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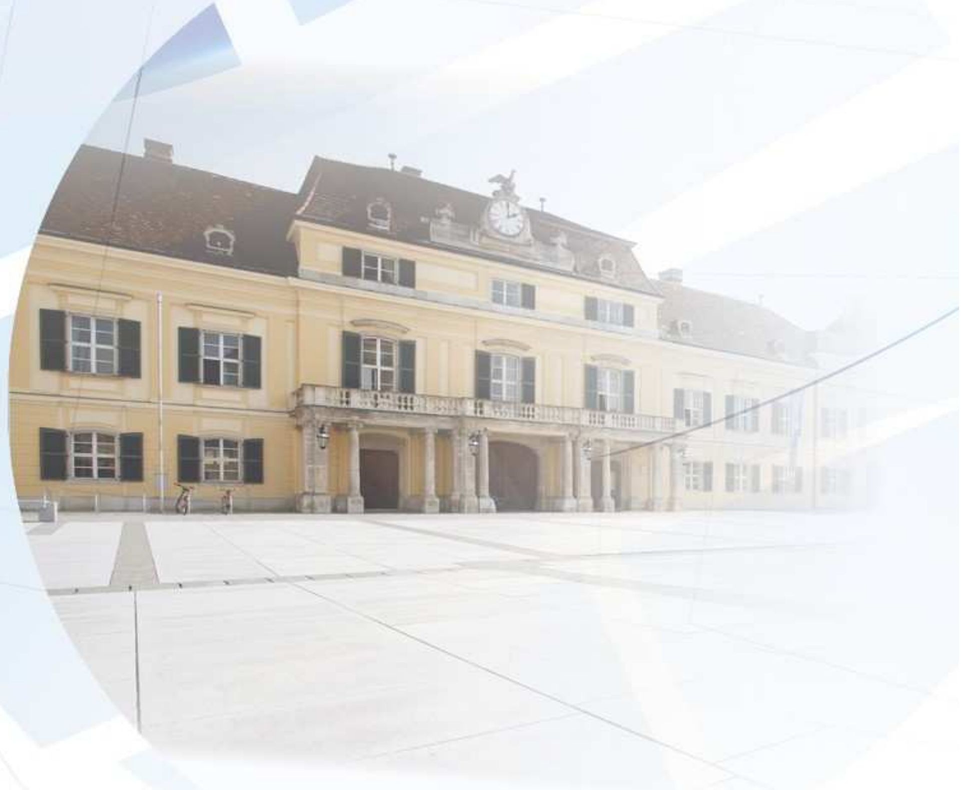

Working Paper

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Foreword

The paper deals with approximate and exact controllability of the wave equation with interior pointwise control acting along the curve specified in advance in the system's spatial domain. The structure of the control input is dual to the structure of the observations describing the measurements of velocity and gradient of the solution of the dual system, obtained from the single pointwise moving sensor. A relevant formalization of such a control problem is discussed, based on transposition. For any given time-interval $[0,T]$ the existence of the curves providing approximate controllability in $H^{-[n/2]-1}(\Omega) \times H^{-[n/2]-2}(\Omega)$ (where n stands for the spatial dimension) is established with controls from $L^2_{n+1}(0,T)$. The same curves ensure exact controllability in $L^2(\Omega) \times H^{-1}(\Omega)$ if controls are allowed to be selected in $L^{\infty\prime}_{n+1}(0,T)$. Required curves can be constructed to be continuous on $[0,T)$.

The used techniques are based on the observability results involving a priori energy estimates of an instantaneous type.

Controllability of the Wave Equation with Pointwise Moving Control

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Key words: controllability; wave equation; moving pointwise control.

Mathematics Subject Classification: 93, 35.

1. Introduction and Problem Formulation.

Let Ω be a bounded domain of an n-dimensional Euclidean space R^n with a boundary $\partial\Omega$. In Ω we consider the following initial-boundary value problem for the wave equation of an arbitrary dimension:

$$y_{tt} = \Delta y + \mathbf{L}(\hat{x}(\cdot))\hat{v} \qquad \text{in } (0,T)\times\Omega = Q, \tag{1.1}$$

$$y \mid_{\Sigma} = 0, \quad \Sigma = \partial\Omega \times (0,T),$$

$$y \mid_{t=0} = 0, \qquad y_t \mid_{t=0} = 0.$$

In (1.1) the control input is described by the term $\mathbf{L}(\hat{x}(\cdot))\hat{v}$, where $\hat{v} = (v_1, \ldots, v_{n+1})' \in V$ is an $(n+1)$-dimensional control and $V = L^2_{n+1}(0,T)$ or $L^{\infty\prime}_{n+1}(0,T)$. The linear operator $\mathbf{L}(\hat{x}(\cdot))$ specifies the structure of control input along the curve $\hat{x}(\cdot)$,

$$t \rightarrow \hat{x}(t) \in \Omega, \quad t \in [0,T], \tag{1.2}$$

selected in advance.

Most of the work on controllability of distributed parameter systems is devoted to the systems with controls which are distributed either over the system's spatial domain or over the boundary of this domain. Because of their infinite-dimensional nature, distributed controls are not always readily implemented in practice. Therefore, the considerable attention has been given to pointwise controls which are technically easier to realize. To our knowledge the results dealing in this context with controllability of distributed parameter systems of hyperbolic type have been restricted only by the case of *stationary* pointwise control, namely, when

$$\mathbf{L}(\hat{x}(t)) = v(t)\,\delta(x - \bar{x}), \quad \text{i.e.} \quad \hat{x}(\cdot) \equiv \bar{x}.$$

The regularity of the hyperbolic time-invariant systems with stationary pointwise control has been studied in [16, 8, 11, 19]. Exact controllability of the wave equation in the space Φ', where Φ is defined as the completion of the space of smooth initial conditions ϕ^0, ϕ^1 with $\phi^0 = 0$ on $\partial\Omega$ in the norm $(\int_0^T \phi^2(\bar{x}, t)dt)^{1/2}$ (with ϕ being the solution of the wave equation associated with ϕ^0, ϕ^1) has been established by means of the Hilbert Uniqueness Method in [11]. A number of results concerning approximate controllability for the one and two dimensional wave equations with stationary pointwise control has been obtained in [5] on the basis on non-harmonic analysis.

Due to finite speed of propagation, the stationary pointwise controls are able to ensure exact controllability only if T is big enough. Another difficulty arising here is that related to the multiplicity of the eigenvalues of the system in question, which is often either unknown or unbounded. Introduction of moving pointwise controls allows to overcome this difficulty. Furthermore, this type of control does not depend upon the speed of propagation: for any required time-interval a special curve for control can be found.

