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A Non-Equilibrium Evolutionary Economic Theory

Self-Organization of Markets & The Approach to Equilibrium

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A Non-Equilibrium Evolutionary Economic Theory

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The paper "A Non-Equilibrium Evolutionary Economic Theory" is the revised version of "A Computable Economic Progress Function".

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FOREWORD

These two papers represent the product of the first phase of a collaborative research project to explore the foundations of economics from an evolutionary perspective. In these two papers, the authors exploit the irreversibility of simple exchange transactions to derive a computable "progress function" (which they interpret as a stock of useful information) and a "liquidity" function. In the second paper they use this analytical machinery to explore the approach to equilibrium in both the "pure exchange" case and the "growth" case using simulation techniques.

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A NON-EQUILIBRIUM EVOLUTIONARY ECONOMIC THEORY

by Robert U. Ayres[1] and Katalin Martinás[2]

Abstract

Modifying some of the canonical assumptions of general equilibrium theory, in this paper we derive a computable economic progress function *Z* for any economic unit (EU) with bounded rationality (BR). The progress function depends only on the internal economic state of the unit, as measured by possessions: goods, money and (for individuals) the value of future labor and leisure. In the absence of depreciation and *aging* the progress function is non-decreasing. It does not presume utility maximization or general equilibrium. Thus, the underlying theory is essentially in the *evolutionary* tradition.

Arguments are presented for interpreting the progress function as a stock of economically useful information.

Introduction

There are good arguments to suspect that the economic system evolves irreversibly, in some sense, at the macro-level. Not only is this notion consistent with the second law of thermodynamics [Georgescu-Roegen 71]; it is also suggested by the analogy with biological evolution that has been noted a number of times [Faber & Proops 86; Ayres 88b]. Finally, it makes strong intuitive sense that economic progress should follow in parallel with the irreversible increase in human knowledge, especially technology.

However, the foregoing notions of macro-irreversibility are limited in their applicability to micro-economics. Indeed, micro-economics at present is basically a timeless, static, equilibrium theory in which irreversibility plays no central role. We believe, that it should play such a role, as we argue hereafter.

There is another type of *micro* irreversibility in economics; namely, the irreversibility of pairwise exchange transactions [Martinás 89]. This follows from the condition that no economic agent will undertake an economic activity leaving him/her less well off. This is essentially *bounded rationality* (BR) — sometimes called *satisficing* — in the sense of Herbert Simon [Simon 55, 59, 82]. In terms of transactions in the market domain the argument is simple: In the first place, if A is willing to trade apples for oranges with B, A will not be willing to trade in the reverse direction (oranges for apples). He does not trade for the sake of trading. In the second place, the trade will not take place unless both A and B are *better off* in their own terms. This is essentially a restatement of Edgeworth's first principle of economics: "Every agent is actuated by self-interest" [Edgeworth 81 p.16].

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Transactional irreversibility, in the above sense, is implicit in Walras' Law and the *tâtonnement* process, but it was first formulated explicitly as *Ville's axiom, viz.*

> "No (price) path exists which moves always in the preferred direction but ends at its starting point" [Ville 51]

The Ville axiom was originally set forth as a necessary condition for the existence of a differentiable total utility function, depending only on quantities of exchangeable goods. Ville's axiom is applicable at the level of an economic system (i.e. a market) for which a unique price is defined for each commodity. The corresponding axiom in our case can be stated:

> There is no spontaneous transaction between two economic decision-makers resulting in negative (or zero) surplus values for either party.

Nothing is assumed at this stage about prices.

The stronger principles of profit maximization and utility maximization, on which most of neo-classical economics was built, are not necessary to obtain our result. Irreversibility of transactions follows from the condition that neither party will undertake a transaction leaving him/her less well off. Bounded rationality (BR) is sufficient to guarantee irreversibility. Yet the implications of transactional irreversibility were never explored by the pioneers of utility theory, such as Jevons, Walras, Pareto, Fisher and Edgeworth.

It has been pointed out by Mirowski, for instance, that the analogy between physical science and *moral science* was very clear to the early neo-classicists [Mirowski 84]. Mirowski cites a variety of evidence supporting this assertion [Mirowski 89]. For instance Stanley Jevons (1905) stated that "The notion of value is to our science what that of energy is to mechanics" [ibid], although Mirowski contends that Jevons misunderstood the physics. Walras wrote in 1862 of his intention to try to create "a science of economic forces analogous to the science of astronomical forces...the analogy is complete and striking." Later he wrote an article entitled "Économique et Mécanique" full of analogies (some erroneous) between mechanics and economics [ibid]. Fisher included a table showing the *concordance* between physics and economic variables in his 1926 book [ibid].

For a final instance, Edgeworth discusses this analogy at length in the opening chapter of his 1881 book "Mathematical Psychics", where the following passage is to be found:

> "The application of mathematics to the world of soul is countenanced by the hypothesis that every psychical phenomenon is the concomitant, and in some sense, the other side of a physical phenomenon. The particular hypothesis adopted in these pages, that Pleasure is the concomitant of Energy. **Energy** may be regarded as the central idea of Mathematical Physics; **maximum energy** the object of the principal investigations in that science. By aid of this conception we reduce into scientific order physical phenomena, the complexity of which may be compared with the complexity which appears so formidable in Social Science".

For Jevons, Walras, Pareto, Edgeworth (and the other neo-classicists) the central problem of mathematical economics (or psychics) was to determine the conditions for maximization of pleasure (or utility). In particular, neo-classical economics since Walras has focussed intensively on the conditions for existence of a general equilibrium, rather than on the properties of non-equilibrium states and the approach to equilibrium. In this respect, the neo-classical program was very different from ours.

