_1, \ldots, h_q) : X \mapsto \mathbf{R}^q$$

defined on an open neighborhood of L.

Let K be the subset of L defined by the constraints

$$K := \{x \in L \mid g_i(x) \geq 0, \ i = 1, \ldots, p \ \& \ h_j(x) = 0, \ j = 1, \ldots, q\}$$

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of active constraints.

Proposition 4.1 *Let us posit the following transversality condition at a given $x \in K$:*

$$\begin{cases} \exists\, \vartheta_0 \in \Theta(X) \text{ such that } \overset{\circ}{h}(x)\vartheta_0 = 0 \text{ and} \\ \forall\, i \in I(x), \ \overset{\circ}{g}_i(x)\vartheta_0 > 0 \end{cases}$$

Then a transition $\vartheta \in \Theta(X)$ belongs to the contingent transition set to K at x if and only if ϑ satisfies the constraints

$$\forall\, i \in I(x), \ \overset{\circ}{g}_i(x)\vartheta \geq 0 \ \& \ \forall\, j = 1, \ldots, q, \ \overset{\circ}{h}_j(x)\vartheta = 0$$

Proof — We observe that $T_K(x) = X$ whenever $I(x) = \emptyset$ and that, otherwise, inclusion

$$T_K(x) \subset \{\vartheta \in \Theta(X) \mid \forall\, i \in I(x),\ \overset{\circ}{g}_i(x)\vartheta \geq 0\}$$

holds true when g is mutable at x.

Assume now that the *constraint qualification assumption* holds true and prove the other inclusion.

Let ϑ satisfy $\overset{\circ}{g}_i(x)\vartheta \geq 0$ for any $i \in I(x)$. For $i \notin I(x)$, strict inequalities $g_i(x) > 0$ imply that for some $\alpha > 0$, we have

$$\forall\, h \in [0,\alpha],\ \forall\, i \notin I(x),\ g_i(x+hu) \geq 0$$

Consider first the case when $\overset{\circ}{g}_i(x)\vartheta > 0$ for any $i \in I(x)$. Then

$$\forall\, i \in I(x),\ \ g_i(x+hu) = g_i(x+hu) - g_i(x) = \overset{\circ}{g}_i(x)\vartheta + h\varepsilon_i(h)$$

where $\varepsilon_i(h)$ converges to 0 with h. This implies that $g_i(x+hu) \geq 0$ for h small enough and all $i \in I(x)$, and thus, for all $i = 1,\ldots,p$. Then such an element u belongs to the contingent transition set $T_K(x)$.

Consider now the general case. By assumption, we deduce that for any $\beta \in]0,1[$, the transition ϑ_β defined by $\vartheta_\beta(h,x) := \vartheta(1-\beta)h, \vartheta_0(\beta h,x)$ satisfies strict inequalities $\overset{\circ}{g}_i(x)\vartheta_\beta > 0$ for any $i \in I(x)$ since

$$\overset{\circ}{g}_i(x)\vartheta_\beta = (1-\beta)\,\overset{\circ}{g}_i(x)\vartheta + \beta\,\overset{\circ}{g}_i(x)\vartheta_0$$

Therefore, by what precedes, it belongs also to the contingent transition set $T_K(x)$. Letting β converge to 0, we infer that the limit ϑ of the ϑ_β's belongs also to the contingent transition set $T_K(x)$. □

5 Calculus of Tangent Transition Sets

5.1 Adjacent and Circatangent Transition Sets

Let $K \subset X$ be a subset of a metric space X and $x \in K$ belong to K.

We observe that the **contingent transition set** $T_K(x)$ is defined by

$$T_K(x) := \{\vartheta \in \Theta(X) \mid \liminf_{h \to 0+} d_K(\vartheta(h,x))/h = 0\}$$

We introduce the following concepts of **directional lim inf and lim sup**: Let $\varphi : [0,1] \times X \mapsto \mathbf{R}$ and $\psi : [0,h] \mapsto X$ be two single valued- maps and $x := \psi(0) = \lim_{h\to 0+} \psi(h)$. We set

$$\liminf_{h\mapsto 0+,\ y\equiv\psi(h)} \varphi(h,y) := \sup_{\varepsilon>0} \inf_{h\in]0,\varepsilon],\ y\in B(\psi(h),\varepsilon h)} \varphi(h,y)$$

and

$$\limsup_{h\mapsto 0+,\ x\equiv\psi(h)} \varphi(h,x) := \inf_{\varepsilon>0} \sup_{h\in]0,\varepsilon],\ y\in B(\psi(h),\varepsilon h)} \varphi(h,y)$$

As in the case of tangent cones to subsets of normed spaces, we introduce other concepts of transition sets:

Definition 5.1 *Let $K \subset X$ be a subset of a metric space X and $x \in K$ belong to K.*

1. — the **adjacent transition set** $T^\flat_K(x)$ *is defined by*

$$T^\flat_K(x) := \{\vartheta \in \Theta(X) \mid \lim_{h\to 0+} \frac{d_K(\vartheta(h,x))}{h} = 0\}$$

2. — the **circatangent transition set** $C_K(x)$ *is defined by*

$$C_K(x) := \{\vartheta \in \Theta(X) \mid \lim_{h\to 0+, x'\to_K x} \frac{d_K(\vartheta(h,x'))}{h} = 0\}$$

where $\to_K$ denotes the convergence in K.