The main aim of this paper is to study – on the basis of a relevant formalization – both approximate and exact controllability of the system (1.1)-(1.2) in the case when the control operator $\mathbf{L}(\hat{x}(\cdot))$ is dual to the observation operator $\mathbf{G}(\hat{x}(\cdot))$ representing at every instant of time the values of the velocity and gradient of the solution φ of the homogeneous dual system along the curve $\hat{x}(t)$, $t \in [0,T]$, namely,

$$\mathbf{G}(\hat{x}(\cdot))\varphi = \begin{pmatrix} \mathbf{G}_x(\hat{x}(\cdot))\varphi_x \\ \mathbf{G}_t(\hat{x}(\cdot))\varphi_t \end{pmatrix}, \tag{1.3}$$

$$\mathbf{G}_x(\hat{x}(\cdot))\varphi_x = (\varphi_{x_1}(\hat{x}(\cdot), \cdot), \ldots, \varphi_{x_n}(\hat{x}(\cdot), \cdot))', \tag{1.4}$$

$$\mathbf{G}_t(\hat{x}(\cdot))\varphi_t = \varphi_t(\hat{x}(\cdot), \cdot). \tag{1.5}$$

We stress that the observations of type (1.3)-(1.5) provide the information about the *energy* E of the *conservative* dual system which is one of its most important physical characteristics,

$$E = \int_\Omega (\varphi_t^2(x,t) + \varphi_x^2(x,t))dx = \text{const}, \quad t \in [0,T],$$

$$\varphi_x = (\varphi_{x_1}, \ldots, \varphi_{x_n})', \quad \varphi_x^2 = \| \varphi_x \|^2_{R^n} .$$

This makes such a type of observations be of considerable interest and, consequently, leads to the structure of control input that is of our interest in this paper. Thus, the *aim of our study* is controllability of the system (1.1), (1.2) with the control operators that, by duality, can formally be represented as follows,

$$\mathbf{L}(\hat{x}(\cdot))\hat{v} = \mathbf{L}_x(\hat{x}(\cdot))v^{(n)} + \mathbf{L}_t(\hat{x}(\cdot))v_{n+1}, \tag{1.6}$$

$$\mathbf{L}_x(\hat{x}(\cdot))v^{(n)} = \nabla\, \delta(x - \hat{x}(\cdot)) \circ v^{(n)}, \quad v^{(n)} = (v_1, \ldots, v_n)', \tag{1.7}$$

$$\mathbf{L}_t(\hat{x}(\cdot))v_{n+1} = \frac{\partial}{\partial t}\, (\delta(x - \hat{x}(\cdot)) \circ v_{n+1}), \tag{1.8}$$

where the symbol "$\circ$" corresponds the duality associated with V (see the next section for details). This type of control – being finite-dimensional at every moment of time – allows, however, to obtain a number of both approximate and exact controllability results. For any given time interval $[0,T]$ the existence of the curves providing the system (1.1)-(1.2), (1.6)-(1.8) with approximate controllability in $H^{-[n/2]-1}(\Omega)\times H^{-[n/2]-2}(\Omega)$ is established for the case when controls belong to $L^2_{n+1}(0,T)$ (Theorems 2.1 and 2.2). It is also proved that if controls are selected from $L^{\infty\prime}_{n+1}(0,T)$, the same curves ensure exact controllability in $L^2(\Omega)\times H^{-1}(\Omega)$ (Theorem 2.3). The used techniques are based on the "energy" estimates for solutions that are treated here as ones of an instantaneous type.

Remark 1.1. The control operators (1.7), (1.8) are similar (to some extent) to those used in

[4] by L. F. Ho, who studied the problem of exact controllability for the one-dimensional wave equation with locally distributed controls.

Remark 1.2. Martin in [14] suggested an approach based on the results of Sakawa [18] (based, in turn, on harmonic analysis) to the controllability and observability problems for the one-dimensional parabolic systems with scanning sensors (for observability of the one-dimensional heat equation see also [10]). The optimal control problems for the parabolic systems with different types of moving and scanning controls have been studied in [15, 1], see also [6].

Remark 1.3. The details about the space $L^{\infty\prime}_{n+1}(0,T)$ can be found in [3].

The paper is organized as follows. The next section lists the main results of the paper: Theorems 2.1-2.3. On the basis of transposition [13] Section 3 discusses the regularity of the solutions of the system (1.1). It is shown that although the initial-boundary value problem (1.1) with the non-smooth right-hand sides (1.6)-(1.8) is allowed to have the solutions discontinuous in time in the spaces of interest, the mapping between them and the terminal conditions is *injective*. A similar situation with boundary controls has been discussed in [9]. The proofs of Theorems 2.1-2.2 are given in Section 4. Theorem 2.3 is proved in Section 5. The schemes of all the proofs are based on the framework of the general duality theory of observation and control and on the observability results involving the estimates in the norm of $L^{\infty}_{n+1}(0,T)$, obtained in [7].

We proceed now with the formulation of the main results.

2. Main results

Let us recall first two well-known definitions of controllability.

Definition 2.1. The system (1.1) is said to be approximately controllable in the Hilbert space H if its attainable set

$$Y = \{ \ \{y\mid_{t=T},\ y_t\mid_{t=T}\} \ \mid \ y \ \text{satisfies} \ (1.1)\}$$

at $t = T$ is dense in H.

Definition 2.2. The system (1.1) is said to be exactly controllable in the Hilbert space H if $Y = H$.

The formulation of the main results requires a number of auxiliary notations. We begin by the introduction of the adjoint system associated with (1.1):

$$\varphi_{tt} = \Delta\varphi + f \qquad \text{in } Q, \tag{2.1}$$

$$\varphi \mid_{\Sigma} = 0,$$

$$\varphi \mid_{t=T} = \varphi_0, \qquad y_t \mid_{t=T} = \varphi_1,$$

assuming that $\{\varphi_0,\ \varphi_1,\ f\}$, (we use the notation $\{(\cdot),(\cdot),(\cdot)\} = ((\cdot),(\cdot),(\cdot))'$) belong to the Hilbert space

$$\Phi_0 \times \Phi_1 \times F = H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega) \times H_D^{[n/2]+1}(Q), \tag{2.1$'$}$$

$$H_D^s(\Omega) = \{\phi \mid \phi \in H^s(\Omega),\ \phi\mid_{\partial\Omega} = \ldots = \Delta^{[(s-1)/2]}\phi \mid_{\partial\Omega} = 0\},$$

$$H_D^s(Q) = \{f \mid f \in H^s(Q),\ f\mid_{\partial\Omega\times[0,T]} = \ldots = \Delta^{[(s-1)/2]} f \mid_{\partial\Omega\times[0,T]} = 0\}.$$

Remark 2.1. The system (2.1), (2.1)$'$ admits a unique solution from the space $H^{[n/2]+2}(Q)$ and the following estimate holds true (see [17] for details):

$$\| \sum_{p=0}^{[n/2]+2} \frac{\partial^p \varphi}{\partial t^p} \|_{H^{[n/2]+2-p}(\Omega)} \leq$$

$$\leq \text{ const } (\| \varphi_0 \|_{H^{[n/2]+2}(\Omega)} + \| \varphi_1 \|_{H^{[n/2]+1}(\Omega)} + \| f \|_{H^{[n/2]+1}(Q)}) \quad \forall t \in [0,T].$$

