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Contents

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V.M. Veliov

1 Introduction

The present paper is a continuation of the paper [28] and, in particular, contains the detailed proofs of the statements from the latter.

The main problem that is considered here is the following one. Let the differential inclusion

$$\dot{x} \in G(x,t) + v(t) \tag{1.1}$$

presents an uncertain control system. Namely, let $G : \mathbf{R}^n \times [t_0, T] \Rightarrow \mathbf{R}^n$ be a set-valued mapping interpreted as the set of all control forces that can be applied at the position (x,t), and let $v(t) \in \mathbf{R}^n$ presents an uncertain external force. The value of v at the moment t is known only to belong to a given set $V(t) \subset \mathbf{R}^r$:

$$v(t) \in V(t),\ t \in [t_0, T]. \tag{1.2}$$

Given an estimation

$$x(t_0) \in X_0 \subset \mathbf{R}^n \tag{1.3}$$

of the initial position $x(t_0)$, a target set $M_T \subset \mathbf{R}^n$ and a state (viability) constraint

$$x(t) \in Y(t) \subset \mathbf{R}^n,\ t \in [t_0, T], \tag{1.4}$$

the aim is to control the system (1.1) by selecting appropriate velocities from $G(x,t)$ in such a way that whatever are the initial position (1.3) and the measurable realization $v(\cdot)$ of the uncertainty (1.2) the corresponding trajectory $x(\cdot)$ satisfies the state constraint (1.4) ant reaches the target M_T at T:

$$x(T) \in M_T. \tag{1.5}$$

Exact knowledge of the current state $x(t)$ (and possibly of the current value of the uncertainty $v(t)$) is supposed when choosing the velocity from $G(x,t)$ at the current moment t.

The rigorous formulation of the problem will be given in Section 7.

Problems of the above type have been investigated by many authors starting from the basic works of Krasovskii (see [14]) in the framework of the differential games. A crucial role in these investigations has been played by the notions of weak invariance (viability property) and invariance of set-valued mappings with respect to a differential inclusion [2, 13, 15]

$$\dot{x} \in F(x,t),\ t \in [t_0, T]. \tag{1.6}$$

The above mentioned properties ware characterized under different suppositions in the terms of appropriate infinitesimal objects like tangent cones or contingent derivatives (see e.g. [2, 10, 11, 12, 18, 25]). An alternative approach, providing a suitable basis for constructive theory involves certain generalized differential equations called funnel equations [16, 17, 22, 23, 27]. The latter approach is exploited in the present paper for characterization of the weak invariance (viability) property with respect to (1.6), if F is measurable in t and Lipschitz continuous in x. The approach is extended to the state constrained case and to the case of weak invariance with respect to certain families of differential inclusions, arising whenever F depends on uncertain parameters as in (1.1).

To be precise, let us consider a family of inclusions

$$0 \in g_\alpha(\dot{x}, x, t);\ \alpha \in \mathcal{A}, \tag{1.7}$$

where $\mathcal{A}$ is an abstract set of parameters, $x \in \mathbf{R}^n$, $t \in [t_0, T]$.

Definition 1. The set-valued mapping $W(\cdot) : [t_0, T] \Rightarrow \mathbf{R}^n$ is weakly invariant with respect to the inclusion $0 \in g_\alpha(\dot{x}, x, t)$ if for every $\tau \in [t_0, T]$ and $x \in W(\tau)$ there exists an absolutely continuous function $x(\cdot) : [\tau, T] \mapsto \mathbf{R}^n$ such that $x(\tau) = x$, (1.7) is satisfied for a.e. $t \in [\tau, T]$ and $x(t) \in W(t)$ for every $t \in [\tau, T]$. $W(\cdot)$ is weakly invariant w.r. to the family (1.7) if it is weakly invariant w.r. to everyone of the inclusions in this family.

In particular, if (1.7) consists of inclusion (1.6) and the pointwise inclusion (1.4), where $\operatorname{graph} Y$ is closed, the weak invariance of $W(\cdot)$ means that $W(\cdot)$ is weakly invariant w.r. to (1.6) and $W(t) \subset Y(t)$, $t \in [t_0, T]$. We shall consider also families of differential inclusions

$$\dot{x} \in G(x,t) + v(t)\,, \quad v(\cdot) \in \mathcal{V}, \tag{1.8}$$

together with (1.4), where an uncertain function $v(\cdot)$ from a given set of functions $\mathcal{V}$ $(= \mathcal{A})$ stays for the parameter α. In particular, we are interested in the family corresponding to the uncertain system (1.1),(1.2), where

$$\mathcal{V} = \{v(\cdot)\,;\; v(\cdot) - \text{measurable}, v(t) \in V(t) \text{ for a.e. } t \in [t_0, T]\}. \tag{1.9}$$

Definition 2. The set-valued mapping $W(\cdot) : [t_0, T] \Rightarrow \mathbf{R}^n$ has the stable invariance property w.r. to the differential inclusion

$$\dot{x} \in \Phi(x,t) \tag{1.10}$$

if there exist a constant $\delta > 0$ and an integrable function $\mu(\cdot) : [t_0, T] \mapsto [0, +\infty]$ such that for each $\tau \in [t_0, T]$ and $x \in W(\tau) + \delta\mathcal{B}$ ($\mathcal{B}$ is the unit ball in $\mathbf{R}^n$) every solution $x(\cdot)$ of (1.10) starting from x at the moment τ exists up to the moment T and satisfies

$$\operatorname{dist}(x(t), W(t)) \leq \exp\Big(\int_\tau^t \mu(s)\, ds\Big) \operatorname{dist}(x, W(\tau)). \tag{1.11}$$

If the above requirements are satisfied for $\delta = 0$ only, the tube $W(\cdot)$ will be (merely) called invariant.

In Section 5 we prove that for a mapping $W(\cdot)$ that is weakly invariant w.r. to (1.6) there exists a sub-mapping $\Phi(x,t) \subset F(x,t)$ such that $W(\cdot)$ has the stable invariance property w.r. to the "closed-loop" inclusion (1.10). Since the regulation mappings Φ that come in use may be discontinuous (u.s.c.) and non-convex valued, Definition 2 requires a theorem for existence of a solution to the closed-loop inclusion (1.10). Such is proven in Section 4. All weakly invariant tubes $W(\cdot)$ are characterized in Section 3 as solutions to certain funnel equations. The funnel inequalities introduced there as approximate versions of the funnel equations and the lemma of the type of Filippov given in Section 2 serve as approximation tools. The regulation mappings corresponding to a solution tube of such a funnel inequality ensure "approximate" stable invariance of the tube (Section 6). In Section 7 we give a necessary and sufficient condition for solvability of the problem of guaranteed control formulated above, in terms of the solution of a boundary value problem for the corresponding funnel equation. An "approximate" regulation map is defined there, that uses an approximation of a solution to this boundary value problem.

In Section 8 we show that the closed-loop inclusion (1.10), despite of the discontinuity and the non-convexity of Φ, can be treated by means of finite difference approximations, at least in the case of time-independent weakly invariant sets (viability domains).

For the sake of simplicity and coherence we suppose that the following conditions concerning (1.4) and (1.6) are satisfied throughout the paper, despite that some of them can be relaxed.

A1. $F : \mathbf{R}^n \times [t_0, T] \Rightarrow \mathbf{R}^n$ is convex and compact valued, $t \Rightarrow F(\cdot, x)$ is measurable for every $x \in \mathbf{R}^n$. For every compact set $S \subset \mathbf{R}^n$ there are a constant $m = m(S)$ and an integrable upper semicontinuous function $\lambda(\cdot) = \lambda(\cdot\,; S) : [t_0, T] \mapsto \mathbf{R} \cup \{+\infty\}$ such that

$$
\begin{aligned}
|F(x,t)| &\leq m \qquad\qquad \text{(local boundedness)},\\
F(t,x') &\subset F(x,t) + \lambda(t)|x - x'| \ \text{(local Lipschitz condition)}
\end{aligned}
$$

for every $t \in [t_0, T]$ and $x, x' \in S$.

A2. For every compact set $S_0 \subset \mathbf{R}^n$ there is a compact $S = S(S_0)$ such that if $x(\cdot)$ is a solution of (1.6) on an interval $[t_1, t_2] \subset [t_0, T]$ and $x(\tau) \in S_0$ for some $\tau \in [t_1, t_2]$, then $x(t) \in \operatorname{int} S$ for every $t \in [t_1, t_2]$.

A3. grapf $Y = \{(t, x)\,;\ x \in Y(t)\}$ is closed.

We shall consider compact valued tubes $W(\cdot)$ only. Under the above conditions every compact valued weakly invariant tube $W(\cdot)$ with closed graph satisfies

1. $W(\cdot)$ is compact valued and upper semicontinuous;
2. there is a constant M such that $W(s) \subset W(t) + M(t - s)\mathcal{B}$ for every $s, t \in [t_0, T], s \leq t$.

Further we denote by $\mathcal{W}[t_0, T]$ the class of all mappings $W(\cdot) : [t_0, T] \Rightarrow \mathbf{R}^n$ satisfying 1) and 2) (each $W(\cdot)$ with its own constant M).

Whenever we consider the perturbed differential inclusion (1.8) we suppose that

A4. $\mathcal{V}$ is given by (1.9), where $V(\cdot) \in \mathcal{W}[t_0, T]$ is convex valued; the mapping $F(x, t) = G(x, t) + V(t)$ satisfies conditions **A1** and **A2**.

2 Lemma of the type of Filippov

In this section we present an extension of a result due to Filippov [9], which estimates the uniform distance between an absolutely continuous function $y(\cdot)$ and the set of trajectories of the differential inclusion (1.6), by means of constant times the "discrepancy"

$$\int_{t_0}^{T} \mathrm{dist}\,(\dot{y}(t), F(y(t),t))\,dt\,.$$

Instead of a single-valued function $y(\cdot)$ we take a mapping $X(\cdot) \in \mathcal{W}[t_0,T]$ and use as a local measure of the discrepancy of $X(\cdot)$ the quantity

$$\rho_{X(\cdot)}(t) = \overline{\lim_{h\to 0+}}\, \frac{1}{h} H^{+}\Big(X(t-h),\ \bigcup_{x\in X(t)} (x - hF(x,t))\Big)\,, \tag{2.1}$$

where

$$H^{+}(P,Q) = \inf\{\alpha;\ P \subset Q + \alpha\mathcal{B}\}$$

is the Hausdorff semi-distance from P to Q.

Lemma 2.1 *Let conditions* **A1** *and* **A2** *be satisfied. Then for every $X(\cdot) \in \mathcal{W}[t_0,T]$ the function $\rho_{X(\cdot)}(\cdot)$ is integrable on $[t_0,T]$. Moreover h can be replaced in (2.1) with $1/k$, $k \to +\infty$.*

Thanks to the above lemma we can define

$$\rho_{X(\cdot)} = \int_{t_0}^{T} \rho_{X(\cdot)}(t)\,dt\,.$$

In the case of a single-valued $X(t) = \{y(t)\}$ with absolutely continuous $y(\cdot)$ the value $\rho_{X(\cdot)}$ coincides with the "discrepancy" in the Filippov's lemma.

Theorem 2.1 *(Extension of the Filippov's lemma.) Let conditions* **A1** *and* **A2** *be fulfilled and let $X(\cdot) \in \mathcal{W}[t_0,T]$. Suppose that for some compact set $S \subset \mathbf{R}^n$*

$$X(t) + \exp\Big(\int_{t_0}^{T} \lambda(s)\,ds\Big)\rho_{X(\cdot)}\mathcal{B} \subset \mathrm{int}\,S\,,\ t \in [t_0,T]\,,$$

where $\lambda(\cdot) = \lambda(\cdot; S)$ is a Lipschitz constant corresponding to S according to **A1**. *Then there exists a weakly invariant tube $W(\cdot) \in \mathcal{W}[t_0,T]$ w.r. to (1.6) such that*

$$X(t) \subset W(t) \subset X(t) + \exp\Big(\int_{t_0}^{T} \lambda(s)\,ds\Big)\rho_{X(\cdot)}\mathcal{B} \tag{2.2}$$

for every $t \in [t_0,T]$.

The essential part of the proof consists of the following proposition that we use in the sequel.

Proposition 2.1 *If the conditions of Theorem 1 are fulfilled, then for every* $\tau \in [t_0, T)$ *and* $x \in X(\tau)$ *there is a Lipschitz continuous selection* $x(\cdot)$ *of* $X(\cdot)$ *on* $[\tau, T]$ *such that* $x(\tau) = x$ *and*

$$\mathrm{dist}(\dot{x}(t), F(x(t), t)) \leq \rho_{X(\cdot)}(t) \tag{2.3}$$

for a.e. $t \in [\tau, T]$.

Remark. The proof of the proposition shows also that the Lipschitz constant of $x(\cdot)$ can be estimated by a constant that is independent of τ and x.

