International Institute for Applied Systems Analysis
IIASA www.iiasa.ac.at

Viability Theorems Applied to the Leontieff Model

Takacs, T.

IIASA Working Paper

WP-91-053

December 1991

Working Paper

Viability Theorems Applied to the Leontieff model

Tibor Takács

WP-91-53
December 1991

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

Viability Theorems Applied to the Leontieff model

Tibor Takács

WP-91-53
December 1991

IIASA International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria
Telephone: +43 2236 715210 □ Telex: 079 137 iiasa a □ Telefax: +43 2236 71313

Foreword

An economic application of viability theory is presented.The continuous-time Leontieff model is considered with a reference trajectory. The paper examines assumptions under which the economy can be kept around this trajectory. In the model the scarcity of goods is also taken into consideration.

Acknowledgements

This paper was made during the Young Scientists' Summer Program of IIASA in 1991. I am very thankful for the support of the Institute.

Viability Theorems Applied to the Leontieff model

*Tibor Takács**

1 Introduction

Viability theory is a mathematical theory for examining of evolution of different systems. It can therefore be applied also to economic systems. The first economic application is owed to J. P. Aubin [3], who examined the decentralized evolution of allocations. In the present paper the well-known dynamic Leontieff model is investigated from the viewpoint of viability theory. It is examined whether the economy can be kept around a reference trajectory. Under adequate assumptions some viability theorems can be applied.
In this paper the same symbols and definitions are used as in [2] and [4]. Matrices, sets and set valued maps are denoted by capital letters. $\rightsquigarrow$ stands for the set valued mapping, $'$ for the (time) derivative and I denotes the identity matrix. If a vector space X is given the unit ball around the origin is denoted by B_X. The unit ball around some ξ is $\xi + B_X$.

2 Problem statement and assumptions

The continuous-time dynamical Leontieff model is considered:

$$\tilde{x}(t) = A\tilde{x}(t) + B\tilde{x}'(t) + \tilde{c}(t)$$

where
$\tilde{x}_i(t)$: production of the i^{th} branch/good in t
$\tilde{c}_i(t)$: consumption of the i^{th} branch/good in t $\qquad (i = 1, \ldots, n)$
A: the input-output Leontieff matrix
B: the capital coefficient matrix $\qquad (A, B \in R^{n \times n})$
Assumption 1. A reference trajectory (x_r,c_r) can be determined for which

$$x_r(t) = Ax_r(t) + Bx_r'(t) + c_r(t)$$

Set $x(t) := \tilde{x}(t) - x_r(t)$, $c(t) := \tilde{c}(t) - c_r(t)$.
Assumption 2. Either B^{-1} exists or the system will be transformed applying Luenberger's and Arbel's approach. The non-singularity of B is realistic only in the case, when model is aggregated enough, otherwise some rows of B may be zero (there are products which are not used for production). This is the well-known singularity problem of the Leontieff model. In the first case we have the system:

$$x'(t) \in B^{-1}(I - A)x(t) - B^{-1}C(x(t))$$

*economist, Central Mining Development Institute, H-Budapest, III. Mikoviny S. u. 2-4.