Let $\mathbf{S}(t),\ t \in [0,T]$ denote the operator representing the general solution of the system (2.1),

$$\mathbf{S}(\cdot) : \Phi_0 \times \Phi_1 \times F \to W, \quad \mathbf{S}(\cdot)\{\varphi_0,\varphi_1,f\} = \mathbf{S}_1(\cdot)\{\varphi_0,\varphi_1\} + \mathbf{S}_2(\cdot)f = \varphi,$$

where W is the Hilbert space consisting of all the solutions of (2.1) taken on $(0,T)$ endowed with the norm

$$\| \varphi \|_W = \left(\| \varphi_0 \|^2_{H^{[n/2]+2}(\Omega)} + \| \varphi_1 \|^2_{H^{[n/2]+1}(\Omega)} + \| f \|^2_{H^{[n/2]+1}(Q)}\right)^{1/2}.$$

Set next

$$\mathbf{S}_x(\cdot): \Phi_0 \times \Phi_1 \times F \to W_x, \quad \mathbf{S}_x(\cdot)\{\varphi_0, \varphi_1, f\} = \varphi_x,$$

$$\mathbf{S}_t(\cdot): \Phi_0 \times \Phi_1 \times F \to W_t, \quad \mathbf{S}_t(\cdot)\{\varphi_0, \varphi_1, f\} = \varphi_t,$$

where W_x, W_t are the spaces for the gradients and the velocities of the solutions of (2.1) associated with W.

The controllability problem for the system (1.1), (1.6)-(1.8) – being dual for the observability problem with scanning observations (1.3)-(1.5) – requires the introduction of controls through the respective functionals. Therefore, we introduce (formally for a while) two linear continuous functionals defined by controls $\hat{v} \in V$ on $\Phi_0 \times \Phi_1 \times F$:

$$I_x(v^{(n)}, \varphi) = < v^{(n)}, \ \mathbf{G}_x(\hat{x}(\cdot))\varphi_x >, \tag{2.2}$$

$$I_t(v_{n+1}, \varphi) = < v_{n+1}, \ \mathbf{G}_t(\hat{x}(\cdot))\varphi_t >, \tag{2.3}$$

Remark 2.2. Everywhere in this paper the symbol $< u^*, u >$ is used to describe the duality between the Banach space U $(U \ni u)$ and its dual U' $(U' \ni u^*)$ – the space of the linear continuous functional on U. In more complicated cases we shall write explicitly – $< u^*, u >_U$.

In accordance with (2.2), (2.3), the operator $\mathbf{L}_x(\hat{x}(\cdot))$ is defined by the following identity:

$$< v^{(n)}, \ \mathbf{G}_x(\hat{x}(\cdot))\varphi_x > = < \mathbf{L}_x(\hat{x}(\cdot))v^{(n)}, \ \mathbf{S}(\cdot)\{\varphi_0, \varphi_1, f\} > \tag{2.4}$$

$$\forall \hat{v} \in V, \quad \forall \{\varphi_0, \varphi_1, f\} \in \Phi_0 \times \Phi_1 \times F.$$

In turn, the operator $\mathbf{L}_t(\hat{x}(\cdot))$ is defined by

$$< v_{n+1}, \ \mathbf{G}_t(\hat{x}(\cdot))\varphi_t > = < \mathbf{L}_t(\hat{x}(\cdot))v_{n+1}, \ \mathbf{S}(\cdot)\{\varphi_0, \varphi_1, f\} > \tag{2.5}$$

$$\forall \hat{v}(\cdot) \in V, \quad \forall \{\varphi_0, \varphi_1, f\} \in \Phi_0 \times \Phi_1 \times F,$$

so that

$$\{\mathbf{L}_x(\hat{x}(\cdot)),\ \mathbf{L}_t(\hat{x}(\cdot))\} : V \ \rightarrow \ W'. \tag{2.5$'$}$$

Remark 2.3. The observations (1.4)-(1.5) and, consequently, the control operators (2.4)-(2.5)$'$ require a considerable regularity of the solutions of the dual problem (2.1) increasing with the growth of n. We shall discuss this in the next section.

Theorem 2.1 (Approximate Controllability). Let the boundary $\partial\Omega$ be of the class $C^{[n/2]+2}$. Then there exists a class of curves $\hat{x}(\cdot)$ continuous on $[0,T)$ that make the system (1.1), (1.2), (1.6), (2.4)-(2.5)$'$ with $V = L^2_{n+1}(0,T)$ be approximate controllable in $H^{-[n/2]-1}(\Omega) \times H^{-[n/2]-2}(\Omega)$.

Denote by $\{\lambda_k\}_{k=1}^{\infty}$, $\{\omega_k(\cdot)\}_{k=1}^{\infty}$ the sequences of eigenvalues and respective eigenfunctions (orthonormalized in $L^2(\Omega)$) for the spectral problem

$$\Delta\omega_k(\cdot) \ = \ -\lambda_k\, \omega_k(\cdot), \quad \omega_k(\cdot) \in H_D^{[n/2]+2}(\Omega),$$

$$< \omega_k(\cdot), \omega_m(\cdot) > \ = \ \delta_{km},$$

so that

$$\lambda_{k+1} \ \geq \ \lambda_k; \ \ \lambda_k \rightarrow +\infty, \ \ k \rightarrow +\infty\, ; \ \ \delta_{km} = \begin{cases} 1, & k = m, \\ 0, & k \neq m. \end{cases}$$

Let $L^2_{(i)}(\Omega)$ $(i = 1, \ldots)$ stand for the subspace of $L^2(\Omega)$ spanned by the functions

$$\omega_k(\cdot), \quad k = 1, \ldots i.$$

Recalling that the eigenfunctions are orthogonal in any of the spaces $H_D^{[n/2]+s}(\Omega)$, $s = 1,2$ with respect to the scalar product

$$[v_1, v_2] = \sum_{k=1}^{\infty} \lambda_k^{[n/2]+s} v_{1k}\, v_{2k}, \tag{2.6}$$

$$v_{1k} = \int_\Omega v_1(x)\omega_k(x)\, dx, \quad v_{2k} = \int_\Omega v_2(x)\omega_k(x)\, dx,$$

we can introduce the (orthogonal) projection operator $\mathbf{P}_{(i)}$ from $H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)$ endowed with the norm associated with (2.6) on its finite-dimensional subspace

$$(H_D^{[n/2]+2}(\Omega) \bigcap L^2_{(i)}(\Omega)) \times (H_D^{[n/2]+1}(\Omega) \bigcap L^2_{(i)}(\Omega)).$$

Denote the latter by $H_{(i)}$.

The next result may be interpreted as an intermediate one between approximate and exact controllability. In fact, Theorem 2.2 establishes exact controllability of the projections of (1.1) on all the finite-dimensional subspaces dual to those spanned by eigenfunctions. In order to do this, it suffices in every particular case to construct moving controls only on the part of the interval $[0, T]$, namely, where the curve $\hat{x}(\cdot)$ is continuous.