Proofs.

Proof of Lemma 2.1. Let M be the constant corresponding to $X(\cdot)$ according to the second requirement from the definition of the set $\mathcal{W}[t_0, T]$ (see page 4). Let S be a compact set containing $X(t)$ for $t \in [t_0, T]$ and let m the constant corresponding to S according to supposition **A1**. Then

$$\rho_{X(\cdot)} \leq \overline{\lim_{h \to 0+}} \frac{1}{h} H^+(X(t-h), X(t)) + \sup_{x \in X(t)} |F(x,t)| \leq M + m \tag{2.4}$$

and $\rho_{X(\cdot)}(\cdot)$ is bounded.

Now let us prove the last statement of the lemma. Denote for brevity

$$\mathcal{F}(X; s, t) = \bigcup_{x \in X} (x - (t-s)F(x,t)) \tag{2.5}$$

and also

$$\varphi(t,h) = H^+(X(t-h), \mathcal{F}(X(t); t-h, t)).$$

For $0 < h' < h''$ one can estimate

$$\begin{aligned}
\varphi(t,h'') &\leq H^+(X(t-h') + M(h''-h')\mathcal{B}, \mathcal{F}(X(t); t-h'', t)) \\
&\leq H^+(X(t-h'), \mathcal{F}(X(t); t-h', t)) + M(h''-h') + m(h''-h') \\
&= \varphi(t,h') + (M+m)(h''-h').
\end{aligned}$$

Hence

$$\frac{\varphi(t,h'')}{h''} \leq \frac{\varphi(t,h')}{h''} + \frac{(M+m)(h''-h')}{h''} \leq \frac{\varphi(t,h')}{h'} + \frac{(M+m)(h''-h')}{h''}.$$

Suppose that $h_i \to 0$ is a monotone decreasing sequence such that

$$\rho_{X(\cdot)}(t) = \lim_{i\to+\infty} \frac{1}{h_i}\varphi(t,h_i).$$

Let k_i be such an integer that $h_i \in [\frac{1}{k_i+1}, \frac{1}{k_i})$. Then using the last inequality one can estimate

$$\frac{\varphi(t,h_i)}{h_i} \leq (k_i+1)\varphi(t,(k_i+1)^{-1}) + \frac{M+m}{k_i}.$$

Hence

$$\rho_{X(\cdot)}(t) = \overline{\lim_{k\to+\infty}}\, k\varphi(t,\frac{1}{k}),$$

which proves the last statement of the lemma.

As a consequence, the measurability of $\rho_{X(\cdot)}(\cdot)$ would follow from measurability of the function $\varphi(t) = \varphi(t,1/k)$ for every fixed k. To prove the latter, fix a finite or countable family of measurable functions $\{g_i(\cdot)\}_i$ such that $X(t) = \mathrm{cl}\,\{g_i(t)\}_i$ (Castaing representation of $X(\cdot)$, which is measurable thanks to the upper semicontinuity). Then

$$\mathcal{F}(X(t);t-1/k,t) = \mathrm{cl}\bigcup_i (g_i(t) - \frac{1}{k}F(g_i(t),t))$$

and the measurability of the mapping $t \Rightarrow \mathcal{F}(X(t);t-1/k,t)$ follows from [3, Theorem 4.2]. It remains to mention that measurability of two mappings $P(\cdot)$ and $Q(\cdot)$ implies measurability of $H^+(P(\cdot),Q(\cdot))$, as it follows from Corollary 8.2.13 and Lemma 8.2.12 [3]. Q.E.D.

Proof of Theorem 2.1. Proposition 2.1 makes the proof straightforward. Actually, let $x(\cdot)$ be as in Proposition 2.1. Then (2.3) and the Filippov's lemma (in the form presented in [6]) imply the existence of a trajectory $y(\cdot) = y[\tau,x](\cdot)$ of (1.6) on $[\tau,T]$ such that

$$|y(t) - x(t)| \leq \exp\left(\int_{t_0}^{T} \lambda(s)\,ds\right)\rho_{X(\cdot)}. \tag{2.6}$$

Now define

$$W(t) = \bigcup\{y[\tau,x](t)\,;\ \tau \in [t_0,T),\ x \in X(\tau)\}.$$

By its definition $W(\cdot)$ is weakly invariant with respect to (1.6), and (2.2) apparently follows from (2.6), which proves the theorem. Q.E.D.

Proof of Proposition 2.1. Let S be as in the formulation of Theorem 2.1 and let m and $\lambda(\cdot)$ be the constants from supposition **A1**. Take arbitrarily $\tau \in [t_0, T)$ and $x \in X(\tau)$. According to [29, Proposition 4.1] there is a set $\Lambda_1 \subset [\tau, T]$ of measure zero such that every $t \in (\tau, T] \setminus \Lambda_1$ is a Lebesque point to $F(x, \cdot)$ for every $x \in S$. That is, (because of the uniform boundedness of F)

$$\lim_{h \to 0+} H(\frac{1}{h} \int_{t-h}^{t} F(x,s)\, ds, F(x,t)) = 0 \tag{2.7}$$

for every $t \in [\tau, T] \setminus \Lambda_1$ and $x \in S$. Here and further

$$H(P,Q) = \min\{H^+(P,Q), H^+(Q,P)\}$$

denotes the Hausdorff distance between the compact sets P and Q in $\mathbf{R}^n$. Condition **A1** implies that if for some t (2.7) holds for every $x \in S$ and if $\lambda(t)$ is finite and t is a Lebesque point for $\lambda(\cdot)$, then the convergence in (2.7) is uniform in $x \in S$. Hence, one can replace the second argument of H^+ in (2.1) with $\mathcal{F}(t-h,t)$, where

$$\mathcal{F}(s,t) = \bigcup_{x \in X(t)} (x + \int_t^s F(x,\theta)\, d\theta), \tag{2.8}$$

possibly changing $\rho_{X(\cdot)}(\cdot)$ on a set of measure zero. For brevity further we omit the subscript in $\rho_{X(\cdot)}(\cdot)$ and $\rho_{X(\cdot)}$.

Take an arbitrarily small $\varepsilon > 0$. Let Λ_2 be a set of measure zero which contains Λ_1 and such that every $t \in (t_0, T] \setminus \Lambda_2$ is a Lebesque point for $\lambda(\cdot)$ and $\rho(\cdot)$ and $\lambda(t)$ is finite. Then for every $t \in (\tau, T] \setminus \Lambda_2$ there is $h_\varepsilon(t) \in (0, \varepsilon)$ such that

$$|\rho(t) - \frac{1}{h} \int_{t-h}^{t} \rho(s)\, ds| \le \frac{\varepsilon}{2}$$

and

$$\frac{1}{h} H^+(X(t-h), \mathcal{F}(t-h,t)) \le \rho(t) + \frac{\varepsilon}{2}$$

for every $h \in (0, h_\varepsilon(t))$ ($\mathcal{F}$ is defined by (2.8)). Combining the above two inequalities we obtain that

$$X(t-h) \subset \mathcal{F}(t-h,t) + \int_{t-h}^{t} \rho(s)\, ds\, \mathcal{B} + h\varepsilon\mathcal{B} \tag{2.9}$$

for every $t \in [\tau, T] \setminus \Lambda_2$ and $h \in (0, h_\varepsilon(t))$.

Let $\{\Delta_i\}$ be a finite or countable collection of open (in $[\tau, T]$) intervals, such that

$$\Lambda_2 \cup \{\tau, T\} \subset \bigcup_i \Delta_i\,, \quad \sum_i \operatorname{meas} \Delta_i < \varepsilon.$$

For $t \in \Gamma = [\tau, T] \setminus \bigcup_i \Delta_i$ define

$$a_\varepsilon(t) = \max\{\tau, t - h_\varepsilon(t)\}\,, \quad b_\varepsilon(t) = \min\{T, t + \varepsilon(t - a_\varepsilon(t))\}.$$

The open (in $[\tau, T]$) intervals $\{\Delta_i\}$, $\{(a_\varepsilon(t), b_\varepsilon(t))\}_{t\in\Gamma}$ form an open covering of $[\tau, T]$. Let $\{a_i, b_i)\}_{i=1}^p$ be a finite subcovering. One can assume that it is ordered and minimal, i.e. that

$$a_{i+1} > a_i,\ b_{i+1} > b_i,\ b_{j-1} < a_{j+1},\ i = 1, \dots, p,\ j = 2, \dots, p-1.$$

Now we shall define a function $x_\varepsilon(\cdot) : [\tau, T] \to \mathbf{R}^n$, $x_\varepsilon(\tau) = x$, in the following way.

By definition $a_1 = \tau$ and $[a_1, b_1]$ is some of the intervals from $\{\Delta_i\}$. Define $x_\varepsilon(b_1)$ as an arbitrary element from the set $\mathcal{P}_{X(b_1)}x$, where we use the notation

$$\mathcal{P}_Y x = \{y \in Y;\ |x - y| = \operatorname{dist}(x, Y)\} \tag{2.10}$$

for the projection of x on the closed set Y. Define $x_\varepsilon(\cdot)$ as linear on $[a_1, b_1]$. Thus $x_\varepsilon(\cdot)$ is Lipschitz continuous on $[\tau, b_1]$ with a Lipschitz constant M.

Suppose that $x_\varepsilon(\cdot)$ is already defined on $[\tau, b_{i-1}]$ as a Lipschitz continuous function with a Lipschitz constant $M + 2m + \varepsilon$, and that $x_\varepsilon(b_j) \in X(b_j)$, $j = 1, \dots, i-1$. We shall extend it on $[\tau, b_i]$ preserving the above properties. Three different cases will be considered.

1) Let (a_i, b_i) be one of the intervals from $\{\Delta_i\}$. Then define $x_\varepsilon(b_i)$ as an arbitrary point from $\mathcal{P}_X(b_i)x_\varepsilon(b_{i-1})$ end extend it as linear on $[b_{i-1}, b_i]$. Clearly, the Lipschitz constant is M on $[b_{i-1}, b_i]$.

2) Let (a_i, b_i) be an interval $(a_\varepsilon(t_i), b_\varepsilon(t_i))$ for some $t_i \in \Gamma$ and let $t_i \le b_{i-1}$. Then we extend $x_\varepsilon(\cdot)$ to $[\tau, b_i]$ exactly as in the case 1)—as linear with a Lipschitz constant M on $[b_{i-1}, b_i]$.

3) Finally, let $(a_i, b_i) = (a_\varepsilon(t_i), b_\varepsilon(t_i))$ and $t_i > b_{i-1}$. Then, according to (2.9)

$$X(b_{i-1}) \subset \mathcal{F}(b_{i-1}, t_i) + \int_{b_{i-1}}^{t_i} \rho(s)\, ds\, \mathcal{B} + \varepsilon(t_i - a_i)\mathcal{B}.$$

Hence $x_\varepsilon(b_{i-1})$ can be presented as

$$x_\varepsilon(b_{i-1}) = x_i - y_i + \int_{b_{i-1}}^{t_i} \rho(s)\, ds\, \psi_i + \varepsilon(t_i - a_i)\eta_i, \tag{2.11}$$

where

$$x_i \in X(t_i) \tag{2.12}$$

$$y_i \in \int_{b_{i-1}}^{t_i} F(x_i, s)\, ds \tag{2.13}$$

$$|\psi_i| \le 1, |\eta_i| \le 1. \tag{2.14}$$

Set $x_\varepsilon(t_i) = x_i$ and extend $x_\varepsilon(\cdot)$ linearly on $[b_{i-1}, t_i]$. Then (2.11) together with (2.4) implies that $x_\varepsilon(\cdot)$ is Lipschitz continuous with a Lipschitz constant $M + 2m + \varepsilon$ on $b_{i-1}, t_i]$. Finally, extend $x_\varepsilon(\cdot)$ on $[t_i, b_i]$ exactly as in the case 1) as linear with a Lipschitz constant M and such that $x_\varepsilon(b_i) \in X(b_i)$.

Continuing in the same way we define $x_\varepsilon(\cdot)$ on the whole interval $[\tau, T]$. $x_\varepsilon(\cdot)$ is piece-wise linear with a Lipschitz constant $M + 2m + \varepsilon$ and satisfies $x_\varepsilon(b_i) \in X(b_i)$, $i = 1, \dots, p$. Hence $x_\varepsilon(t) \in S$ for $t \in [\tau, T]$ and for all sufficiently small ε. Let $x(\cdot)$ be a condensation point of $\{x_k(\cdot)\}$ in the uniform metric, $x(\cdot)$ being thus Lipschitz continuous with a constant $L = M + 2m$. From the upper semicontinuity of $X(\cdot)$ it follows that $x(\cdot)$ is a selection of $X(\cdot)$. Thus, in order to complete the proof of the proposition it remains to verify (2.3).