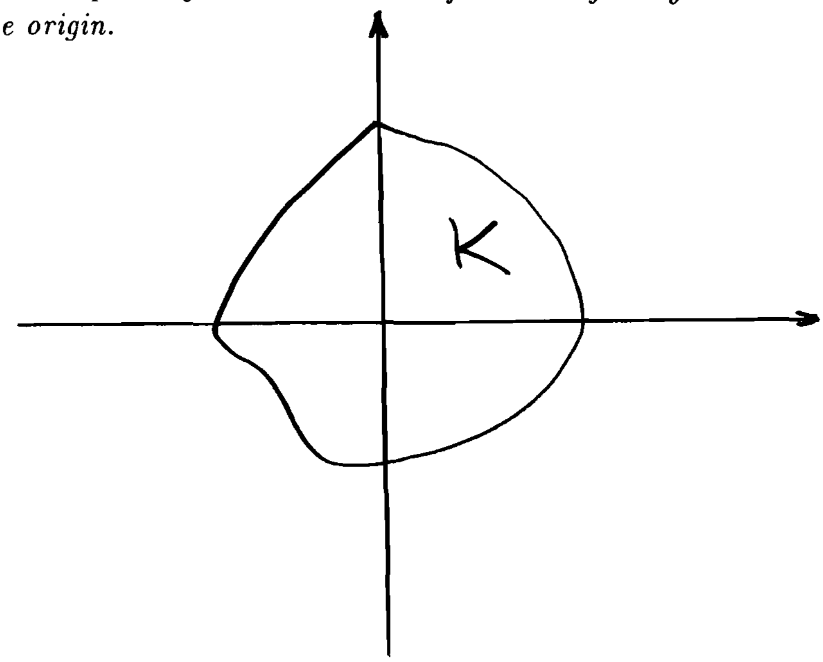
where
$C(x(t))$ is the set of the admissible consumption functions, i.e. $c(t) \in C(x(t))$.
In the second case the continuous time model can be transformed in the same way as the discrete time one in [5]. Let assume that we have the model

$$Bx'(t) = (I - A)x(t) - Ic(t)$$

and the zero rows of B can be found in the lower block. Then the partitioned form of the system is:
$\begin{pmatrix} U \\ 0 \end{pmatrix} = B, \begin{pmatrix} G \\ H \end{pmatrix} = I - A, \begin{pmatrix} J \\ D \end{pmatrix} = I.$

Now we assume, that the inverse of matrix $\begin{pmatrix} U \\ H \end{pmatrix}$ exists, and is partitioned as (EF).

Let $y := Ux$, so that $y' = Ux'$. $Dc = Hx$. $\begin{pmatrix} U \\ H \end{pmatrix} x = \begin{pmatrix} y \\ Dc \end{pmatrix}$ then $x = Ey + FDc$, $Ux' = G(Ey + FDc) - Jc$, $y' = GEy + (GFD - J)c$. If B had l zero rows, we have now an l-dimensional system i.e. $y \in R^l$. The viability theorems can be applied for y in this case.
In this model consumption is used as a control although it is normally the purpose of the economic activity. We can however formulate such questions as: 'How can the economy develop in a certain direction while an acceptable consumption is guaranteed?'. In this model consumption is used as a control. The viability theorems will provide a law of its evolution in order to keep the economy around the reference trajectory.
Now we formulate the problem.
The purpose is to keep the system around the reference trajectory i.e. to keep x in a neighbourhood of the origin.



Here K is considered as the viability set.
Assumption 3. $x_0 = x(t_0) \in K$. It will be later examined when this assumption is not satisfied.
Assumption 4. $x_0 \neq 0$. Otherwise system is obviously in the equilibrium point, i.e economy is at (x_r, c_r).
Assumption 5. $C(x(t)) = C$ and given by $-\alpha \leq c(t) \leq \alpha$ for some $\alpha \in R^n$ and $\alpha_i \geq 0$ for all i. The following viability theorems are however valid even if C depends on x. In this case it has to be assumed that the graph of C is closed, it is lower semicontinuous with

convex values and has linear growth. The dependence on x can be easily interpreted, as the dynamic Leontieff model does not take into consideration the amortization explicitly, i.e. it is included in c.

Assumption 6. $K := rB_{R^n}$ for some positive $r \in R$, where B_{R^n} is unit ball of R^n, and $\text{dom}(F) := r'B_{R^n}$ for some reasonable $r \ll r'$.

3 Viability properties of the model

In this section we use propositions and theorems whose proofs can be found either in [2] or [4].