Theorem 2.2 (Exact Controllability in Finite Dimensions). Let the boundary $\partial\Omega$ be of the class $C^{[n/2]+2}$. Then the assertion of Theorem 2.1 holds true for controls $\hat{v}$ vanishing in some neighborhood of $t = T$. Moreover, let the spaces $H_D^{[n/2]+s}(\Omega)$, $s = 1,2$ be endowed with the norm associated with (2.6) and let $H_{(i)}$ be identified with its dual. Then for any given sequence $0 < \delta_i \to 0$ there exists such a class of curves $\hat{x}(\cdot)$ that for any $i = 1, \ldots$ the projection of the attainable set Y of the system (1.1)-(1.2), (1.6), (2.4)-(2.5)$'$ on $H'_{(i)}$ with controls $\hat{v} \in L^2_{n+1}(0,T)$ vanishing in $[T - \delta_i, T]$ coincides with $H'_{(i)}$, i.e.,

$$\mathbf{P}^*_{(i)} Y = H'_{(i)} = H_{(i)}, \quad i = 1, \ldots .$$

Remark 2.4. Although in Theorem 2.2 the basis of the eigenfunctions has been considered, the result remains true (by applying *Galerkin's* method) for any basis in $H_D^{[n/2]+2}(\Omega)$, orthonormalized in $L^2(\Omega)$.

The following remark plays an important role in the statement of Theorem 2.3.

Remark 2.5. Let H_1 and H_2 be a pair of the Hilbert spaces such that H_1 embeds continuously into H_2 and let H_1 be dense in H_2. In this situation we can identify H_2' with some subspace of H_1'.

The following theorem states the main result of the paper with respect to exact controllability.

Theorem 2.3 (Exact Controllability). Let the boundary $\partial\Omega$ be of the class $C^{[n/2]+2}$. Then there exists a class of curves $\hat{x}(\cdot)$ continuous on $[0,T)$ that make the system (1.1), (1.2), (1.6), (2.4)-(2.5)$'$ with $V = L^{\infty\prime}_{n+1}(0,T)$ be exactly controllable in $L^2(\Omega) \times H^{-1}(\Omega)$.

Corollary 2.1. Let all the conditions of Theorem 2.3 be fulfilled. Then, in order to drive the system (1.1) to the terminal state $\{y\,|_{t=T},\ y_t\,|_{t=T}\} \in L^2(\Omega) \times H^{-1}(\Omega)$, it suffices to use bounded controls satisfying

$$\| \hat{v} \|_{L^{\infty\prime}_{n+1}(0,T)} \leq \ M_0\ E^{*1/2}(y\,|_{t=T}, y_t\,|_{t=T}),$$

where $E^{*1/2}(\cdot,\cdot)$ is the norm dual to the *energy* one, namely, to

$$E^{1/2}(\varphi\,|_{t=T}) \ = \ E^{1/2}(\varphi\,|_{t=T}, \varphi_t\,|_{t=T}) \ = \ (\int_\Omega (\varphi_x^2(x,T) + \varphi_t^2(x,T))dx)^{1/2}$$

and the constant M_0 is defined by the estimate (4.3), (4.7) providing exact observability of the dual system.

Remark 2.6. Due to Poincaré's inequality,

$$\| \varphi(\cdot,t) \|_{L^2(\Omega)} \leq \ \text{const}\ \| \varphi_x(\cdot,t) \|_{L^2(\Omega)},$$

the "energy" norm is equivalent to the standard one of the space $H_0^1(\Omega)$.

3. Auxiliary results.

In this section the regularity of the solutions of the systems (1.1), (1.6)-(1.8) is discussed, based on *transposition* [13]. Special attention is given to the treatment of the terminal conditions. Indeed, the non-smoothness of control terms might generate, in general, the discontinuity in time of the solutions in the spaces of interest. It will be shown, however, that the mapping between the triplet $\{y\mid_{t=0}, y_t\mid_{t=0}, y\}$ and the terminal conditions is injective.

Consider the following auxiliary problem

$$\varphi_{tt} = \Delta\varphi + f \qquad \text{in } Q, \tag{3.1}$$

$$\varphi\mid_{\Sigma} = 0,$$

$$\varphi\mid_{t=T} = 0, \qquad \varphi_t\mid_{t=T} = 0,$$

assuming the boundary $\partial\Omega$ to be of the class $C^{[n/2]+2}$.

Denote by X_Δ the space of all the solutions of (3.1) when f ranges $H_D^{[n/2]+1}(Q)$. Endow X_Δ with the norm

$$\|\varphi\|_{X_\Delta} = \|f\|_{H_D^{[n/2]+1}(Q)}\,.$$

Note $X_\Delta \subset W$. ¿From Remark 2.1 it follows that the operator

$$\varphi \longrightarrow \frac{\partial^2\varphi}{\partial t^2} + \Delta\varphi$$

is an isomorphism of X_Δ onto $H_D^{[n/2]+1}(Q)$. Following [13], we shall establish that if

$$\mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{L}_t(\hat{x}(\cdot))v_{n+1} \in X'_\Delta$$

the following identities

$$< \bar{y}, \varphi_{tt} - \Delta\varphi > = < \mathbf{L}_x(\hat{x}(\cdot))v^{(n)}, \varphi >, \quad \forall\,\varphi \in X_\Delta, \tag{3.2}$$

$$< \bar{\bar{y}}, \varphi_{tt} - \Delta\varphi > = < \mathbf{L}_t(\hat{x}(\cdot))v_{n+1}, \varphi >, \quad \forall\,\varphi \in X_\Delta, \tag{3.2$'$}$$

define by transposition the unique solutions from $H^{-[n/2]-1}(Q)$ of two following initial-boundary value problems:

$$\begin{cases} \bar{y}_{tt} = \Delta\bar{y} + \mathbf{L}_x(\hat{x}(\cdot))v^{(n)} & \text{in } Q, \\ \bar{y}\,|_\Sigma = 0, \quad \bar{y}\,|_{t=0} = 0, \quad \bar{y}_t\,|_{t=0} = 0, \end{cases} \tag{3.3}$$

$$\begin{cases} \bar{\bar{y}}_{tt} = \Delta\bar{\bar{y}} + \mathbf{L}_t(\hat{x}(\cdot))v_{n+1} & \text{in } Q, \\ \bar{\bar{y}}\,|_\Sigma = 0, \quad \bar{\bar{y}}\,|_{t=0} = 0, \quad \bar{\bar{y}}_t\,|_{t=0} = 0. \end{cases} \tag{3.4}$$

Theorem 3.1. Let $\hat{x}(t)$ be an arbitrary measurable function on $[0,T]$ such that $\hat{x}(t) \in \Omega$ *a.e.* on $[0,T]$. Let the boundary $\partial\Omega$ be of class $C^{[n/2]+2}$. Then the problems (3.3) and (3.4) with $V = L^2_{n+1}(0,T)$ or $L^{\infty\prime}_{n+1}(0,T)$ admit unique solutions from the space $H^{-[n/2]-1}(Q)$ and the mapping

$$\hat{v} \to \bar{y} + \bar{\bar{y}}$$

is linear continuous from $L^2_{n+1}(0,T)$ or $L^{\infty\prime}_{n+1}(0,T)$ into $H^{-[n/2]-1}(Q)$.