Let $\{x_k(\cdot) = x_{\varepsilon_k}\}_k$ be a subsequence that uniformly converges to $x(\cdot)$. According to the definition of $x_k(\cdot)$ there are points

$$\tau = s_0'' \le s_1' \le s_1'' \le \dots \le s_m'' \le s_{m+1}' = T$$

(depending on k) such that

1. Every interval $[s_i', s_i'']$ is of length at most ε_k and

   $$|x_k(s_i'') - x_k(s_i') - y_i| \le \int_{s_i'}^{s_i''} \rho(s)\, ds + \varepsilon(s_i'' - s_i'),$$

   where

   $$y_i \in \int_{s_i'}^{s_i''} F(x_k(s_i''), s)\, ds$$

   (see (2.11),(2.12),(2.13),(2.14)).

2. The total length of the intervals $[s_i'', s_{i+1}']$, $i = 1, \dots, m$ is at most

   $$\varepsilon + \sum_{j \in J} (b_j - t_j) \le \varepsilon + \sum_{j \in J} \varepsilon (t_j - a_j) \le \varepsilon + 2\varepsilon(T - \tau) = c\varepsilon,$$

   since the covering $\{a_i, b_i\}$ of $[\tau, T]$ is minimal. Here J is the set of those indexes j for which (a_j, b_j) from the definition of $x_{\varepsilon_k}(\cdot)$ is of the type $(a_{\varepsilon_k}(t_j), b_{\varepsilon_k}(t_j))$.

Let $t \in (\tau, T)$ be a Lebesque point of $\rho(\cdot)$ at which $\dot{x}(t)$ exists. Then for an arbitrarily fixed $\delta > 0$ one can find $h > 0$ so small that

$$\left| \dot{x}(t) - \frac{x(t+h) - x(t)}{h} \right| < \delta.$$

Fix k so large that

$$\varepsilon_k < \delta h, \quad \|x_k(\cdot) - x(\cdot)\|_C < \delta h. \tag{2.15}$$

Let $p,\ q \in \{1, \ldots, m\}$ be such that

$$s'_p > t, \quad s''_q < t + h, \quad s'_p - t \le (c+1)\delta h, \quad t + h - s''_q \le (c+1)\delta h.$$

Then

$$\left| \dot{x}(t) - \frac{x_k(s''_q) - x_k(s'_p)}{h} \right| < (2(c+1)L + 3)\delta. \tag{2.16}$$

For every $i \in \{p, \ldots, q\}$ we have

$$|x_k(s''_i) - x(t)| \le |x_k(s''_i) - x(s''_i)| + |x(s''_i) - x(t)| \le \delta h + Lh = c'h.$$

Hence

$$H\Big(\int_{s'_i}^{s''_i} F(x_k(s''_i), s)\, ds, \int_{s'_i}^{s''_i} F(x(t), s)\, ds \Big) \le c'h \int_{s'_i}^{s''_i} \lambda(s)\, ds.$$

Using this relation and the property 1) we conclude that there is

$$\tilde{y}_i \in \int_{s'_i}^{s''_i} F(x(t), s)\, ds$$

such that

$$|x_k(s''_i) - x_k(s'_i) - \tilde{y}_i| \le \int_{s'_i}^{s''_i} \rho(s)\, ds + \varepsilon_k(s''_i - s'_i) + c'h \int_{s'_i}^{s''_i} \lambda(s)\, ds. \tag{2.17}$$

Moreover,

$$|x_k(s'_i) - x_k(s''_{i-1})| \le (L + \varepsilon_k)(s'_i - s''_{i-1}),$$

which combined with (2.17) gives

$$|x_k(s''_q) - x_k(s'_p) - \sum_{i=p}^{q} \tilde{y}_i| \le \sum_{i=p}^{q} \Bigg[\int_{s'_i}^{s''_i} \rho(s)\, ds + \varepsilon_k(s''_i - s'_i) \tag{2.18}$$

$$+ c'h \int_{s'_i}^{s''_i} \lambda(s)\, ds + (L + \varepsilon_k) \sum_{i=p+1}^{q} (s'_i - s''_{i-1}) \Bigg].$$

Using the property 2) and (2.15) we estimate the right-hand side by

$$\int_t^{t+h} \rho(s)\, ds + ch \int_t^{t+h} \lambda(s)\, ds + \delta h^2 + 2Lc\delta h. \tag{2.19}$$

Moreover, we have

$$\sum_{i=p}^{q} \tilde{y}_i \in \sum_{i=p}^{q} \int_{s'_i}^{s''_i} F(x(t), s)\, ds \subset \int_t^{t+h} F(x(t), s)\, ds + (3c+2)m\delta h \mathcal{B}. \tag{2.20}$$

Combining (2.16)–(2.20) we estimate

$$\operatorname{dist}(\dot{x}(t), \frac{1}{h}\int_t^{t+h} F(x(t), s)\, ds) \le |\dot{x}(t) - \frac{1}{h}\sum_{i=p}^{q} \tilde{y}_i| + (3c+2)m\delta$$

$$\le \left| \frac{x_k(s_q'') - x_k(s_p')}{h} - \frac{1}{h}\sum_{i=p}^{q} \tilde{y}_i \right| + (2(c+1)L + 3 + (3c+2)m)\delta$$

$$\le \frac{1}{h}\int_t^{t+h} \rho(s)\, ds + c\int_t^{t+h} \lambda(s)\, ds + \bar{c}\delta,$$

where $\bar{c}$ is a constant. Here we can take the limit in h tending to zero, supposing that t is a Lebesque point of $\rho(\cdot)$ and $F(x, \cdot)$ for every $x \in S$ (we use again [29, Proposition 4.1]). This gives

$$\operatorname{dist}(\dot{x}(t), F(x(t), t)) \le \rho(t) + \bar{c}\delta,$$

which implies (2.3) since $\delta > 0$ is arbitrary. Q.E.D.

3 Characterization of the viability property

In this section we characterize the weakly invariance property with respect to a differential inclusion or with respect to certain families of inclusions (containing point-wise inclusions and/or uncertain differential inclusions) by means of *funnel equations.*

First consider differential inclusion

$$\dot{x} \in F(x, t), \quad t \in [t_0, T] \tag{3.1}$$

supposing **A1** and **A2** . If $X(\cdot) \in \mathcal{W}[t_0, T]$ satisfies the "funnel" equation

$$\lim_{h \to 0^+} \frac{1}{h} H^+\Big(X(t-h), \bigcup_{x \in X(t)} (x - hF(x, t))\Big) = 0 \text{ for a.e.} t \in (t_0, T]\,, \tag{3.2}$$

then $\rho_{X(\cdot)} = 0$ in (2.1) and Theorem 2.1 implies that $X(\cdot)$ is a weakly invariant tube for (3.1). The inverse is also true.

Theorem 3.1 $X(\cdot) \in \mathcal{W}[t_0, T]$ *is weakly invariant w.r. to (3.1) if and only if (3.2) is satisfied.*

The approximate version of (3.2) has the form of a "funnel" inequality:

$$\overline{\lim_{h\to 0+}} \frac{1}{h} H^+\Big(X(t-h), \bigcup_{x\in X(t)} (x - hF(x,t))\Big) \le \rho(t) \tag{3.3}$$

for a.e. $t \in (t_0, T]$, where $\rho = \int_{t_0}^{T} \rho(t)\,dt$ is presumably "small". The role of (3.3), and the other funnel inequalities below, for "approximate" regulation will become clear in Section 7.

Now let us consider the state constrained case: (3.1) together with

$$x(t) \in Y(t), \quad t \in [t_0, T], \tag{3.4}$$

supposing that **A1** - **A3** are satisfied. In this case the funnel inequality analogous to (3.3) has the form

$$\overline{\lim_{h\to 0+}} \frac{1}{h} H^+\Big(X(t-h), \bigcup_{x\in X(t)\cap Y(t)} (x - hF(x,t))\Big) \le \rho(t) \tag{3.5}$$

for a.e. $t \in (t_0, T]$. If $X(\cdot) \in \mathcal{W}[t_0, T]$ is a solution to (3.5), then $X(t) \subset Y(t), t \in [t_0, T]$ ($X(\cdot)$, being u.s.c., is continuous in a dense subset and graph Y is closed). If S is a compact set such that $X(t) \subset S$, for $t \in [t_0, T]$, and λ is the Lipschitz constant corresponding to $S + \mathcal{B}$ according to **A1**, then Theorem 2.1 implies the existence of an weakly invariant tube $W(\cdot) \in \mathcal{W}[t_0, T]$ w.r. to (3.1) such that (2.2) is fulfilled with $\rho_{X(\cdot)} = \int_{t_0}^{T} \rho(t)\,dt$, provided that $\rho_{X(\cdot)}$ is sufficiently small.

From here we conclude that if $X(\cdot) \in \mathcal{W}[t_0, T]$ satisfies the equation

$$\lim_{h\to 0+} \frac{1}{h} H^+\Big(X(t-h), \bigcup_{x\in X(t)\cap Y(t)} (x - hF(x,t))\Big) = 0 \tag{3.6}$$

for a.e. $t \in (t_0, T]$, then $X(\cdot)$ is weakly invariant w.r. to (3.1),(3.4). The inverse is also true.

Theorem 3.2 *$X(\cdot) \in \mathcal{W}[t_0, T]$ is weakly invariant w.r. to (3.1), (3.4) if and only if (3.6) is satisfied.*

Finally, consider the family consisting of (3.4) and the inclusions

$$\dot{x} \in G(x,t) + v(x,t)\,, \quad v(\cdot) \in \mathcal{V} \tag{3.7}$$

supposing that **A3** and **A4** hold. The corresponding funnel inequality has the form

$$\overline{\lim_{h\to 0+}} \frac{1}{h} H^+\Big(X(t-h) + hV(t-h), \bigcup_{x\in X(t)\cap Y(t)} (x - hG(x,t))\Big) \le \rho(t) \tag{3.8}$$

for a.e. $t \in (t_0, T]$. Let us fix an arbitrary $v(\cdot) \in \mathcal{V}$. Then the right-hand side of the corresponding inclusion in (3.7) satisfies (3.5) and applying Theorem 2.1 we obtain that $X(t) \subset Y(t)$ and if $\rho_{X(\cdot)} = \int_{t_0}^{T} \rho(t)\, dt$ is sufficiently small, then for every $v(\cdot) \in \mathcal{V}$ there is an weakly invariant tube $W(\cdot) = W_{v(\cdot)}(\cdot) \in \mathcal{W}[t_0, T]$ w.r. to the corresponding inclusion in (3.7) such that (2.2) is satisfied. As a consequence, if $X(\cdot)$ satisfies the funnel equation

$$\lim_{h \to 0+} \frac{1}{h} H^+\Big(X(t-h) + hV(t-h), \bigcup_{x \in X(t) \cap Y(t)} (x - hG(x,t))\Big) = 0 \qquad (3.9)$$

for a.e. $t \in (t_0, T]$, then $X(\cdot)$ is weakly invariant w.r. to the family (3.4), (3.7). Moreover, the inverse holds:

Theorem 3.3 *$X(\cdot) \in \mathcal{W}[t_0, T]$ is weakly invariant w.r. to the family (3.4),(3.7) if and only if (3.9) is satisfied.*

Further we shall use also the following proposition.

Proposition 3.1 *If $X(\cdot) \in \mathcal{W}[t_0, T]$ satisfies (3.3) with an integrable $\rho(\cdot)$ such that the conditions of Theorem 2.1 are fulfilled, then it satisfies also the inequality*

$$\overline{\lim_{h \to 0+}} \frac{1}{h} H^+\Big(X(t), \bigcup_{x \in X(t+h)} (x - hF(x,t))\Big) \le \rho(t) \qquad (3.10)$$

for a.e. $t \in [t_0, T]$.

Equations of the type of (3.2) and (3.6) were introduced in [17] for linear differential inclusions.

Proofs

Proof of Theorem 3.1. We have to prove only the necessity of (3.2). Let $X(\cdot) \in \mathcal{W}[t_0, T]$ be weakly invariant w.r. to (3.1). Let S be a compact set containing $X(t)$, $t \in [t_0, T]$. Take a point $\tau \in (t_0, T]$ which is a Lebesque point of $F(x, \cdot)$ for every $x \in S$ and at which the function $\lambda(\cdot)$ corresponding to S according to **A1** is finite. Then

$$\lim_{h \to 0+} \sup_{x \in S} H\left(\frac{1}{h} \int_{\tau-h}^{\tau} F(x,s)\, ds, F(x,\tau)\right) = 0. \qquad (3.11)$$

According to [29, Proposition 4.1] almost every τ is such. For an arbitrary $\bar{x} \in X(\tau - h)$ there is a trajectory $x(\cdot)$ of (3.1) on $[\tau - h, T]$ such that $x(\tau - h) = \bar{x}$ and $x(t) \in X(t)$ on $[\tau, T]$. Thus $x = x(\tau) \in X(\tau)$ and

$$\bar{x} \in x - \int_{\tau-h}^{\tau} F(x(s), s)\, ds \subset x - hF(x, \tau) +$$

$$\left(hH^{+}\left(\frac{1}{h}\int_{\tau-h}^{\tau} F(x, s)\, ds, F(x, \tau)\right) + m\int_{\tau-h}^{\tau} (\tau - s)\lambda(s)\, ds\right) \mathcal{B},$$

where m and $\lambda(\cdot)$ correspond to S according to **A1**. The last inclusion together with (3.11) implies (3.2). Q.E.D.

