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FOREWORD

We study convergence of value-functions associated to control systems with a singular perturbation. In the nonlinear case, we prove new convergence results: the limit of optimal costs of the perturbed system is an optimal cost for the reduced system. We furthermore provide an estimation of the rate of convergence when the reduced system has solutions regular enough.

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SINGULAR PERTURBATIONS IN NON LINEAR OPTIMAL CONTROL SYSTEMS

Marc Quincampoix & Huilong Zhang

Introduction

We shall study the following singularly perturbed control system for almost all $t \in [0,T]$ and T fixed

$$(1) \qquad \begin{cases} \dfrac{dx_\varepsilon(t)}{dt} & = f(x_\varepsilon(t), y_\varepsilon(t), v(t)) \qquad x_\varepsilon(0) = x_0 \\ \varepsilon \dfrac{dy_\varepsilon(t)}{dt} & = g(x_\varepsilon(t), y_\varepsilon(t), v(t)) \qquad y_\varepsilon(0) = y_0 \end{cases}$$

The state-variable x and y belong to some finite dimensional vector-space X and Y. The control $v(t)$ belongs to some compact convex subset U included in some finite dimensional space Z.

These equations are used to model a system with a slow variable $x(\cdot)$ and a fast variable $y(\cdot)$. It is possible to refer to [7] for numerous examples and applications. Since the works of Tychonoff [9], the convergence of solution of (1) (when $\varepsilon \longrightarrow 0$) has been studied by many authors (cf [4], [10], [8],...).

Our main goal is to study the convergence of an optimal cost associated with (1). With any solution $(x_\varepsilon(t), y_\varepsilon(t), v(t))$ to (1) we associate the following cost

$$J^\varepsilon(v) = h(x_\varepsilon(T))$$

We define V_ε the *value-function* which is the infimum of J^ε over all solution to (1).

We wish to underline that the results of this paper are still available for the following cost

$$\tilde{J}^{\varepsilon}(v) = \int_0^T l(x_\varepsilon(s), y_\varepsilon(s), v(s))\, ds + h(x_\varepsilon(T))$$

We can reduce the problem with the integral cost $\tilde{J}$ into a new one with only final state cost. Actually, let us transforme $\mathrm{Inf}\tilde{J}(x_\varepsilon, y_\varepsilon, v)$ into $\mathrm{Inf} I(x_\varepsilon, y_\varepsilon, z_\varepsilon, v)$ where

$$\frac{dz_\varepsilon(t)}{dt} = l(x_\varepsilon(t), y_\varepsilon(t), v(t)), \qquad z_\varepsilon(0) = 0$$

and

$$I(x_\varepsilon(T), y_\varepsilon(T), z_\varepsilon(T), v(T)) = h(x_\varepsilon(T)) + z_\varepsilon(T)$$

So, by adding the dimension of $x(\cdot)$, we get a new equivalent system with no integral part in the cost. In all this paper we can assume that $l = 0$ and

$$J^{\varepsilon}(v) = h(x_\varepsilon(T))$$

In the same way to a solution to

$$(2) \qquad \begin{cases} \dfrac{dx(t)}{dt} &= f(x(t), y(t), v(t)) \qquad x(0) = x_0 \\ 0 &= g(x(t), y(t), v(t)) \qquad y(0) = y_0 \end{cases}$$

we associate the following cost

$$J(v) = h(x(T))$$

and the corresponding value-function V_0.

Our goal is to prove the following results under suitable assumptions (the notations are defined successively in the paper).

1st main result

- Convergence of value-functions:

$$V_\varepsilon \longrightarrow V_0$$

- Rate of convergence. If for any trajectory of the limit system, we have

$$\left\| \frac{dy}{dt} \right\|_{L^2} < +\infty$$

then

$$|V_\varepsilon - V_0| \le c\sqrt{\varepsilon}$$

2nd main result

If $V_\varepsilon \longrightarrow V_0$ and

$$x_\varepsilon^*, y_\varepsilon^* \quad \text{optimal trajectory of } E_\varepsilon$$

$$x^*, y^* \quad \text{optimal trajectory of } E_0$$

then

$$x_\varepsilon^* \longrightarrow x^* \quad \text{in} \quad H_1$$

$$y_\varepsilon^* \longrightarrow y^* \quad \text{in} \quad L^2_{\text{weak}}$$

Furthermore, if $\left\| \frac{dy}{dt} \right\|_{L^2} < +\infty$ then

$$||x_\varepsilon^* - x^*||_{H^1} \le k\sqrt{\varepsilon}$$

$$||y_\varepsilon^* - y^*||_{L^2} \le k\sqrt{\varepsilon}$$

The purpose of the paper is to generalize well-known results in linear case (cf [11] for instance) to nonlinear case. In the nonlinear case, there exists some work of Binding [4] but with no estimation of the rate of convergence. We also want to refer to the book of Bensoussan [3], because our goal is to obtain similar results without assumptions concerning adjoint variables.

1 Perturbed and reduced control system

1.1 Problems and assumptions

It's almost classical that (1) and (2) can be translated into the equivalent differential inclusion problems (see [2]).

$$\left(\frac{dx_\varepsilon(t)}{dt}, \varepsilon\frac{dy_\varepsilon(t)}{dt}\right) \in H\left(x_\varepsilon(t), y_\varepsilon(t)\right) \tag{3}$$

and

$$\left(\frac{dx_\varepsilon(t)}{dt}, 0\right) \in H(x, y) \tag{4}$$

where

$$H\left(x(t), y(t)\right) = \left\{\left(f(x,y,v), g(x,y,v)\right) | v \in U\right\}$$

We denote $S(\varepsilon, x_0, y_0)$ as the set of $(x_\varepsilon(\cdot), y_\varepsilon(\cdot))$ absolutely continous solutions on $[0,T]$ to (1), et $S(x_0, y_0)$ as the set of $(x(\cdot), y(\cdot))$ absolutely continous solutions to (2) on $[0,T]$. We define

$$R(x) = \{\,(y,v)\,|\,g(x,y,v) = 0\}$$

in this way, we transform (2) into

$$x'(t) \in f(x(t), R(x(t))) \tag{5}$$

We need the following assumptions concerning (1), (2), (3) and (4).

Assumption 1.1

(i) f, g are k-Lipschitz with respect to (x,y,v)

(ii) h is l-Lipschitz

(iii) $H(x,y)$ is a set valued map k-Lipschitz with compact convex nonempty values and with k linear growth.