We wish to show that irreversibility, as defined above, together with the assumption of a universal medium of exchange (money), is sufficient to prove the existence of a *progress function* that is non-decreasing except for long-run depreciation and aging effects. Nevertheless to include this irreversibility into economic theory we have to modify some of the *canonical* assumptions of General Equilibrium Theory. Indeed, the exercise yields an unexpected reward in terms of suggesting new micro-economic models and new interpretations of existing data, as will be seen below. It also confirms the analogy between social science and physical science [Edgeworth ibid], in one respect, at least: our non-decreasing *progress function* is closely analogous to the familiar non-decreasing **entropy** function of thermodynamics. Yet, its derivation from axiomatic economic first principles is rigorous. It involves no thermodynamical reasoning.

Micro-foundations

Assume the existence of an **economic unit** (EU), which may (or may not) be one of many units which together constitute an **economic system** (ES). An EU is defined for our purposes as the smallest entity with an implicit or explicit decision-making rule with the property that no economic transaction occurs that leaves the EU *worse off* than before the transaction. This rule is termed **bounded rationality** (BR) for convenience, hereafter. If the EU's are part of an ES (which involves some further assumptions, including a set of rules governing exchange transactions) then a system-wide medium-of-exchange can be assumed. It will be called **money**.

By assumption, an EU may be capable of any of three types of transactions: **production**, **consumption**, and **exchange**. An EU would normally be either a **firm** or an individual[3]. EU's may interact with each other only in binary fashion, via exchanges of goods[4], using a **medium of exchange** (not necessarily money). An aggregation of EU's is an Economic System (ES). We do not assume the existence of an unique posted market price known to all EU's for pairwise economic exchanges between EU's. In each individual exchange there is a money transfer corresponding to the goods transfer. Prices, in our theory, are defined only for specific transactions. (It will be shown in a subsequent paper that in a static pure exchange model, pairwise exchange prices converge to a unique equilibrium price). Nevertheless, market prices for an ES can be defined only in the equilibrium limit; in the non-equilibrium case individual transaction prices vary among transactors and over time.

None of the components of an EU are themselves EU's. This means that the employees of a firm are not *components* of the firm; they are independent EU's who contract voluntarily with it to sell their labor to the firm in exchange for wages. A firm may be credited with a certain amount of potential labor only in the sense that it has explicit (or implicit) contracts with a certain workforce. The only owners and sellers of labor are individuals, while (by convention) the only producers of goods are firms[5]. Thus organizations are assumed to have an existence and a decision-making rule (BR) independent of the identities of their members (who may come and go).

It is assumed that no transaction of any type occurs in the absence of an explicit decision to act, based on the BR decision-rule. The criterion for a positive decision is that the EU not be left *worse off* than it was before. This can be restated in more precise terms. We assume that *well-offness* (welfare) is a function of the **economic state** of the EU. The latter is determined by a set of **observables**. Examples of observables include stocks of money, durable goods, and raw materials, potential output of labor per period, skill level of the labor, money income (wage and non-wage) per period, consumption of goods per period, consumption of unpriced environmental services per period,

[3] In the real world an EU might also be a government agency, co-op, a foundation, a commune, a criminal organization, a church, or some other entity.

[4] In general, services can also be exchanged for goods (or other services). However we restrict ourselves at this stage to transactions involving only tangible goods or labor. The extension to other services will be considered later.

[5] A self-employed person is therefore a worker who does not sell his labor to any other firm, but uses it himself for production purposes. This confusion of roles in practice creates no conceptual difficulty.

stock of economically useful knowledge and (in the case of an individual human being) physical health and life-expectancy. The observables are, of course, variables of the system.

The above assumptions make it possible to introduce the concept of internal worth or simply *worth*. The worth of the i^{th} good is V_i

$$V_i = V_i(X_1, X_2, \cdots, M) \tag{1}$$

where X_i stands for the quantity of the i^{th} good, and M for the money. The obvious alternatives, value and utility, have already been established in economic theory, and have acquired conventional meanings. It is important for us to distinguish *worth* from both value and utility (although at first sight the three words have similar connotations to the non-specialist). The economic worth of a good is defined for (and by) each EU, and known only to the EU. Assuming BR, an EU agrees to sell a good if and only if its worth is less than the price offered by another EU; conversely, an EU agrees to buy the good if and only if the worth of the good received equals or exceeds the price offered. Similar logic can be applied for the production (consumption) decision.

In the special case where the EU is indifferent to whether the transaction takes place, or not, the internal worth of the exchanged goods must be exactly equal to the money price. (In this case, and only this case, the exchange is **reversible**). The above definition is basically the same as that used conventionally for **value** in microeconomics [Debreu 59], with three differences:

(i) It presumes only **bounded rationality (BR)**, not perfect rationality. In the standard case, producers maximize profit, while consumers maximize utility. In our case, BR is applicable to both consumption and production decisions.

(ii) It depends only on the **internal state** (quantities of goods and money) of an economic unit (EU), and is independent of the economic system (ES) to which the EU belongs, except insofar as money is somehow created by the larger economic system. In the case of general equilibrium, by contrast, value is system-determined.

(iii) Internal worth does not determine the actual **path** of the economic process. It only determines whether the process is possible or not in each specific case. However, the **rate** at which an economic process takes place, for instance, involves additional factors, including technological capabilities and constraints, and individual characteristics. The real process cannot be described without specifying these factors and constraints.

Any production, consumption or exchange transaction will result in a change in the economic state of the EU through the change of quantities of goods and money. For instance, a production decision will involve a conversion of raw materials and actual labor into finished goods for sale. An exchange transaction is a sale of goods for money at an agreed price. A consumption transaction is only possible for individuals: it may be a conversion of potential labor (leisure time) into actual labor at an agreed money wage, or an exchange of money for either *consumable* goods (food, clothing, shelter, medicines, etc) or for *consumer durables*. A consumption transaction could also be an addition to money savings or a subtraction from money savings for current expenses. It is important to note that, since each EU may have a different decision rule, two EU's would be likely to assign a different worth for each good, even if both were in the same economic state.