We shall say that a subset $K \subset X$ is **derivable** *at $x \in \overline{K}$ if and only if $T^\flat_K(x) = T_K(x)$ and* **tangentially regular** *at x if $T_K(x) = C_K(x)$.*

We see at once that

$$C_K(x) \subset T^\flat_K(x) \subset T_K(x)$$

If X is a metric space, these tangent transition sets to K and the closure $\overline{K}$ of K do coincide and

$$\text{if } x \in \text{Int}(K), \text{ then } C_K(x) = \Theta(X)$$

It is very convenient to use the following characterization of these transition sets in terms of sequences.

$$\begin{cases} \vartheta \in T^\flat_K(x) \text{ if and only if } \forall\ h_n \to 0+,\ \exists\, \varepsilon_n \to 0+ \\ \exists\ x_n \in K \to x \text{ such that } d(\vartheta(h_n,x),x_n) \le \varepsilon_n h_n \end{cases}$$

and

$$\begin{cases} \vartheta \in C_K(x) \text{ if and only if } \forall\ h_n \to 0+,\ \forall\, y_n \to_K x, \\ \exists\, \varepsilon_n \to 0+,\ \exists\, x_n \in K \to x \text{ such that } d(\vartheta(h_n, y_n), x_n) \le \varepsilon_n h_n \end{cases}$$

Proposition 5.2 *The circatangent transition set $C_K(x)$ satisfies the following properties*

$$C_K(x) \circ C_K(x) \subset C_K(x)$$

and

$$C_K(x) \circ T_K(x) \subset T_K(x) \ \&\ C_K(x) \circ T_K^{\flat}(x) \subset T_K^{\flat}(x)$$

Proof — Let ϑ_1 and ϑ_2 belong to $C_K(x)$. To prove that $\vartheta_1 \circ \vartheta_2$ belongs to this transition set, let us choose any sequence $h_n > 0$ converging to 0 and any sequence of elements $y_n \in K$ converging to x. There exists a sequence of elements $y_{2\,n} \in K$ converging to x such that the elements $d(\vartheta_2(h_n, y_n), y_{2\,n})/h_n$ converges to 0. But since the sequence $y_{2\,n}$ does also converge to x in K, there exists a sequence of elements $y_{1\,n}$ converging to x such that $d(\vartheta_1(h_n, y_{2\,n}), y_{1\,n})/h_n$ converges to 0. Therefore, we deduce that h small enough,

$$\begin{cases} d(\vartheta_1 \circ \vartheta_2(h_n, y_n), y_{1\,n}) = d(\vartheta_1(h_n, \vartheta_2(h_n, y_{1\,n}))) \\ \le d(\vartheta_1(h_n, \vartheta_2(h_n, y_n)), \vartheta_1(h_n, y_{2\,n})) + d(\vartheta_1(h_n, y_{2\,n}), y_{1\,n}) \\ \le l d(\vartheta_2(h_n, y_n), y_{2\,n}) + d(\vartheta_1(h_n, y_{2\,n}), y_{1\,n}) \end{cases}$$

This implies that $\vartheta_1 \circ \vartheta_2$ belongs to $C_K(x)$.

The proof of the two other inclusions is analogous and left as an exercise. □

Unfortunately, the price to pay for enjoying this property of the circatangent transition sets is that they may often be reduced to the trivial transition set $\{1\}$.

But we shall show in just a moment that the circatangent transition set and the contingent transition set do coincide at those points x where K is sleek, i.e., where the set-valued map $x \rightsquigarrow T_K(x)$ is lower semicontinuous. Hence the transition set $C_K(x)$ can be seen as a "regularization" of the contingent transition set $T_K(x)$.

5.2 External Contingent Transition Sets

We recall that we have set

$$D_\uparrow \boldsymbol{d}_K(x)(\vartheta) \ := \ \liminf_{h\to 0+} \frac{\boldsymbol{d}_K(\vartheta(h,x)) - \boldsymbol{d}_K(x)}{h}$$

We observe that when $x \in K$, a transition ϑ is contingent to K at x if and only if $D_\uparrow \boldsymbol{d}_K(x)(\vartheta) \leq 0$.

Definition 5.3 *Let K be a subset of a metric space X and x belong to X. We extend the notion of contingent transition set to the one of* external contingent transition set *to K at points outside of K in the following way:*

$$T_K(x) := \{\vartheta \in \Theta(X) \mid D_\uparrow \boldsymbol{d}_K(x)(\vartheta) \leq 0\}$$

We point out an easy but important relation between the external contingent transition set at a point and the contingent transition set at its projection:

Lemma 5.4 *Let K be a closed subset of a metric space and $\Pi_K(y)$ be the set of projections of y onto K, i.e., the subset of $z \in K$ such that $\boldsymbol{d}(z,y) = \boldsymbol{d}_K(y)$. Then, for any mutation ϑ, the following inequalities*

$$D_\uparrow \boldsymbol{d}_K(y)(\vartheta) \ \leq \ \inf_{z\in\Pi_K(y)} \boldsymbol{d}_\infty(\vartheta, T_K(z)) + (\|\vartheta\|_\Lambda - 1)\boldsymbol{d}_K(y)$$

hold true.

Proof — Choose $z \in \Pi_K(y)$ and $\tau \in T_K(z)$. Then, for h small enough, using estimate (??) on primitives,

$$\left\{\begin{array}{l} \dfrac{\boldsymbol{d}_K(\vartheta(h,y)) - \boldsymbol{d}_K(y)}{h} \leq \dfrac{\boldsymbol{d}(\vartheta(h,y),\tau(h,z)) - \boldsymbol{d}(y,z)}{h} + \dfrac{\boldsymbol{d}_K(\tau(h,z))}{h} \\ \leq \dfrac{\int_0^h e^{(\|\vartheta\|_\Lambda - 1)(h-s)}\boldsymbol{d}_\infty(\vartheta,\tau)ds + \boldsymbol{d}(y,z)(e^{(\|\vartheta\|_\Lambda-1)h} - 1)}{h} \\ + \dfrac{\boldsymbol{d}_K(\tau(h,z))}{h} \end{array}\right.$$

Since z belongs to K and $\tau \in T_K(z)$, the above inequality implies that

$$D_\uparrow \boldsymbol{d}_K(y)(\vartheta) \ \leq \ \boldsymbol{d}_\infty(\vartheta,\tau) + (\|\vartheta\|_\Lambda - 1)\boldsymbol{d}_K(y) \ \square$$

5.3 Sleek Subsets

Definition 5.5 (Sleek Subsets) *We shall say that a subset K of X is* **sleek** *at $x \in K$ if the set-valued map*

$$K \ni x' \rightsquigarrow T_K(x') \subset \Theta(X) \text{ is lower semicontinuous at } x$$

and that it is **sleek** *if and only if it is sleek at every point of K.*

Theorem 5.6 *Let K be a closed subset of a finite dimensional vector-space X. Consider a set-valued map F from K to $\Theta(X)$ satisfying*

$$\begin{cases} i) & F \text{ is lower semicontinuous at } x \text{ and bounded} \\ ii) & \exists\, \delta > 0 \text{ such that } \forall\, z \in B_K(x,\delta),\ F(z) \subset T_K(z) \end{cases}$$

Then $F(x) \subset C_K(x)$.