(iv) $\exists\, c \in \mathbf{R}^+$

(iv) $x \mapsto f(x, R(x))$ is convex valued.

$$\begin{cases} |f(x,y,v)| \le c(1+|x|+|y|) & \forall v \in U, \forall x, y \\ |g(x,y,v)| \le c(1+|x|+|y|) & \forall v \in U, \forall x, y \end{cases}$$

1.2 Existence of optimal solutions

We shall state an easy proposition furnishing existence of optimal solutions which is classical in the linear case and also in the case (cf [3]):

$$g_y(x,y,v) \leq -\nu I$$

Proposition 1.2 *If*

$$(6) \qquad < g(x,y_1,v) - g(x,y_2,v), y_1 - y_2 > \leq -\nu ||y_1 - y_2||^2 \quad \forall x, y_1, y_2, v$$

and with assumptions 1.1, then there exists at least an optimal solution to (2). Furthermore, for any control $u(\cdot)$ there exists an unique solution to (2).

PROOF.

Let us notice that, thanks to (6), for each fixed (x,v) there exists an *unique* y such that $0 = g(x,y,v)$. Furthermore thanks to the compactness of U, for any x, y is bounded by some constant which does not depends on v. On the other hand, because the dynamics is continuous, R is closed compact valued. Thanks to [2], chapter 5.4.3 we deduce that R is Lipschitz[1]. Since h is continuous, and the set of solutions to $x'(t) \in F(x(t), R(x(t))$ is compact there exists an optimal solution [2]. The uniqueness of solution to (2), when $v(\cdot)$ is given, follows from standard argument of differential equation theory (cf [2] for instance).

This completes the proof. □

2 Convergence

2.1 Convergence of optimal cost

We denote by V^ϵ (*resp.* V^0) the optimal cost of the system (1) (*resp.* *(2)*). Let us state the following

[1] It is easy to notice that a pseudo- Lipschitz map with compact values is Lipschitz

[2] Let's recall (cf [2]) that when Φ is Lipschitz with convex compact values, the set of solution of

$$\begin{cases} x'(t) \in \Phi(x(t)) \\ x(0) = x_0 \end{cases}$$

is compact in $W^{1,1}$.

Proposition 2.1 *Under Assumptions (1.1), consider an optimal control* $u(\cdot)$ *for the reduced problem (2). If furthermore*

$$(7)\qquad \begin{cases} < g(x,y_1,u(t)) - g(x,y_2,u(t)), y_1 - y_2 > \leq -\nu|y_1 - y_2|^2 \\ \text{with} \quad \nu > 0, \quad \text{for } \forall x, y_1, y_2 \quad \text{and} \quad t \leq T \end{cases}$$

then

$$(8)\qquad \limsup_{\varepsilon \to 0} V^\varepsilon \leq V^0$$

Before proving this proposition, following the idea of [3], we have

Lemma 2.2 *Consider an optimal control* $u(\cdot)$ *for the reduced problem (2). Under assumptions of Proposition 2.1, if* $\bar{x}_\varepsilon(\cdot), \bar{y}_\varepsilon(\cdot)$ *is a solution of*

$$(9)\qquad \begin{cases} \dfrac{d\bar{x}_\varepsilon(t)}{dt} &= f(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)) \qquad \bar{x}_\varepsilon(0) = x_0 \\ \varepsilon\dfrac{d\bar{y}_\varepsilon(t)}{dt} &= g(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)) \qquad \bar{y}_\varepsilon(0) = y_0 \end{cases}$$

then

$$(10)\qquad \lim_{\varepsilon \to 0} J^\varepsilon(u) = V^0$$

Proof. According to (1.1), we have by multiplying the first equation of (9) by $\bar{x}_\varepsilon$

$$(11)\qquad \frac{1}{2}\frac{d}{dt}|\bar{x}_\varepsilon(t)|^2 \leq c|\bar{x}_\varepsilon(t)|\,(1 + |\bar{x}_\varepsilon(t)| + |\bar{y}_\varepsilon(t)|)$$

for the same reason

$$\begin{aligned} \varepsilon\frac{1}{2}\frac{d}{dt}|\bar{y}_\varepsilon(t)|^2 &= < g(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)), \bar{y}_\varepsilon(t) > \\ &= \langle g(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)) - g(\bar{x}_\varepsilon(t), 0, u(t)), \bar{y}_\varepsilon(t)\rangle + \langle g(\bar{x}_\varepsilon(t), 0, u(t)), \bar{y}_\varepsilon(t)\rangle \\ &\leq -\nu|\bar{y}_\varepsilon(t)|^2 + c(1 + |\bar{x}_\varepsilon(t)|)\,|\bar{y}_\varepsilon(t)| \end{aligned}$$

Integrating it from 0 to t we obtain

$$\frac{1}{2}\varepsilon|\bar{y}_\varepsilon(t)|^2 + \nu\,||\bar{y}_\varepsilon||^2_{L_2[0,t]} \leq c\int_0^t (|\bar{y}_\varepsilon(s)| + |\bar{x}_\varepsilon(s)|\,|\bar{y}_\varepsilon(s)|)\,ds + \frac{\varepsilon}{2}|y_0|^2$$

by inequality of Cauchy-Schwarz

$$\nu \, ||\bar{y}_\varepsilon||^2_{L_2[0,t]} \leq c \, ||\bar{x}_\varepsilon||_{L_2[0,t]} \, ||\bar{y}_\varepsilon||_{L_2[0,t]} + c\sqrt{t} \, ||\bar{y}_\varepsilon||_{L_2[0,t]} + \frac{\varepsilon}{2} \, |y_0|^2$$

By standard arguments concerning zeroes of second order equations.

$$||\bar{y}_\varepsilon||_{L_2[0,t]} \leq K \left(1 + ||\bar{x}_\varepsilon||_{L_2[0,t]}\right) \tag{12}$$

where K is bounded constant. With (11) we get

$$\begin{aligned} \frac{1}{2} \, |\bar{x}_\varepsilon(t)|^2 - \frac{1}{2} \, |x_0|^2 \;&\leq\; c \left(\int_0^t |\bar{x}_\varepsilon(s)| \, ds + ||\bar{x}_\varepsilon||^2_{L_2[0,t]} + \int_0^t |\bar{x}_\varepsilon(s)| \, |\bar{y}_\varepsilon(s)| \, ds \right) \\ &\leq\; ct + 2c \, ||\bar{x}_\varepsilon||^2_{L_2[0,t]} + c \, ||\bar{x}_\varepsilon||_{L_2[0,t]} \, ||\bar{y}_\varepsilon||_{L_2[0,t]} \end{aligned}$$

because $|\bar{x}_\varepsilon(t)| \leq 1 + |\bar{x}_\varepsilon(t)|^2$. By (12)

$$\begin{aligned} \frac{1}{2} \, |\bar{x}_\varepsilon(t)|^2 \;&\leq\; ct + 2c \, ||\bar{x}_\varepsilon||^2_{L_2[0,t]} + cK \left(||\bar{x}_\varepsilon||_{L_2[0,t]} + ||\bar{x}_\varepsilon||^2_{L_2[0,t]} \right) + \frac{1}{2} \, |x_0|^2 \\ &\leq\; (2c + 2cK) \, ||\bar{x}_\varepsilon||^2_{L_2[0,t]} + \left(ct + cK + \frac{1}{2} \, |x_0|^2 \right) \end{aligned}$$

