



Isaacs' Equations for Value- Functions of Differential Games

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Working Paper

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FOREWORD

The authors study value functions of a differential game with payoff which depends on the state at a given end time. They consider differential games with feedback strategies and with nonanticipating strategies. They prove that value-functions are solutions to some Hamilton-Jacobi-Isaacs equations in the viscosity and contingent sense. For these two notions of strategies, with some regularity assumptions, The authors prove that value-functions are the unique solution of Isaacs' equations.

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Isaacs' equations for value-functions of differential games

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1 Introduction

Let us consider the following differential game:

$$(1) \quad \begin{cases} i) & x'(t) = f(t, x(t), u(t), v(t)), t \in [t_0, T] \\ ii) & u(t) \in U \quad v(t) \in V \end{cases}$$

The two players act on the state $x(\cdot)$ by choosing controls, u for the first player and v for the second one. The goal of the first player is to maximize at the given end time T the payoff $g(x(T))$, the second player wants to minimize it.

Let us recall that the game with the following payoff:

$$(2) \quad g(x(T)) + \int_0^T L(t, x(t), u(t), v(t)) dt$$

may be reduced to the above one. In fact, by the simple change of variable $z := (x, y)$, we obtain the new game

$$(3) \quad \begin{cases} i) & (x'(t), y'(t)) = (f(t, x(t), u(t), v(t)), L(t, x(t), u(t), v(t))) \\ ii) & u(t) \in U \\ iii) & v(t) \in V \end{cases}$$

with the payoff $G(x(T), y(T)) := g(x(T)) + y(T)$ which is equal to (2).

Since Isaacs (cf [14]), it is well-known that the value-function satisfies a partial differential equation (the Isaacs' equation) when the game is regular enough. The solutions of this equation have been studied by Isaacs himself in C^1 case (see [14]), lipschitz solutions have been studied for example in [15], and in [9], [5], [17] for viscosity solutions. However such regularity is not always the case (see also for instance [12], [4], [6]... for control systems).

We introduce two notions of strategies and we prove that the associated value-functions are solutions to Isaacs' equation without any assumptions concerning the regularity of g . In this paper, we mainly state results (see [13] for more detailed proofs).

2 Feedback strategies of differential games

Consider a function $f : [0, T] \times \mathbb{R}^n \times U \times V \mapsto \mathbb{R}^n$ where U and V are two complete separable metric spaces. Let us denote by $t \mapsto x(t, t_0, x_0, u(\cdot), v(\cdot))$ the solution to (1) corresponding to controls $u(\cdot), v(\cdot)$, starting from x_0 at

time t_0 (i.e. such that $x(t_0) = x_0$). We shall need the following assumptions:

$$(4) \quad \left\{ \begin{array}{l} i) \quad f \text{ is continuous} \\ ii) \quad \forall (t, x) \in [0, T] \times \mathbb{R}^n, \forall (u, v) \in U \times V \text{ the sets} \\ \quad \quad f(t, x, u, V), f(t, x, U, v) \text{ are compact and convex} \\ iii) \quad \forall (t, x, u, v) \in [0, T] \times \mathbb{R}^n \times U \times V, \\ \quad \quad \text{functions } f(t, x, \cdot, v) \text{ and } f(t, x, u, \cdot) \text{ are} \\ \quad \quad l\text{-Lipschitz, where } l > 0 \\ iv) \quad \forall R > 0, \exists c_R \in L^1(0, T) \text{ such that for almost all} \\ \quad \quad t \in [0, T] \text{ and for all } (u, v) \in U \times V, f(t, \cdot, u, v) \\ \quad \quad \text{is } c_R(t)\text{-Lipschitz on } B(0, R). \\ v) \quad \exists k \in L^1(0, T) \text{ such that for almost all } t \in [0, T] \\ \quad \quad \sup_{u \in U} \sup_{v \in V} \|f(t, x, u, v)\| \leq k(t)(1 + \|x\|) \end{array} \right.$$

We call *feedback strategy* for the first player any function $\varphi : [0, T] \times X \mapsto U$ such that $(t, x) \mapsto f(t, x, \varphi(t, x), V)$ is upper semicontinuous with respect to (t, x) . We denote by Φ the set of feedback strategies for the first player. In a similar way we can define feedback strategies for the second player and Ψ the set of such strategies. We denote by \mathcal{U} (respectively \mathcal{V}) the set of measurable functions $[0, T] \mapsto U$ (respectively $[0, T] \mapsto V$).