Proof of Theorem 3.2. The sufficiency of (3.6) has already been proven. If $X(\cdot)$ is weakly invariant w.r. to (3.1),(3.4), then it is weakly invariant w.r. to (3.1) and (3.2) is fulfilled, which together with $X(t) \subset Y(t)$ implies (3.6). Q.E.D.

Proof of Theorem 3.3. Again we have to prove only the necessity of (3.9). Let $X(\cdot) \in \mathcal{W}[t_0, T]$ be weakly invariant w.r. to (3.4),(3.7). As in the proof of Theorem 3.1, take an arbitrary $t \in (t_0, T]$ that is a Lebesque point of $F(x, \cdot)$ for every $x \in S$ (S is a compact containing $X(t)$, $t \in [t_0, T]$). Let M_V be the constant corresponding to $V(\cdot)$ according to the definition of $\mathcal{W}[t_0, T]$ (see page 4) Take an arbitrary $v \in V(t-h)$. From Proposition 2.1 applied to the mapping $F \equiv \{0\}, X(\cdot) = V(\cdot)$ it follows that there is a selections $v(\cdot)$ of $V(\cdot)$ on $[t - h, T]$ which is Lipschitz continuous with a constant M_V (since $\rho_{V(\cdot)}(s) \equiv M_V$). Then exactly as in the proof of Theorem 3.1 we obtain

$$H^{+}\left(X(t-h), \bigcup_{x \in X(t)} (x - hF(x, t) - hv(t)\right) \leq$$

$$hH^{+}\left(\frac{1}{h}\int_{\tau-h}^{\tau} F(x, s)\, ds, F(x, \tau)\right) + m\int_{\tau-h}^{\tau} (\tau - s)\lambda(s)\, ds,$$

$$\leq h\theta(h),$$

where θ is independent of $v(\cdot)$ and tends to zero with h. Hence

$$H^{+}\left(X(t-h) + hv, \bigcup_{x \in X(t)} (x - hF(x, t))\right) \leq h\theta(h) + M_V h^2$$

end since $v \in V(t - h)$ is arbitrary we obtain (3.9). Q.E.D.

Proof of Proposition 3.1. Let $t \in [t_0, T)$ be a Lebesque point of $\lambda(\cdot)$, $\rho(\cdot)$ and $F(x, \cdot)$, $x \in S$, at which $\lambda(t)$ and $\rho(\cdot)$ are finite. Take an arbitrary $x \in X(t)$. According to Proposition 2.1 there is a Lipschitz continuous selection $x(\cdot)$ of $X(\cdot)$ such that

$$\operatorname{dist}(\dot{x}(s), F(x(s), s)) \leq \rho(s)$$

for a.e. $s \in [t, T]$. According to the remark after Proposition 2.1 the Lipschitz constant L of $x(\cdot)$ can be assumed independent of $x \in X(t)$. Moreover, thanks to **A1**

$$o_1(h) = H(F(x,t), \int_t^{t+h} F(x,s)\, ds)$$

can also be thought independent of $x \in X(t) + hL\mathcal{B}$ (see the first paragraph of the proof of Proposition 2.1). Thus we have

$$x(t+h) \in x(t) + \int_t^{t+h} F(x(s), s)\, ds + \int_t^{t+h} \rho(s)\, ds\mathcal{B}$$

$$\subset x(t) + \int_t^{t+h} F(x(t+h), s)\, ds + \int_t^{t+h} \lambda(s)|x(s) - x(t+h)|\, ds\mathcal{B} + \int_t^{t+h} \rho(s)\, ds\mathcal{B}$$

$$\subset x(t) + hF(x(t+h), t) + (h\rho(t) + o_1(h) + Lh^2\lambda(t) + o(h))\mathcal{B},$$

where $o(h)/h$ and $o_1(h)/h$ tend to zero with h. This implies the desired result, since $x(t+h) \in X(t+h)$. Q.E.D.

4 An existence result

This section deals with existence of a solution in a class of differential inclusions with u.s.c. and, possibly, non-convex-valued right-hand sides. Well known examples show that such a differential inclusion does not have a solution, in general. Existence results requiring some additional properties like monotonicity or cyclic monotonicity were proven in [1, 4, 5]. Here we focus on a specific class of differential inclusions that will turn out to contain the closed-loop inclusions of the type of (1.10) that will be considered in the next section. A brief comparison with the above mentioned results will be given in this section before the proofs.

Let us start with the differential inclusion (1.6) supposing that conditions **A1** and **A2** are satisfied. We also introduce two functions $\psi : \mathbf{R}^n \times \mathbf{R} \mapsto (-\infty, +\infty)$ and $\varphi : \mathbf{R}^n \times \mathbf{R} \mapsto (-\infty, +\infty]$ that are supposed to satisfy the following conditions:

i) ψ is bounded and there are constants L_x and L_t such that

$$\psi(x',t') - \psi(x,t) \leq L_x\|x - x'\| + L_t(t' - t) \tag{4.1}$$

for every $x, x' \in \mathbf{R}^n$, $t, t' \in [t_0, T]$, $t' \geq t$.

ii) φ satisfies

$$\overline{\lim_{\substack{x' \to x \\ t' \to t^+}}} \varphi(x',t') \leq \varphi(x,t) \tag{4.2}$$

for every $x \in \mathbf{R}^n, t \in [t_0, T)$.

Further we denote by

$$D^-\psi(x,t;y) = \underline{\lim}_{h \to 0^+} \frac{\psi(x + hy, t + h) - \psi(x,t)}{h}$$

the lower Dini derivative of ψ in the direction of $(y, 1)$. Because of (4.1) $D^-\psi(x,t;y) \leq L_x\|y\| + L_t$.

Now define

$$\Phi(x,t) = \{y \in F(x,t);\ D^-\psi(x,t;y) \leq \varphi(x,t)\} \tag{4.3}$$

and consider the differential inclusion

$$\dot{x} \in \Phi(x,t)\,. \tag{4.4}$$

Theorem 4.1 *Let differential inclusion (1.6) satisfy conditions* **A1** *and* **A2**, *let $S_0 \subset \mathbf{R}^n$ be a compact set and $S = S(S_0)$ be as in* **A2** *and let $\Phi(\cdot,t)$ defined by (4.3) be nonempty-valued on S for almost every $t \in [t_0,T]$. Claim: for each $\tau \in [t_0,T)$ and $x \in S_0$ differential inclusion (4.4) with initial condition $x(\tau) = x$ has a solution on $[\tau, T]$, every solution is extendible up to the moment T and the set of all solutions is compact in $C[\tau, T]$.*

Clearly the above theorem is interesting only if the function ψ is non-differentiable and non-convex with respect to x (otherwise Φ would be convex valued). Of this type are the applications given below.

Let a tube $W(\cdot) \in \mathcal{W}[t_0,T]$ be given, let $S \subset \mathbf{R}^n$ be a compact set containing $W(t)$ in its interior for $t \in [t_0,T]$, and let $\lambda(\cdot)$ be the Lipschitz constant corresponding to S according to **A1**. Define

$$\psi(x,t) = \mathrm{dist}(x, W(t))\,, \tag{4.5}$$

$$\varphi(x,t) = \begin{cases} \lambda(t)\mathrm{dist}(x, W(t)) & \text{if } x \notin W(t), \\ +\infty & \text{if } x \in W(t). \end{cases} \tag{4.6}$$

Proposition 4.1 *Let (1.6) satisfy conditions* **A1** *and* **A2** *and let* $W(\cdot) \in \mathcal{W}[t_0, T]$ *be weakly invariant. Then the claim in Theorem 4.1 holds for differential inclusion (4.4) with* Φ *given by (4.3), (4.5) and (4.6).*

The above proposition outlines one application of Theorem 4.1 that will be used in the next section. If $W(t) \equiv W$ is a closed set then for ψ given by (4.5) and $x \notin W$ we have

$$D^-\psi(x,t;y) = \min_{z \in \mathcal{P}_W x} \langle \frac{x-z}{|x-z|}, y \rangle,$$

where

$$\mathcal{P}_W x = \{w \in W;\ |x-w| = \mathrm{dist}(x,W)\}$$

is the projection of x on W. Now one can define the function φ in a different way as follows:

$$\varphi(x,t) = \begin{cases} \max_{z \in \mathcal{P}_W x} \min_{y \in F(x,t)} \langle \frac{x-z}{|x-z|}, y \rangle & \text{if } x \notin W(t), \\ +\infty & \text{if } x \in W(t). \end{cases} \tag{4.7}$$

The right-hand side of (4.4) now takes the form

$$\Phi(x,t) = \{y \in F(x,t);\ \langle z_0 - x, y \rangle \ \geq \min_{z \in \mathcal{P}_W x} \max_{\xi \in F(x,t)} \langle z - x, \xi \rangle$$

$$\text{for some } z_0 \in \mathcal{P}_W x\} \tag{4.8}$$

and is apparently non-empty. Theorem 4.1 can be applied to obtain the following proposition.

Proposition 4.2 *Let (1.6) satisfy conditions* **A1** *and* **A2** *and let* W *be closed. Then the claim in Theorem 4.1 holds for differential inclusion (4.4) with* Φ *given by (4.8).*

In the particular case $F(x,t) = \mathrm{dist}(x,W)\mathcal{B}$ (4.4) becomes

$$\dot{x} = \mathcal{P}_W x - x$$

and Proposition 4.2 implies existence of a solution to this inclusion. This result follows also from [4, Proposition 2] and [1, Theorem] in combination, since the right-hand side is sum of a cyclically monotone operator and a continuous function. However, a mapping Φ given by (4.8) is non necessarily of this type (even in the time-invariant case), as the following example shows

Example. Let $n = 2$, $W = \{(2,0),(0,2)\}$ and $F(x) = \mathrm{co}\{(0,1),(-1,1.25)\}$. Here the corresponding mapping Φ defined by (4.8) does not contain any sub-mapping that is sum of a cyclically monotone one and a continuous function.

Proofs

Proof of Theorem 4.1. Let τ and x_0 be fixed as in the formulation of the theorem. Let $S = S(S_0)$ be a compact set such that every trajectory $x(\cdot)$ of (1.6) on $[\tau, T]$ starting from x_0 satisfies $x(t) \in \operatorname{int} S$ for $t \in [\tau, T]$. We shall modify the mappings F, Φ and φ in the following way. Take a compact set $\tilde{S}$ such that $S \subset \operatorname{int} \tilde{S}$. Let $\gamma(\cdot)$ be a Lipschitz continuous function on $\mathbf{R}^n$ such that

$$\gamma(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin \tilde{S} \end{cases}$$

and let

$$\tilde{F}(x,t) = \gamma(x) F(x,t)$$

$$\tilde{\varphi}(x,t) = \begin{cases} \varphi(x,t) & \text{if } x \in S \\ +\infty & \text{if } x \notin S. \end{cases}$$

Because of **A1** and the above definitions, there are a constant $\tilde{m}$ and an integrable function $\tilde{\lambda}(\cdot)$ such that $\tilde{F}$ is bounded by $\tilde{m}$ in $\mathbf{R}^n \times [t_0, T]$ and is Lipschitz continuous w.r. to x for fixed t with a Lipschitz constant $\tilde{\lambda}(t)$. Define

$$\tilde{\Phi}(x,t) = \{y \in \tilde{F}(x,t);\ D^- \psi(x,t;y) \le \tilde{\varphi}(x,t)\} \tag{4.9}$$

Obviously $\tilde{\Phi}(\cdot, t)$ is also nonempty-valued, now on $\mathbf{R}^n$, for all $t \in [\tau, T] \setminus \Lambda_0$, where $\operatorname{meas} \Lambda_0 = 0$.

For each x and $t \in [\tau, T] \setminus \Lambda_0$ fix arbitrarily some $y(x,t) \in \tilde{\Phi}(x,t)$. Denote by Λ a set of measure zero that contains Λ_0 and such that every $t \in [\tau, T) \setminus \Lambda$ is a Lebesque point (from the right) of $\tilde{F}(x, \cdot)$ for every $x \in \mathbf{R}^n$ (we use [29, Proposition 4.1]).