We can then apply the inequality of Grõnwall to get

$$||\bar{x}_\varepsilon||^2_{L_2[0,t]} \leq M, \qquad \forall t \in [0,T] \tag{13}$$

Consequently, we verify because of (12) that $|\bar{x}_\varepsilon(t)|$ and $||\bar{x}_\varepsilon||_{L_2[0,t]}$ are bounded. The first equation of (9) implies also

$$\left|\left| \frac{d\bar{x}_\varepsilon}{dt} \right|\right|^2_{L_2[0,t]} \leq M \tag{14}$$

so there exists $\bar{x}, \bar{y}$ such that

$$\begin{aligned} &\bar{x}_\varepsilon \longrightarrow \bar{x} \qquad \text{weakly in } \; H^1[0,T] \; \text{ and, thus, strongly in } \; L^2[0,T] \\ &\bar{y}_\varepsilon \longrightarrow \bar{y} \qquad \text{weakly in } \; L^2[0,T] \end{aligned} \tag{15}$$

We claim that

Lemma 2.3 *Under assumptions of Lemma 2.2, we have*

$$\bar{y}_\varepsilon \longrightarrow \bar{y} \quad \textit{strongly in} \quad L^2$$

and $(\bar{x}, \bar{y}, u)$ is a solution of (2), thus an optimal solution.

According to Lemma 2.3, we have

$$\lim_{\varepsilon \to 0} J^\varepsilon(u) = V^0 \tag{16}$$

This is precisely the assertion of Lemma 2.2 □

PROOF. of Lemma 2.3. Here we follow the method of MINTY explicited in BENSOUSSAN [3] Chapter V Section 1.3.

We first notice that thanks to (7), the maps $A_\varepsilon : z(\cdot) \mapsto -g(x_\varepsilon(\cdot), z(\cdot), u(\cdot))$ and $A : z(\cdot) \mapsto -g(\bar{x}(\cdot), z(\cdot), u(\cdot))$ are monotone maps from L^2 into itself (because these maps are also lipschitzean thanks to similar property concerning g). Furthermore $\varepsilon \bar{y}'_\varepsilon = A_\varepsilon(\bar{y}_\varepsilon)$

Thanks to the monotonicity property, we have

$$\begin{cases} \forall\, z \in L^2[0,T] \\ < A_\varepsilon(\bar{y}_\varepsilon) - A_\varepsilon(z), \bar{y}_\varepsilon - z >_{L^2} \geq 0 \end{cases} \tag{17}$$

In one hand, for any $\eta \in C^\infty$ such that its support is contained in $]0,T[$, we obtain, by integrating by parts $< A_\varepsilon(\bar{y}_\varepsilon), \eta >_{L^2} = \varepsilon < \bar{y}_\varepsilon, \eta' >_{L^2}$ which converges to 0. Hence $\varepsilon \bar{y}'_\varepsilon = A_\varepsilon(\bar{y}_\varepsilon)$ converge weakly to 0.

In the other hand $< A_\varepsilon(\bar{y}_\varepsilon), \bar{y}_\varepsilon >_{L^2} = -\frac{\varepsilon}{2}(\bar{y}_\varepsilon(T) - y_0)$. Hence, we can have passing to the limit in (17) (it is possible because $\bar{x}_\varepsilon$ converges strongly, for any z in L^2, $A_\varepsilon(z)$ converges to $A(z)$ in L^2).

$$\forall\, z \in L^2[0,T], < -A(z), \bar{y} - z >_{L^2} \geq 0$$

In this inequality, we replace z by $\bar{y} + \lambda\eta$, where $\lambda < 0$ and $\eta \in L^2$. Dividing by λ, we obtain for every η, $0 \leq < -A(\bar{y}), \eta >_{L^2}$. Thus $A(\bar{y}) = 0$, this is to say that $(\bar{x}, \bar{y}, u)$ is a solution of (2).

Let us prove now that $\bar{y}_\varepsilon$ converges strongly. Replacing z by $\bar{y}$ in (17), and thanks to (7) we have

$$< A_\varepsilon(\bar{y}_\varepsilon) - A_\varepsilon(\bar{y}), \bar{y}_\varepsilon - \bar{y} >_{L^2} \geq \nu \|\bar{y} - \bar{y}_\varepsilon\|^2_{L^2} \geq 0 \tag{18}$$

We know that $< A_\varepsilon(\bar{y}_\varepsilon), \bar{y}_\varepsilon >_{L^2}$ converges to 0. So it is for $< A_\varepsilon(\bar{y}_\varepsilon), \bar{y} >_{L^2}$ because $A_\varepsilon(\bar{y}_\varepsilon$ converges weakly to 0. Hence, passing to the limit in (18), we obtain that $\bar{y}_\varepsilon$ converges strongly to $\bar{y}$. The proof is complete.

□

Remark 2.4 *Comparing with the result in Section 1.3 of chapter V in [3], we do not need the assumptions (1.9), (1.10) and (1.11) which guarantees the uniqueness of the optimal solution of the limit problem (2). The assumption 1.14 of [3] is also weakened by (7).*

The Proposition 2.1 is an immediate consequence of Lemma 2.2.
Proof of Proposition 2.1. We take u_ε such that

$$J^\varepsilon(u_\varepsilon) \le J^\varepsilon(u) \tag{19}$$

We note that such u_ε exists for any $\varepsilon > 0$, because if for certain $\varepsilon > 0$, we have $J^\varepsilon(v) > J^\varepsilon(u)$ for any $v \in U$, then we can chose $u_\varepsilon = u$ to get (19). We then have for such u_ε

$$\limsup_{\varepsilon\to 0} J^\varepsilon(u_\varepsilon) \le J(u) \tag{20}$$

in view of the Lemma 2.2. It is sufficient to remark that

$$V^\varepsilon \le J^\varepsilon(u_\varepsilon)$$

to obtain (8). □

To obtain the convergence of optimal cost, we should prove the following

Proposition 2.5 *Under the same assumption as Proposition 2.1 we have*

$$\liminf_{\varepsilon\to 0} V^\varepsilon \ge V^0$$

We use the same idea as in the proof of Proposition 2.1. For any $\varepsilon > 0$, there exists u_ε such that for any measurable control $v \in U$

$$\begin{cases} J^\varepsilon(u_\varepsilon) & \le & J^\varepsilon(u) \\ J^\varepsilon(u_\varepsilon) & \le & J^\varepsilon(v) \end{cases} \tag{21}$$

We recall that u is an optimal solution of (2). We need the following lemma, it will be proved later.