In particular, if K is sleek at $x \in K$, then the contingent and circatangent transition sets coincide: $T_K(x) = C_K(x)$.

Proof — Let us take $x \in K$ and $\vartheta \in F(x)$, assumed to be different from 0. Since F is lower semicontinuous at x, we can associate with any $\varepsilon > 0$ a number $\eta \in]0, \delta[$ such that $\boldsymbol{d}_\infty(\vartheta, F(z)) \leq \boldsymbol{d}_\infty(\vartheta, F(x)) + \varepsilon = \varepsilon$ for any $z \in B_K(x, \eta)$ (because $\boldsymbol{d}_\infty(\vartheta, F(x)) = 0$). Therefore, for any $y \in B(x, \eta/4)$ and $\tau \leq \eta/4\|\vartheta(x)\|$, the inequality: $\forall\, z \in \Pi_K(\vartheta(\tau, y))$,

$$\begin{cases} \boldsymbol{d}(z,x) \leq \boldsymbol{d}(z, \vartheta(\tau,y)) + \boldsymbol{d}(\vartheta(\tau,y),x) \leq 2\boldsymbol{d}(\vartheta(\tau,y),x) \\ \leq 2\boldsymbol{d}(\vartheta(\tau,x),x) + 2l\boldsymbol{d}(x,y) \leq \eta \end{cases}$$

implies that

$$\begin{cases} \boldsymbol{d}_\infty(\vartheta, T_K(z)) & \leq & \boldsymbol{d}_\infty(\vartheta, F(z)) \\ & \leq & \boldsymbol{d}_\infty(\vartheta, F(x)) + \varepsilon \ = \ \varepsilon \end{cases}$$

We set $g(\tau) := \boldsymbol{d}_K(\vartheta(\tau, y))$ and $c := \|\vartheta\|_\Lambda - 1$. By Lemma 5.4, we obtain

$$\begin{cases} \liminf_{h \to 0+} \left(g(\tau+h) - g(\tau)\right)/h \ = \ D_\uparrow \boldsymbol{d}_K(\vartheta(\tau,y))(v) + cg(\tau) \\ \leq \ \boldsymbol{d}_\infty(\vartheta, T_K(z)) \ \leq \ \varepsilon \end{cases}$$

The function g being Lipschitz, it is almost everywhere differentiable, so that $g'(t) \leq \varepsilon + cg(t)$ for almost all t small enough. Gronwall's Lemma implies that

Table 1: **Properties of Contingent Transition Sets.**

(1)	▷	If $K \subset L$ and $x \in \overline{K}$, then $T_K(x) \subset T_L(x)$
(2)	▷	If $K_i \subset X$, $(i = 1, \ldots, n)$ and $x \in \overline{\bigcup_i K_i}$, then $T_{\bigcup_{i=1}^n K_i}(x) = \bigcup_{i \in I(x)} T_{K_i}(x)$ where $I(x) := \{i \mid x \in \overline{K_i}\}$
(3)	▷	If $K_i \subset X_i$, $(i = 1, \ldots, n)$ and $x_i \in \overline{K_i}$, then $T_{\prod_{i=1}^n K_i}(x_1, \ldots, x_n) \subset \prod_{i=1}^n T_{K_i}(x_i)$
(4)	▷	If g is mutable, if $K \subset X$, $x \in \overline{K}$ and $M \subset Y$, then $\overset{\circ}{g}(x)(T_K(x)) \subset T_{g(K)}(g(x))$; $T_{g^{-1}(M)}(x) \subset \overset{\circ}{g}(x)^{-1} T_M(g(x))$
(5)	▷	If $K_i \subset X$, $(i = 1, \ldots, n)$ and $x \in \overline{\bigcap_i K_i}$, then $T_{\bigcap_{i=1}^n K_i}(x) \subset \bigcap_{i=1}^n T_{K_i}(x)$

$$d_K(\vartheta(\tau, y)) = g(h) = g(h) - g(0) \leq \varepsilon \frac{e^{ch} - 1}{c}$$

for any $y \in B(x, \eta/4)$ and $\tau \leq \eta/4\|\vartheta(x)\|$. This shows that v belongs to $C_K(x)$.

By taking $F(x) = T_K(x)$, we deduce that $T_K(x) \subset C_K(x)$ whenever K is sleek at $x \in K$, and thus, that they coincide. □

We derive from the Inverse Function Theorem the basic results of the calculus of tangent transition sets.

Theorem 5.7 *Let X be a complete metric space and Y a Banach space. Consider a single-valued maps f from X to Y, closed subsets $L \subset X$ and $M \subset Y$ and elements $x_0 \in K := L \cap f^{-1}(M)$.*

We assume that f is strictly mutable on a neighborhood of x_0 and verify the following stability assumption*:*

there exist constants $c > 0$, $\alpha \in [0,1[$ *and* $\eta > 0$ *such that*

$$\left\{ \begin{array}{l} \forall\, x \in L \cap B(x_0, \eta),\ \forall\, y \in M \cap B(y_0, \eta) \\ B_Y \subset \left(\overset{\circ}{f}(x) T_L(x) - T^{\flat}_M(y)\right) \cap c B_X) + \alpha B_Y \end{array} \right. \tag{5.1}$$