It is convenient, for what follows, to distinguish between **extensive** variables which, in some sense, measure the *size* of the system, and **intensive** variables, which are ratios and which measure characteristics that are independent of size. Stocks of goods and money are examples of extensive variables. Intensive variables can be ratios of extensive variables. Worth (defined above) is another example.

The rate of change of extensive variables can always be expressed in terms of flows[6]:

$$\frac{\Delta X_i^\alpha}{\Delta t} = J_i^\alpha + S_i^\alpha \tag{2}$$

where J_i^α is net imports of the i^{th} commodity (imports minus exports) and S_i^α is the net production (production minus consumption) of the i^{th} commodity within the EU. (Here it is also assumed, for convenience, that durable goods have lifetimes long compared to the reference period, and that consumables are used only by human beings in their role as **worker-consumers** for the satisfaction of biological subsistence requirements). By similar logic, one can write

$$\frac{\Delta M^\alpha}{\Delta t} = -\sum_i P_i^\alpha J_i^\alpha + I^\alpha \tag{3}$$

where P_i^α is the money cost (market price) of one unit of the i^{th} good or commodity and I^α is the net financial inflow, i.e. the difference between credits, subsidies, interest or dividends received, loans or investments from outside the EU (e.g. dividends or capital gains) and debits (interest or dividends paid, taxes paid, losses on external investments, etc.).

For convenience, introduce a new notation $J_i^{\alpha\beta}$ which is interpreted as the flow of commodity i from unit β to unit α, where

$$J_i^{\alpha\beta} = -J_i^{\beta\alpha} \tag{4}$$

i.e. every flow to the EU can be identified by origin. For conserved quantities (i.e. goods)

$$J_i^{\alpha\beta} = -J_i^{\alpha\beta} \tag{5}$$

For non-conserved quantities (e.g. knowledge) equation (5) does not hold. Technology transfer increases the knowledge stock of the recipient, without reducing that of the donor.

Let $I^{\alpha\beta}$ be the (non-trade) financial flows from unit β to unit α and $P_i^{\alpha\beta}$ is the price of the i^{th} good in the exchange between the α^{th} and β^{th} EU.
So the money flow can be written as

$$I^\alpha = -\sum_{\beta i} P_i^{\alpha\beta} J_i^{\alpha\beta} + \sum_\beta I^{\alpha\beta} \tag{6}$$

The above assumptions imply that $S_i^\alpha = S_i^\alpha(X_1^\alpha, X_2^\alpha \cdots)$, $J_i^{\alpha\beta} = J_i^{\alpha\beta}(X_1^{\alpha\beta}, X_2^{\alpha\beta} \cdots)$, and $P_i^{\alpha\beta} = P_i^{\alpha\beta}(X_1^{\alpha\beta}, X_2^{\alpha\beta} \cdots)$. These relations characterize the EU, so they can be *determined* experimentally, at least in a *gedanken* sense. In a subsequent paper we will discuss some further constraints on these functions.

[6] The general bookkeeping equation for any conserved quantity X (such as a physical commodity) is

$$\frac{dX}{dt} = F + G$$

where F is a generalized *current* (inflow) that crosses the boundary of the economic unit and G is a generalized *source* (or, with a negative sign, a sink). By assumption X can be any commodity that can be bought, sold, produced or consumed, including money or shares of stock.

With one exception, the above assumptions are basically familiar to economists. The exception is our more extreme form of decentralization of the exchange process, without a *market price* in the usual sense. In our case (as will be seen later) bounded rationality (BR) acts like a sort of *invisible hand,* at least to the extent that in the static pure exchange case it leads to an equilibrium.

A Progress-Function for Firms

A **productive** EU (i.e. a firm) can possess and accumulate two kinds of wealth, viz. **material goods** and **monetary assets**. The former consists of capital goods, raw materials, inventory of work-in-progress and unsold final goods. The latter comprises investments, loan portfolios, bank accounts and cash.

The change in wealth ΔW^{α} of the α^{th} EU during a time period Δt can now be expressed by the accounting balance

$$\Delta W^{\alpha} = \sum_i V_i^{\alpha} \Delta X_i^{\alpha} + \Delta M_i^{\alpha} \tag{7}$$

where V_i is the worth expressed in monetary units — of the i^{th} material good or commodity, X_i is the stock of the i^{th} material good (or commodity in the EU) and M^{α} is the quantity of monetary assets held by the α^{th} EU. It would be convenient (our problem would be solved) if ΔW^{α}, ΔX_i^{α} and ΔM^{α} could simply be converted into perfect differentials, resulting in an integrable expression. However, in its present form this is not possible.

One way to formulate the problem is to note that the change of wealth $\int \Delta W$ is **not** a well-behaved, differentiable function of X_i, M alone, but depends also on other factors (variable parameters). In other words, the change in level of wealth after a finite time is **path dependent**; it depends on the particular **sequence** of transactions that is followed in X_i, M space. In mathematical language, what is needed is an **integrating factor**. But for the expression (7) as it stands, it cannot even be proved that such a factor exists. While $\Delta W \geq 0$ for all spontaneous processes (a version of Walras' Law[7]), there exist non-market economic processes arising from cooperative behavior (e.g. taxes) such that $\Delta W < 0$ is possible.

The next step, therefore, is to seek a mathematical transformation into a form such that the existence of an integrating factor is provable. This means manipulating the expressions into a form that explicitly reflects some additional information about the nature of economic transactions that is not explicitly reflected in (7). To be specific, we seek an expression that explicitly reflects the **bounded rationality** of exchanges and the consequent (micro) **irreversibility** of economic transactions.