Lemma 2.6 *Assume that assumptions (1.1), (7) hold true and that u_ε is constructed by (21). Then there exists $(\bar{x}, \bar{y}, \bar{u})$ a solution to (2) such that*

$$x_\varepsilon \longrightarrow \bar{x} \quad \textit{weakly in } H^1[0,T] \textit{ and, thus,strongly in } L^2[0,T]$$

$$y_\varepsilon \longrightarrow \bar{y} \quad \textit{strongly in } L^2[0,T]$$

PROOF OF PROPOSITION 2.5. By definition

$$J^{\varepsilon}(u_{\varepsilon}) = h(x_{\varepsilon}(T))$$

According to (ii) of Assumption (1.1) and Lemma 2.6

$$\lim_{\varepsilon \to 0} h(x_{\varepsilon}(T)) = h(\bar{x}(T)) = J(\bar{u})$$

This means that $\bar{u}$ is also an optimal solution of (2). The second inequality of (21) gives us

$$J^{\varepsilon}(u_{\varepsilon}) \leq \inf_{v} J^{\varepsilon}(v) + \varepsilon = V^{\varepsilon} + \varepsilon$$

passing to limit, we get

$$\liminf_{\varepsilon \to 0} J^{\varepsilon}(u_{\varepsilon}) \leq \liminf_{\varepsilon \to 0} V^{\varepsilon}$$

Consequently

$$V^{0} = J(\bar{u}) \leq \liminf_{\varepsilon \to 0} V^{\varepsilon}$$

PROOF OF LEMMA 2.6. Let us set

$$x_{\bar{\varepsilon}}(t) = x_{\varepsilon}(t) - x(t), \qquad y_{\bar{\varepsilon}}(t) = y_{\varepsilon}(t) - y(t)$$

where $(x_{\varepsilon}, y_{\varepsilon})$ (*resp.* (x, y)) is the pair of trajectories with respect to u_{ε} (*resp.* u). We get the differential system

$$(22)\begin{cases} \dfrac{dx_{\bar{\varepsilon}}(t)}{dt} = f(x_{\varepsilon}(t), y_{\varepsilon}(t), u_{\varepsilon}(t)) - f(x(t), y(t), u(t)) & x_{\bar{\varepsilon}}(0) = 0 \\ \varepsilon\dfrac{dy_{\bar{\varepsilon}}(t)}{dt} = g(x_{\varepsilon}(t), y_{\varepsilon}(t), u_{\varepsilon}(t)) - g(x(t), y(t), u(t)) & y_{\bar{\varepsilon}}(0) = 0 \end{cases}$$

From the second equation

$$\varepsilon\frac{dy_{\bar{\varepsilon}}(t)}{dt} = g(x_{\varepsilon}(t), y_{\varepsilon}(t), u_{\varepsilon}(t)) - g(x(t), y_{\varepsilon}(t), u(t)) + g(x(t), y_{\varepsilon}(t), u(t)) - g(x(t), y(t), u(t))$$

Multiplying this equation by $y_\varepsilon(t)$ we get

$$\varepsilon\frac{1}{2}\frac{d}{dt}\left|y_{\bar{\varepsilon}}(t)\right|^2 \leq k\left(\left|x_\varepsilon(t)-x(t)\right|+\left|u_\varepsilon(t)-u(t)\right|\right)\left|y_\varepsilon(t)\right|-\nu\left|y_{\bar{\varepsilon}}(t)\right|^2$$

$$+y(t)g\left(x(t),y_\varepsilon(t),u(t)\right)$$

by integrating and thanks to the linear growth condition

$$\begin{aligned}
\nu\left|\left|y_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]} & \\
&\leq k\int_0^t\left|x_\varepsilon(s)\right|\left|y_\varepsilon(s)\right|ds \\
&\leq +kM\int_0^t\left|y_{\bar{\varepsilon}}(s)\right|ds+k\int_0^t\left|y(s)\right|(1+\left|x(s)\right|+\left|y_\varepsilon(s)\right|)ds
\end{aligned}$$

So, we have, as in the proof of lemma 2.2

$$\left|\left|y_{\bar{\varepsilon}}\right|\right|_{L_2[0,t]} \leq K\left(1+\left|\left|x_{\bar{\varepsilon}}\right|\right|_{L_2[0,t]}\right)$$

by the first equation of (22)

$$\frac{1}{2}\frac{d}{dt}\left|x_{\bar{\varepsilon}}(t)\right|^2 \leq k\left|x_{\bar{\varepsilon}}(t)\right|\left(1+\left|x_{\bar{\varepsilon}}(t)\right|+\left|y_{\bar{\varepsilon}}(t)\right|\right)$$

Integrating it to get

$$\begin{aligned}
\frac{1}{2}\left|x_{\bar{\varepsilon}}(t)\right|^2 &\leq k\int_0^t\left|x_{\bar{\varepsilon}}(s)\right|ds+k\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]}+\int_0^t\left|x_{\bar{\varepsilon}}(s)\right|\left|y_{\bar{\varepsilon}}(s)\right|ds \\
&\leq 2k\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]}+\left|\left|x_{\bar{\varepsilon}}\right|\right|_{L_2[0,t]}\left|\left|y_{\bar{\varepsilon}}\right|\right|_{L_2[0,t]}+kt \\
&\leq 2k\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]}+K\left|\left|x_{\bar{\varepsilon}}\right|\right|_{L_2[0,t]}\left(1+\left|\left|x_{\bar{\varepsilon}}\right|\right|_{L_2[0,t]}\right)+kt \\
&\leq (2k+K)\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]}+K\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]}+kt+K \\
&\leq (2k+2K)\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]}+(K+kt)
\end{aligned}$$

So

$$\left|\left|x_{\bar{\varepsilon}}\right|\right|^2_{L_2[0,t]} \leq M, \qquad \forall t\in[0,T]$$

Finally we observe that

$$|x_{\bar{\varepsilon}}(t)|, \quad |y_{\bar{\varepsilon}}(t)|, \quad \frac{d}{dt}|x_{\bar{\varepsilon}}(t)|^2$$

are bounded, and there exists a subsequence such that

$$x_\varepsilon \longrightarrow \bar{x} \quad \text{weakly in } H^1[0,T] \text{ and strongly in } L^2[0,T]$$

$$y_\varepsilon \longrightarrow \bar{y} \quad \text{weakly in } L^2[0,T]$$

we can prove also that y_ε converge strongly in L_2 to $\bar{y}$ by using the same method as in Lemma 2.2. Hence limits solutions satisfies $(\bar{x}'(t), 0) \in H(\bar{x}(t), \bar{y}(t))$, so there exists $\bar{u}$ such that $0 = g(\bar{x}(t), \bar{y}(t), \bar{u}(t))$.