The viability set K is closed and convex, therefore sleek as well. We recall, that if an $x_1 \in int(K)$ then $T_K(x_1) = R^n$, where $T_K(x_1)$ denotes the contingent cone to K at x_1. As C does not depend on x, the regulation map is

$$R_K(x_1) = \{c \in C\}$$

when $x_1 \in int(K)$, because for all $c \in C$,

$$B^{-1}(I - A)x_1 - B^{-1}c \in T_K(x_1)(= R^n)$$

If an $x_2 \in \partial K$ (i.e. it lies on the boundary of K) then $T_K(x_2)$ is the tangent cone to K at x_2.

$$T_K(x_2) = \{v \mid \langle v, x_c - x_2 \rangle \leq 0\}$$

where

$\| x_c - x_2 \| = d_K(x_c)$, and d_K denotes the distance from K of some x_c. (See e.g. [2].) Let us assume that $R_K(x_2) \neq \emptyset$. Under the above assumption K is obviously a viability domain.

Proposition 1. K enjoys the viability property.

In the following, let $f(x, c)$ denote the right hand side of the original differential inclusion and set $F(x) := f(x, C)$. According to *Assumption 5.* C is a constant closed set. The map f is continuous, the velocity subsets $F(x)$ are convex, and f and C have linear growth. Therefore K enjoys the viability property (see Theorem 6.1.3 of [2]).

Proposition 2. The control system has slow viable solutions.

If the regulation map is lower semicontinuous with nonempty convex values this proposition holds for our system (Theorem 6.5.3 of [2]). The lower semicontinuity of R_K follows from the following facts (Proposition 6.2.1 of [2]):

i) T_K and C is lower semicontinuous with convex values,

ii) f is continuous

iii) for all x, $c \to f(x, c)$ is affine

iv) for all $x \in R^n$, $\exists \gamma$, δ, c and ρ positive numbers such that for all $\xi \in x + \delta B_{R^n}$

$$\gamma B_{R^n} \subset f(\xi, C \bigcap \rho B_{R^n}) - T_K(\xi)$$

It is reasonable to prescribe that $\| c'(t) \| < \phi$ for some $\phi \in R^+$, in order to avoid greater changes in the consumption.

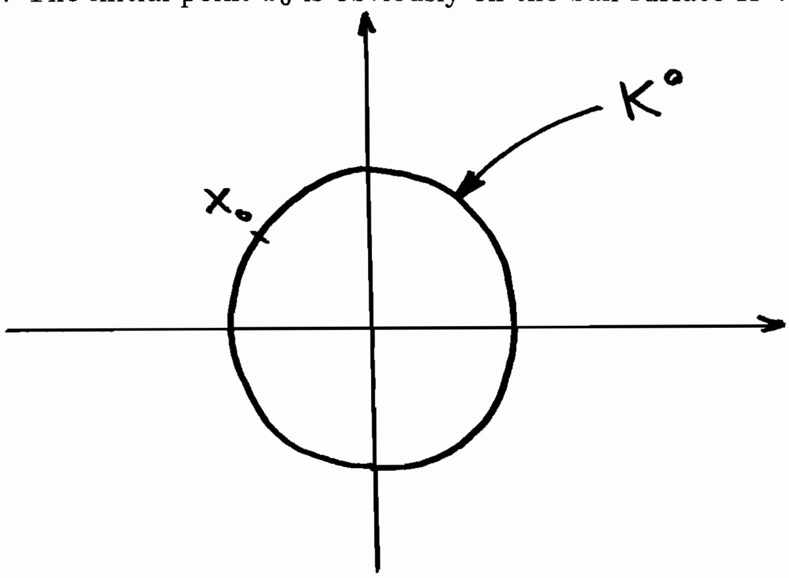
Proposition 3. The system has a ϕ-smooth solution.