Proof. Step 1. We recall first that all the solutions of the system (2.1) (and, in particular (3.1)) belong to $C^1(\bar{Q})$ (see [17] for details). Therefore, the observations (1.3)-(1.5) are well-defined. Furthermore (see Remark 2.1),

$$\mathbf{S}(\cdot):\ H_D^{[n/2]+2}(\Omega)\times H_D^{[n/2]+1}(\Omega)\times H_D^{[n/2]+1}(Q)\ \to\ C([0,T];\ H^{[n/2]+2}(\Omega))\ \subset\ H^{[n/2]+2}(Q),$$

$$\mathbf{S}_x(\cdot):\ H_D^{[n/2]+2}(\Omega)\times H_D^{[n/2]+1}(\Omega)\times H_D^{[n/2]+1}(Q)\ \to\ C_n([0,T];\ H^{[n/2]+1}(\Omega)),$$

$$C_n([0,T];\ H^{[n/2]+1}(\Omega))\ =\ \underbrace{C([0,T];\ H^{[n/2]+1}(\Omega))\ \times\ \ldots\times\ C([0,T];\ H^{[n/2]+1}(\Omega))}_{n},$$

$$\mathbf{S}_t(\cdot):\ H_D^{[n/2]+2}(\Omega)\times H_D^{[n/2]+1}(\Omega)\times H_D^{[n/2]+1}(Q)\ \to\ C([0,T];\ H^{[n/2]+1}(\Omega)),$$

and all the above mappings are continuous.

Step 2. The following assertion deals with the regularity of the control operator $\mathbf{L}(\hat{x}(\cdot))$.

Lemma 3.1. Let all the assumptions of Theorem 3.1 be fulfilled. Then the mapping

$$\hat{v} \rightarrow \mathbf{L}_x(\hat{x}(\cdot))v^{(n)} + \mathbf{L}_t(\hat{x}(\cdot))v_{n+1} \tag{3.5}$$

is linear continuous from $L^2_{n+1}(0,T)$ or $L^{\infty'}_{n+1}(0,T)$ into W'.

Proof of Lemma 3.1. Indeed, from (2.4) we obtain for $V = L^2_{n+1}(0,T)$:

$$< \mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{S}(\cdot)\{\varphi_0, \varphi_1, f\} > = \int_0^T v^{(n)'}(t)\ \mathbf{G}_x(\hat{x}(t))\varphi_x dt.$$

Hence, taking into account that, due to the embedding theorems ([13, 17]),

$$C(\bar{\Omega}) \subset H^{[n/2]+1}(\Omega),$$

we obtain

$$< \mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{S}(\cdot)\{\varphi_0, \varphi_1, f\} >^2 \leq \tag{3.6}$$

$$\leq \| v^{(n)} \|^2_{L^2_n(0,T)} \int_0^T \| (\varphi_{x_1}(\hat{x}(t),t),\ \ldots,\ \varphi_{x_n}(\hat{x}(t),t))' \|^2_{R^n} dt \leq$$

$$\leq \text{ const } \| v^{(n)} \|^2_{L^2_n(0,T)} \int_0^T \| (\varphi_{x_1}(\cdot,t),\ \ldots,\ \varphi_{x_n}(\cdot,t))' \|^2_{H_n^{[n/2]+1}(\Omega)} \leq$$

$$\leq \text{ const } \| v^{(n)} \|^2_{L^2_n(0,T)} \times \| \varphi \|^2_W \quad \forall\, v^{(n)} \in L^2_n(0,T),\ \forall\, \varphi \in W.$$

The last estimate and the similar one for $\mathbf{L}_t(\hat{x}(\cdot))$ yield (3.5) when $V = L^2_{n+1}(0,T)$. In turn, when $V = L^{\infty\prime}_{n+1}(0,T)$, we obtain

$$|< \mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{S}(\cdot)\{\varphi_0, \varphi_1, f\} >| \leq \tag{3.7}$$

$$\leq \| v^{(n)} \|_{L^{\infty\prime}_n(0,T)} \times \| (\varphi_{x_1}(\hat{x}(\cdot),\cdot),\ \ldots,\ \varphi_{x_n}(\hat{x}(\cdot),\cdot))' \|_{L^{\infty}_n(0,T)} \leq$$

$$\leq \text{ const } \| v^{(n)} \|_{L^{\infty\prime}_n(0,T)} \times \| (\varphi_{x_1}(\cdot,\cdot),\ \ldots,\ \varphi_{x_n}(\cdot,\cdot))' \|_{C_n([0,T];\, H^{[n/2]+1}(\Omega))} \leq$$

$$\leq \text{ const } \| v^{(n)} \|_{L_n^{\infty\prime}(0,T)} \times \| \varphi \|_W \quad \forall\, v^{(n)} \in L_n^{\infty\prime}(0,T), \ \forall\, \varphi \in W.$$

The proof of Lemma 3.1 is completed. □

Remark 3.1. From the chains (3.6) and (3.7) it also follows that for $\hat{v} \in V = L^2_{n+1}(0,T)$:

$$\mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{L}_t(\hat{x}(\cdot))v_{n+1} \in H^{-[n/2]-2}(Q), \tag{3.8}$$

and for $\hat{v} \in V = L^{\infty\prime}_{n+1}(0,T)$:

$$\mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{L}_t(\hat{x}(\cdot))v_{n+1} \in H^{-[n/2]-3}(Q). \tag{3.9}$$

Step 3. We have proved that the right-hand sides of the identities (3.2) and (3.2)′ are linear continuous functionals on X_Δ when $V = L^2_{n+1}(0,T)$ or $L^{\infty\prime}_{n+1}(0,T)$. Now the proof of Theorem 3.1 follows by applying the transposition argument [13]. □

We note that Theorems 3.1 has not established the continuity of the solutions of (3.3) and (3.4) in time. However the terminal conditions for the systems (3.3) and (3.4) can be determined as follows.

Multiply both parts of the equations (3.3) and (3.4) by an arbitrary solution of the system (2.1) subjected to (2.1)′ with $f = 0$ and apply formally the Green's formula. This yields

$$< \varphi_1, \bar{y}\,|_{t=T}> - < \varphi_0, \bar{y}_t\,|_{t=T}> = < -\mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \mathbf{S}_1(\cdot)\{\varphi_0,\varphi_1\} >, \tag{3.10}$$

$$< \varphi_1, \bar{\bar{y}}\,|_{t=T}> - < \varphi_0, \bar{\bar{y}}_t\,|_{t=T}> = < -\mathbf{L}_t(\hat{x}(\cdot))v_{n+1},\ \mathbf{S}_1(\cdot)\{\varphi_0,\varphi_1\} > \tag{3.11}$$

$$\forall\, \{\varphi_0,\varphi_1\} \in H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega).$$

The proof of Theorem 3.1 yields the following assertion.