Take an arbitrary $h \in (0, 0.5)$. According to (4.9) for every $x \in \mathbf{R}^n$, $t \notin \Lambda$ there is $\alpha(x,t) > 0$ (depending also on h) such that

$$\alpha(x,t) \le h \tag{4.10}$$

$$\psi(x + \alpha(x,t) y(x,t), t + \alpha(x,t)) - \psi(x,t) \le \alpha(x,t) \tilde{\varphi}(x,t) + h \alpha(x,t) \tag{4.11}$$

$$H\Big(\tilde{F}(x,t), \frac{1}{\alpha(x,t)} \int_t^{t+\alpha(x,t)} \tilde{F}(x,s)\, ds\Big) \le h. \tag{4.12}$$

There is a finite or countable collection of open (relative to $[\tau, T]$) intervals $\{(p_i, q_i)\}$ such that

$$\sum_i (q_i - p_i) \le h \quad \text{and} \quad \Lambda \subset \bigcup_i (p_i, q_i).$$

We shall define a function $x_h(\cdot)$ on $[\tau, T]$ in the following way.

Denote $s_0 = \tau$, $x_h(s_0) = x_0$ and suppose that $x_h(\cdot)$ is already defined on $[\tau, s_k]$, $s_k \in [\tau, T)$. Consider the following two cases.

1) $s_k \in (p_i, q_i)$ for some i. Then define

$$s_{k+1} = q_i, \quad x_h(t) = x_h(s_k) \;\text{ for }\; t \in [s_k, q_i].$$

2) $s_k \notin \cup_i (p_i, q_i)$. Then consider

$$\tilde{\alpha} = \sup\{\alpha = \alpha(x,t)\,;\; |x - x_h(s_k)| \le \tilde{m}|t - s_k|,\; t \notin \Lambda,\; 0 \le t - s_k \le \alpha h\}.$$

The set in the right-hand side is nonempty since it contains $\alpha(x_h(s_k), s_k)$. Hence $\tilde{\alpha} \ge \alpha(x_h(s_k), s_k) > 0$. Let $\tilde{x}_k$, $\tilde{s_k}$ and α_k be such that

$$\begin{gathered} \alpha_k \ge \tilde{\alpha}_k/2, \quad \alpha_k = \alpha(\tilde{x}_k, \tilde{s}_k), \quad \tilde{s}_k \notin \Lambda, \\ |\tilde{x}_k - x_h(s_k)| \le \tilde{m}|\tilde{s}_k - s_k|, \;\; 0 \le \tilde{s}_k - s_k \le \alpha_k h. \end{gathered} \tag{4.13}$$

According to (4.10) $\alpha_k \le h$. Denote

$$y_k = y(\tilde{x}_k, \tilde{s}_k)$$

and let (according to (4.12)) $g_k^h(\cdot)$ be a measurable selection of $\tilde{F}(\tilde{x}_k, \cdot)$ on $[\tilde{s}_k, \tilde{s}_k + \alpha_k]$ such that

$$\left|\frac{1}{\alpha_k}\int_{\tilde{s}_k}^{\tilde{s}_k+\alpha_k} g_k^h(t)\,dt - y_k\right| \le h. \tag{4.14}$$

Then define

$$\begin{gathered} s_{k+1} = \tilde{s}_k + \alpha_k, \; x_h(\tilde{s}_k) = \tilde{x}_k, \; x_h(\cdot) \text{ – linear on } [s_k, \tilde{s}_k], \\ x_h(t) = \tilde{x}_k + \textstyle\int_{\tilde{s}_k}^t g_k^h(s)\,ds \text{ for } t \in [\tilde{s}_k, s_{k+1}]. \end{gathered} \tag{4.15}$$

Repeat the above recursive procedure until for some N it happens that $s_N \ge T - h$. We shall prove that such an integer N actually exists. Suppose the opposite, namely that the sequence $\{s_k\}$ converges to some $\bar{t} < T - h$, and consider the following two cases.

i) $\bar{t} \in (p_i, q_i)$ for some i. Then $s_k \in (p_i, q_i)$ for some (sufficiently large) k and by definition $s_{k+1} = q_i > \bar{t}$, which is a contradiction;

ii) $\bar{t} \notin \cup_i (p_i, q_i)$. Denote $x_k = x_h(s_k)$. From (4.13), (4.15) and **A2** we have

$$|x_{k+1} - x_k| \le |\tilde{x}_k - x_k| + |\tilde{x}_k - x_{k+1}| \le$$

$$\tilde{m}|\tilde{s}_k - s_k| + \tilde{m}|s_{k+1} - \tilde{s}_k| = \tilde{m}|s_{k+1} - s_k|,$$

which means that $\{x_k\}$ is also convergent and for the limit point $\bar{x}$ and every k it holds

$$|\bar{x} - x_k| \le \tilde{m}(\bar{t} - s_k). \tag{4.16}$$

Let $\alpha = \alpha(\bar{x}, \bar{t})$. For a sufficiently large k

$$\bar{t} - s_k \le h\alpha,$$

which together with (4.16) gives $\tilde{\alpha}_k \ge \alpha$ and from (4.13)

$$\alpha_k \ge \alpha/2.$$

On the other hand $\alpha_k \le \bar{t} - s_k \le h\alpha < \alpha/2$ since $h \in (0, 0.5)$, which is a contradiction.

Thus we have $s_N \ge T - h$ for some $N = N(h)$. Let us extend $x_h(\cdot)$ to $[\tau, T]$ as $x_h(t) = x_h(s_N)$, $t \in [s_N, T]$. The family $\{x_h(\cdot)\}_h$ is equicontinuous and uniformly bounded, because of the definition of $\tilde{F}$ (observe that $x_h(\cdot)$ is Lipschitz with a constant $\tilde{m}$ on the intervals $[s_k, \tilde{s}_k]$, according to (4.13)). Let

$$\lim_{k \to +\infty} \|x_{h_k}(\cdot) - x(\cdot)\|_{C[t_0, T]} = 0$$

for a sequence $h_k \to 0$ and a continuous $x(\cdot)$. We shall prove that $x(\cdot)$ is a trajectory of

$$\dot{x} \in \tilde{F}(x, t) \tag{4.17}$$

on $[\tau, T]$. For this purpose let us estimate

$$D = \int_\tau^T \operatorname{dist}(\dot{x}_h(t), \tilde{F}(x_h(t), t))\, dt.$$

Denote by I the set of those i for which $x_h(\cdot)$ is defined in $[s_i, s_{i+1}]$ in the second way ii), and let $\bar{I} = \{0, \dots, N\} \setminus I$ (we set $s_{N+1} = T$). Then using (4.13) we obtain

$$\begin{aligned}
D = &\left[\sum_{i \in I} \left(\int_{s_i}^{\tilde{s}_i} + \int_{\tilde{s}_i}^{s_{i+1}}\right) + \sum_{i \in \bar{I}} \int_{s_i}^{s_{i+1}}\right] \operatorname{dist}(\dot{x}_h(t), \tilde{F}(x_h(t), t))\, dt \\
\le &\sum_{i \in I} [\int_{s_i}^{\tilde{s}_i} \operatorname{dist}(\frac{\tilde{x}_i - x_i}{\tilde{s}_i - s_i}, \tilde{F}(x_h(t), t))\, dt + \int_{\tilde{s}_i}^{s_{i+1}} \operatorname{dist}(g_i^h(t), \tilde{F}(x_h(t), t))\, dt] \\
&+ \sum_{i \in \bar{I}} \int_{s_i}^{s_{i+1}} |\tilde{F}(x_h(t), t)|\, dt \\
\le &\sum_{i \in I} (2\tilde{m}(\tilde{s}_i - s_i) + \int_{\tilde{s}_i}^{s_{i+1}} \operatorname{dist}(g_i^h(t), \tilde{F}(\tilde{x}_i, t))\, dt + \int_{\tilde{s}_i}^{s_{i+1}} \tilde{\lambda}(t)\tilde{m}(t - \tilde{s}_i)\, dt) \\
&+ \sum_{i \in \bar{I}} \tilde{m}(s_{i+1} - s_i)
\end{aligned}$$

$$\le \sum_{i\in I}(2\tilde{m}\alpha_i h + \tilde{m}\alpha_i \int_{\bar{s}_i}^{s_{i+1}} \tilde{\lambda}(t)\,dt) + (\tilde{m}+1)h$$

$$\le 2\tilde{m}(T-\tau)h + h\tilde{m}\int_\tau^T \tilde{\lambda}(t)\,dt + (\tilde{m}+1)h \le ch.$$

According to the Filippov's lemma [3] and the closedness of the set of trajectories of (4.17) in $C[\tau, T]$ this estimate implies that $x(\cdot)$ is a trajectory of (4.17).

We have $x(\tau) = x_0 \in S_0$ and $\tilde{F}(x,t) = F(x,t)$ for $x \in S$, which implies that $x(t) \in \operatorname{int} S$ and that $x(\cdot)$ is a trajectory of (1.6). Moreover, for every sufficiently large k we have $x_{h_k}(t) \in S$, which means that the sign $\tilde{}$ can be removed from the letters Φ, F and φ in the above considerations.

Now we shall prove that $x(\cdot)$ is a trajectory of (4.4). Since (4.4) has to be proven for a.e. t, let us fix an arbitrary $t \in (\tau, T)$ for which $\dot{x}(t)$ exists and belongs to $F(x(t), t)$. Take $\sigma > 0$ and let $\omega(\sigma)$ be such that

$$x(t+\sigma) = x(t) + \sigma\dot{x}(t) + \sigma\omega(\sigma). \tag{4.18}$$

Let k be so large that $x_k(\cdot) = x_{h_k}(\cdot)$ satisfies

$$\|x(\cdot) - x_k(\cdot)\|_C \le \sigma^2, \quad h_k \le \sigma^2.$$

We shall estimate

$$\Delta = \psi(x(t+\sigma), t+\sigma) - \psi(x(t), t).$$

Let p and q (depending on k) be such that

$$t \le s_p < \ldots < s_q \le t+\sigma, \quad t+\sigma - s_q \le h_k, \quad s_p - t \le h_k.$$

Then using (4.1) we obtain

$$\Delta \le \psi(x_k(s_q), s_q) - \psi(x_k(s_p), s_p) + L_x(|x(t+\sigma) - x_k(s_q)| + |x(t) - x_k(s_p)|)$$
$$+ L_t(|t+\sigma - s_q| + |s_p - t|)$$
$$\le \psi(x_k(s_q), s_q) - \psi(x_k(s_p), s_p) + c\sigma^2,$$

where $c = 2L_x(\tilde{m}+1) + 2L_t$.

Let J be the set of those indexes i from $\{p, \ldots, q-1\}$ for which $x_k(\cdot)$ is defined in the second way ii), and let $\bar{J} = \{p, \ldots, q-1\} \setminus J$. We have

$$\Delta \le \sum_{i=p}^{q-1}(\psi(x_k(s_{i+1}), s_{i+1}) - \psi(x_k(s_i), s_i)) + c\sigma^2$$

$$
\begin{aligned}
&\le \sum_{i\in J}[(\psi(x_k(s_{i+1}), s_{i+1}) - \psi(\tilde{x}_i, \tilde{s}_i)) + (\psi(\tilde{x}_i, \tilde{s}_i) - \psi(x_k(s_i), s_i))] \\
&\quad + \sum_{i\in J}(\psi(x_k(s_{i+1}, s_{i+1}) - \psi(x_k(s_i), s_i)) + c\sigma^2 \\
&\le \sum_{i\in J}(\psi(\tilde{x}_i + \int_{\tilde{s}_i}^{s_{i+1}} g_i^{h_k}(s)\, ds, s_{i+1}) - \psi(\tilde{x}_i, \tilde{s}_i)) \\
&\quad + \sum_{i\in J}(L_x|\tilde{x}_i - x_k(s_i)| + L_t|s_i - \tilde{s}_i|) \\
&\quad + \sum_{i\in J}(L_x|x_k(s_{i+1}) - x_k(s_i)| + L_t|s_{i+1} - s_i|) + c\sigma^2 \\
&\le \sum_{i\in J}(\psi(\tilde{x}_i + \alpha_i y_i, \tilde{s}_i + \alpha_i) - \psi(\tilde{x}_i, \tilde{s}_i)) \\
&\quad + \sum_{i\in J}(L_x(\tilde{m}+1)h_k\alpha_i + L_t h_k \alpha_i) + L_t h_k + c\sigma^2 \\
&\le \sum_{i\in J}(\alpha_i\varphi(\tilde{x}_i, \tilde{s}_i) + h_k\alpha_i) + ((L_x(\tilde{m}+1) + L_t)(T-\tau) + L_t + c)\sigma^2 \\
&\le \sum_{i\in J}\alpha_i\varphi(\tilde{x}_i, \tilde{s}_i) + c_1\sigma^2.
\end{aligned}
$$

Since $\varphi(\cdot)$ satisfies (4.2), for the given t there is a monotone decreasing function $\theta(\sigma) \to 0$ with $\sigma \to 0$ such that

$$
\varphi(x(t), t) \ge \varphi(x', t') - \theta(|x(t) - x'| + |t - t'|)
$$

for every (x', t') in a neighborhood of $(x(t), t)$, $t' \ge t$. Hence

$$
\begin{aligned}
\Delta &\le \sum_{i\in J}\alpha_i\varphi(x(t), t) + \sum_{i\in J}\alpha_i\theta(|\tilde{x}_i - x(t)| + |\tilde{s}_i - t|) + c_1\sigma^2 \\
&\le \sigma\varphi(x(t), t) + \sigma\theta(c_2\sigma) + c_3\sigma^2. \qquad (4.19)
\end{aligned}
$$

Observe that (4.19) holds even if $\varphi(x(t), t)$ is negative, because one can easily estimate $0 \le \sigma - \sum_{i\in J}\alpha_i \le 4\sigma^2$, but in this case c_3 depends on the value of $\varphi(x(t), t)$ (otherwise $c_3 = c_1$).