C is a closed constant set. The function f is continuous and has linear growth. Let ϕ be

a positive constant and let R^ϕ denote the ϕ-smooth regulation map. Then (see Theorem 7.2.8 of [2]) for any initial state $x_0 \in Dom(R^\phi)$ and any initial control $c_0 \in R^\phi(x_0)$, there exists a ϕ-smooth state-control solution starting at (x_0, c_0), where x is regulated by the consumption c starting at c_0, and for all $t \geq 0, c(t) \in R^\phi(x(t))$.
Solution of the differential equation system can be expressed according to the Cauchy-formula for inhomogenous linear systems (see e.g. [7]). The consumption $c \equiv 0$ is a punctuated equilibrium.
Proposition 4. There exist heavy viable solutions.
We are going to determine a heavy viable solution on a sphere contained in K. Let $L{:=}1/r_{x_0}^2 I$ a symmetric, positive definite linear operator, where $r_{x_0} = d(x_0, 0)$. Let us consider the sphere

$$K^o := \{x \in R^n \mid \langle Lx, x\rangle = 1\}$$

Then $K^o \subset K$. The initial point x_0 is obviously on the ball surface K^o.



We assume that for all x, there is a c, such that

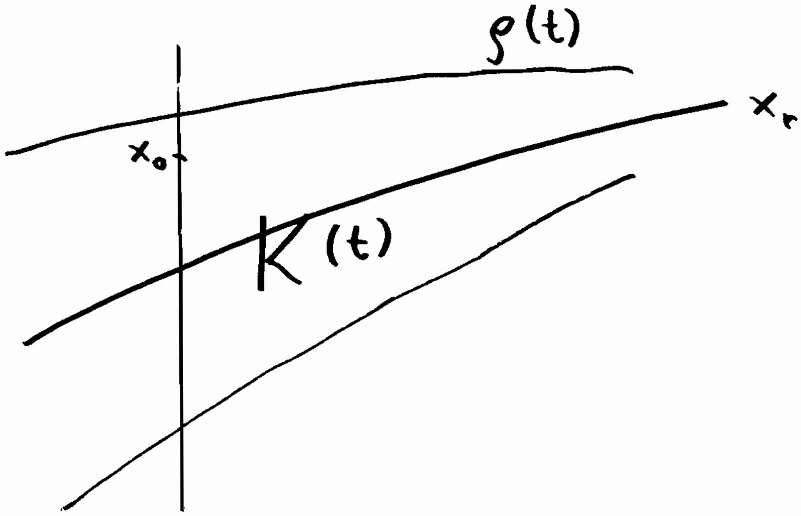
$$x^* L B^{-1}((I - A)x - c) = 0$$

(where $*$ denotes the transpose).
If we take into consideration, that for all $x \in K^o$, $(B^{-1})^* x \neq 0$, Proposition 7.6.1 of [2] can be applied. It means that there exist heavy viable solutions on the sphere determined by L. They can be explicitely given as well: they are solutions to the system of the differential equation system
i) $x' = B^{-1}(I - A)x + (-B^{-1})c$
ii) $c' = \frac{1}{(\|(-B^{-1})^* x\|)^2}\,(-B^{-1})^* x p(x, c)$, where
$p(x,c) = x^*(I - A)^*((B^{-1})^*(B^{-1})(I-A)x + x^*(B^{-1})(I-A)(B^{-1})(I-A)x + c^*((B^{-1})^*)(B^{-1})$ $c - 2x^*(I - A)^*((B^{-1})^*)(B^{-1})c - x^*(B^{-1})(I - A)(B^{-1})c$.
Now we omit *Assumption 3*. This is the case when $x_0 \notin K = rB_{R^n}$. A time-dependent set will be considered as a viability set in the following way. Set

$$K(t) = \rho(t) B_{R^n}$$

and
i) $x_0 \in K(t_0)$
ii) $\rho{:}R \to R$, $\rho(t) \in C^1$

iii) $\rho'(t) < 0$
iv) $\lim_{t\to\infty} \rho(t) = r$.



We can prescribe the velocity of convergence as well:
v) $\frac{\|\rho'(t)\|}{\|\rho(t)\|} > \lambda$ for some $\lambda \in R^+$.
Now Proposition 11.3.2 of [2] can be applied.
For all $t \geq 0$ and $x \in K(t)$, there exists a $z \in B_{R^n}$ so that $x = \rho(t)z$, and if it satisfies
i) $(F(\rho(t)z) - \rho'(t)z) \bigcap (T_{B_{R^n}}(z)) \neq \emptyset$ if $t < T$ (for a certain $T < \infty$),
ii) $(F(\rho(t)z)) \bigcap (T_{B_{R^n}}(z)) \neq \emptyset$ if $t = T$, then $K(t)$ is a viability tube on $[0, T]$. (The notation F is used as the same before.)
It is easy to check, that F is a Marchaud-map. Then $K(t)$ enjoys the viability property (Theorem 11.1.3 of [2]).