Theorem 3.2. Let $\hat{x}(\cdot)$ be an arbitrary measurable function such that $\hat{x}(t) \in \Omega$ *a.e.* on $[0,T]$. Let the boundary $\partial\Omega$ be of class $C^{[n/2]+2}$. Then the mapping

$$\hat{v} \rightarrow \{\bar{y} + \bar{\bar{y}},\ \bar{y}\,|_{t=T} + \bar{\bar{y}}\,|_{t=T},\ \bar{y}_t\,|_{t=T} + \bar{\bar{y}}_t\,|_{t=T}\},$$

is linear continuous from $L^2_{n+1}(0,T)$ or $L^{\infty\prime}_{n+1}(0,T)$ into $H^{-[n/2]-1}(Q) \times H^{-[n/2]-1}(\Omega) \times H^{-[n/2]-2}(\Omega)$.

The above type of terminal conditions admits natural interpretation, namely – the identities (3.10) and (3.11) imply the coincidence of the solutions of the equations (3.3) and (3.4) evolving from zero - initial conditions and the solutions of the same equations evolving in backward time from the terminal conditions specified by (3.10) or (3.11).

Indeed, consider, for example, the system (3.3) and let $\hat{y}$ be the solution of the following initial-boundary value problem

$$\hat{y}_{tt} = \Delta\hat{y} + \mathbf{L}_x(\hat{x}(\cdot))v^{(n)} \qquad \text{in } Q, \tag{3.12}$$

$$\hat{y}\mid_\Sigma = 0, \quad \hat{y}\mid_{t=T} = \bar{y}\mid_{t=T}, \quad \hat{y}_t\mid_{t=T} = \bar{y}_t\mid_{t=T},$$

which is treated in the sense of the identity

$$< \hat{y}, \hat{\varphi}_{tt} - \Delta\hat{\varphi} > = < \mathbf{L}_x(\hat{x}(\cdot))v^{(n)}, \hat{\varphi} > + \tag{3.13}$$

$$+ < \hat{\varphi}_t\mid_{t=T}, \bar{y}\mid_{t=T}> - < \hat{\varphi}\mid_{t=T}, \bar{y}_t\mid_{t=T}>, \quad \forall\, \hat{\varphi} \in \hat{X}_\Delta.$$

In the above $\hat{X}_\Delta$ consists of all the solutions of the equation (3.1) subjected to zero - initial condition when f ranges $H_D^{[n/2]+1}(Q)$, namely,

$$\hat{\varphi}_{tt} = \Delta\hat{\varphi} + f \qquad \text{in } Q, \quad f \in H_D^{[n/2]+1}(Q), \tag{3.14}$$

$$\hat{\varphi}\mid_\Sigma = 0, \quad \hat{\varphi}\mid_{t=0} = 0, \quad \hat{\varphi}_t\mid_{t=0} = 0.$$

The following Lemma shows that

$$\bar{y} = \hat{y}. \tag{3.15}$$

Lemma 3.2. Let the boundary $\partial\Omega$ be of class $C^{[n/2]+2}$ and $\bar{y}$, $\bar{\bar{y}}$ be the solutions of the problems (3.3) and (3.4) with $V = L^2_{n+1}(0,T)$ or $L^{\infty\prime}_{n+1}(0,T)$. Then the mappings

$$\bar{y} \rightarrow \{\bar{y}\mid_{t=T},\ \bar{y}_t\mid_{t=T}\}, \qquad \bar{\bar{y}} \rightarrow \{\bar{\bar{y}}\mid_{t=T},\ \bar{\bar{y}}_t\mid_{t=T}\},$$

where the terminal conditions are defined by (3.10) and (3.11) are injective.

Proof. We give the proof for the problem (3.3), (3.10). Observe that $\hat{\varphi}$ admits the following representation:

$$\hat{\varphi} = \varphi^1 + \varphi^2,$$

where

$$\begin{cases} \varphi^1_{tt} = \Delta\varphi^1 \quad \text{in} \quad Q, \\ \varphi^1\mid_{\Sigma} = 0, \quad \varphi^1\mid_{t=T} = \hat{\varphi}\mid_{t=T}, \quad \varphi^1_t\mid_{t=T} = \hat{\varphi}_t\mid_{t=T}, \end{cases}$$

$$\begin{cases} \varphi^2_{tt} = \Delta\varphi^2 + f \quad \text{in} \quad Q, \\ \varphi^2\mid_{\Sigma} = 0, \quad \varphi^2\mid_{t=T} = 0, \quad \varphi^2_t\mid_{t=T} = 0. \end{cases}$$

Thus, one can rewrite (3.13) as follows

$$< \hat{y}, \varphi^2_{tt} - \Delta\varphi^2 > = < \mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \varphi^1 + \varphi^2 > + \tag{3.16}$$

$$+ < \varphi^1_t\mid_{t=T}, \bar{y}\mid_{t=T}> - < \varphi^1\mid_{t=T}, \bar{y}_t\mid_{t=T}>, \quad \forall\, \hat{\varphi} \in \hat{X}_{\Delta}.$$

Let us recall that the set of all the pairs $\{\varphi^1\mid_{t=T},\ \varphi^1_t\mid_{t=T}\}$ embeds into $H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)$ (see Remark 2.1). Hence, we deduce from (3.16), (3.13) and (3.10)

$$< \hat{y}, \varphi^2_{tt} - \Delta\varphi^2 > = < \mathbf{L}_x(\hat{x}(\cdot))v^{(n)},\ \varphi^2 >, \qquad \forall\, \varphi^2 \in X_{\Delta}.$$

This completes the proof. □

4. Proofs of Theorems 2.1-2.2.

The proofs of Theorems 2.1-2.3 are based on the observability results obtained in [7].

Proof of Theorem 2.1. Consider the observability problem described by the system equation (2.1) subjected to (2.1)′ with $f = 0$ assuming that an output $z(t) = \{z_1(t), z_2(t)\}$ represents the values of the gradient and velocity of the solution along the curve $\hat{x}(\cdot)$, so that

$$z_1(t) = \begin{pmatrix} \varphi_{x_1}(\hat{x}(t), t)) \\ \vdots \\ \varphi_{x_n}(\hat{x}(t), t)) \end{pmatrix}, \tag{4.1}$$

$$z_2(t) = \varphi_t(\hat{x}(t), t). \tag{4.2}$$

The following observability result for the system (2.1), (2.1)′ with $f = 0$, (4.1)-(4.2) has been obtained in [7]: there exists such a class of curves $\hat{x}(t) \in \Omega$, $t \in [0, T]$ continuous on $[0, T)$ that the following estimate

$$M_0 \parallel \{\mathbf{G}_x(\hat{x}(\cdot))\varphi_x, \ \mathbf{G}_t(\hat{x}(\cdot))\varphi_t\} \parallel_{L^\infty_{n+1}(0,T)} \geq \ E^{1/2}(\varphi(\cdot, T), \varphi_t(\cdot, T)), \tag{4.3}$$

$$M_0 = \text{const} \ > \ 0$$

is fulfilled for any solution of the system (2.1) with $f = 0$, (2.1)′.