We assume furthermore the following crucial condition which allows to define the value function of the game:

There exists a pair of feedback strategies $(\varphi^*, \psi^*) \in \Phi \times \Psi$ such that for any measurable control $u(\cdot)$, there exists a unique solution to

$$x'(t) = f(t, x(t), u(t), \psi^*(t, x(t)))$$

such that $x(t_0) = x_0$ and we denote by $x(\cdot, t_0, x_0, u(\cdot), \psi^*(\cdot, \cdot))$ this solution.

In a similar way, we assume also the existence and unicity of $x(\cdot, t_0, x_0, \varphi^*(\cdot, \cdot), v(\cdot))$ and $x(\cdot, t_0, x_0, \phi^*(\cdot, \cdot), \psi^*(\cdot, \cdot))$.

$$(5) \quad \left\{ \begin{array}{l} \forall (t_0, x_0), \forall (u(\cdot), v(\cdot)) \in \mathcal{U} \times \mathcal{V} \\ g(x(T, t_0, x_0, u(\cdot), \psi^*(\cdot, \cdot))) \leq g(x(T, t_0, x_0, \phi^*(\cdot, \cdot), \psi^*(\cdot, \cdot))) \\ \leq g(x(T, t_0, x_0, \phi^*(\cdot, \cdot), v(\cdot))) \end{array} \right.$$

Definition 2.1 *If (5) is satisfied, we call*

$$W(t_0, x_0) := g(x(T, t_0, x_0, \phi^*(\cdot, \cdot), \psi^*(\cdot, \cdot)))$$

the value function¹ of the differential game with feedback strategy.

3 Contingent solutions to Isaacs' equations

Consider the following contingent² inequalities:

$$(6) \quad \begin{cases} \Theta(T, \cdot) = g(\cdot) \text{ and } \forall (t, x) \in \text{Dom}(\Theta) \\ i) \text{ if } t \in [0, T[, \text{ then} \\ \sup_{u \in U} \inf_{v \in V} D_1 \Theta(t, x)(1, f(t, x, u, v)) \leq 0 \\ ii) \text{ if } t \in [0, T[, \text{ then} \\ \sup_{u \in U} \inf_{v \in V} D_1 \Theta(t, x)(1, f(t, x, u, v)) \geq 0 \end{cases}$$

$$(7) \quad \begin{cases} \Theta(T, \cdot) = g(\cdot) \text{ and } \forall (t, x) \in \text{Dom}(\Theta) \\ i) \text{ if } t \in [0, T[, \text{ then} \\ \inf_{v \in V} \sup_{u \in U} D_1 \Theta(t, x)(1, f(t, x, u, v)) \leq 0 \\ ii) \text{ if } t \in [0, T[, \text{ then} \\ \inf_{v \in V} \sup_{u \in U} D_1 \Theta(t, x)(1, f(t, x, u, v)) \geq 0 \end{cases}$$

A such $\Theta : [0, T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ is called contingent solution to the Isaacs' equation. We can prove *without any assumptions on g* the following:

Proposition 3.1 *Assume that (4) and (5) hold true, then the value function satisfies (6)i) and (7)ii).*

¹This definition is very related to the one of Pierre Bernhard (see [7]).

²Recall the definition of the contingent epiderivative of $\Theta : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ at $x_0 \in \text{Dom}(\Theta)$,

$$D_1 \Theta(x_0)(u) = \liminf_{h \rightarrow 0^+, v \rightarrow u} \frac{\Theta(x_0 + hv) - \Theta(x_0)}{h}$$

or equivalently $\text{Epi}D_1 \Theta(x_0) = T_{\text{Epi}\Theta}(x_0, \Theta(x_0))$, where *Epi* states for the epigraph. In a similar way for the contingent hypoderivative of Θ at $x_0 \in \text{Dom}(\Theta)$ is defined by $D_1 \Theta(x)(u) := -D_1(-\Theta)(x)(u)$, and the contingent derivative of Θ at $x_0 \in \text{Dom}(\Theta)$ is defined by:

$$\text{Graph}D\Theta(x_0) = T_{\text{Graph}\Theta}(x_0, \Theta(x_0)).$$

Proof — Let us prove (6)i). Fix $\bar{u} \in U$ and $h > 0$. Define x_h a solution of $x'(t) = f(t, x(t), \bar{u}, \psi^*)$ $t \in [t_0, t_0 + h]$ such that $x(t_0) = x_0$. Let us introduce

$$u_h(t) := \begin{cases} \bar{u} & \text{if } t \in [t_0, t_0 + h] \\ \varphi^*(t, x(t, t_0 + h, x_h(t_0 + h), \varphi^*(\cdot, \cdot), \psi^*(\cdot, \cdot))) & \\ \text{if } t > t_0 + h \end{cases}$$