Then

$$T^{\flat}_L(x_0) \cap \overset{\circ}{f}(x_0)^{-1} T_M(f(x_0) \subset T_{L \cap f^{-1}(M)}(x_0)$$

$$T^{\flat}_L(x_0) \cap \overset{\circ}{f}(x_0)^{-1} T^{\flat}_M(f(x_0)) = T^{\flat}_{L \cap f^{-1}(M)}(x_0)$$

and

$$C_L(x_0) \cap \overset{\circ}{f}(x_0)^{-1} C_M(f(x_0)) \subset C_{L \cap f^{-1}(M)}(x_0)$$

Proof — Let us prove for instance the inclusion for the circatangent transition sets. Consider the closed subset

$$K := L \cap f^{-1}(M)$$

and take any sequence of elements $x_n \in K$ which converges to x. Let us pick any transition $\vartheta \in C_L(x_0)$ such that $\overset{\circ}{f}(x_0)\vartheta \in C_M(f(x_0))$. Hence for any sequences $h_n > 0$ and $x_n \in K$ converging to 0 and x_0 respectively, there exist sequences $\widehat{x}_n \in L$ and $\widehat{y}_n \in M$ converging to x and $f(x_0)$ respectively such that

$$d(\vartheta(h_n, x_n), \widehat{x}_n) \leq \alpha_n h_n \ \ \& \ \ \|y_n + h_n \overset{\circ}{f}(x_0)\vartheta - \widehat{y}_n\| \leq \beta_n h_n$$

We now apply Theorem 3.2.

The pair $(\widehat{x}_n, \widehat{y}_n)$ belongs to $L \times M$ and

$$(f \ominus \mathbf{1})(\widehat{x}_n, \widehat{y}_n) \text{ converges to } 0$$

because f is continuous at x.

Therefore, by Theorem 3.2, there exist $l > 0$ and a solution $(\widehat{x}_n, \widehat{y}_n) \in L \times M$ to the equation $\widehat{y}_n = f(\widehat{x}_n)$ such that

$$\left\{ \begin{array}{l} \max\left(d(\vartheta(h, x_n), \widehat{x}_n), \|f(x_n) + h_n \overset{\circ}{f}(x_0)\vartheta - \widehat{y}_n\|\right) \\ \leq l h_n \|(f(\widehat{x}_n) - \widehat{y}_n\| \end{array} \right.$$

Hence

$$d(\vartheta(h_n, x_n), K) \leq d(\vartheta(h_n, x_n), \widehat{x}_n) \leq l \varepsilon_n h_n$$

which means that the transition ϑ belongs to the circatangent transition set to K at x_0. □

6 Contingent Mutations of Set-Valued Maps

6.1 Definition

We have already introduced the concept of contingent mutation of the solutions to the invariant manifold problem, to define them as solutions of partial mutational equations.

We adapt the concepts of contingent derivatives of set-valued map from a normed space to another one to set-valued maps from a metric space to another one by following the same strategy, defining geometrically mutations of set-valued maps from the choice of tangent transition sets to the graphs.

First, we observe that

$$\Theta(X) \times \Theta(Y) \subset \Theta(X \times Y)$$

Definition 6.1 *Let X, Y be metric spaces and $F : X \rightsquigarrow Y$ be a set-valued map.*

The contingent mutation $\mathring{D} F(x,y)$ *of F at $(x,y) \in$ Graph(F) is the set-valued map from $\Theta(X)$ to $\Theta(Y)$ defined by*

$$\tau \in \mathring{D} F(x,y)(\vartheta) \text{ if and only if } (\vartheta, \tau) \in T_{\mathrm{Graph}(F)}(x,y)$$

When $F := f$ is single-valued, we set $\mathring{D} f(x) := \mathring{D} f(x, f(x))$.

We shall say that F is sleek *at $(x,y) \in \mathrm{Graph}(F)$ if the map*

$$\mathrm{Graph}(F) \ni (x', y') \rightsquigarrow \mathrm{Graph}(\mathring{D} F(x', y'))$$

is lower semicontinuous at (x,y) (i.e., if the graph of F is sleek at (x,y)).

It is said to be derivable *at (x,y) if $\mathrm{Graph}(F)$ is derivable at that point.*

The set-valued map F is sleek *(respectively* derivable*) if it is sleek (respectively derivable) at every point of its graph.*

Therefore, a transition $\tau \in \Theta(Y)$ belongs to the contingent mutation $\mathring{D} F(x,y)(\vartheta)$ if and only if there exist sequences $h_n > 0$ converging to $0+$, x_n and $y_n \in F(x_n)$ converging to x and y respectively such that

$$\begin{cases} d_X(\vartheta(h_n, x), x_n) \le \alpha_n h_n \\ d_Y(\tau(h_n, x), y_n) \le \beta_n h_n \end{cases}$$

One can restate this in the following form:

$$\liminf_{h\to 0+, x_h \equiv \vartheta(h,x)} \frac{d_Y(\tau(h,y), F(x_h))}{h} = 0$$

in the sense that for any $\varepsilon > 0$, for every $\eta > 0$, there exist $h \in]0, \eta[$ and x_h satisfying $d(\vartheta(h,x), x_h) \le \eta h$ such that

$$\frac{d_Y(\tau(h,y), F(x_h))}{h} \le \varepsilon$$

Naturally, if we can embed subspaces $\Theta_0(X)$ and $\Theta_0(Y)$ of transitions into the spaces $\Theta(X)$ and $\Theta(Y)$ respectively, then we can restrict the contingent mutation to a set-valued map $\overset{\circ}{D} F(x,y) : \Theta_0(X) \rightsquigarrow \Theta_0(Y)$ in the obvious way.

In particular, if X and Y are normed spaces, we find again the concept of contingent mutations, by embedding X and Y in $\Theta(X)$ and $\Theta(Y)$ respectively.