To accomplish the desired transformation for convenience one can set $\Delta t = 1$ and insert (2) and (3) into (7). This yields, after combining terms,

[7] The usual statement of Walras' Law is that the vector product of market prices **P** and excess demand **E** is always equal to zero in a pure exchange economy, even when equilibrium has not been established.

$$\Delta W^{\alpha} = \sum_i \left(V_i^{\alpha} - P_i^{\alpha}\right) J_i^{\alpha} + \sum_i V_i^{\alpha} S_i^{\alpha} + I^{\alpha} \tag{8}$$

Assuming both trade and production decisions are governed by bounded rationality (BR), it follows that the first two terms on the r.h.s. of (8) are non-negative. In other words, the flows are *uni-directional*, reflecting the irreversibility caused by BR. Specifically

$$\sum_i \left(V_i^{\alpha} - P_i^{\alpha}\right) J_i^{\alpha} \geq 0 \tag{9}$$

is equivalent to asserting that trades only occur when there is an economic benefit to the EU. Similarly

$$\sum_i V_i^{\alpha} S_i^{\alpha} \geq 0 \tag{10}$$

is equivalent to asserting that a unit of output is only produced (net) if the internal worth to the EU is non-negative. It follows from (9) and (10) that, for the α^{th} EU

$$\Delta W^{\alpha} \geq I^{\alpha} \tag{11}$$

What we seek is an integrating function $T(X_i, M)$ such that, for each EU independently,

$$\lim_{\Delta t \to 0} \frac{\Delta W}{\Delta t} = T \frac{dZ}{dt} \tag{12}$$

It was proved a number of years ago by Carathéodory [Carathéodory 09] that such an integrating function T exists for an irreversible process. The rigorous proof depends on characterizing the irreversibility as follows: namely, in the near neighborhood of **every** point in the state-space there is another point arbitrarily nearby that **cannot** be reached by any reversible process when there is no net financial inflow ($I^{\alpha} = 0$). A reversible process in our case is one such that $\sum_i (V_i - P_i) J_i = 0$ and $\sum_i V_i S_i = 0$. The full proof is complicated and not worth repeating here. Its applicability to the economic case, as described above, was first shown by Bródy, Martinás and Sajó [Bródy *et al* 85]. Actually, a similar proof of integrability was given in 1979 by Hurwicz and Richter [Hurwicz & Richter 79], based on the Ville axiom stated previously. The Ville formulation is essentially equivalent to the Carathéodory irreversibility condition[8].

In short, the necessary conditions for (12) are satisfied. It follows that, substituting back into (8) — and dropping the superscript α for convenience —

[8] Hurwicz and Richter showed that the Ville axiom suffices to prove the integrability of an expression corresponding to ΔW for an economic system (ES), rather than an individual economic unit (EU), provided there exists a unique price vector *p(x)* for each bundle of goods *x*. Later we argue that such a price vector need not exist. For this reason, the total utility function for an ES is undefined.

$$TdZ = \sum_i (V_i - P_i)J_i + \sum_i V_i S_i + I \tag{13}$$

and from (7) we have

$$TdZ = \sum_i V_i dX_i + dM \tag{14}$$

For future reference, we note that for voluntary processes ($I \geq 0$) equations (11) and (12) imply $dZ \geq 0$. That is to say, Z is absolutely non-decreasing in this case. (It will be seen later that this no longer holds true when depreciation of durable goods and consumption processes are introduced).

The next step is to characterize and select a function $Z(X_i,M)$. Equation (14) implies that

$$\frac{\partial Z}{\partial X_i} = \frac{V_i}{T} \tag{15}$$

and

$$\frac{\partial Z}{\partial M} = \frac{1}{T} \tag{16}$$

The progress function Z contains essentially the same information as the internal worth function V_i together with the integrating function T. Equation (14) defines only the TdZ product. To define them individually, there are some arbitrary choices to be fixed. First, we want Z to increase in a spontaneous economic process. It follows that T should be positive.

A further requirement on T is that it should be **homogeneous of zeroth order** (to ensure that Z is a first-order homogeneous function). Homogeneity to zeroth order for T means, in effect, that we want it to depend only on **intensive** variables; i.e. ratios of extensive variables. On the other hand, we want Z to be homogeneous to the first-order, meaning that its dependence on *size* (extensive variables) is essentially linear. (It can be shown that additivity of the Z-function is only consistent with first-order homogeneity. The first-order homogeneity condition is

$$\lambda Z = Z(\lambda X_i, \lambda M) \tag{17}$$

Differentiating (17) with respect to λ yields, after straightforward manipulation and setting $\lambda = 1$,

$$Z = \sum_i \frac{V_i X_i}{T} + \frac{M}{T} \tag{18}$$

An implication of differentiability is that

$$dZ = \sum_i \frac{V_i}{T} dX_i + \frac{1}{T} dM + \sum_i X_i d\left(\frac{V_i}{T}\right) + Md\left(\frac{1}{T}\right) \tag{19}$$

But (14) also holds, whence by matching terms we can derive another equation for either T or V_i, viz.

$$0 = \sum_i X_i d\left(\frac{V_i}{T}\right) + Md\left(\frac{1}{T}\right) \tag{20}$$

When T and V_i are specified Z is determined by (18).