Then by the very definition of W ,

$W(t_0 + h, x_h(t_0 + h)) = g(x(T, t_0 + h, x(t_0 + h), \varphi^*(\cdot, \cdot), \psi^*(\cdot, \cdot)))$ which is equal to $g(x(T, t_0, x_0, u_h(\cdot), \psi^*(\cdot, \cdot)))$ and by (5), $g(x(T, t_0, x_0, u_h(\cdot), \psi^*(\cdot, \cdot))) \leq g(x(T, t_0, x_0, \phi^*(\cdot, \cdot), \psi^*(\cdot, \cdot))) = W(t_0, x_0)$. Hence

$$\liminf_{h \rightarrow 0^+} \frac{W(t_0 + h, x_h(t_0 + h)) - W(t_0, x_0)}{h} \leq 0$$

Thanks to our assumptions and by the Mean Value Theorem, there exists $h_n \rightarrow 0^+$ such that:

$$\frac{x_{h_n}(t_0 + h_n) - x_0}{h_n} \rightarrow w \in f(t_0, x_0, \bar{u}, V)$$

This yields $\inf_{v \in V} D_1 W(t_0, x_0)(1, f(t, x, \bar{u}, v)) \leq 0$ and proves (6)i). \square

We can obtain other contingent inequalities with suitable regularity assumptions concerning strategies

Proposition 3.2 *Assume that (4) and (5) hold true. If $(t, x) \mapsto f(t, x, \varphi^*(t, x), V)$ is continuous at (t_0, x_0) , then the value function W satisfies (6)ii). If $(t, x) \mapsto f(t, x, U, \psi^*(t, x))$ is continuous at (t_0, x_0) , then the value function W satisfies (7)i).*

Now we shall state an unicity result³:

Theorem 3.3 *Assume that (4), (5) hold true and that W is continuous.*

- *If $(t, x) \mapsto f(t, x, \varphi^*(t, x), V)$ is continuous, then any lower semicontinuous (l.s.c.) function Θ satisfying (6)i) is larger or equal than W*

³cf the proof in [13]

- If $(t, x) \mapsto f(t, x, U, \psi^*(t, x))$ is continuous, then any upper semicontinuous (u.s.c.) function Θ satisfying (7)ii) is lower or equal than W .

Corollary 3.4 When (4), (5) hold true, and W is continuous and when $(t, x) \mapsto f(t, x, U, \psi^*(t, x))$ and $(t, x) \mapsto f(t, x, \varphi^*(t, x))$ are continuous, the value function W is the unique continuous solution to Hamilton-Jacobi-Isaacs inequalities (6) and (7).

4 Viscosity solutions to Isaacs' equation

We first define the lower and upper Hamiltonians of the differential (1):

$$\begin{aligned} H_-(t, x, p) &:= \max_{v \in V} \min_{u \in U} \langle p, f(t, x, u, v) \rangle \\ H_+(t, x, p) &:= \min_{u \in U} \max_{v \in V} \langle p, f(t, x, u, v) \rangle, \end{aligned}$$

Consider two Hamilton-Jacobi equations:

$$(8) \quad \begin{cases} -\frac{\partial \Theta}{\partial t}(t, x) + H_+(t, x, -\frac{\partial \Theta}{\partial x}(t, x)) = 0 \\ \Theta(T, \cdot) = g(\cdot) \end{cases}$$

$$(9) \quad \begin{cases} -\frac{\partial \Theta}{\partial t}(t, x) + H_-(t, x, -\frac{\partial \Theta}{\partial x}(t, x)) = 0 \\ \Theta(T, \cdot) = g(\cdot) \end{cases}$$

In this section, we give some results concerning viscosity solutions to Hamilton-Jacobi-Isaacs equations. First, we recall the definition of viscosity solution by using sub and super differentials⁴:

Definition 4.1 Consider $H : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}$ Let us recall that the function $\Theta : [0, T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ is a viscosity supersolution to the following Hamilton-Jacobi equation $-\frac{\partial \Theta}{\partial t}(t, x) + H(t, x, -\frac{\partial \Theta}{\partial x}(t, x)) = 0$ if and only if:

$$\forall (t, x) \in \text{Dom}(\Theta), \forall (p_t, p_x) \in \partial_- \Theta(t, x), -p_t + H(t, x, -p_x) \geq 0$$

The function Θ is a viscosity subsolution if and only if:

$$\forall (t, x) \in \text{Dom}(\Theta), \forall (p_t, p_x) \in \partial_+ \Theta(t, x), -p_t + H(t, x, -p_x) \leq 0$$

A function Θ is a viscosity solution if it is a supersolution and a subsolution.