On the other hand, using (4.18) we get

$$
\begin{aligned}
\Delta &= \psi(x(t) + \sigma\dot{x}(t) + \sigma\omega(\sigma), t + \sigma) - \psi(x(t), t) \\
&\ge \psi(x(t) + \sigma\dot{x}(t), t + \sigma) - \psi(x(t), t) - L_x\sigma|\omega(\sigma)|.
\end{aligned}
$$

Combining this with (4.19) we obtain

$$
D^+(x(t), t; \dot{x}(t)) \le \varphi(x(t), t),
$$

which implies that $\dot{x}(t) \in \Phi(x(t), t)$ and the proof of the existence is complete.

Now we shall prove the compactness of the set of trajectories of (4.4) starting from a given point (τ, x_0), $\tau \in [t_0, T)$, $x_0 \in S_0$. Clearly, an absolutely continuous function $x(\cdot)$ is a solution of (4.4) if and only if it is a solution of (1.6) and satisfies the inequality

$$D^- \psi(x(t), t; \dot{x}(t)) \leq \varphi(x(t), t) \ \text{ for a.e. } t \in [\tau, T]. \tag{4.20}$$

Thus we have to prove only that the set of solutions of (1.6) starting from a given point and satisfying (4.20) is closed in $C[\tau, T]$. Let $\{x_k(\cdot)\}$ be a sequence from this set that converges uniformly to $x(\cdot)$ (thanks to **A1** and **A2** $x(\cdot)$ is also absolutely continuous).

Below we apply to the function $\xi(t) = \psi(x(t), t)$ the following assertion: if $\xi(\cdot)$ satisfies the condition

$$\xi(t') - \xi(t) \leq L(t' - t) \ \text{ for } t' \geq t \tag{4.21}$$

then $\dot{\xi}(t)$ exists for a.e. t and

$$\xi(t + h) - \xi(t) \leq \int_t^{t+h} \dot{\xi}(s)\, ds. \tag{4.22}$$

This follows from the fact that ξ is of bounded variation, and thanks to (4.21) can be presented as a sum of a monotone increasing absolutely continuous function and a monotone decreasing function, both known to satisfy (4.22).

If $\varphi(x(t), t) = +\infty$ for some t then (4.20) is fulfilled. Take an arbitrary t for which $\varphi(x(t), t) = a < +\infty$ and $\dot{\xi}(t)$ exists ((4.21) obviously follows from (4.1)) and chose an arbitrary $\varepsilon > 0$.

From (4.2) it follows that $\varphi(x(\cdot), \cdot)$ is measurable and bounded by $a + \varepsilon$ on $[t, t + \delta]$ if δ is sufficiently small, and

$$\varphi(x(t), t) \geq \overline{\lim_{h \to h^+}} \frac{1}{h} \int_t^{t+h} \varphi(x(s), s)\, ds.$$

Thus for all sufficiently small $h > 0$ we have

$$\varepsilon + \varphi(x(t), t) \geq \frac{1}{h} \int_t^{t+h} \varphi(x(s), s)\, ds$$

and

$$\frac{d}{dt} \psi(x(t), t) \leq \frac{\psi(x(t + h), t + h) - \psi(x(t), t)}{h} + \varepsilon.$$

For every such h there is $k_0 = k_0(h)$ such that for $k > k_0$

$$\|x(\cdot) - x_k(\cdot)\|_C \leq h^2.$$

Then

$$D^-\psi(x(t),t;\dot{x}(t)) = \frac{d}{dt}\psi(x(t),t) \le \frac{\psi(x_k(t+h),t+h) - \psi(x_k(t),t)}{h} + 2L_x h + \varepsilon$$

$$\le \frac{1}{h}\int_t^{t+h} \frac{d}{ds}\psi(x_k(s),s)\,ds + 2L_x h + \varepsilon \le \frac{1}{h}\int_t^{t+h} \varphi(x_k(s),s)\,ds + 2L_x h + \varepsilon.$$

Since $\varphi(x_k(s),s) \le a + 2\varepsilon$ for all sufficiently large k we can take limes supremum with respect to k and continue the above inequalities by

$$\frac{1}{h}\int_t^{t+h} \varphi(x(s),s)\,ds + 2L_x h + \varepsilon \le \varphi(x(t),t) + 2L_x h + 2\varepsilon,$$

which implies (4.20). Q.E.D.

Proof of Proposition 4.1. The upper semicontinuity of $\lambda(\cdot)$ and the properties 1) and 2) from the definition of $\mathcal{W}[t_0,T]$ (page 4) imply that ψ and φ defined by (4.5) and (4.6) satisfy (4.1) and (4.2). Thus, in order to apply Theorem 4.1 we have to prove only that $\Phi(\cdot,t)$ is nonempty-valued on S for a.e. $t \in [t_0,T]$.

Let $\tau \in [t_0,T)$ be a Lebesque point of $\lambda(\cdot)$ and $F(x,\cdot)$, $x \in S$ (we use again [29, Proposition 4.1], according to which almost every τ is such). Fix an arbitrary $x \in S$ and denote $\hat{x} = \mathcal{P}_{W(t)}x$. Since $W(\cdot)$ is weakly invariant w.r. to (1.6) there is a trajectory $\hat{x}(\cdot)$ of (1.6) such that $\hat{x}(\tau) = \hat{x}$ and $\hat{x}(t) \in W(t)$ for $t \in [\tau,T]$. Then

$$\hat{x} = \hat{x}(\tau) = x(t) + \int_t^\tau F(\hat{x}(s),s)\,ds$$

$$\subset x(t) + \int_t^\tau F(\hat{x},s)\,ds + m(t-\tau)\int_t^\tau \lambda(s)\,ds\mathcal{B}.$$

Using the Lebesque property of τ we have

$$\varliminf_{h\to 0+} \frac{1}{h}\mathrm{dist}(\hat{x}+h\hat{y}, W(\tau+h)) = 0 \tag{4.23}$$

for some $\hat{y} \in F(\hat{x},\tau)$.

Let $y \in F(x,\tau)$ be such that

$$|y-\hat{y}| \le \lambda(\tau)|x-\hat{x}|. \tag{4.24}$$

Then for any $w_h \in \mathcal{P}_{W(t+h)}(\hat{x}+h\hat{y})$ we have

$$D^-(x,\tau;y) = \varliminf_{h\to 0+} \frac{1}{h}[|x+hy-w_h| - |x-\hat{x}|]$$

$$\leq \varlimsup_{h\to 0+} \frac{1}{h}[|\hat{x} + h\hat{y} - w_h| + |x - \hat{x}| + h|y - \hat{y}| - |x - \hat{x}|]$$

$$\leq \varlimsup_{h\to 0+} \frac{1}{h}\mathrm{dist}(\hat{x} + h\hat{y}, W(\tau + h)) + |y - \hat{y}|$$

$$\leq \lambda(\tau)\mathrm{dist}(x, W(\tau)),$$

which implies $y \in \Phi(x,\tau)$. Q.E.D.

5 The regulation problems

Consider the differential inclusion

$$\dot{x} \in F(x(t),t), \quad t \in [t_0, T] \tag{5.1}$$

supposing that conditions **A1** and **A2** are satisfied. Let $W(\cdot) \in \mathcal{W}[t_0,T]$ be a weakly invariant tube w.r. to (5.1) and let $S \subset \mathbf{R}^n$ be a compact set containing $W(t)$, $t \in [t_0,T]$, in its interior. We focus on the following problem. Find a sub-mapping (set-valued feedback control) $\Phi(x,t) \subset F(x,t)$ such that the tube $W(\cdot)$ has the stable invariance property w.r. to the closed-loop inclusion

$$\dot{x} \in \Phi(x,t). \tag{5.2}$$

The ideas for the solution of this problem go back to Krasovskii (see [14]) in the framework of the differential games. A formal solution in the framework of the differential inclusions was given in [18] in the case of a convex valued tube $W(\cdot)$. We stress the fact that the stable invariance property is not implied in general by the invariance property (which corresponds to $\delta = 0$ in Definition 2, Section 1). The reason is that discontinuous non-convex valued mappings Φ are involved and neither the standard existence theory nor the standard perturbation analysis for differential inclusions are applicable. That is not the case in [18] where Φ is convex valued due to the convexity of $W(t)$.

Let us define the mapping Φ as in (4.3) with ψ and φ given by (4.5) and (4.6).

Theorem 5.1 *Under the suppositions* **A1** *and* **A2** *the tube $W(\cdot)$ has the stable invariance property w.r. to (5.2).*

The definition of stable invariance requires two things: 1) existence and extendibility of the solutions starting "near" graph $W(\cdot)$; 2) relation (1.11). 1) follows from Proposition 4.1, while 2) follows from the following proposition.

Proposition 5.1 *Let under the conditions of Theorem 5.1 $x(\cdot)$ be a trajectory of (5.2) on some interval $[t_1, t_2]$ and let $x(t) \in int\, S$ for $t \in [t_1, t_2]$. Then*

$$\text{dist}(x(t), W(t)) \le \exp\left(\int_{t_1}^{t} \lambda(s)\, ds\right) \text{dist}(x(t_1), W(t_1)).$$

The definition of the regulation mapping Φ by (4.3), (4.5), (4.6) is not quite explicit since it uses the Lipschitz constant $\lambda(\cdot)$ and the Dini derivative of ψ. However for practical purposes it is often sufficient to know some non-empty subset of $\Phi(x,t)$. Such can be defined in a constructive way as the set of all "extremal" directions from $F(x,t)$ with respect to $W(t)$ (extremal aiming strategy of Krasovskii). Namely, define

$$\Phi_e(x,t) = \bigcup_{z \in \mathcal{P}_{W(t)}x} \text{Arg} \max_{y \in F(x,t)} \langle z - x, y \rangle \,, \tag{5.3}$$

where Argmax means the set of all points at which max is attained. Clearly Φ_e is non-empty valued, u.s.c. in x and $\Phi(x,t) = F(x,t)$ if $x \in W(t)$.

Proposition 5.2 *Let* **A1**, **A2** *be satisfied, let $W(\cdot) \in \mathcal{W}[t_0, T]$ be a weakly invariant tube w.r. to (5.1) and let $S \subset \mathbf{R}^n$ be a compact set containing $W(t)$ in its interior, $t \in [t_0, T]$. Then there is a set $\Omega \subset [t_0, T]$ of measure zero such that*

$$\Phi_e(x,t) \subset \Phi(x,t) \tag{5.4}$$

for every $x \in S$ and $t \in [t_0, T] \setminus \Omega$, where $\Phi_e(x,t)$ is defined by (5.3) and the regulation mapping Φ is given by (4.3), (4.5), (4.6).

Since Φ_e coincides with F on graph $W(\cdot)$, the existence and extendibility of the solutions of the next inclusion starting from $W(t_0)$ is ensured and we have

Corollary 5.1 *Under the assumption of Proposition 5.2 the tube $W(\cdot)$ is invariant with respect to the inclusion*

$$\dot{x} \in \Phi_e(x,t)\,.$$

In the case of a constant weakly invariant tube $W(t) \equiv W$ (viability set) one can define as in (4.8)

$$\Phi_0(x,t) = \Big\{ y \in F(x,t);\ \max_{z_0 \in \mathcal{P}_W x} \langle z_0 - x, y \rangle \ \ge \min_{z \in \mathcal{P}_W x} \max_{\xi \in F(x,t)} \langle z - x, \xi \rangle \Big\}\,. \tag{5.5}$$

Obviously $\Phi_e(x,t) \subset \Phi_0(x,t)$.From propositions 4.2 and 5.1 we obtain

Corollary 5.2 *W has the stable invariance property w.r. to the inclusion*

$$\dot{x} \in \Phi_0(x,t)\,. \tag{5.6}$$

The difference between Φ_e and Φ_0 can be illustrated by the following example in $\mathbf{R}^2$.