There are an infinite number of possible functional forms for T, Z. Among them, one of the simplest expressions for T satisfying all the required conditions is

$$T = \frac{M}{\sum_i g_i X_i} \tag{21}$$

where the g_i are coefficients yet to be specified. Substituting (21) into (20), differentiating and collecting terms one obtains an integrable equation for $\frac{V_i}{T}$. Integrating (and multiplying by T) yields

$$V_i = \left[\frac{g_i M}{\sum_i g_i X_i}\right]\left[\ln\left(\frac{M}{X_i}\right) + c_i\right] \tag{22}$$

where c_i is a constant of integration. Substituting (22) back into (18) completely defines the **form** of Z, viz.

$$Z = \sum_i g_i X_i \left[\ln\left(\frac{M}{X_i}\right) + c_i\right] = \sum_i g_i X_i \ln\left(\frac{M}{k_i X_i}\right) \tag{23}$$

(since the g_i have not yet been defined). It only remains to find a consistent interpretation of the coefficients g_i and k_i and the quantities X_i. The physical interpretation of these terms is deferred until after the next section.

An important caveat that must be emphasized is that (21) is only one possible form. In fact, there is no guarantee that this particular form is the correct one in any given case. Nor is it necessarily true (or even likely) that all EU's in the real world will be characterized by a T-function (or the corresponding V-, Z-functions) having the same form. The actual form would have to be determined by experiment or observation on a case-by-case basis. However, for idealized models involving transactions among **indistinguishable** (i.e. interchangeable) EU's, it is clear that the mathematical form of T and Z should also be indistinguishable, hence **identical** for our purposes.

Finally, given that Z has the useful property — to be proved later — that its maximum corresponds to a static equilibrium, then the standard second order condition (declining marginal internal worth) holds, namely:

$$\frac{\partial V_i}{\partial X_i} \leq 0 \tag{24}$$

A Progress Function for Individuals

It is now appropriate to generalize the foregoing argument. As noted already, individual worker-consumers, as EU's, differ from firms in three ways. First, they sell only **labor** (man-hours) in exchange for money wages. Second, they **consume** (or collect) goods, rather than producing them. Evidently, a complete economic system (ES) must consist of both types of EU, namely firms (buyers of labor and producers of goods) and worker-consumers (sellers of labor and buyers of goods). Third — anticipating the existence of a progress function (or wealth function) Z for individuals — it is clear that additional variables related to biological condition (e.g. health and life expectancy) may be involved.

Bearing in mind these key differences, one need not repeat the entire derivation above. It is sufficient to note the changes. Thus, equation (7) is also applicable to individuals, with the addition of a term L on the r.h.s. Here L represents the present monetary worth of future labor (wages) plus any other kinds of personal attributes related to biological condition, such as physical health, vigor, appearance, athletic ability, learning and experience. Since these are incommensurable in themselves, one must distinguish their quantities from their monetary worth. It is mathematically convenient to define a new variable K and a parameter a, as follows:

$$K = \frac{\alpha M}{L} \tag{25}$$

Equation (7) is unchanged in form for individuals, except that only one *commodity* (labor) is produced and exported, and all others are imported, either for consumption or accumulation. Since no commodities are internally produced (except leisure time and labor), the *source* terms are all zero or negative with these two exceptions. Equation (3) and the interpretation of I as a financial flow are unchanged. (Note that I does not include wage income). The analog of (8) can now be rewritten (again dropping the superscript α), as follows:

$$\Delta W = (P_L - V_L)J_L + \sum_i (V_i - P_i)J_i + \sum_i V_i S_i + K\Delta L + I \tag{26}$$

where P_L and V_L are respectively the unit price of labor (i.e. the wage rate) and the private worth of leisure time, S_i represents consumption of the i^{th} commodity (still excepting labor and leisure time), while K is defined by (25). The first term represents the surplus producer worth of labor sold by the EU, while the second term represents the surplus consumer worth of all purchased commodities, whether consumed for metabolic purposes, or accumulated. The third term represents the *surplus worth* of consumption for metabolic purposes (i.e. to maintain good health) and being alive. Since health depreciates very rapidly in the absence of food, clothing and shelter, it is perfectly consistent with bounded rationality for individuals to try to replenish these necessities and even increase them. As before, assuming the EU has bounded rationality (BR) it follows that the first three terms on the r.h.s. must be non-negative. It follows that

$$\Delta W > \sum_i V_i S_i + K\Delta L + I \tag{27}$$

But the first term is non-negative, as argued previously, and the second term is non-negative during the first several decades of life (discussed later). It follows again that $\Delta W > I$, the analog of (11). As argued before, equation (12) holds for EU's if there exist feasible economic transactions under BR such that $I = 0$. This condition was met for producers. Its analog for the worker-consumer also holds true, at least if (and when) ΔL is non-negative.