⁴Recall the definition of the subdifferential of $\phi : \mathbb{R}^n \mapsto \overline{\mathbb{R}}$, at $x_0 \in \text{Dom}(\phi)$
 $\partial_- \phi(x_0) := \{ p \in \mathbb{R}^n \mid \liminf_{x \rightarrow x_0} \frac{\phi(x) - \phi(x_0) - \langle p, x - x_0 \rangle}{\|x - x_0\|} \geq 0 \}$ and the super differential of ϕ at x_0 is given by: $\partial_+ \phi(x_0) := -\partial_- (-\phi)(x_0)$.

We can prove *without any assumptions* concerning g the following existence result:

Proposition 4.2 *Assume that (4) holds true, then the value-function W is a supersolution to (8) and a subsolution to (9).*

But when the value-function W is continuous, we have the more precise

Proposition 4.3 *Assume (4). Then if W is continuous and $(t, x) \mapsto f(t, x, U, \psi^*(t, x))$ is continuous, then W is a viscosity solution to (8).*

If W is continuous and $(t, x) \mapsto f(t, x, \varphi^(t, x), V)$ is continuous, then W is a viscosity solution to (9).*

Theorem 4.4 *Let assumptions of Corollary 3.4 hold true. Let $\Theta : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$ be continuous. Then Θ is the value function of the game if and only if it is a viscosity supersolution to (8) and a viscosity subsolution to (9).*

Corollary 4.5 *Let us assume (4), (5) and let $\Theta : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function. If we assume the following Isaacs' condition:*

$$(10) \quad \forall (t, x, p), \quad H_-(t, x, p) = H_+(t, x, p),$$

then the value function is the unique viscosity solution to (9) (or equivalently (8)).

These results follow from results of the previous section and from the following section.

5 Comparison between viscosity and continuous solutions to Hamilton Jacobi Isaacs equations

Proposition 5.1 *Consider $\Theta : [0, T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$ verifying (6) (respectively (7)). Then Θ is a viscosity solution to (8) (respectively to (9)).*

This result is a consequence of the following

Lemma 5.2 *Consider $\Theta : [0, T] \times \mathbb{R}^n \mapsto \overline{\mathbb{R}}$.*

- If Θ satisfies (6)i), then it is a supersolution of (8).
- If Θ satisfies (6)ii), then it is a subsolution of (8).
- If Θ satisfies (7)i), then it is a supersolution of (9).
- If Θ satisfies (7)ii), then it is a subsolution of (9).

Proof — Let us prove the first statement. If Θ satisfying (6)i) then⁵:

$$(11) \quad \begin{cases} \forall (p_t, p_x) \in \partial_- \Theta(t, x), \forall (u, v) \in U \times V, \\ D_t \Theta(t, x)(1, f(t, x, u, v)) \geq p_t + \langle p_x, f(t, x, u, v) \rangle \end{cases}$$

Then by taking the “supinf” of this inequality, we prove that Θ is a supersolution to (8). The proofs of the other statements are similar. \square

When value functions are continuous, the notions of contingent and viscosity solutions of Isaacs’ equations are equivalent.

Theorem 5.3 *Let $\Theta : [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}$ be a continuous function and let (4) hold true. Then Θ satisfies the contingent inequalities (6) (respectively (7)) if and only if it is a viscosity solution to the Hamilton-Jacobi-Isaacs equation (8) (respectively (9)).*

Lemma 5.4 *If (4) holds true.*

- Any l.s.c. function Θ is a supersolution of (8) if and only if it satisfies (6)i).
- Any l.s.c. function Θ is a supersolution of (9) if and only if it satisfies (7)i).
- Any u.s.c. function Θ is a subsolution of (8) if and only if it satisfies (6)ii).