We shall meet often the case when X is a metric space and Y is a normed space. In this case, $v \in Y$ belongs to the contingent mutation $\overset{\circ}{D} F(x,y)(\vartheta)$ if and only if there exist sequences $h_n > 0$ converging to $0+$, x_n and $y_n \in F(x_n)$ converging to x and y respectively such that

$$\begin{cases} d_X(\vartheta(h_n,x), x_n) \le \alpha_n h_n \\ \|y + h_n v - y_n\| \le \beta_n h_n \end{cases}$$

Consider now the case when X is a normed space and Y is a metric space. Then a transition $\tau \in \Theta(Y)$ belongs to the contingent mutation $\overset{\circ}{D} F(x,y)(u)$ if and only if there exist sequences $h_n > 0$ converging to $0+$, x_n and $y_n \in F(x_n)$ converging to x and y respectively such that

$$\begin{cases} \|x + h_n u - x_n\| \le \alpha_n h_n \\ d_Y(\tau(h_n,x), y_n) \le \beta_n h_n \end{cases}$$

One can restate this in the following form:

$$\liminf_{h\to 0+, x_h \equiv x+hu} \frac{d_Y(\tau(h,y), F(x_h))}{h} = 0$$

Example Let X be a finite dimensional vector-space, E be a finite dimensional vector-space and $Y := \mathcal{K}(E)$.

Then the contingent mutation $\overset{\circ}{D} P(x)(u)$ is a set of Lipschitz maps $f : E \mapsto E$ such that there exist $h_n \to 0+$, x_n satisfying $\frac{x_n - x}{h_n} \to u$ such that

$$\vartheta_f(h_n, P(x)) \subset B(P(x_n), \beta_n h_n) \;\;\&\;\; P(x_n) \subset B(\vartheta_f(h_n, P(x)), \beta_n h_n)$$

If $P(\cdot)$ is locally Lipschitz, this boils down to

$$\vartheta_f(h_n, P(x)) \subset B(P(x+h_n u), \beta_n h_n) \;\;\&\;\; P(x+h_n u) \subset B(\vartheta_f(h_n, P(x)), \beta_n h_n) \quad \square$$

We can easily compute the mutation of the inverse of a set-valued map F (or even of a non injective single-valued map): **The contingent mutation of the inverse of a set-valued map F is the inverse of the contingent mutation:**

$$\overset{\circ}{D}(F^{-1})(y,x) = \overset{\circ}{D} F(x,y)^{-1}$$

The restriction $F := f|_K$ of a single-valued map f to a subset $K \subset X$ provides an example of a set-valued map defined by

$$f|_K(x) := \begin{cases} f(x) & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases}$$

for which we obtain the following formula:

If f is strictly mutable around a point $x \in K$, then **the contingent mutation of the restriction of f to K is the restriction of the mutation to the contingent transition set:**

$$\overset{\circ}{D}(f|_K)(x) = \overset{\circ}{D}(f|_K)(x, f(x)) = \overset{\circ}{f}(x)|_{T_K(x)}$$

Actually, this follows from the useful

Proposition 6.2 *Let X be a metric space, Y be a normed space, f be a single-valued map from an open subset $\Omega \subset X$ to Y, $M : X \rightsquigarrow Y$ be a set-valued map and $L \subset X$. Define the set-valued map $F : X \rightsquigarrow Y$ by:*

$$F(x) := \begin{cases} f(x) - M(x) & \textit{when } x \in L \\ \emptyset & \textit{when } x \notin L \end{cases}$$

If f is strictly mutable at $x \in \Omega \cap \mathrm{Dom}(F)$, then for every $y \in F(x)$,

$$\overset{\circ}{D} F(x,y)(u) \subset \begin{cases} \overset{\circ}{f}(x)u - \overset{\circ}{D} M(x, f(x)-y)(u) & \text{when } u \in T_L(x) \\ \emptyset & \text{when } u \notin T_L(x) \end{cases}$$

Equality holds true when we assume that either L or M is derivable at x and M is Lipschitz at x.

In particular if M is constant, then

$$\forall u \in T_L(x), \quad \overset{\circ}{D} F(x,y)(u) = \overset{\circ}{f}(x)u - T_M(f(x)-y)$$

Proof — Let v belong to $\overset{\circ}{D} F(x,y)(\vartheta)$. Then there exist $h_n > 0$ converging to 0 and sequences x_n and y_n converging to x and y respectively such that for every n

$$d_X(\vartheta(h_n,x),x_n) \le \alpha_n h_n \ \ \& \ \ \|y + h_n v - y_n\| \le \beta_n h_n$$

and

$$x_n \in L_n \ \ \& \ \ y_n \in f(x_n) - M(x_n)$$

Since

$$f(x_n) = f(x) + h_n(\overset{\circ}{f}(x)\vartheta + e(h_n))$$

where $e(h_n)$ converges to 0 with h_n we get

$$\|f(x) - y + h_n(\overset{\circ}{f}(x)\vartheta - v) - (f(x_n) - y_n)\| \le \varepsilon_n h_n$$

so that $\overset{\circ}{f}(x)\vartheta - v \in \overset{\circ}{D} M(x, f(x)-y)(\vartheta)$.

Conversely, assume for instance that M is derivable, that the transition u belongs to $T_L(x)$ and that $\overset{\circ}{f}(x)u - v$ belongs to $\overset{\circ}{D} M(x, f(x)-y)(u)$. Hence, there exist a sequence $h_n > 0$ converging to 0 and sequences $\overline{x}_n \in L_n$, $x_n \in L$ and $y_n \in M(x_n)$ converging to x, x and $f(x) - y$ respectively, such that

$$d_X(\vartheta(h_n,x),x_n) \le \alpha_n h_n$$

$$d_X(\vartheta(h_n,x),x_n) \le \alpha_n h_n \ \ \& \ \ \|f(x) - y + h_n(\overset{\circ}{f}(x)\vartheta - v) - z_n\| \le \beta_n h_n$$

Set $y_n := f(x_n) - z_n \in f(x_n) - M(x_n)$. Since M is Lipschitz,

$$M(x_n) \subset M(\overline{x}_n) + \lambda d(x_n, \overline{x_n})$$

Hence, there exists e_n such that $\|e_n\| \leq \varepsilon_n$ converges to 0 such that $d(x_n, \overline{x_n}) \leq \varepsilon_n h_n$. Hence $y_n + e_n \in f(x_n) - M(\overline{x}_n)$ where $\overline{x}_n \in L$. and

$$\|y + h_n v - y_n - e_n\| \; = \; \|f(x) - f(x_n) + h_n \overset{\circ}{f}(x)\vartheta + h_n b_n\| \; \leq h_n(\beta_n + \varepsilon_n)$$

we infer that v belongs to $\overset{\circ}{D} F(x, y)(\vartheta)$.