From Proposition 2.1 and 2.5, one get the first result

Theorem 2.7 *Under Asssumptions 1.1 and (7), we have the cost convergence*

$$V^\varepsilon \longrightarrow V^0$$

2.2 Rate of the convergence

The result can be improved if the limit problem (2) satisfies extra regularity condition. Let's state at first

Lemma 2.8 *Under the assumptions of Proposition 2.1, we suppose furthermore that there exists an optimal trajectory $(x(\cdot), y(\cdot)) \in S(0, x_0, y_0)$ such that*

$$\left\|\frac{dy}{dt}\right\|_{L_2[0,t]} < \infty. \tag{23}$$

Then

$$||\bar{x}_\varepsilon - x||_{H_1} \leq c\sqrt{\varepsilon}$$

$$||\bar{y}_\varepsilon - y||_{L_2} \leq c\sqrt{\varepsilon}$$

Proof. Let us set

$$x_1^\varepsilon(t) = \bar{x}_\varepsilon(t) - x(t), \qquad y_1^\varepsilon(t) = \bar{y}_\varepsilon(t) - y(t)$$

It follows from (9) and (2) that

$$
(24)\begin{cases}
\dfrac{dx_1^\varepsilon(t)}{dt} = f(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)) - f(x(t), y(t), u(t)), & x_1^\varepsilon(0) = 0 \\
\varepsilon\dfrac{dy_1^\varepsilon(t)}{dt} = \varepsilon\dfrac{d\bar{y}_\varepsilon(t)}{dt} - \varepsilon\dfrac{dy(t)}{dt}, & y_1^\varepsilon(0) = 0
\end{cases}
$$

Rewrite the second differential equation by

$$
\begin{aligned}
\varepsilon\frac{dy_1^\varepsilon(t)}{dt} &= g\left(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)\right) - \varepsilon\frac{dy(t)}{dt} \\
&= g\left(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)\right) - g\left(\bar{x}_\varepsilon(t), y(t), u(t)\right) \\
&\quad + g\left(\bar{x}_\varepsilon(t), y(t), u(t)\right) - g\left(x(t), y(t), u(t)\right) - \varepsilon\frac{dy(t)}{dt}
\end{aligned}
$$

Taking the scalar product of this equation with y_1^ε, we obtain

$$
(25) \qquad \frac{1}{2}\varepsilon\frac{d}{dt}\left|y_1^\varepsilon(t)\right|^2 + \nu\left|y_1^\varepsilon(t)\right|^2 \le \varepsilon\left|\frac{dy(t)}{dt}\right|\left|y_1^\varepsilon(t)\right| + k\left|x_1^\varepsilon(t)\right|\left|y_1^\varepsilon(t)\right|
$$

Doing the same calculation for the first equation in (24) to get at once

$$
(26) \qquad \begin{aligned}
\frac{dx_1^\varepsilon(t)}{dt} &= f(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)) - f\left(x(t), \bar{y}_\varepsilon(t), u(t)\right) \\
&\quad + f\left(x(t), \bar{y}_\varepsilon(t), u(t)\right) - f(x(t), y(t), u(t))
\end{aligned}
$$

and then

$$
(27) \qquad \frac{1}{2}\frac{d}{dt}\left|x_1^\varepsilon(t)\right|^2 \le k\left(\left|x_1^\varepsilon(t)\right|^2 + \left|x_1^\varepsilon(t)\right|\left|y_1^\varepsilon(t)\right|\right)
$$

Integrating this inequality and (25) from 0 to t

$$
\begin{cases}
\dfrac{1}{2}\left|x_1^\varepsilon(t)\right|^2 \le k\left(\displaystyle\int_0^t \left|x_1^\varepsilon(s)\right|^2 ds + \int_0^t \left|x_1^\varepsilon(s)\right|\left|y_1^\varepsilon(s)\right| ds\right) \\
\dfrac{1}{2}\varepsilon\left|y_1^\varepsilon(t)\right|^2 + \nu\displaystyle\int_0^t \left|y_1^\varepsilon(s)\right|^2 ds \le \varepsilon\int_0^t \left|\frac{dy(s)}{ds}\right|\left|y_1^\varepsilon(s)\right| ds + k\int_0^t \left|x_1^\varepsilon(s)\right|\left|y_1^\varepsilon(s)\right| ds
\end{cases}
$$

By inequality of Schwarz

$$\begin{cases} \dfrac{1}{2}\left|x_1^\varepsilon(t)\right|^2 \leq k\left(||x_1^\varepsilon||^2_{L_2[0,t]} + ||x_1^\varepsilon||_{L_2[0,t]}\,||y_1^\varepsilon||_{L_2[0,t]}\right) \\ \dfrac{1}{2}\varepsilon\left|y_1^\varepsilon(t)\right|^2 + \nu\,||y_1^\varepsilon||^2_{L_2[0,t]} \leq \varepsilon\left|\left|\dfrac{dy}{ds}\right|\right|_{L_2[0,t]}||y_1^\varepsilon||_{L_2[0,t]} + k\,||x_1^\varepsilon||_{L_2[0,t]}\,||y_1^\varepsilon||_{L_2[0,t]} \end{cases}$$

from the second equation

$$\nu\,||y_1^\varepsilon||^2_{L_2[0,t]} \leq ||y_1^\varepsilon||_{L_2[0,t]}\left(\varepsilon\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]} + k\,||x_1^\varepsilon||_{L_2[0,t]}\right)$$

we obtain

$$||y_1^\varepsilon||_{L_2[0,t]} \leq \frac{\varepsilon}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]} + \frac{k}{\nu}\,||x_1^\varepsilon||_{L_2[0,t]} \tag{28}$$

by the first equation

$$\begin{aligned} \frac{1}{2}\left|x_1^\varepsilon(t)\right|^2 &\leq k\left(||x_1^\varepsilon||^2_{L_2[0,t]} + ||x_1^\varepsilon||_{L_2[0,t]}\left(\frac{\varepsilon}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]} + \frac{k}{\nu}\,||x_1^\varepsilon||_{L_2[0,t]}\right)\right) \\ &\leq \left(\frac{k^2}{\nu} + k\right)||x_1^\varepsilon||^2_{L_2[0,t]} + \varepsilon\frac{k}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]}||x_1^\varepsilon||_{L_2[0,t]} \\ &\leq \left(\frac{k^2}{\nu} + k\right)||x_1^\varepsilon||^2_{L_2[0,t]} + \varepsilon\frac{k}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]}\left(1 + ||x_1^\varepsilon||^2_{L_2[0,t]}\right) \\ &= \left(\frac{k^2}{\nu} + k + \varepsilon\frac{k}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]}\right)||x_1^\varepsilon||^2_{L_2[0,t]} + \varepsilon\frac{k}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]} \end{aligned}$$