Remark 4.1. The estimate (4.3) represents the result of [8] formulated in the terms of terminal conditions (that is the same for the conservative system).

Observe that the identities (3.10) and (3.11) yield

$$< \varphi_1, \ y \mid_{t=T}> - < \varphi_0, \ y_t \mid_{t=T}> \ = \ - < \mathbf{S}_1^*(\cdot)\mathbf{L}_x(\hat{x}(\cdot))v^{(n)}, \{\varphi_0, \varphi_1\} > \ - \tag{4.4}$$

$$- \ < \mathbf{S}_1^*(\cdot)\mathbf{L}_t(\hat{x}(\cdot))v_{n+1}, \{\varphi_0, \varphi_1\} >$$

$$\forall \ \hat{v} \in L^2_{n+1}(0, T), \quad \forall \{\varphi_0, \varphi_1\} \in H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega),$$

where

$$\mathbf{S}_1^*(\cdot) : W' \ \rightarrow \ H^{-[n/2]-2}(\Omega) \times H^{-[n/2]-1}(\Omega).$$

Combining (4.3) and (4.4) implies, due (2.4)-(2.5), that the attainable set Y of (1.1), namely,

$$Y = \{\{y_0, y_1\} \mid \{y_1, -y_0\} \ = \mathbf{S}_1^*(\cdot)\mathbf{L}_t\hat{x}(\cdot))v_{n+1} \ + \ \mathbf{S}_1^*(\cdot)\mathbf{L}_x(\hat{x}(\cdot))v^{(n)}, \ \ V = L^2_{n+1}(0,T)\},$$

is dense in $H^{-[n/2]-1}(\Omega) \times H^{-[n/2]-2}(\Omega)$ if (4.3) is verified for $\hat{x}(\cdot)$.
This completes the proof of the Theorem 2.1. □

Proof of Theorem 2.2.

Step 1. Let us recall that the general solution of the system (2.1), (2.1)′ with $f = 0$ admits the following representation

$$\varphi(x,t) = \sum_{k=1}^{\infty} c_k(t)\, \omega_k(x), \tag{4.5}$$

where

$$c_k(t) = \varphi_{0k} \cos\sqrt{\lambda_k}\,(t-T) \ + \ \frac{\varphi_{1k}}{\sqrt{\lambda_k}} \sin\sqrt{\lambda_k}\,(t-T), \quad k = 1, \dots ,$$

$$\varphi_{0k} = \int_\Omega \varphi_0(x)\, \omega_k(x) dx, \quad \varphi_{1k} = \int_\Omega \varphi_1(x)\, \omega_k(x) dx.$$

The series (4.5) converges with its first derivatives with respect to x and t in $C(\bar{\Omega})$ uniformly over $t \in [0,T]$ and the following estimate holds [17]:

$$\max_{t\in[0,T]}\{\| \varphi(\cdot,t) \|_{C(\bar{\Omega})}, \ \| \varphi_{x_s}(\cdot,t) \|_{C(\bar{\Omega})}, \ \| \varphi_t(\cdot,t) \|_{C(\bar{\Omega})}\} \ \leq$$

$$\leq \ c_0 \,(\| \varphi_0 \|^2_{H^{[n/2]+2}(\Omega)} \ + \ \| \varphi_1 \|^2_{H^{[n/2]+1}(\Omega)})^{1/2}, \quad s = 1, \dots, n, \quad c_0 = \text{const}.$$

The following assertion was proved in [7] (although it was not explicitly formulated):

Theorem 4.1. Let the closed time-interval $\tau \subset [0,T]$, a constant $\gamma > 0$, $\gamma < (2\sqrt{n+1}\, c_0 \text{ meas}^{1/2}\{\Omega\})^{-1}$ and an integer i be given. Then there exists a finite number of pairs $\{x^i_{(j)}, t^j_i\}_{j=1}^{J(i,\gamma)} \subset \Omega \times \tau$ such that an arbitrary measurable curve $\hat{x}(t)$, $t \in [0,T]$ continuous on τ and satisfying

$$\hat{x}(t^j_i) = x^i_{(j)}, \quad j = 1, \ldots, J(i,\gamma),$$

ensures exact observability of the system (2.1), (2.1)$'$ with $f = 0$, (4.1)-(4.2) on $H_{(i)}$ with the estimate

$$E^{1/2}(\varphi_0, \varphi_1) \leq \frac{\text{meas}^{1/2}\{\Omega\}}{1 - 2\sqrt{n+1} c_0 \gamma \text{ meas}^{1/2}\{\Omega\}} \| (\varphi_x(\hat{x}(\cdot), \cdot), \varphi_t(\hat{x}(\cdot), \cdot))' \|_{C_{n+1}(\tau)} . \tag{4.6}$$

Remark 4.2. There exist infinitely many set of pairs $\{x^i_{(j)}, t^j_i\}_{j=1}^{J(i,\gamma)}$ that can form the skeleton of a curve $\hat{x}(\cdot)$ required in Theorem 4.1.

Remark 4.3. As it follows from (4.6),

$$M_0 = \frac{\text{meas}^{1/2}\{\Omega\}}{1 - 2\sqrt{n+1} c_0 \gamma \text{ meas}^{1/2}\{\Omega\}}. \tag{4.7}$$

Step 2. Select an arbitrary sequence of positive numbers $\{\delta_s\}_{s=1}^{\infty}$

$$\delta_i \to 0 \quad \text{when} \quad i \to 0.$$

Set $\tau_i = [0,\ T - \delta_i]$ and take any curve $\hat{x}(\cdot)$ such that the assertion of Theorem 4.1 holds true (how to do this is described in [7]) for all $\tau = \tau_i$, $i = 1, \ldots$. Then, the equality

$$(\varphi_{x_1}(\hat{x}(t), t),\ \ldots,\ \varphi_{x_n}(\hat{x}(t), t),\ \varphi_t(\hat{x}(t), t)) = 0 \quad a.e. \text{ in } \quad [0,\ T - \delta_i]$$

implies that $\varphi \equiv 0$ in $\Omega \times [0,T]$ if

$$\{\varphi \mid_{t=T},\ \varphi_t \mid_{t=T}\} \in L^2_{(i)}(\Omega) \times L^2_{(i)}(\Omega).$$