4 Decentralization of the system

Let us consider the system transformed by Luenberger's and Arbel's method:

$$y'(t) = GEy(t) + (GFD - J)c(t)$$

(Here F denotes the second block of $\begin{pmatrix} U \\ H \end{pmatrix}$ again.) In this system $y(t)(= Ux(t))$ can be interpreted as the total stock of capital goods required to produce the output $x(t)$ (see [5] p.994). It is assumed that all data of the model is given in terms of some currency. If we take into consideration that $x(t) \in rB_{R^n}$ (see *Assumption 6.*), then $y \in L \subset R^l$, where L is closed and convex. Set

$$h(y(t)) = 1^* y(t) := z(t)$$

i.e $z(t)$ is the sum of the required capital. Let Y, X and s denote the income, the volume of taxes and the saving ratio respectively. Then

$$z' = s(Y - X) + X = sY + (1 - s)X$$

(see e.g. [1] p.226). We assume that z is in linear relationship with Y

$$Y = \alpha z(t) + \beta$$

and $z(t) \in M$, where $M = [-\kappa, \kappa], \kappa \in R^+$.
(Linearity allows us to interprete also the variable z as deviation from a reference path.)
Now we obtain the equation:

$$z'(t) = s\alpha z(t) + (1-s)X(t) + s\beta$$

M is considered as a viability set for this equation expressing a scarcity condition. The following conditions are satisfied:
i) L and M are sleek,
ii) $h \in C^1$ (continuously differentiable) and
iii) for all $x \in L \bigcap h^{-1}(M)$, $h'(y)T_L(y) \bigcap T_M(h(y)) \neq \emptyset$.
Set

$$R_L(y) = \{c \in C \mid GEy(t) + (GFD - J)c(t) \in T_L(y)\}$$

and let us introduce the following notation:

$$f(y,c) := GEy(t) + (GFD - J)c(t) \qquad (\in R^l)$$

Furthermore, set

$$R_h(y,z;X) = \{c \in R_L(y) \mid \sum_{i=1}^{l} f_i(y,c) = s\alpha z(t) + (1-s)X(t) + s\beta$$

First a viable solution should be found for z in M, i.e. scarcity of capital is taken into consideration. The same theorems and propositions can be applied as presented above. It remains then to study the evolution of the system $y' = f(y,c)$ through the regulation map $R_h(y,z;X)$. This is called decentralization in viability theory.

5 Physical scarcity of goods

We can take into account the scarcity in physical units as well. Let us consider the system

$$x' = \mu(Ax + c), \qquad c \in C$$

i.e. it is assumed that change of production is proportional to the total consumption. It is also assumed, that $\mu < 0$. We recall, that variables are deviations from reference values, therefore the system tends to the reference path (if total consumption is below the reference values, elements of x are increasing, and vice versa). If the rate of proportion is different for the different goods/branches, the diagonal matrix $< \mu_i > \quad (i = 1, \dots, n)$ can be used with negative diagonal elements.
Set $y := Bx$, y is the total capital stock on hand in t (see [5]). The production process of y is: $y = z - Az$ i.e. $z = (I - A)^{-1}y$. Then $z = (I - A)^{-1}Bx, \quad z \in R^n$. Without loss of generality we can also assume, that the first m components refer to the raw or basic materials. Set $S := (I_m, 0) \in R^{m \times n}$, and $w := Sz$. Then

$$w = S(I - A)^{-1}Bx \qquad (\in R^n)$$

Let the linear system

$$w' = A_w w + B_w v$$

describe the dynamics of w. The control v may include taxes, subsides in prices, interest rates etc. Set $h := S(I-A)^{-1}Bx$. In this case we have the regulation map:

$$R_h(x, w; v) = \{c \in R_K(x) \mid \mu S(I-A)^{-1}B(Ax+c) = A_w w + B_w v\}$$

where

$$R_K = \{c \in C \mid \mu(Ax+c) \in T_K(x)\}$$

6 Annex: Definitions

In this section we repeat the definitions of viability theory used in this paper.