Applying inequality of Grōnwall

$$\left|x_1^\varepsilon(t)\right|^2 \leq \varepsilon\frac{2k}{\nu}\left|\left|\frac{dy}{dt}\right|\right|_{L_2[0,t]}\left(-\frac{1}{A_\varepsilon} + \frac{1}{A_\varepsilon}e^{A_\varepsilon T}\right)$$

Where $A_\varepsilon = \dfrac{2k^2}{\nu} + 2k + \varepsilon\dfrac{2k}{\nu}\left|\left|\dfrac{dy}{dt}\right|\right|_{L_2[0,t]}$, obviously A_ε and $B_\varepsilon = -\dfrac{1}{A_\varepsilon} + \dfrac{1}{A_\varepsilon}e^{A_\varepsilon T}$ are bounded by a constant which is independent from ε. Finally, we have

$$\left|x_1^\varepsilon(t)\right|^2 \leq c\varepsilon, \qquad ||x_1^\varepsilon||^2_{L_2[0,t]} \leq c\varepsilon$$

It result from (28) that $||y_1^\varepsilon||_{L_2[0,t]} \leq c\sqrt{\varepsilon}$. Using (26), we get

$$\left\|\frac{dx_1^\varepsilon}{dt}\right\|_{L_2[0,t]} \leq ||a^\varepsilon||_{L_2[0,t]} + ||b^\varepsilon||_{L_2[0,t]}$$

Where

$$\begin{aligned} a^\varepsilon(t) &= f(\bar{x}_\varepsilon(t), \bar{y}_\varepsilon(t), u(t)) - f(x(t), \bar{y}_\varepsilon(t), u(t)) \\ b^\varepsilon(t) &= f(x(t), \bar{y}_\varepsilon(t), u(t)) - f(x(t), y(t), u(t)) \end{aligned}$$

with (1.1) we see

$$||a^\varepsilon||_{L_2[0,t]} \leq k\,||x_1^\varepsilon||_{L_2[0,t]}, \quad ||b^\varepsilon||_{L_2[0,t]} \leq k\,||y_1^\varepsilon||_{L_2[0,t]}$$

to get

$$\left\|\frac{dx_1^\varepsilon}{dt}\right\|_{L_2[0,t]} \leq c\sqrt{\varepsilon}$$

and the proof is complete. □

The following proposition is an immediate consequence of this result.

Proposition 2.9 *Assume (1.1), (7) and (23), we have inequality*

$$V^\varepsilon \leq V^0 + c\sqrt{\varepsilon}$$

PROOF. Thanks to Assumption (23), we can improve the (10) in Lemma 2.2 into

(29) $$\left|J^\varepsilon(u) - V^0\right| \leq c\sqrt{\varepsilon}$$

Indeed, since h is Lipschitz, we have

$$|J^\varepsilon(u) - J(u)| = |h(\bar{x}_\varepsilon(T)) - h(x(T))| \leq c\sqrt{\varepsilon}$$

So we get

$$V^\varepsilon \leq J^\varepsilon(u) \leq V^0 + c\sqrt{\varepsilon}$$

□

To get $\left|V^\varepsilon - V^0\right| \leq c\sqrt{\varepsilon}$, we have to prove $V^0 \leq V^\varepsilon + c\sqrt{\varepsilon}$. We state

Proposition 2.10 *Under assumptions*[3] *(1.1), (7) and if for any* $(x(\cdot), y(\cdot), v(\cdot))$ *solution of (2), we have*

$$\left\| \frac{dy}{dt} \right\|_{L_2[0,t]} \leq M \tag{30}$$

then

$$V^0 \leq V^\varepsilon + c\sqrt{\varepsilon} \tag{31}$$

PROOF. For any $\varepsilon > 0$, we note that there exists $w_\varepsilon \in U$ such that

$$\begin{cases} J(w_\varepsilon) & \leq & J(u_\varepsilon^*) \\ J(w_\varepsilon) & \leq & J(v) + \varepsilon, \qquad \forall v \in U \end{cases} \tag{32}$$

Where $u_\varepsilon^*(\cdot)$ is an optimal control of (1). Let's denote by $(\tilde{x}_\varepsilon(\cdot), \tilde{y}_\varepsilon(\cdot), w_\varepsilon(\cdot))$ the solution of (2) with respect to w_ε and $(\widehat{x}_\varepsilon(\cdot), \widehat{y}_\varepsilon(\cdot), u_\varepsilon^*(\cdot))$ be the solution of (2) with respect to $u_\varepsilon^*(\cdot)$. To get the convergence of $(\widehat{x}_\varepsilon(\cdot), \widehat{y}_\varepsilon(\cdot))$, we set

$$x_\varepsilon^\sharp(t) = \tilde{x}_\varepsilon(t) - x(t), \qquad y_\varepsilon^\sharp(t) = \tilde{y}_\varepsilon(t) - y(t)$$

so

$$\begin{cases} \dfrac{dx_\varepsilon^\sharp(t)}{dt} = f(\tilde{x}_\varepsilon(t), \tilde{y}_\varepsilon(t), w_\varepsilon(t)) - f(x(t), y(t), u(t)) \quad x_\varepsilon^\sharp(0) = 0 \\ 0 = g(\tilde{x}_\varepsilon(t), \tilde{y}_\varepsilon(t), w_\varepsilon(t)) - g(x(t), y(t), u(t)) \quad y_\varepsilon^\sharp(0) = 0 \end{cases}$$

Multiplying the first equation by $x_\varepsilon^\sharp(t)$, we have

$$\frac{1}{2}\frac{d}{dt}\left|x_\varepsilon^\sharp(t)\right|^2 \leq A\left(1 + \left\|x_\varepsilon^\sharp\right\|_{L_2[0,t]}^2 + \left\|x_\varepsilon^\sharp\right\|_{L_2[0,t]}\left\|y_\varepsilon^\sharp\right\|_{L_2[0,t]}\right)$$

since f is Lipschitz and U is bounded. By the second equation we have

$$\begin{aligned} 0 &= < g(\tilde{x}_\varepsilon(t), \tilde{y}_\varepsilon(t), w_\varepsilon(t)) - g(\tilde{x}_\varepsilon(t), y(t), w_\varepsilon(t)), y_\varepsilon^\sharp(t) > \\ &\quad + < g(\tilde{x}_\varepsilon(t), y(t), w_\varepsilon(t)) - g(x(t), y(t), u(t)), y_\varepsilon^\sharp(t) > \\ &\leq -\nu\left|y_\varepsilon^\sharp(t)\right|^2 + k\left(\left|x_\varepsilon^\sharp(t)\right| + |w_\varepsilon(t) - u(t)|\right)\left|y_\varepsilon^\sharp(t)\right| \end{aligned}$$

[3]Let us notice that here we do not need existence of optimal control, consequently assumption (1.1) is not useful to prove the Proposition.