Example.

$$F(0,0) = \mathrm{co}\left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right\},\ W = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}.$$

In this case

$$\Phi_e(0,0) = \left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}\right\}, \tag{5.7}$$

$$\Phi_0(0,0) = \mathrm{co}\left\{\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}0\\2\end{pmatrix}, \begin{pmatrix}0.5\\1\end{pmatrix}\right\} \cup \left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}. \tag{5.8}$$

Proofs.

Theorem 5.1 is a consequence of propositions 4.1 and 5.1. In the proof of Proposition 5.1 we use the auxiliary result presented in Lemma 5.1 below, versions of which have been used by many authors. We present the proof of our version for completeness.

Lemma 5.1 *Let $\lambda(\cdot)$, $\mu(\cdot)$ and $\varepsilon(\cdot)$ be nonnegative integrable functions on $[t_0,T]$ and $g(\cdot)$ be defined as*

$$g(t) = \exp\left(\int_{t_0}^{t} \lambda(s)\,ds\right)(a + \int_{t_0}^{t} \varepsilon(s)\,ds),$$

where a is a nonnegative constant. Let $f(\cdot) : [t_0,T] \mapsto \mathbf{R}$ be an arbitrary function satisfying the conditions

i) $f(t_0) = a$;
ii) for every $s,t \in [t_0,T]$, $s \leq t$,

$$f(t) - f(s) \leq \int_{s}^{t} \mu(\tau)\,d\tau$$

iii)for almost every $t \in [t_0,T)$ for which $f(t) > g(t)$

$$\varliminf_{h\to 0+} \frac{f(t+h) - f(t)}{h} \leq \lambda(t)f(t) + \varepsilon(t).$$

Then $f(t) \leq g(t)$ for every $t \in [t_0,T]$.

Proof. Clearly the function $\tilde{f}(t) = \max\{f(t), 0\}$ satisfies the above conditions i) – iii) (since $g(\cdot)$ is nonnegative) and $\tilde{f}(t) \le g(t)$ implies $f(t) \le g(t)$. Therefore we may assume that $f(t) \ge 0$ for every $t \in [t_0, T]$. Then ii) implies that $f(\cdot)$ is of bounded variation.

Using well known facts from the theory of the functions with bounded variation we can present $f(\cdot) = f_1(\cdot) + f_2(\cdot)$, where $f_1(\cdot)$ is an absolutely continuous function and $f_2(\cdot)$ is monotone decreasing function that is differentiable almost everywhere and $\dot{f}_2(t) = 0$ for a.e. $t \in [t_0, T]$. Actually, $f(\cdot)$ can be presented as

$$f(t) = f(t_0) + \bigvee_{[t_0,t]}{}^{+} f - \bigvee_{[t_0,t]}{}^{-} f,$$

where $\bigvee^{+}$ and $\bigvee^{-}$ denote the positive and the negative variation on $[t_0, t]$, respectively. Both functions in the right-hand side are monotone increasing, the firs one is absolutely continuous, as it follows from ii). The second function can be presented as a sum of an absolutely continuous function and a monotone increasing function having its derivative equal to zero for a.e. t. The latter function (with $-$ sign) is just the function $f_2(\cdot)$.

Suppose that $f(t) > g(t)$ for some $t > t_0$. From ii) and the continuity of $g(\cdot)$ it follows that there is an interval $(\tau, t]$ in which $f(s) > g(s)$ and $f(\tau) = g(\tau)$. For $\theta \in [\tau, t]$ we have

$$f(\theta) - f(\tau) = f_1(\theta) - f_1(\tau) + f_2(\theta) - f_2(\tau) \le \int_\tau^t \dot{f}_1(s)\, ds$$

$$= \int_\tau^t \dot{f}(s)\, ds \le \int_\tau^t (\lambda(s) f(s) + \varepsilon(s))\, ds.$$

The Gronwall inequality together with $f(\tau) = g(\tau)$ implies $f(t) \le g(t)$, which is a contradiction. Q.E.D.

Proof of Proposition 5.1. We shall apply Lemma 5.1 to the function

$$f(t) = \operatorname{dist}(x(t), W(t)), \quad t \in [t_1, t_2].$$

Condition i) is fulfilled with $a = \operatorname{dist}(x(t_1), W(t_1))$. Condition ii) follows from the absolute continuity of $x(\cdot)$ and the definition of $\mathcal{W}[t_0, T]$ (page 4). iii) follows from (4.3),(4.5),(4.6) and

$$\underline{\lim}_{h \to 0+} \frac{f(t+h) - f(t)}{h} = D^{-}\psi(x(t), t; \dot{x}(t)) \le \lambda(t) f(t),$$

which holds for each t at which $\dot{x}(t)$ exists and satisfies (5.2) and $x(t) \notin W(t)$ (the last is apparently fulfilled if $f(t) > g(t)$). Then the proposition follows from Lemma 5.1 with $\varepsilon(t) \equiv 0$. Q.E.D.

Proof of Proposition 5.2. In the proof it will be convenient to use the modification in the form of (3.10) of the funnel equation (3.2). Under the conditions of Proposition 3.1 (applied for $\rho(t) = 0$) $\mathcal{W}[t_0, T](\cdot)$ satisfies for a.e. t the equation

$$\lim_{h \to 0+} \frac{1}{h} H^+\Big(X(t), \bigcup_{x \in W(t+h)} (x - hF(x,t))\Big) = 0. \tag{5.9}$$

Now let $x \in S$ and let $t \in [t_0, T)$ be such that (5.9) holds. We can suppose that $x \notin W(t)$, since otherwise the claim is trivial. Take an arbitrary $y \in \Phi_e(x,t)$ and let $z \in \mathcal{P}_{W(t)} x$ be such that

$$\langle z - x, y \rangle \geq \langle z - x, \tilde{y} \rangle \text{ for every } \tilde{y} \in F(x,t). \tag{5.10}$$

From (5.9) we obtain the presentation

$$z = x_h - h\xi_h + o(h), \quad x_h \in W(t+h), \; \xi_h \in F(x_h, t+h), \quad o(h)/h \to 0.$$

Since $|x_h - x| \leq \operatorname{dist}(x, W(t)) + hm + o(h)$ we have

$$\begin{aligned}
\frac{1}{h}[\psi(x + hy, t+h) - \psi(x,t)] &\leq \frac{1}{h}[|x + hy - x_h| - |x - z|] \\
= \frac{1}{h}[|x - z + hy - h\xi_h - o(h)| - |x - z|] &\leq \frac{\langle x - z, y - \xi_h \rangle}{|x - z|} + \frac{o(h)}{h} \\
&\leq \frac{\langle x - z, y - \tilde{\xi}_h \rangle}{\operatorname{dist}(x, W(t))} + \lambda(t)(\operatorname{dist}(x, W(t)) + hm) + \frac{o(h)}{h} \\
&\leq \varphi(x,t) + h\lambda(t)m + o(h)/h,
\end{aligned}$$

where $\tilde{\xi}_h \in F(x,t)$ is such that

$$|\tilde{\xi}_h - \xi_h| \leq H(F(x_h, t), F(x,t)).$$

Taking the limit we obtain $D^- \psi(x, t; y) \leq \varphi(x,t)$. Q.E.D.

6 Approximate regulation

The results presented in the preceding section have their "approximate" analogs. The essence of the issue is the following. Usually there is not a priori given any weakly invariant tube, such should be found first (by solving the corresponding funnel equation from Section 3 or based on some other viability conditions like in [24] or whatsoever) and then used for constructing of a regulation mapping. So the best one can hope is to use some "approximation" of a weakly invariant tube. In this section we show that a proper meaning of "approximation" to a weakly invariant tube can be given in the terms of the funnel inequalities introduced in Section 3, and that the constructions of regulation mappings from Section 5 when applied to such approximate weakly invariant tubes result in reasonable approximate solutions to the regulation problem. In this sense the regulation mappings from Section 5 are correct (with respect to approximations).

To be specific, let us consider the following regulation problem: given the differential inclusion (5.1), the state constraint (1.4), an initial set X_0 and a target $M_T \subset \mathbf{R}^n$ find a regulation mapping $\Phi(x,t) \subset F(x,t)$ such that any solution to the closed-loop inclusion (5.2) starting from a point of X_0 at t_0 satisfies the state constraint (1.4) and hits the target M_T at T.

Suppose that conditions **A1** - **A3** are satisfied and that M_T is compact. Then according to **A2** there is a compact set $\tilde{S}$ that contains the values of all the trajectories of (5.1) on $[t_0, T]$ which reach the set M_T at T. Let $S = \tilde{S} + c\mathcal{B}$ and let $\lambda(\cdot)$ be the Lipschitz constant from **A1** corresponding to S.

Let $W(\cdot) \in \mathcal{W}[t_0, T]$ be a solution to the funnel inequality (3.5) satisfying also the end condition

$$X(T) \subset M\ .$$

where $\rho = \int_{t_0}^{T} \rho(t)\, dt$ is such that

$$(\rho + \delta) \exp\left(\int_{t_0}^{T} \lambda(t)\, dt\right) \leq c$$

for some $\delta \geq 0$. Define the regulation mapping Φ as in (4.3),(4.5),(4.6), but with a slightly modified φ:

$$\varphi(x,t) = \begin{cases} \lambda(t)\mathrm{dist}(x, W(t)) + \rho(t) & \text{if } x \notin W(t), \\ +\infty & \text{if } x \in W(t). \end{cases} \tag{6.1}$$

Theorem 6.1 *Let $X_0 \subset W(t_0) + \delta\mathcal{B}$. Then every solution $x(\cdot)$ of differential inclusion (5.2), with Φ defined by (4.3), (4.5), (6.1), starting from a point from X_0 exists up to the moment T and satisfies*

$$\operatorname{dist}(x(t), Y(t)) \leq \exp\left(\int_{t_0}^{t} \lambda(s)\, ds\right)\left(\operatorname{dist}(x(t_0), W(t_0)) + \int_{t_0}^{t} \rho(s)\, ds\right)$$

for every $t \in [t_0, T]$, and

$$\operatorname{dist}(x(t), M) \leq \exp\left(\int_{t_0}^{T} \lambda(s)\, ds\right)\left(\operatorname{dist}(x(t_0), W(t_0)) + \int_{t_0}^{T} \rho(s)\, ds\right)$$

Moreover, as in Section 5 the set $\Phi(x,t)$ defined above contains all "extremal" directions of $F(x,t)$ with respect to $W(t)$:

Proposition 6.1 *Under the suppositions of this section there is a set $\Omega \subset [t_0, T]$ of measure zero such that the set $\Phi_e(x,t)$ defined by (5.3) is contained in $\Phi(x,t)$ (defined in Theorem 6.1) for every $x \in S + \mathcal{B}$ and $t \in [t_0, T] \setminus \Omega$.*

Proofs

Proof of Theorem 6.1. The existence and extendibility requirement follows from Proposition 4.1. We need to prove only a corresponding modification of Proposition 5.1. As in the proof of the latter we can apply again Lemma 5.1, this time for the functions

$$f(t) = \operatorname{dist}(x(t), W(t)), \quad \varepsilon(t) = \rho(t).$$

to obtain the estimations in Theorem 6.2. Q.E.D.

Proof of Proposition 6.1. The proof is similar to that of Proposition 5.2. Equation (5.9) should be replaced with (3.10) and all consequent changes are obvious. Q.E.D.

7 Guaranteed control of uncertain systems

In this section we return to the problem of guaranteed control of an uncertain system from which we started in Section 1. Consider again the model of an uncertain control system

$$\dot{x} \in G(x,t) + v(t), \quad t \in [t_0, T], \tag{7.1}$$

$$x(t_0) \in X_0, \tag{7.2}$$

$$v(\cdot) \in \mathcal{V}, \tag{7.3}$$

$$x(t) \in Y(t), \;\; t \in [t_0, T], \tag{7.4}$$

$$x(T) \in M_T, \tag{7.5}$$

interpreted as in Section 1. Throughout this section we suppose that conditions **A3** and **A4** are satisfied and that X_0 is compact. Our aim is to define a control strategy (regulation map) $\Phi(x,t,v) \subset G(x,t)$, $x \in \mathbf{R}^n$, $t \in [t_0, T]$ such that for every $v(\cdot) \in \mathcal{V}$ and $x_0 \in X_0$

1) every solution of the differential inclusion

$$\dot{x} \in \Phi(x,t,v(t)) + v(t) \tag{7.6}$$

starting from x_0 is extendible to $[t_0, T]$;

2) every solution of (7.6) on $[t_0, T]$ starting from x_0 satisfies (7.4) and (7.5).