Therefore for any $\hat{v} \in L^2_{n+1}(0,\ T)$ such that $\hat{v} \equiv 0$ on $[T - \delta_i, T]$, due to the duality relations, namely,

$$\int_0^{T-\delta_i} \hat{v}'(t)\{\mathbf{G}(\hat{x}(\cdot))\varphi_x, \mathbf{G}_t(\hat{x}(\cdot))\varphi_t\}\, dt =$$

$$= < \mathbf{S}_1^*(\cdot)\mathbf{L}_x(\hat{x}(\cdot))v^{(n)}, \{\varphi_0, \varphi_1\} > + < \mathbf{S}_1^*(\cdot)\mathbf{L}_t(\hat{x}(\cdot))v_{n+1}, \{\varphi_0, \varphi_1\} >,$$

we obtain (identifying $H_{(i)}$ with its dual) the coincidence of two following finite-dimensional spaces

$$\mathbf{P}_{(i)}^* Y = \mathbf{P}_{(i)}^*\{\{y_0, y_1\} \mid \{y_1, -y_0\} = \mathbf{S}_1^*(\cdot)\mathbf{L}_t(\hat{x}(\cdot))v_{n+1} + \mathbf{S}_1^*(\cdot)\mathbf{L}_x(\hat{x}(\cdot))v^{(n)},$$

$$\hat{v} \in L^2_{n+1}(0, T), \ \hat{v} \equiv 0 \text{ when } t \in [T - \delta_i, T]\} = H'_{(i)} = H_{(i)}.$$

This completes the proof of Theorem 2.2. □

5. Proof of Theorem 2.3.

The scheme of the proof of Theorem 2.3 is, in fact, as much the same as the proof given in [2] (pp. 194-195) for the duality relations of the general type. The proof given below is adapted for our particular problem and is based on the estimate (4.3) involving $L^\infty_{n+1}(0,T)$–norm and the regularity results of Section 3.

Step 1. Set

$$Z(0,T) = \{z \mid z(\cdot) = \{\varphi_x(\hat{x}(\cdot), \cdot), \varphi_t(\hat{x}(\cdot), \cdot)\}, \ \{\varphi_0, \varphi_1\} \in H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega), \ f = 0\},$$

i.e., $Z(0,T)$ is the set of all the possible outputs of the system (2.1), (2.1)$'$ with $f = 0$, (1.3)-(1.5). Due to the embedding theorems [13, 17], $Z(0,T) \subset L^\infty_{n+1}(T)$. Designate by $Z^\infty(0,T)$ the completion of $Z(0,T)$ in the norm of $L^\infty_{n+1}(0,T)$, so that $(L^\infty_{n+1}(0,T))' = Z^{\infty\prime}(0,T)$.

Denote

$$\mathbf{P} = \mathbf{G}(\cdot)\mathbf{S}_1(\cdot): H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega) \rightarrow Z(0,T).$$

Let us assume that the curve $\hat{x}(\cdot)$ has already been constructed according to [4] in order to verify (4.3). Then, due to the estimate (4.3), the operator $\mathbf{P}$ is injective and the inverse operator into $H_0^1(\Omega) \times L^2(\Omega)$, i. e.,

$$\mathbf{P}^{-1} : Z(0,T) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$$

is bounded on $Z(0,T)$. Hence, without loss of generality, we may consider $\mathbf{P}^{-1}$ on the completion of $Z(0,T)$ in the norm of $L_{n+1}^{\infty}(0,T)$, namely,

$$\mathbf{P}^{-1} : Z^{\infty}(0,T) \rightarrow H_0^1(\Omega) \times L^2(\Omega).$$

Endow the range of $\mathbf{P}^{-1}$ with the "energy" norm.

Step 2. The inequality

$$E^{1/2}(\mathbf{P}^{-1}z) \leq M_0 \parallel z \parallel_{L_{n+1}^{\infty}(0,T)}, \tag{5.1}$$

which is valid for any $z \in Z^{\infty}(0,T)$, allows to introduce the dual bounded operator $\mathbf{P}^{-1*}$,

$$\mathbf{P}^{-1*} : H^{-1}(\Omega) \times L^2(\Omega) \rightarrow L_{n+1}^{\infty\prime}(0,T),$$

through the following identity

$$< \mathbf{P}^{-1*}\psi, z > = < \psi, \mathbf{P}^{-1}z >_E \tag{5.2}$$

$$\forall\, \psi \in H^{-1}(\Omega) \times L^2(\Omega) \quad \forall z \in Z^{\infty}(0,T),$$

where $< (\cdot), (\cdot) >_E$ stands for duality relation between $H_0^1(\Omega) \times L^2(\Omega)$ and $H_0^{-1}(\Omega) \times L^2(\Omega)$. associated with the "energy" norm (see Remark 2.6).

Step 3. Observe that, in order to study exact controllability of the system (1.1), (1.2), (2.7) one has to analyze the range of the bounded operator $\hat{\mathbf{P}}^*$,

$$\hat{\mathbf{P}}^* : L^{\infty\prime}_{n+1}(0,T), \quad \rightarrow \quad H^{-[n/2]-2}(\Omega) \times H^{-[n/2]-1}(\Omega),$$

which is defined through the identity

$$< \hat{\mathbf{P}}^* z^*, \{\varphi_0, \varphi_1\} > = < z^*, \mathbf{P}\{\varphi_0, \varphi_1\} >_{H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)} \tag{5.3}$$

$$\forall\, z^* \in L^{\infty\prime}_{n+1}(0,T), \quad \forall\, \{\varphi_0, \varphi_1\} \in H^{[n/2]+2}(\Omega) \times H^{[n/2]+1}(\Omega).$$

Step 4. Take an arbitrary $\hat{\psi} \in H^{-1}(\Omega) \times L^2(\Omega)$, and set

$$\hat{z}^* = \mathbf{P}^{-1*}\hat{\psi}. \tag{5.4}$$

Then, due to (5.3),

$$< \hat{z}^*, \mathbf{P}\{\varphi_0, \varphi_1\} > = < \hat{\mathbf{P}}^*(\mathbf{P}^{-1*}\hat{\psi}), \ \{\varphi_0, \varphi_1\} >_{H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)}, \tag{5.5}$$

$$\forall\, \{\varphi_0, \varphi_1\} \in H^{[n/2]+2}(\Omega) \times H^{[n/2]+1}(\Omega).$$

In turn, from (5.2) it follows

$$< \hat{z}^*, \mathbf{P}\{\varphi_0, \varphi_1\} > = < \mathbf{P}^{-1*}\hat{\psi}, \ \mathbf{P}\{\varphi_0, \varphi_1\} > = < \hat{\psi}, \ \{\varphi_0, \varphi_1\} >_{H_0^1(\Omega) \times L^2(\Omega)} \tag{5.6}$$

$$\forall\, \{\varphi_0, \varphi_1\} \in H^{[n/2]+2}(\Omega) \times H^{[n/2]+1}(\Omega).$$

Finally, recalling that $H_D^{[n/2]+2}(\Omega) \times H_D^{[n/2]+1}(\Omega)$ is dense in $H_0^1(\Omega) \times L^2(\Omega)$ and combining (5.5) and (5.6), we obtain the conclusion of Theorem 2.3. □

In turn, combining (5.4) and (5.1) yields Corollary 2.1.

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