So by integrating we get

$$\left\|y_\varepsilon^\sharp\right\|_{L_2[0,t]} \le K\left(1+\left\|x_\varepsilon^\sharp\right\|_{L_2[0,t]}\right)$$

Therefore $\left\|x_\varepsilon^\sharp\right\|_{L_2[0,t]}, \dfrac{d}{dt}\left|x_\varepsilon^\sharp(t)\right|^2$ and $\left\|y_\varepsilon^\sharp\right\|_{L_2[0,t]}$ are bounded, so there exists $(\bar{x}(\cdot), \bar{y}(\cdot), \bar{u}(\cdot))$ solution of (2) such that

$$\tilde{x}_\varepsilon \longrightarrow \bar{x} \qquad \text{weakly in } H^1[0,T] \text{ and strongly in } L^2[0,T]$$

$$\tilde{y}_\varepsilon \longrightarrow \bar{y} \qquad \text{strongly in } L^2[0,T]$$

and $\bar{u}(\cdot)$ is optimal control of (2) by the construction of $w_\varepsilon(\cdot)$. Indeed as we know

$$\begin{aligned} J(\bar{u}) &= h(x(T)) = \lim_{\varepsilon\to 0} h(\tilde{x}_\varepsilon(T)) = \lim_{\varepsilon\to 0} J(w_\varepsilon) \\ &\le \lim_{\varepsilon\to 0}(J(v)+\varepsilon) = J(v) \quad \forall v \in U \end{aligned}$$

To obtain inequality (31). it is sufficient to prove that under the condition (30) we have

$$\|x_\varepsilon^* - \hat{x}_\varepsilon\|_{H_1} \le c\sqrt{\varepsilon}, \quad \|y_\varepsilon^* - \hat{y}_\varepsilon\|_{L_2[0,t]} \le c\sqrt{\varepsilon}$$

Where $(x_\varepsilon^*(\cdot), y_\varepsilon^*(\cdot), u_\varepsilon^*(\cdot))$ is an optimal solution of (1).

We use the same method as Lemma 2.8. Set

$$x_\varepsilon^\flat(t) = x_\varepsilon^*(t) - \tilde{x}_\varepsilon(t), \qquad y_\varepsilon^\flat(t) = y_\varepsilon^*(t) - \tilde{y}_\varepsilon(t),$$

They are solution of differential equation system

$$\left\{\begin{aligned} \frac{dx_\varepsilon^\flat(t)}{dt} &= f(x_\varepsilon^*(t), y_\varepsilon^*(t), u_\varepsilon^*(t)) - f(\hat{x}_\varepsilon(t), \hat{y}_\varepsilon(t), u_\varepsilon^*(t)), && x_\varepsilon^\flat(0) = 0 \\ \varepsilon\frac{dy_\varepsilon^\flat(t)}{dt} &= \varepsilon\frac{dy_\varepsilon^*(t)}{dt} - \varepsilon\frac{d\hat{y}_\varepsilon(t)}{dt}, && y_\varepsilon^\flat(0) = 0 \end{aligned}\right.$$

(33)

Rewrite the second differential equation by

$$\begin{aligned}
\varepsilon\frac{dy^b_\varepsilon(t)}{dt} &= g\left(x^*_\varepsilon(t), y^*_\varepsilon(t), u^*_\varepsilon(t)\right) - \varepsilon\frac{d\widehat{y}_\varepsilon(t)}{dt} \\
&= g\left(x^*_\varepsilon(t), y^*_\varepsilon(t), u^*_\varepsilon(t)\right) - g\left(x^*_\varepsilon(t), \widehat{y}_\varepsilon(t), u^*_\varepsilon(t)\right) \\
&\quad + g\left(x^*_\varepsilon(t), \widehat{y}_\varepsilon(t), u^*_\varepsilon(t)\right) - g\left(\widehat{x}_\varepsilon(t), \widehat{y}_\varepsilon(t), u^*_\varepsilon(t)\right) - \varepsilon\frac{d\widehat{y}_\varepsilon(t)}{dt}
\end{aligned}$$

Taking the scalar products of this equation with $y^b_\varepsilon(t)$, we obtain

$$\frac{1}{2}\varepsilon\frac{d}{dt}\left|y^b_\varepsilon(t)\right|^2 + \nu\left|y^b_\varepsilon(t)\right|^2 \leq \varepsilon\left|\frac{d\widehat{y}_\varepsilon(t)}{dt}\right|\left|y^b_\varepsilon(t)\right| + k\left|x^b_\varepsilon(t)\right|\left|y^b_\varepsilon(t)\right| \tag{34}$$

Doing the same calculation for the first equation in (33) to get at once

$$\begin{aligned}
\frac{dx^b_\varepsilon(t)}{dt} &= f(x^*_\varepsilon(t), y^*_\varepsilon(t), u^*_\varepsilon(t)) - f\left(\widehat{x}_\varepsilon(t), y^*_\varepsilon(t), u^*_\varepsilon(t)\right) \\
&\quad + f\left(\widehat{x}_\varepsilon(t), y^*_\varepsilon(t), u^*_\varepsilon(t)\right) - f(\widehat{x}_\varepsilon(t), \widehat{y}_\varepsilon(t), u^*_\varepsilon(t))
\end{aligned} \tag{35}$$

and then

$$\frac{1}{2}\frac{d}{dt}\left|x^b_\varepsilon(t)\right|^2 \leq k\left(\left|x^b_\varepsilon(t)\right|^2 + \left|x^b_\varepsilon(t)\right|\left|y^b_\varepsilon(t)\right|\right) \tag{36}$$