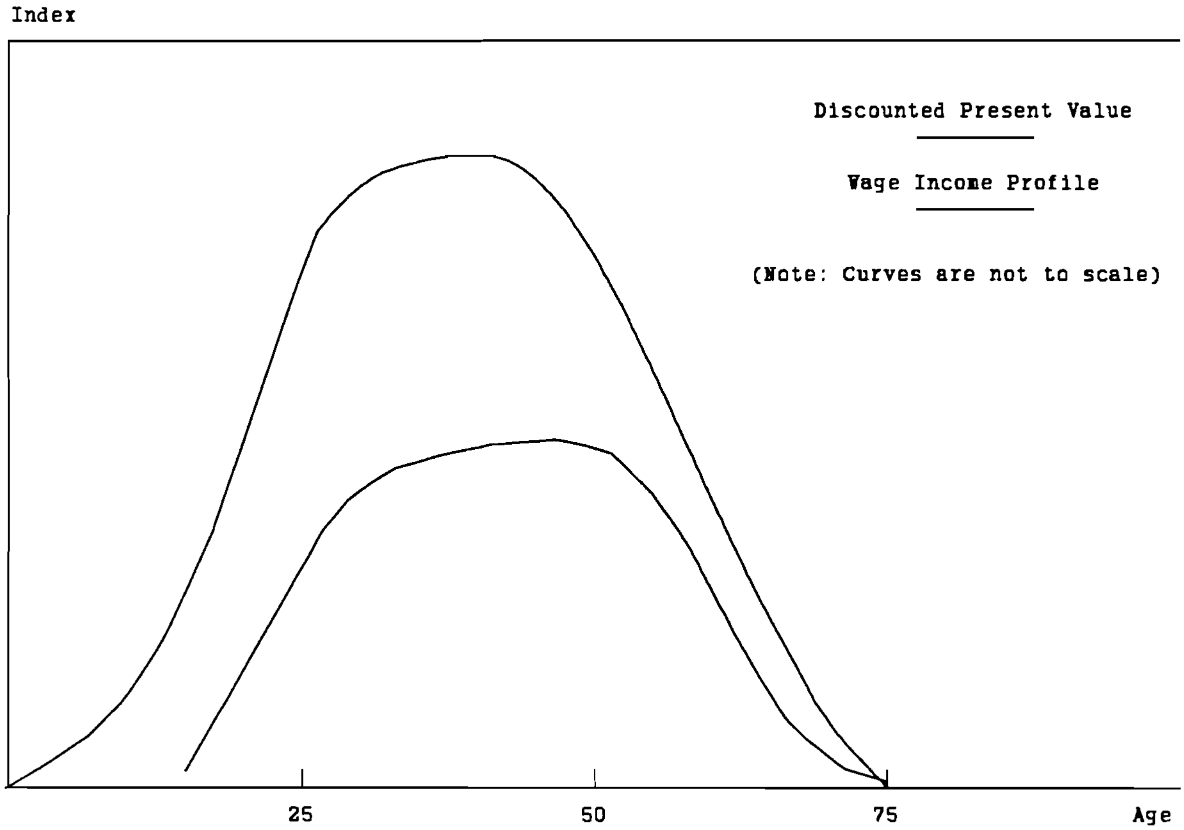

Figure 1 : **Discounted Present Value of Time & Income Profile**

Making use of results explained in detail in the Appendix to this paper, it is fairly easy to show that the **form** of the labor income component of the function L depends mainly on the **age** of the worker-consumer and, to a lesser extent, on his/her personal discount rate[9]. As shown schematically in Figure 1, L increases from zero to a peak like an elongated S-curve, but declines sharply after age 55. (The shape depends on educational level; it reaches a peak somewhere around age 30 for unskilled and semi-skilled workers and increases with educational level and wage level to about 70 for the highest paid segment [Ghez & Becker 75]). The discounted present worth curve is shifted to the left by a few years (depending on the discount rate). Setting

$$\lim_{\Delta t \to 0} \Delta L = \dot{L} \tag{28}$$

it can be seen that $\dot{L}$ has the form of a parabola (inverted U), reaching a maximum positive worth around age 25, thence decreasing to zero when L is at its peak, and finally becoming negative.

Clearly $\dot{L}$ is **not** non-negative during the entire worker life-cycle. On the other hand, it **is** non-negative during roughly the first two thirds of it and only becomes negative as workers approach the age of retirement from the work-force. During this period, at least, we can exploit the Carathéodory theorem and assert the existence of an integrating factor T and a differentiable, non-decreasing progress function Z. We think it is not unreasonable to assume that **whatever progress function is appropriate**

[9] Its magnitude, on the other hand, depends on sex, skill, prevailing wage rate in the locality, and so on.

for worker-consumers during the early part of the life cycle (i.e. when $\dot{L} > 0$) would still be applicable thereafter. We make this assumption, albeit with some qualms.

Proceeding, then, we substitute (12) into (26) obtaining

$$TdZ = (P_L - V_L)J_L + \sum_i (V_i - P_i)J_i + \sum_i V_i S_i + KdL + I \tag{29}$$

Extending the sum over the index i to include labor and leisure time as a commodity, using (7), we get

$$TdZ = \sum_i V_i dX_i + dM + KdL \tag{30}$$

An implication of differentiability is

$$TdZ = \sum_i \frac{\partial Z}{\partial X_i} dX_i + \frac{\partial Z}{\partial M} dM + \frac{\partial Z}{\partial L} KdL \tag{31}$$

Matching terms between (31) and (30) yields equations (15) and (16) as before, plus

$$\frac{\partial Z}{\partial L} = \frac{K}{T} \tag{32}$$

The homogeneity condition (analogous to (17)) is

$$\lambda Z = Z(\lambda X_i, \lambda M, \lambda L) \tag{33}$$

Differentiating (33) with respect to λ and setting $\lambda = 1$ yields

$$Z \quad \sum_i \frac{\partial Z}{\partial \lambda X_i} X_i + \frac{\partial Z}{\partial \lambda M} M + \frac{\partial Z}{\partial \lambda L} L = \sum_i \frac{V_i X_i}{T} + \frac{M}{T} + \frac{KL}{T} \tag{34}$$

Differentiating (34)

$$dZ = \frac{\sum_i V_i dX_i + dM + KdL}{T} + \sum_i X_i d\left(\frac{V_i}{T}\right) + Md\left(\frac{1}{T}\right) + Ld\left(\frac{K}{T}\right) \tag{35}$$

comparing (35) with (30) yields the expression

$$0 = \sum_i X_i d\left(\frac{V_i}{T}\right) + Md\left(\frac{1}{T}\right) + Ld\left(\frac{K}{T}\right) \tag{36}$$

which is analogous to (20). The equation can be solved for V_i by inserting the previous expression (21) for T and (25). Carrying out the indicated differentiations and the same manipulations as in the previous case, yields the final result:

$$V_i = g_i T[\ln M + a \ln L - (1 + a)\ln N_i + c_i] \tag{37}$$

whence

$$Z = \sum_i g_i X_i [\ln M + a \ln L - (1 + a)\ln K_i X_i] \tag{38}$$

For the case of a single product,

$$Z = gX \ln\left(\frac{ML^a}{X^{1+a}}\right) \tag{39}$$

Notice that if we neglect L (e.g. by setting $a = 0$), the previous result (23) is obtained. Again, we emphasize that T need not take the simple form we have chosen for convenience. However, for models in which the EU's are indistinguishable within a class, one can at least be sure that the T-, V-, and Z-functions will have the same form.