⁵Let us recall (see [2] chapter 6 for instance) that we have the following equivalent definition for the subdifferential of a function ϕ

$$\partial_- \phi(x_0) = \{ p \mid \forall q \in \mathbb{R}^n, D_t \phi(x_0)(q) \geq \langle p, q \rangle \}$$

- Any u.s.c. function Θ is a subsolution of (9) if and only if it satisfies (7)ii).

Proof of Lemma — We already know, thanks to Proposition 5.1 and Lemma 5.2, that contingent solutions are viscosity solutions. Let us prove the converse implication.

Assume that Θ is a supersolution to (8), i.e.:

$$(12) \quad \forall (p_t, p_x) \in \partial_- \Theta(t, x), \sup_{u \in U} \inf_{v \in V} p_t + \langle p_x, f(t, x, u, v) \rangle \leq 0$$

Hence, for any $u \in U$, $\inf_{v \in V} p_t + \langle p_x, f(t, x, u, v) \rangle \leq 0$. But we know, (cf [12]) that $(p_t, p_x) \in \partial_- \Theta(t, x)$ if and only if $(p_t, p_x, -1)$ belongs to the normal cone $(T_{Epi\Theta}(t, x, \Theta(t, x)))^-$. We claim that

$$(13) \quad \forall u \in U, \{1\} \times f(t, x, u, V) \times \{0\} \cap co(T_{Epi\Theta}(t, x, \Theta(t, x))) \neq \emptyset$$

Assume for a moment that is false, then, by the separation theorem we should have:

$$(14) \quad \begin{cases} \exists (p_t, p_x, q) \in (T_{Epi\Theta}(t, x, \Theta(t, x)))^-, \exists u \in U \text{ such that} \\ \forall v \in V, p_t + \langle p_x, f(t, x, u, v) \rangle > q > 0 \end{cases}$$

This is a contradiction with (12). So

$$\begin{cases} \forall (t, x, y) \in Epi(\Theta), \text{ for all } u \in U \\ \{1\} \times f(t, x, u, V) \times \{0\} \cap co(T_{Epi\Theta}(t, x, y)) \neq \emptyset \end{cases}$$

and we can deduce from⁶ Theorem 3.2.4 in [1], that $\{1\} \times f(t, x, u, V) \times \{0\} \cap T_{Epi\Theta}(t, x, \Theta(t, x)) \neq \emptyset$, for any $(t, x) \in Dom\Theta$. This implies the following contingent equation:

$$\forall (t, x) \in Dom(\Theta), \forall u \in U, \inf_{v \in V} D_1 \Theta(t, x)(1, f(t, x, u, v)) \leq 0$$

⁶Let us recall a duality result in viability theory (due to Ushakov see for instance Theorem 3.2.4 in [1]). Consider a closed set $K \subset \mathbf{R}^n$ and let F be u.s.c set-valued map with compact convex values. Then the following two statements are equivalent:

- i) $\forall x \in K, F(x) \cap T_K(x) \neq \emptyset$
- ii) $\forall x \in K, F(x) \cap co(T_K(x)) \neq \emptyset$

where co is the closed convex hull.

Let us prove the third statement. Assume that Θ satisfies

$$(15) \quad \forall (p_t, p_x) \in \partial_+ \Theta(t, x), \sup_{u \in U} \inf_{v \in V} p_t + < p_x, f(t, x, u, v) > \geq 0$$

We claim that

$$(16) \quad \exists u \in U, \{1\} \times f(t, x, u, V) \times \{0\} \subset \text{co}(T_{\text{Hypo}\Theta}(t, x, \Theta(t, x)))$$

If (16) is not satisfied, by the separation Theorem

$$(17) \quad \begin{cases} \forall u \in U, \exists v \in V \text{ such that} \\ \exists (p_t, p_x, q) \in (T_{\text{Hypo}\Theta}(t, x, \Theta(t, x)))^-, \\ p_t + < p_x, f(t, x, u, v) > < 0 \end{cases}$$

This is a contradiction with (15). Then, thanks to (4), and since (cf [2] p.130),

$$\liminf_{(t', x', y') \rightarrow (t, x, \Theta(t, x))} \text{co}(T_{\text{Hypo}\Theta}(t', x', y')) \subset T_{\text{Hypo}\Theta}(t, x, \Theta(t, x))$$

we can deduce that $\{1\} \times f(t, x, u, V) \times \{0\} \subset T_{\text{Hypo}\Theta}(t, x, \Theta(t, x))$ and consequently (7)ii) holds true. The proofs are similar for the other statements. \square

6 Nonanticipating strategies

We shall define value-functions for a concept of strategy studied by Elliot-Kalton (see also [9]). We denote by $\mathcal{U}(t)$ (respectively by $\mathcal{V}(t)$) the set of measurable functions $u : [t, T] \mapsto U$ (respectively $v : [t, T] \mapsto V$).