Integrating this inequality and (34) from 0 to t

$$\begin{cases}
\frac{1}{2}\left|x^b_\varepsilon(t)\right|^2 \leq k\left(\int_0^t \left|x^b_\varepsilon(s)\right|^2 ds + \int_0^t \left|x^b_\varepsilon(s)\right|\left|y^b_\varepsilon(s)\right| ds\right) \\
\frac{1}{2}\varepsilon\left|y^b_\varepsilon(t)\right|^2 + \nu\int_0^t \left|y^b_\varepsilon(s)\right|^2 ds \leq \varepsilon\int_0^t \left|\frac{d\widehat{y}_\varepsilon(s)}{ds}\right|\left|y^b_\varepsilon(s)\right| ds + k\int_0^t \left|x^b_\varepsilon(s)\right|\left|y^b_\varepsilon(s)\right| ds
\end{cases}$$

By inequality of Schwarz

$$\begin{cases}
\frac{1}{2}\left|x^b_\varepsilon(t)\right|^2 \leq k\left(\left\|x^b_\varepsilon\right\|^2_{L_2[0,t]} + \left\|x^b_\varepsilon\right\|_{L_2[0,t]}\left\|y^b_\varepsilon\right\|_{L_2[0,t]}\right) \\
\frac{1}{2}\varepsilon\left|y^b_\varepsilon(t)\right|^2 + \nu\left\|y^b_\varepsilon\right\|^2_{L_2[0,t]} \leq \varepsilon\left\|\frac{d\widehat{y}_\varepsilon}{ds}\right\|_{L_2[0,t]}\left\|y^b_\varepsilon\right\|_{L_2[0,t]} + k\left\|x^b_\varepsilon\right\|_{L_2[0,t]}\left\|y^b_\varepsilon\right\|_{L_2[0,t]}
\end{cases}$$

from the second equation

$$\nu \left\|y_\varepsilon^b\right\|^2_{L_2[0,t]} \leq \left\|y_\varepsilon^b\right\|_{L_2[0,t]} \left(\varepsilon \left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]} + k \left\|x_\varepsilon^b\right\|_{L_2[0,t]}\right)$$

we obtain

(37) $$\left\|y_\varepsilon^b\right\|_{L_2[0,t]} \leq \frac{\varepsilon}{\nu} \left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]} + \frac{k}{\nu} \left\|x_\varepsilon^b\right\|_{L_2[0,t]}$$

by first equation

$$\begin{aligned}
\frac{1}{2}\left|x_\varepsilon^b(t)\right|^2 &\leq k \left(\left\|x_\varepsilon^b\right\|^2_{L_2[0,t]} + \left\|x_\varepsilon^b\right\|_{L_2[0,t]} \left(\frac{\varepsilon}{\nu}\left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]} + \frac{k}{\nu}\left\|x_\varepsilon^b\right\|_{L_2[0,t]}\right)\right) \\
&\leq \left(\frac{k^2}{\nu} + k\right)\left\|x_\varepsilon^b\right\|^2_{L_2[0,t]} + \varepsilon\frac{k}{\nu}\left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]} \left\|x_\varepsilon^b\right\|_{L_2[0,t]} \\
&\leq \left(\frac{k^2}{\nu} + k\right)\left\|x_\varepsilon^b\right\|^2_{L_2[0,t]} + \varepsilon\frac{k}{\nu}\left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]} \left(1 + \left\|x_\varepsilon^b\right\|^2_{L_2[0,t]}\right) \\
&= \left(\frac{k^2}{\nu} + k + \varepsilon\frac{k}{\nu}\left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]}\right)\left\|x_\varepsilon^b\right\|^2_{L_2[0,t]} + \varepsilon\frac{k}{\nu}\left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]}
\end{aligned}$$

Applying the inequality of Grõnwall

$$\left|x_\varepsilon^b(t)\right|^2 \leq \varepsilon\frac{2k}{\nu}\left\|\frac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]} \left(-\frac{1}{A_\varepsilon} + \frac{1}{A_\varepsilon}e^{A_\varepsilon T}\right)$$

Where $A_\varepsilon = \dfrac{2k^2}{\nu} + 2k + \varepsilon\dfrac{2k}{\nu}\left\|\dfrac{d\widehat{y}_\varepsilon}{dt}\right\|_{L_2[0,t]}$, obviously A_ε and $B_\varepsilon = -\dfrac{1}{A_\varepsilon} + \dfrac{1}{A_\varepsilon}e^{A_\varepsilon T}$ are bounded by a constant which is independent to ε. Finally, we have

$$\left|x_\varepsilon^b(t)\right|^2 \leq c\varepsilon, \qquad \left\|x_\varepsilon^b\right\|^2_{L_2[0,t]} \leq c\varepsilon$$

It result from (37) that $\left\|y_\varepsilon^b\right\|_{L_2[0,t]} \leq c\sqrt{\varepsilon}$. Using (35), we get

$$\left\|\frac{dx_\varepsilon^b}{dt}\right\|_{L_2[0,t]} \leq \left\|a^\varepsilon\right\|_{L_2[0,t]} + \left\|b^\varepsilon\right\|_{L_2[0,t]}$$

Where

$$a^\varepsilon(t) = f(x_\varepsilon^*(t), y_\varepsilon^*(t), u_\varepsilon^*(t)) - f(\tilde{x}_\varepsilon(t), y_\varepsilon^*(t), u_\varepsilon^*(t))$$

$$b^\varepsilon(t) = f(\tilde{x}_\varepsilon(t), y_\varepsilon^*(t), u_\varepsilon^*(t)) - f(\tilde{x}_\varepsilon(t), \tilde{y}_\varepsilon(t), u_\varepsilon^*(t))$$

with (1.1) we see

$$\|a^\varepsilon\|_{L_2[0,t]} \leq k \left\|x_\varepsilon^b\right\|_{L_2[0,t]}, \quad \|b^\varepsilon\|_{L_2[0,t]} \leq k \left\|y_\varepsilon^b\right\|_{L_2[0,t]}$$

to get

$$\left\|\frac{dx_\varepsilon^b}{dt}\right\|_{L_2[0,t]} \leq c\sqrt{\varepsilon}$$

this gives us

$$|J^\varepsilon(u_\varepsilon^*) - J(u_\varepsilon^*)| \leq c\sqrt{\varepsilon}$$

The inequality (31) is then proved by noting

$$V^0 = J(\bar{u}) \leq J(u_\varepsilon^*)$$

□

By Proposition 2.9 and Proposition 2.10, we conclude this section by stating the following result

Theorem 2.11 *Under assumptions (1.1), (7) and (30) we obtain the convergence rate*

$$\left|V^\varepsilon - V^0\right| \leq c\sqrt{\varepsilon}$$

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