6.2 Adjacent and Circatangent Mutations

Naturally, we can also associate with any other concept of tangent transition set a concept of mutation.

Definition 6.3 *Let X, Y be metric spaces and $F : X \rightsquigarrow Y$ be a set-valued map.*

1. — the adjacent mutation $\overset{\circ}{D}^{\flat} F(x, y)$ *is the set-valued map from $\Theta(X)$ to $\Theta(Y)$ defined by*

$$\tau \in \overset{\circ}{D} F^{\flat}(x, y)(\vartheta) \text{ if and only if } (\vartheta, \tau) \in T^{\flat}_{\mathrm{Graph}(F)}(x, y)$$

2. — the circatangent mutation $\overset{\circ}{C} F(x, y)$ *is the set-valued map from $\Theta(X)$ to (Y) defined by*

$$\tau \in \overset{\circ}{C} F(x, y)(\vartheta) \text{ if and only if } (\vartheta, \tau) \in C_{\mathrm{Graph}(F)}(x, y)$$

When $F := f$ is single-valued, we set

$$\overset{\circ}{D}^{\flat} f(x) := \overset{\circ}{D}^{\flat} f(x, f(x)), \;\; \overset{\circ}{C} f(x) := \overset{\circ}{C} f(x, f(x))$$

We see at once that

$$\forall\, u, \;\; \overset{\circ}{C} F(x, y)(\vartheta) \subset \overset{\circ}{D}^{\flat} F(x, y)(\vartheta) \subset \overset{\circ}{D} F(x, y)(\vartheta)$$

6.3 Chain Rules

We derive from the calculus of tangent transition sets the associated calculus of mutations of set-valued maps. We begin naturally by the chain rule for computing mutations of the composition product of a set-valued map $G : X \rightsquigarrow Y$ and a set-valued map $H : Y \rightsquigarrow Z$.

We shall need the following result:

Proposition 6.4 *Let X, Y be metric spaces, $F : X \rightsquigarrow Y$ be a set-valued map and K be a subset of X. Assume that F is Lipschitz around some $x \in K$. Then, for any $y \in F(x)$, we have*

$$\overset{\circ\,\flat}{D} F(x,y)(T_K(x)) \subset T_{F(K)}(y)$$

As a consequence, we deduce that when M is a subset of Y and $y \in M$, then

$$T_{F^{\ominus}(M)}(x) \subset \overset{\circ\,\flat}{D} F(x,y)^{\ominus}(T_M(y)) \tag{6.1}$$

Proof — Take the transition ϑ in $T_K(x)$ and $\tau \in \overset{\circ\,\flat}{D} F(x,y)(\vartheta)$. Then there exist sequences $h_n > 0$ converging to 0, $x_{1n} \in K$ and x_{2n} converging to x and $y_n \in F(x_{2n})$ converging to y such that

$$\begin{cases} d(\vartheta(h,x), x_{1n}) \leq \alpha_{1n} h_n \\ d(\vartheta(h,x), x_{2n}) \leq \alpha_{2n} h_n \\ d(\tau(h,y), y_n) \leq \beta_n h_n \end{cases}$$

Since F is Lipschitz around x with a Lipschitz constant l, we deduce that

$$y_n \in B(F(x_{1n}), l h_n d(u_{1n}, u_{2n})$$

so that there exists another sequence $y_n^\star$ converging to y such that

$$y_n^\star \in F(x_{1n}) \subset F(K)$$

and

$$d(\tau(h,y), y_n^\star) \leq \beta_n^\star h_n$$

This implies that the transition τ belongs to the contingent transition set to $F(K)$ at y.

Consider now $K := F^{\ominus}(M)$. Since $F(F^{\ominus}(M))$ is contained in M, we deduce that

$$\overset{\circ\,\flat}{D} F(x,y)\left(T_{F^{\ominus}(M)}(x)\right) \subset T_{F(F^{\ominus}(M))}(y) \subset T_M(y)$$

from which formula (6.1) ensues. □

Remark — Naturally, we can show in the same way that for a Lipschitz map F the formula

$$\overset{\circ}{D} F(x,y) T_K^\flat(x) \subset T_{F(K)}(y)$$

is also true whenever $y \in F(x)$. □

We begin by the following simple result:

Theorem 6.5 *Let us consider metric spaces X, Y, Z, a set-valued map $G : X \rightsquigarrow Y$ and a set-valued map $H : Y \rightsquigarrow Z$.*

1. — Let us assume that H is Lipschitz around y, where $y \in G(x)$. Then, for any $z \in H(y)$, we have

$$\begin{cases} \mathring{D}^{\flat} H(y,z) \circ \mathring{D}\, G(x,y) \subset \mathring{D}\,(H \circ G)(x,z) \\ \\ \mathring{D}\, H(y,z) \circ \mathring{D}^{\flat}\, G(x,y) \subset \mathring{D}\,(H \circ G)(x,z) \end{cases}$$

2. — If $G := g$ is single-valued and strictly mutable at x, we obtain

$$\forall\, z \in H(g(x)),\ \mathring{D}\,(Hg)(x,z)(u) \subset \mathring{D}\, H(g(x),z)(\mathring{g}\,(x)u)$$

and the equality holds true when H is Lipschitz around $g(x)$.