Theorem 7.1 *A regulation map Φ satisfying 1) and 2) exists if and only if the funnel equation (3.9) has a solution $W(\cdot) \in \mathcal{W}[t_0, T]$ satisfying the boundary conditions*

$$W(T) \subset M_T, \tag{7.7}$$

$$X_0 \subset W(t_0). \tag{7.8}$$

If $W(\cdot)$ is as in the above theorem according to Theorem 3.3 $W(\cdot)$ is weakly invariant with respect to the family of inclusions (7.1) (parametrized by (7.3)) and (7.4) (that is, $W(\cdot)$ is a stable bridge in the terminology of Krasovskii). Then one can apply the constructions from Section 5 for $F = G + v$ in order to obtain a regulation map solving the problem formulated above. Namely, if S is a compact set containing $W(t)$, $t \in [t_0, T]$, in its interior and $\lambda(\cdot)$ is the Lipschitz constant of F corresponding to S (see **A1**, page 4), define

$$\Phi(x,t,v) = \{y \in G(x,t)\,;\; D^{-}\mathrm{dist}(x, W(t); y+v) \leq \lambda(t)\mathrm{dist}(x, W(t))\} \tag{7.9}$$

for $x \notin W(t)$ and $\Phi(x,t,v) = G(x,t)$ for $x \in W(t)$. According Theorem 5.1 $W(\cdot)$ has the stable invariance property w.r. to (7.4),(7.6) for every $v(\cdot) \in \mathcal{V}$, which, together with (7.7) and (7.8), implies 1) and 2).

The above definition of the regulation map Φ requires knowledge of the current value of the disturbance $v(\cdot)$. However, the set $\Phi(x,t,v)$ contains a subset that is independent of v. Actually, the mapping

$$\Phi_e(x,t) = \bigcup_{z\in\mathcal{P}_{W(t)}x} \operatorname{Arg}\max_{u\in G(x,t)} \langle z-x, y\rangle$$

is such as it follows from (5.3) and Proposition 5.2 applied for $F = G + v$.

The main point of this section is to elaborate the concept of stable invariance and funnel inequalities in order to obtain "approximate" regulation maps on the basis of a known approximation to an weakly invariant mapping $W(\cdot)$. We use the results from Section 6 to obtain the following.

Theorem 7.2 *Let $W(\cdot) \in \mathcal{W}[t_0,T]$ be a solution to the funnel inequality (3.8) with an integrable function $\rho(\cdot)$. Let $W(\cdot)$ satisfy the boundary conditions*

$$W(T) \subset M_T$$

$$X_0 \subset W(t_0) + \delta\mathcal{B}.$$

Denote

$$\gamma(t) = \exp\Big(\int_{t_0}^t \lambda(s)\,ds\Big)\Big(\delta + \int_{t_0}^t \rho(s)\,ds\Big).$$

where $\lambda(\cdot)$ is the Lipschitz constant of F (see **A1***, page 4) corresponding to a compact set S containing $W(t)$, $t\in[t_0,T]$ in its interior, end let $W(t)+\gamma(t)\mathcal{B}\subset S$, $t\in[t_0,T]$. Then there is a regulation map $\Phi(x,t,v)\subset G(x,t)$ such that for each $v(\cdot)\in\mathcal{V}$ and $x_0\in X_0$ every solution $x(\cdot)$ of (7.6) starting from x_0 is extendible to $[t_0,T]$ and satisfies*

$$x(t) \in Y(t) + \gamma(t)\mathcal{B},$$

$$x(T) \in M_T + \gamma(T)\mathcal{B}.$$

A regulation map Φ can be defined as in (7.9) but with the function φ given by (6.1) (that is, with $\lambda(t)\mathrm{dist}(x(t),W(t)) + \rho(t))$ in the right-hand side of (7.9).

Proposition 6.1 and the independence of Φ_e of v give that $\Phi_e(x,t)\subset\Phi(x,t,v)$ for every x, t and $v\in V(t)$.

Remark. Clearly, in the above consideration we suppose that the state x of (7.1) is directly and exactly observable. The much more complicated case of incomplete/inaccurate measurement will be considered in a forthcoming paper.

8 Discretization of regulation inclusions

The regulation mappings Φ defined in Section 6 are non-convex valued and only u.s.c. in x, in general. This gives rise to the question how to simulate numerically the trajectories of the closed-loop system (1.10).

It is known [7, 21, 30] that the set-valued version of the Euler discretization scheme is convergent when applied to a differential inclusion with continuous and convex valued right-hand side. This means that the mapping "$h \mapsto \text{Traj}(h)$" is continuous at $h = 0$ in appropriate natural Hausdorff metric, where h is the discretization step, Traj(0) is the set of trajectories of the differential inclusion (starting from certain initial point) and Traj(h), $h > 0$, is the set of trajectories of the corresponding discrete inclusion. Upper semicontinuity of the above mapping was proven under different conditions (weaker than continuity of the right-hand side) and for different classes of discretization schemes (see [8, 19, 20, 26] and the bibliographies there). However, the convexity of the right-hand side plays a crucial role in the above mentioned results. In this section we claim upper semicontinuity of the mapping "$h \mapsto \text{Traj}(h)$" for the closed-loop differential inclusion (1.10), provided that the regulation mapping Φ is defined by (4.8) (that is, in the case of a constant set W).

So we consider the differential inclusion

$$\dot{x} \in \Phi(x,t)\,,\ x(t_0) = x_0\,, \tag{8.1}$$

with Φ given by

$$\Phi(x,t) = \{y \in F(x,t);\ \min_{z \in \mathcal{P}_W x} \langle x - z, y\rangle \ \le \max_{z \in \mathcal{P}_W x} \min_{\xi \in F(x,t)} \langle x - z, \xi\rangle\} \tag{8.2}$$

supposing that conditions **A1** and**A2** are fulfilled, that W is closed and that, in addition, F is continuous with respect to t. Given an integer N, introduce the uniform grid $t_k^N = t_0 + kh$, $h = (T - t_0)/N$, and consider the finite difference formula

$$x_{k+1}^N = x_k^N + h y_k^N\,,\ y_k^N \in \Phi(x_k^N, t_k^N)\,,\ k = 1, \ldots, N\,,$$

with the initial condition $x_0^N = x_0$. The choice of the velocity $y_k^N \in \Phi(x_k^N, t_k^N)$ at every step is arbitrary. Denote by $x^N(\cdot)$ the piecewise linear interpolation of the points (t_k^N, x_k^N), $k = 0, \ldots, N$.

Theorem 8.1 *Under the suppositions of this section the sequence $\{x^N(\cdot)\}_N$ is precompact in $C[t_0, T]$ and every condensation point $x(\cdot)$ is a solution of (8.1).*

Remark. The convergence of the Euler scheme for (8.1) in the case of a tube $W(\cdot)$ dependant on the time (say, with Φ defined by (4.3),(4.5),(4.6)) is still an open problem.

Proof of Theorem 8.1. Since $x^N(\cdot)$ is an Euler approximation corresponding to the differential inclusion (1.6), the condensation point $x(\cdot)$ exists and is a solution to (1.6) (see e.g. [7, Corollary]). Thus we have to prove for a.e t the inequality

$$\min_{z\in\mathcal{P}_W x}\langle x(t)-z,\dot{x}(t)\rangle \le \varphi_0(x(t),t) \tag{8.3}$$

where φ_0 is defined (c.f.(4.7) as

$$\varphi_0(x,t) = \max_{z\in\mathcal{P}_W x}\min_{\xi\in F(x,t)}\langle z-x,\xi\rangle.$$

Let S be a compact set containing $x(t)$, $t\in[t_0,T]$, in its interior and let m correspond to S according to **A1**. Take arbitrarily $t\in(t_0,T)$ such that $\dot{x}(t)$ exists. If $x(t)\in W$ then both sides of (8.3) are equal to zero. Suppose that $x(t)\notin W$ and let $\sigma_1>0$ be such that $t+\sigma_1\le T$, $\sigma_1\le 1$ and

$$\operatorname{dist}(x(s),W)>0 \text{ for every } s\in[t,t+\sigma_1].$$

Obviously φ_0 is u.s.c., therefore there exists a monotone increasing function $\omega(\cdot)$: $[0,+\infty)\mapsto[0,+\infty)$, $\lim_{\alpha\to 0^+}\omega(\alpha)=0$ such that

$$\varphi_0(x,s)\le\varphi_0(x(t),t)+\omega(|x-x(t)|+|s-t|) \tag{8.4}$$

for $x\in S$, $s\in[t,t+\sigma_1]$.

Suppose that (8.3) is not fulfilled and $\gamma>0$ is such that

$$\Omega = \min_{z\in\mathcal{P}_W x(t)}\langle x(t)-z,\dot{x}(t)\rangle \ge \varphi_0(x(t),t)+\gamma. \tag{8.5}$$

Take an arbitrary $\sigma\in(0,\sigma_1]$ such that

$$\omega((m+2)\sigma)+(m^2+2)\sigma^2\le\gamma/2 \tag{8.6}$$

and fix N such that $h<\sigma^2$ and $\|x(\cdot)-x^N(\cdot)\|\le\sigma^2$. Since $x(t)\in\operatorname{int}S$ we can suppose that $x^N(s)\in S$ for $s\in[t,t+\sigma]$. Let p and q (both depending on N and σ) be such that

$$t\in(t_p^N,t_{p+1}^N],\quad t+\sigma\in(t_q^N,t_{q+1}^N].$$

Since $y_k^N \in \Phi(x_k^N, t_k^N)$, there is $z_k^N \in \mathcal{P}_W x_k^N$ such that

$$\langle x_k^N - z_k^N, y_k^N \rangle \le \varphi_0(x_k^N, t_k^N), \quad k = 0, \ldots, N.$$

From (8.4) and (8.5) we obtain for $k = p, \ldots, q$

$$\begin{aligned} \langle x_k^N - z_k^N, y_k^N \rangle &\le \varphi_0(x(t), t) + \omega(|x_k^N - x(t)| + |t_k^N - t|) \\ &\le \varphi_0(x(t), t) + \omega(m\sigma + \sigma^2 + \sigma) \\ &\le \Omega - \gamma + \omega((m+2)\sigma) \le \Omega - \gamma/2. \end{aligned} \tag{8.7}$$

Now consider

$$\begin{aligned} \Delta &= |x(t_q^N) - z_q^N|^2 - |x(t_p^N) - z_p^N|^2 \ge |x_q^N - z_q^N|^2 - |x_p^N - z_p^N|^2 + c\sigma^2 \\ &= \sum_{k=p}^{q-1} (|x_{k+1}^N - z_{k+1}^N|^2 - |x_k^N - z_k^N|^2) + c\sigma^2 \\ &\le \sum_{k=p}^{q-1} (|x_{k+1}^N - z_k^N|^2 - |x_k^N - z_k^N|^2) + c\sigma^2, \end{aligned} \tag{8.8}$$

where c is a constant and we have used that $z_k^N \in \mathcal{P}_W x_k^N$. From (8.7)

$$\begin{aligned} |x_{k+1}^N - z_k^N|^2 &= |x_k^N + h y_k^N - z_k^N|^2 = |x_k^N - z_k^N|^2 + 2h\langle x_k^N - z_k^N, y_k^N \rangle + h^2 |y_k^N|^2 \\ &\le |x_k^N - z_k^N|^2 + 2h\Omega - h\gamma + h^2 m^2, \end{aligned}$$

which combined with (8.6) and (8.8) gives

$$\Delta \le 2\sigma\Omega - \sigma\gamma + o_1(\sigma). \tag{8.9}$$

On the other hand

$$\begin{aligned} \Delta &= |x(t_p^N) + (t_q^N - t_p^N)\dot{x}(t) - z_q^N|^2 - |x(t_p^N) - z_p^N|^2 + o_2(\sigma) \\ &= |x_p^N + (t_q - t_p)\dot{x}(t) - z_q^N|^2 - |x_p^N) - z_p^N|^2 + o_3(\sigma) \\ &\le |x_p^N - z_q^N|^2 - |x_p^N - z_p^N|^2 + 2\sigma\langle x_p^N - z_q^N, \dot{x}(t) \rangle + o_4(\sigma) \\ &\le 2\sigma\langle x(t) - z_q^N, \dot{x}(t) \rangle + o_5(\sigma), \end{aligned}$$

where $o_i(\sigma)$ are functions such that $o_i(\sigma)/\sigma \to 0$ when $\sigma \to 0^+$. Combining with (8.9) we obtain that

$$\langle x(t) - z_q^N, \dot{x}(t) \rangle \le \Omega - \gamma/2 + \theta(\sigma), \tag{8.10}$$

where $\theta(\sigma) \to 0$. If z is a condensation point of $\{z_q^N\}$ when σ tends to zero and correspondingly, N tends to $+\infty$ (notice that q depends on N and σ), then $z \in \mathcal{P}_W x(t)$ and (8.10) implies

$$\Omega - \gamma/2 > \langle x(t) - z, \dot{x}(t) \rangle,$$

which contradicts the definition of Ω by (8.5). Q.E.D.

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