Interpretation

It is now appropriate to seek reasonable economic interpretations of the expressions for T, V and Z. The integrating factor T was defined by (21) as the ratio of money assets to a weighted sum of goods assets in the EU. This ratio has an obvious interpretation as **liquidity**. For a producer, this makes the expression (22) for V_i easy to interpret. The internal worth of the i^{th} good to a producer is directly proportional to the liquidity of the EU, and directly proportional to the weight of that good in its inventory. This much is entirely in accord with intuition.

The logarithmic term is less obvious. It says that, for fixed liquidity T, the internal worth of any good to a producer decreases logarithmically the more of that good is on hand, whereas the worth of the good as a function of money increases logarithmically. Yet, on reflection, few corporate chief executives would find this rule counter-intuitive. In general, money in the bank is preferable to inventory, always provided there are goods available on the market to buy.

The extension of the internal worth concept to individuals (37) is fairly straightforward. The constant a is clearly a measure of the individual's preferences between money and other kinds of wealth (time, health, life itself). Liquidity appears in much the same way as before.

It would be a natural mistake, in view of (12) to interpret the product TZ as wealth. But it must be recalled that the integral $\int \Delta W$ is path dependent. It was precisely for this reason that we had to find an integrating factor. On the other hand, Z is not wealth, either; among other problems, it cannot have units of money (TZ does). How, then, shall we interpret Z?

It is easy to see that Z is at a local maximum when the EU reaches a condition such that (under BR) it cannot improve its economic state by engaging in further economic activities. This final state of non-activity is, in fact, can be interpreted as an equilibrium for the EU. Thus all economic activity can be interpreted as an approach to (Pareto)-equilibrium. The local maximum would not, in general, constitute a global maximum in the absence of further specific constraints.

It is important to emphasize here that Z is *not* a classical utility function U, although the utility function (when it exists) is also maximized at the equilibrium point. The utility function for an EU is better interpreted as the *difference* between the progress function before an exchange and after it. The relationship between Z and U is discussed in more detail later.

We believe that it is natural to interpret Z as a stock of **economically useful information**, because every good can be expressed in informational terms [Ayres 87a, 87b]. In fact, one can assert that each manufactured good is characterized by a distance from thermodynamic equilibrium, hence a certain quantity of *embodied* information:

$$X_i = \frac{H_i}{b_i} \tag{40}$$

where H_i is measured in *bits* and the coefficient b_i has dimensions of information/quantity[10]. Substituting (40) in (18) we get

$$Z = \sum_i \left(\frac{V_i H_i}{b_i}\right)\frac{1}{T} + \frac{M}{T} \tag{41}$$

Defining

$$w_i = \frac{V_i}{b_i} \tag{42}$$

we can rewrite Z

$$Z = \sum_i \frac{w_i H_i}{T} + \frac{M}{T} \tag{43}$$

There is no **a priori** restriction on the dimensionality of Z. We do know that TZ has the dimension of money. If we choose to express Z in *bits* of information, then T has the dimensions of money/information and w also has dimensions of money/information. Here T may be interpreted as the unit worth of a standard type of information, while w_i is the relative unit worth of the i^{th} type. Thus $\frac{w_i}{T}$ is the relative worth of the i^{th} type of information. One of the w_i can be chosen arbitrarily, for convenience.[11]

The index i over *commodities* or *goods* could equally well be considered as an index over types of information embodied in materials, structures, organizations, etc. These include thermodynamic information, morphological information, symbolic information, organizational information and so on. In this context, money M and the worth of expected future labor (and life), L can also be viewed as special kinds of information.

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[10] We note that information H can be defined directly in terms of entropy and interpreted as "distance from thermodynamic equilibrium"

$$H_i = S_{i_0} - S_i$$

where S_{i_0} is the entropy in the equilibrium state and S_i is the entropy in the actual state. (The symbol S is normally used for entropy in thermodynamics, and must not be confused with the earlier usage in equation (2). The natural unit for H is the "bit".

[11] It is interesting to note that in thermodynamics a similar situation exists. There the temperature scale is arbitrary, and one is free to choose two points on the scale for convenience. (In the case of the Celsius scale, the zero point is set by the freezing point of water and the 100 point is set by the boiling point of water).

The Time Dependence of *Z*

It should be noted at the outset that, as defined, *Z* has no explicit time dependence. Changes over time occur, of course, but only through changes in the stocks of goods (X_i) and money *M* belonging to the EU, as expressed in equations (2) and (3). The source term for money in the economic unit is zero, by definition, but the *flow* term has two components, viz. the net worth (price times quantity) of the commodity imported across the borders of the EU and the net financial inflow from non-trade transactions (credits, subsidies, taxes, interest or dividends on external investments, etc).

Let us now construct a time dependent equation for the progress function *Z* for an economic unit, using the above. Differentiating equation (18) we obtain

$$\frac{dZ}{dt} = \sum_i \frac{V_i}{T}\frac{dX_i}{dt} + \frac{dM}{dt}\frac{1}{T} \tag{44}$$

The next step is to substitute the general expressions (2) for $\frac{dX_i}{dt}$ and (3) for $\frac{dM}{dt}$, which yields

$$\frac{dZ}{dt} = \left[\sum_i (V_i - P_i)J_i + \sum_i V_i S_i + I\right]\frac{1}{T} \tag{45}$$

We can interpret the first term as the trade surplus, the second term as the production/consumption surplus and the third term as the financial in(out)flow.