Definition 1. Let X and Y be metric spaces. The graph of a set valued map $F : X \rightsquigarrow Y$ is defined by

$$Graph(F) := \{(x, y) \in X \times Y \mid y \in F(x)\}$$

A set valued map is said to be nontrivial if its graph is non empty.

Definition 2. A set valued map $F : X \rightsquigarrow Y$ is called

- upper semicontinuous at $x \in Dom(F)$ if for any neighbourhood $\mathcal{U}$ of $F(x)$, $\exists \eta$ such that for all $x^* \in x + \eta B_X, F(x^*) \subset \mathcal{U}$.

It is said to be upper semicontinuous if it is upper semicontinuous at any point $x \in Dom(F)$.

- lower semicontinuous at $x \in Dom(F)$ if for any $y \in F(x)$ and for any sequence of elements $x_n \in Dom(F)$ converging to x, there exists a sequence of elements $y_n \in F(x)$ converging to y. It is said to be lower semicontinuous if it is lower semicontinuous at every point $x \in Dom(F)$.

- continuous at x if it is both upper and lower semicontinuous at x, and that it is continuous if it is continuous at every point of $Dom(F)$.

Definition 3. Let $F(x)$ be a set valued map. We introduce the following notation:

$$\|F(x)\| := \sup_{y \in F(x)} \|y\|$$

F has linear growth if there exists a positive constant c such that

$$\|F(x)\| \leq c(\|x\| + 1)$$

The same definition is used for single valued maps with $\|f(x)\|$.

Definition 4. F is a Marchaud-map if it is nontrivial, upper semicontinuous, has compact convex images and linear growth.

Definition 5. Let X be a Banach space, K be a nonempty subset of X and $x \in K$. The contingent cone to K at x is defined by

$$T_K(x) = \{v \in X \mid \liminf_{h \to +0} \frac{d_K(x+hv)}{h} = 0\}$$

where d_K denotes the distance of y to K.

Definition 6. A subset K of X is sleek at $x^* \in K$ if the set valued map $K \ni x^* \rightsquigarrow T_K(x^*)$ is lower semicontinuous at x. K is sleek if it is sleek at every point of it.

Let us consider the following initial-value problem:

for almost all $t \in [0, T]$,

$$x'(t) \in F(x(t)), \quad \& \quad x(0) = x_0$$

Definition 7. Let K be a subset of the domain of F. K enjoys the local viability property for the set valued map F if for any initial state $x_0 \in K$ there exists a $T > 0$ and a solution on $[0, T]$ to the above differential inclusion starting at x_0 and remaining in K (i.e. there exists a viable solution on this interval). It enjoys the global viability property (or simply the viability property) if we can take $T = \infty$.
Definition 8. Let $F : X \rightsquigarrow X$ be a nontrivial set valued map. A subset $K \subset Dom(F)$ is a viability domain of F if for all $x \in K$,

$$F(x) \bigcap T_K(x) \neq \emptyset$$

Definition 9. Let K be a subset of the domain of a set valued map $F : X \rightsquigarrow X$. We shall say that the largest closed viability domain contained in K (which may be empty) is the viability kernel of K under F.
Definition 10. A control system denoted by (U, f) is defined by
- a feedback set valued map $U : X \rightsquigarrow Z$
- a map $f : Graph(U) \rightarrow X$ describing the dynamics of the system. The evolution of the system (U, f) is governed by the differential inclusion:
i) for almost all t, $x'(t) = f(x(t), u(t))$
ii) where $u(t) \in U(x(t))$.
Definition 11. Consider a system (U, f) described by a feedback map U and dynamics f. We associate with any subset $K \subset Dom(U)$ the state regulation map, or simply the regulation map $R_K : K \rightsquigarrow Z$ defined by the following relation:
for all $x \in K$