Firstly let us recall the definition of *nonanticipating strategies*.

Definition 6.1 *We call nonanticipating strategy for the first player any function $\alpha : \mathcal{V}(t) \mapsto \mathcal{U}(t)$ such that*

$$\begin{aligned} \forall t \in [0, T], \forall (v, \bar{v}) \in \mathcal{V}(t), \forall s \in [0, T], \\ v \equiv \bar{v} \text{ a. e. in } [t, s] \Rightarrow \alpha(v) \equiv \alpha(\bar{v}) \text{ a. e. in } [t, s] \end{aligned}$$

and we denote by $\Gamma(t)$ the set of such nonanticipating strategies.

We call nonanticipating strategy for the second player any function $\beta : \mathcal{U}(t) \mapsto \mathcal{V}(t)$ such that

$$\begin{aligned} \forall t \in [0, T], \forall (u, \bar{u}) \in \mathcal{U}(t), \forall s \in [0, T], \\ u \equiv \bar{u} \text{ a. e. in } [t, s] \Rightarrow \beta(u) \equiv \beta(\bar{u}) \text{ a. e. in } [t, s] \end{aligned}$$

and we denote by $\Delta(t)$ the set of such nonanticipating strategies.

This notion of strategies enables us to define the two value-functions:

Definition 6.2 Consider the upper value-function of the game:

$$\Phi(t_0, x_0) := \inf_{\beta \in \Delta(t_0)} \sup_{u(\cdot) \in \mathcal{U}(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta(u)))$$

and the lower value-function:

$$\Psi(t_0, x_0) := \sup_{\alpha \in \Gamma(t_0)} \inf_{v(\cdot) \in \mathcal{V}(t_0)} g(x(T, t_0, x_0, \alpha(v), v(\cdot)))$$

Proposition 6.3 Assume that (4) holds true. If g is continuous, then Ψ and Φ are continuous.

Proof — We shall prove that Ψ is continuous⁷ at some t_1, x_1 . Consider $\varepsilon > 0$, x_2 and $0 \leq t_1 \leq t_2 \leq T$. By the very definition of the value-function Ψ , there exists $\alpha \in \Gamma(t_1)$ such that

$$(18) \quad \Psi(t_1, x_1) \leq \inf_{v(\cdot) \in \mathcal{V}(t_1)} g(x(T, t_1, x_1, \alpha(v(\cdot)), v(\cdot))) + \varepsilon$$

Fix $\bar{v} \in V$. For any $v(\cdot) \in \mathcal{V}(t_2)$, we define $\underline{v}(s) =$

$$\begin{cases} \bar{v} & \text{if } s \in [t_1, t_2] \\ v(s) & \text{if } s \in [t_2, T] \end{cases}$$

and for any α we define $\underline{\alpha}(v) = \alpha(\underline{v})$.

Hence, there exists $v(\cdot) \in \mathcal{V}(t_2)$ such that $\Psi(t_2, x_2) \geq g(x(T, t_2, x_2, \underline{\alpha}(v), v)) - \varepsilon$ and according to (18), we have $\Psi(t_1, x_1) \leq g(x(T, t_1, x_1, \alpha(v), v)) + \varepsilon$. On the other hand, from Gronwall's Lemma, there exists some $R > 0$ such that

$$\|x(T, t_1, x_1, \underline{\alpha}(v)(\cdot), v(\cdot)) - x(T, t_2, x_2, \alpha(v)(\cdot), v(\cdot))\| \leq R(\|x_1 - x_2\| + (t_1 - t_2))$$

⁷It's easy to extend the proof when g is uniformly continuous and then the value-functions are uniformly continuous too.