Proof — We apply Proposition 6.4 to equality

$$\mathrm{Graph}(H \circ G) = (1 \times H)\,(\mathrm{Graph}(G))$$

for proving the first statement. The second one follows from

$$\begin{cases} \mathrm{Graph}(H \circ g) = (g \times 1)^{-1}\,(\mathrm{Graph}(H)) \\ \\ T_{f^{-1}(K)}(x) \subset \mathring{f}\,(x)^{-1}\,(T_K(f(x))) \end{cases} \square$$

6.4 The Inverse Set-Valued Map Theorem

Theorem 6.6 (Inverse Set-Valued Map Theorem) *Let X be a complete metric space and Y be a normed space. Consider a closed set-valued map $F : X \rightsquigarrow Y$, an element (x_0, y_0) of its graph and let us assume that there exist constants $c > 0$, $\alpha \in [0,1[$ and $\eta > 0$ such that*

$$\begin{cases} \forall\,(x,y) \in \mathrm{Graph}(F) \cap B((x_0,y_0),\eta),\ \forall\, v \in Y, \\ \\ \exists\, \vartheta \in \Theta(X),\ \exists\, w \in Y \text{such that } v \in \mathring{D}\, F(x,y)(\vartheta) + w \\ \\ \text{and } \|\vartheta(x)\| \le c\|v\| \ \&\ \|w\| \le \alpha\|v\| \end{cases}$$

Then y_0 belongs to the interior of the image of F and F^{-1} is pseudo-Lipschitz around (y_0, x_0).

Proof — We apply Theorem 3.1 with X replaced by $X \times Y$, K by Graph(F), f by the projection Π_Y from $X \times Y$ onto Y. We have to prove that the stability assumption implies transversality assumption (5.1) of Theorem 3.2, i.e., that for all $v \in Y$, there exist (u, v) in the contingent transition set $T_{\text{Graph}(F)}(x, y)$ and $w \in Y$ satisfying

$$v = v + w, \quad \max(\|u\|, \|v\|) \le c\|v\|, \quad \|w\| \le \alpha\|v\|$$

This information is provided by our stability assumption since the contingent transition set to the graph is the graph of the contingent mutation and the norm of $v = v - w$ is smaller than or equal to $(1 + \alpha)\|v\|$. □

7 Contingent Epimutations of Extended Functions

Contingent epimutations of extended functions had already been used for characterizing Lyapunov functions. They are also useful in optimization, for deriving necessary conditions such as the Fermat rule.

7.1 Contingent Epimutations

Definition 7.1 *Let X be a metric space, $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ be a nontrivial extended function and x belong to its domain. Then, for any transition $\vartheta \in \Theta(X)$,*

$$\mathring{D}_\uparrow V(x)(\vartheta) := \liminf_{h \to 0+, x_h \equiv \vartheta(h,x)} \frac{V(x_h) - V(x)}{h}$$

is the contingent epimutation *of V at x in the direction ϑ.*

The function V is said to be contingently epimutable *at x if its contingent epimutation never takes the value $-\infty$.*

We define in a symmetric way the contingent hypomutation $\mathring{D}_\downarrow V(x)$ from $\Theta(X)$ to $\mathbf{R} \cup \{\pm\infty\}$ of $V : X \mapsto \mathbf{R} \cup \{\pm\infty\}$ at a point x of its domain by

$$\mathring{D}_\downarrow V(x)(\vartheta) := -\mathring{D}_\uparrow (-V)(x)(u) = \limsup_{h \to 0+, x_h \equiv \vartheta(h,x)} \frac{V(x_h) - V(x)}{h}$$

We could also have defined the contingent epimutation of a function by taking the contingent transition set to its epigraph, since we have proved that

$$\lambda \ge \mathring{D}_\uparrow V(x)(\vartheta) \text{ if and only if } (\vartheta, \lambda) \in T_{\mathcal{E}p(V)}(x, V(x))$$

7.2 Epimutation of a Marginal Function

Let us consider a marginal function of the form

$$V(K) := \inf_{x \in K} U(x)$$

and we set

$$K_U := \{x \in K \quad \text{such that} \quad U(x) = V(K)\}$$

Proposition 7.2 *Assume that U is lower semicontinuous and that K is compact. Then, for any Lipschitz-Marchaud map Φ, we have*

$$\mathring{D}_\uparrow V(K)(\Phi) \leq \inf_{x \in K_U} \inf_{v \in \Phi(x)} D_\uparrow^\flat U(x)(v)$$

If we assume furthermore that U is uniformly Fréchet differentiable, then

$$\lim_{h \to 0+} \frac{V(\vartheta_\Phi(h,K)) - V(K)}{h} = \inf_{x \in K_U} \inf_{v \in \Phi(x)} \langle U'(x), v \rangle$$

Proof

a) Let x be chosen in K_U. Then

$$\frac{V(\vartheta_\Phi(h,K)) - V(K)}{h} \leq \frac{V(\vartheta_\Phi(h,K)) - U(x)}{h}$$

Let us fix $v \in \Phi(x)$. By the existence Theorem ??, there exists a solution $x(\cdot)$ to the differential inclusion $x' \in \Phi(x)$ satisfying

$$\|x(t) - x - tv\| \leq t\varepsilon(t)$$

where $\varepsilon(t)$ converges to 0 with t. Therefore, inequality

$$\frac{V(\vartheta_\Phi(h,K)) - U(x)}{h} \leq \frac{U(x(h)) - U(x)}{h}$$

implies that

$$\limsup_{h \to 0+} \frac{V(\vartheta_\Phi(h,K)) - V(K)}{h} \leq \limsup_{h \to 0+} \frac{U(x(h)) - U(x)}{h} =: D_\uparrow^\flat U(x)(v)$$

b) In order to prove the opposite inequality, let us consider an element $y_h \in \vartheta_\Phi(h,K)$ which minimizes the function U over this subset: $V(\vartheta_\Phi(h,K)) = U(y_h)$. Let $x_h \in K$ and $x_h(\cdot) \in \mathcal{S}(x_h)$ be a solution to the differential inclusion such that $x_h(h) = y_h$. By Theorem ??, there exists a subsequence (again denoted by) which converges uniformly on compact intervals to a solution $x(\cdot) \in \mathcal{S}(x_0)$ where $x_0 \in K$. Since the function U is assumed to be uniformly Fréchet differentiable, there exists $\varepsilon(h)$ converging to 0 such that Therefore,

$$\left\{\begin{array}{l} \frac{V(\vartheta_\Phi(h,K)) - V(K)}{h} \geq \frac{U(y_h) - U(x_h)}{h} \\ = \frac{U(x_h + \int_0^h x'(s)ds) - U(x_h)}{h} \geq \left\langle U'(x_h), \frac{1}{h}\int_0^h x'(s)ds \right\rangle - \varepsilon(h) \end{array}\right.$$