$$R_K(x) := \{u \in U \mid f(x, u) \in T_K(x)\}$$

Controls u belonging to $R_K(x)$ are called viable.
Definition 12. Let $F : X \rightsquigarrow Y$ be upper semicontinuous having closed, convex values and let Y be a Hilbert space. Then $F^\circ(x)$ is a minimal selection of F if

$$F^\circ(x) = \{u \in F(x) \mid \|u\| = min_{y \in F(x)} \|y\|\}$$

Definition 13. Let $r_K^\circ(x(t))$ a minimal selection of some regulation map associated to some f control system and a subset K. The solutions of

$$x'(t) = f(x(t), r_K^\circ(x(t)))$$

are called slow viable solutions of the control system.
Let the system $x'(t) = f(x(t), u(t)), \quad u(t) \in U(x(t))$ given. Let us assume that a non-negative function $u \rightarrow \phi(x, u)$ with linear growth is given. Then the following system is associated to the original one:
i) $x'(t) = f(x(t), u(t))$
ii) $u'(t) \in \phi(x(t), u(t))B$
Definition 14. Any solution $(x(.), u(.))$ of the associated system being viable in $Graph(U)$ is a ϕ-smooth solution to the original control system.
Definition 15. Let us consider a non negative continuous function $(x, u) \rightarrow \phi(x, u)$ with linear growth. We shall denote by R_U^ϕ or simply R^ϕ the set valued map whose graph is the viability kernel of $Graph(U)$ for the associated system. We shall call it the ϕ-growth regulation map to the original control system. If $\phi \equiv 0$, we shall say that R_U^0 is the punctuated regulation map.
Definition 16. Let $F : X \rightsquigarrow Y$ be a set valued map from a normed space X to another

normed space Y and $y \in F(x)$. The contingent derivative $DF(x,y)$ of F at $(x,y) \in Graph(F)$ is the set valued map from X to Y defined by the following relation:

$$Graph(DF(x,y)) = T_{Graph(F)}(x,y)$$

Definition 17. We associate with any control u its viability niche $N^{\phi}(u)$, which is the a (possibly empty) subset of states $x \in Dom(R^{\phi})$ such that

$$0 \in DR^{\phi}(x,u)(f(x,u))$$

When $\phi \equiv 0$, the viability niche $N^{0}(u)$ is called the viability cell of u. A control u is called a punctuated equilibrium if its viability cell is not empty.
Definition 18. Denote by $g^{\circ}_{\phi}(x,u)$ the element of minimal norm of $DR^{\phi}_{U}(f(x,u))$. The solutions of the system
i) $x'(t) = f(x(t),u(t))$
ii) $u'(t) = g^{\circ}_{\phi}(x(t),u(t))$
are heavy viable solutions to the control system (U,f).

References

[1] Arrow, K.J. (1985). *Production and Capital* (The Belknap Press of Harvard University Press Cambridge, Massachusetts, and London, England)

[2] Aubin, J.P. (1991) *Viability Theory* (Birkhäuser, Basel, London)

[3] Aubin, J.P. (1991) *Evolution of Viable Allocations* (Journal of Economic Behaviour and Organization 16 pp.183-215. North-Holland)

[4] Aubin, J.P-Frankowska, H. (1990) *Set Valued Analysis* (Birkhäuser, Basel, London)

[5] Luenberger, D.G.-Arbel, A. (1977) *Singular Dynamic Leontieff Systems* (Econometrica Vol.45.No.4. May)

[6] Martos, B.(1989) *Lineáris rendszerek működőképessége* (Pleriminary study of the Research Institute of Economics of the Hungarian Academy of Sciences)

[7] Tallós, P.(1986) *Differenciálegyenletek* (Publication of the Economic University of Budapest, Hungary)