Since g is continuous, there exists $\delta > 0$ such that for any $(t_2, x_2) \in R([0, 1] \times B)$ we have

$$|g(x(T, t_1, x_1, \underline{\alpha}(v)(\cdot), v(\cdot))) - g(x(T, t_2, x_2, \alpha(v)(\cdot), v(\cdot)))| \leq \varepsilon$$

Hence $\Psi(t_1, x_1) - \Psi(t_2, x_2) \leq 3\varepsilon$. On the other hand for every α : $\Psi(t_1, x_1) \geq \inf_{v(\cdot) \in \mathcal{V}(t_0)} g(x(T, t_1, x_1, \alpha(v), v(\cdot))) \geq \inf_{v(\cdot) \in \mathcal{V}(t_0)} g(x(T, t_2, x_2, \alpha(v), v(\cdot))) - \varepsilon$. Hence $\Psi(t_1, x_1) \geq \Psi(t_2, x_2) - \varepsilon$. We have similar result when $t_2 < t_1$ and for the value-function Φ . \square

7 Solutions to Isaacs equations with nonanticipating strategies

Proposition 7.1 *If (4) holds true, then Φ satisfies (6)i) and Ψ satisfies (7)ii).*

Proof — Fix $\bar{u} \in U$. Consider $\beta_h \in \Delta(t_0)$ such that

$$\sup_{u \in \mathcal{U}(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta_h(u))) \leq \Phi(t_0, x_0) + h^2$$

Let define $\mathcal{U}_h(t_0)$ the subset of measurable controls $u(\cdot) \in \mathcal{U}(t_0)$ such that $u(s) = \bar{u}$ for almost every $s \in [t_0, t_0 + h]$. then

$$(19) \quad \sup_{u \in \mathcal{U}_h(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta_h(u))) \leq \Phi(t_0, x_0) + h^2$$

By the very definition of β_h , there exists some $v(\cdot) \in \mathcal{V}(t_0)$ such that for any $u(\cdot) \in \mathcal{U}_h$, $v(s) = \beta_h(u)(s)$ for almost every $s \in [t_0, t_0 + h]$.

Let $x_h(\cdot)$ denote the solution to $x'(t) = f(t, x(t), \bar{u}, v(t))$ on $[t_0, t_0 + h]$ such that $x_h(t_0) = x_0$. From (19), we deduce

$$\sup_{u \in \mathcal{U}_h(t_0)} g(x(T, t_0 + h, x_h(t_0 + h), u(\cdot), \beta_h(u))) \leq \Phi(t_0, x_0) + h^2$$

Define $\beta \in \Delta(t_0)$ such that for any $u(\cdot) \in \mathcal{U}(t_0)$ we have $\beta(u) := \beta_h(\underline{u})$ with

$$\underline{u}(s) := \begin{cases} \bar{u} & \text{if } s \in [t_0, t_0 + h] \\ u(s) & \text{if } s > t_0 + h \end{cases}$$

Hence $\sup_{u \in \mathcal{U}(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta(u))) \leq \Phi(t_0, x_0) + h^2$ and therefore $\inf_{\beta \in \Delta(t_0)} \sup_{u \in \mathcal{U}(t_0)} g(x(T, t_0, x_0, u(\cdot), \beta(u))) \leq \Phi(t_0, x_0) + h^2$. This proves the following inequality

$$\Phi(t_0 + h, x_h(t_0 + h)) \leq \Phi(t_0, x_0) + h^2.$$

On the other hand, there exists a sequence $h_i \rightarrow 0$ and $\bar{v} \in V$ such that

$$\frac{x_{h_i}(t_0 + h_i) - x_0}{h_i} \rightarrow f(t_0, x_0, \bar{u}, \bar{v})$$

this yields $D_+ \Phi(t_0, x_0)(1, f(t_0, x_0, \bar{u}, \bar{v})) \leq 0$ and consequently (6)i). The proof is similar for the second statement. \square

Proposition 7.2 *If g is continuous, then Φ satisfies (6)ii) and Ψ satisfies (7)ii).*

It is possible to prove that Φ is a viscosity subsolution to (8) and thanks to results of section 5 that it is a contingent solution to (6)ii) (see [13] for the proof).

Corollary 7.3 *If g is continuous, then Φ is a viscosity solution to (8) and Ψ is a viscosity solution to (9).*

Finally we just state an existence result

Proposition 7.4 *Assume that (4) holds true and that g is uniformly continuous. If we assume the Isaacs' condition (10), then $\Phi = \Psi$ and the value-function is the unique uniformly continuous viscosity solution to the Isaacs' equation.*

The proof is based on a theorem of Crandall-Lions concerning the unicity of bounded uniformly continuous solution of Hamilton-Jacobi equations (see [16]).

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