Since

$$v_h := \frac{1}{h}\int_0^h x'(s)ds \in \frac{1}{h}\int_0^h F(x_h(s))ds \subset \overline{\text{co}}(F(K) + \alpha B)$$

for h small enough, we infer that v_h remains in a compact subset, so that it converges to some $v \in F(x_0)$. Consequently, we have proved that

$$\liminf_{h \to 0+} \frac{V(\vartheta_\Phi(h,K)) - V(K)}{h} \geq \langle U'(x_0), v\rangle \geq \inf_{x \in K_U} \inf_{v \in \Phi(x)} \langle U'(x), v\rangle$$

The proof is completed. □

7.3 Fermat and Ekeland Rules

Since we can define the contingent epimutation of any extended function $V : X \mapsto \mathbf{R} \cup \{+\infty\}$, we can extend the "Fermat rule" to any minimization problem.

Theorem 7.3 (Fermat Rule) *Let X be a metric space, $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended function and $x \in \text{Dom}(V)$ a local minimizer of V on X.*

Then x is a solution to the variational inequalities:

$$\forall \vartheta \in \Theta(X), \ 0 \leq \mathring{D}_\uparrow V(x)(\vartheta)$$

Proof — The proof is naturally obvious: We write that for all $\vartheta \in \Theta(X)$, there exists x_n such that

$$d_X(\vartheta(h_n, x), x_n) \leq \alpha_n h_n$$

and
$$0 \leq \frac{V(x_n) - V(x)}{h_n}$$
and we take the lim inf when h converges to 0 and $x_h \equiv \vartheta(h, x)$. □

In the same way, it is easy to derive an epimutational version of Ekeland's Variational Principle:

Theorem 7.4 *Let X be a complete metric space, $V : X \mapsto \mathbf{R}_+ \cup \{+\infty\}$ be a nontrivial lower semicontinuous bounded from below function and $x_0 \in \mathrm{Dom}(V)$ be a given point of its domain. Then, for any $\varepsilon > 0$, there exists a solution $x_\varepsilon \in \mathrm{Dom}(V)$ to:*

$$\begin{cases} i) \quad V(x_\varepsilon) + \varepsilon d(x_\varepsilon, x_0) \leq V(x_0) \\ ii) \quad \forall\, \vartheta \in \Theta(X), \ 0 \leq \mathring{D}_\uparrow V(x_\varepsilon)(\vartheta) + \varepsilon \|\vartheta(x_\varepsilon)\| \end{cases} \tag{7.1}$$

What is not obvious is the use of this Fermat rule for more and more general problems, when the function V is built from other simpler functions and involves constraints.

The search for necessary conditions for a minimum requires quite a rich calculus of contingent epimutations which provides estimates of $\mathring{D}_\uparrow V(x)(u)$. In particular, when constraints (of the type $x \in K$) are involved, the fact that the epimutation of the restriction to K is the restriction of the epimutation to the contingent transition set $T_K(x)$, allows one to write necessary conditions using also contingent transition sets to constraint sets (or in the dual form, using gradients and polars of the contingent transition sets.)

Corollary 7.5 *Let $K \subset X$ be a metric space, $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ be a nontrivial extended function and $x \in K$ a local minimizer of V on K.*

If V is strictly mutable at $x \in K$, then x is a solution:

$$\forall\, \vartheta \in T_K(x), \ 0 \leq \mathring{D}_\uparrow V(x)(\vartheta)$$

Indeed, we have seen that the contingent mutation of the restriction of V to K is the restriction of the mutation to the contingent transition set.

Consider a closed subset L of a complete metric space X and two strictly mutable maps

$$g := (g_1, \ldots, g_p) : X \mapsto \mathbf{R}^p \ \& \ h := (h_1, \ldots, h_q) : X \mapsto \mathbf{R}^q$$

defined on an open neighborhood of L.

Let K be the subset of L defined by the constraints

$$K := \{x \in L \mid g_i(x) \geq 0,\ i = 1, \ldots, p\ \&\ h_j(x) = 0,\ j = 1, \ldots, q\}$$

We denote by $I(x) := \{i = 1, \ldots, p \mid g_i(x) = 0\}$ the subset of active constraints.

Theorem 7.6 *Let us posit the following transversality condition at a given* $x \in K$*:*

$$\begin{cases} \exists\, \vartheta_0 \in \Theta(X) \text{ such that } \overset{\circ}{h}(x)\vartheta_0 = 0 \text{ and} \\ \forall\, i \in I(x),\ \overset{\circ}{g}_i(x)\vartheta_0 > 0 \end{cases}$$

Let $V : X \mapsto \mathbf{R} \cup \{+\infty\}$ *be a nontrivial extended function and* $x \in K$ *a local minimizer of* V *on* K.

If V *is strictly mutable at* $x \in K$*, then* x *is a solution to:*

$$\begin{cases} \forall\, \vartheta \in \Theta(X) \textit{ such that} \\ \forall\, i \in I(x),\ \overset{\circ}{g}_i(x)\vartheta \geq 0\ \&\ \forall\, j = 1, \ldots, q,\ \overset{\circ}{h}_j(x)\vartheta = 0 \\ \text{then } 0 \leq \overset{\circ}{D}_{\uparrow} V(x)(\vartheta) \end{cases}$$

The proof follows from the above proposition and Proposition 4.1.

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Contents