Since the EU is assumed to be characterized by (at least) *bounded rationality* (BR) in its decision-making, the first two terms can be assumed to be positive along the path. (The EU will not knowingly engage in trades or make production decisions resulting in loss). However, the third term is not necessarily subject to BR (taxes, for instance, are involuntary), and consequently can be negative enough to make $\frac{dZ}{dt}$ for the EU negative also. We note once again that, assuming voluntary processes, non-negative I and durable goods, $\frac{dZ}{dt} \geq 0$.

Now let us change focus from the individual economic unit to the economic system as a whole, consisting of a set of interacting economic units using the same monetary unit. The progress function for the system as a whole can be defined as a sum over all EU's in the system

$$Z = \sum_\alpha Z^\alpha \tag{46}$$

The resulting equation for the system as a whole has a similar form to (45), except that the three terms (trade, production and financial) are weighted by the *T*'s of the component EU's:

$$\frac{dZ}{dt} = \sum_{\alpha,i} \frac{\left(V_i^\alpha - P_i^\alpha\right)J_i^\alpha}{T^\alpha} + \sum_{\alpha,i} \frac{V_i^\alpha S_i^\alpha}{T^\alpha} + \sum_\alpha \frac{I^\alpha}{T^\alpha} \tag{47}$$

With the help of equations (3) and (45), combining all terms, we obtain a final expression

$$\frac{dZ}{dt} = \frac{1}{2}\left[\sum_{\alpha,\beta,i} J_i^{\alpha\beta}\left(\frac{V_i^\alpha - P_i^{\alpha\beta}}{T^\alpha} - \frac{V_i^\alpha - P_i^{\alpha\beta}}{T^\beta}\right)\right] + \sum_{\alpha,i}\frac{V_i^\alpha S_i^\alpha}{T^\alpha} + \frac{1}{2}\left[\sum_{\alpha,\beta} I^{\alpha\beta}\left(\frac{1}{T^\alpha} - \frac{1}{T^\beta}\right)\right] \tag{48}$$

The bounded rationality of the EU's guarantees that the first two terms are positive. The third term need not be positive, because the rules governing financial flows are not necessarily governed by the BR of individual EU's, but may be enforced by governments (e.g. taxes) or determined by other exogenous factors.

The function T, as noted previously, can usefully be thought of as **liquidity**. It is interesting to examine the conditions under which $\frac{dZ}{dt}$ is positive. As noted before, the first two terms (trade and production) are always positive by BR.

It is easy to show, however, that the third term is positive if (and only if) the *rules* of the economic system permit financial flows only from units with larger values of T to units with lower values of T. This has a fairly straightforward interpretation in terms of taxes: a regressive tax system that consistently (if inadvertently) transfers wealth from the less liquid to the more liquid units will eventually stop growing. We conjecture that only a tax system that tends to equalize the liquidity (not the wealth) of the units is consistent with continued long term growth.

Equation (48) describes, in principle, the dynamical approach to general equilibrium. We note that our model differs in significant respects from the well-known models of Smale [Smale 76] and Aubin [Aubin 81]. Smale's model presupposes both a unique posted price for each commodity at all times and an all-knowing central planner (or super-auctioneer) with knowledge of the demand functions of each consumer. Aubin dispenses with the auctioneer but still presupposes a unique posted price known to all EU's. Both Smale and Aubin consider only the pure exchange case. By contrast, our model allows production and does not assume either a unique price for all EU's or an auctioneer. Nevertheless, in the static case it converges to the Walrasian equilibrium. Details will be presented in a subsequent paper.

The Final Equilibrium State

If there are no external constraints on growth (e.g. finite resource constraints) there is no necessary limit to growth, hence no final state. However in a closed or isolated economy with limited resources, limits may exist such that net production approaches zero. Under these conditions, trading activity must eventually cease also.

It was pointed out above that Z is a maximum when the EU reaches a condition such that (under BR) it cannot improve its economic state by engaging in further economic activities. This final state of non-activity is, in fact, a **Pareto-optimum**: it can be interpreted as an equilibrium for the ES.

Thus all economic activity can be interpreted as an approach to (Pareto)-equilibrium, although final (static) equilibrium is never actually reached.

From (48) it can be seen that the condition for Pareto optimum is $dZ = 0$. This can occur if $V_i^\alpha = P_i^{\alpha\beta}$ (i.e. $V_i^1 = V_i^2 = \cdots$) or if $J_i^{\alpha\beta} = 0$ and if $\sum_{\alpha,i}\frac{V_i^\alpha S_i^\alpha}{T^\alpha} = 0$ and $I^{\alpha\beta} = 0$ or $I^\alpha = I^\beta$.

The conditions under which a Pareto optimum is also a **global** optimum are that $V_i^1 = V_i^2 = \cdots$ and $T_i^1 = T_i^2 = \cdots$ for all i. In other words, the case of a Pareto optimum that is not a global optimum is characterized by unequal liquidities. We will discuss the approach to equilibrium in a subsequent paper.

If there were no physical depreciation of goods or loss of information, it would be possible to assure (subject to appropriate policy choices) that $\frac{dZ}{dt}$ never decreases. The final state (*bliss*) could remain constant forever. This is not possible, however, for reasons to be discussed presently.

The Relationship Between the Utility Function & the Progress Function

In the present context the economic unit possesses goods (durable and perishable) as well as money. In neoclassical demand theory, one standard formulation (e.g. [Lancaster 87 p.122]) is as follows: the individual consumer is assumed to have a continuous utility function $u(x)$ defined on an n-vector of goods and to be constrained to buy at given prices from a fixed money income m. The n goods of the model are assumed to comprise the consumer's universe, so that he spends all his income on those goods. His actual behavior is assumed to be as if he solved the following classical optimizing problem,

$$\max u(x) \text{ s.}t.\ px = m \tag{49}$$

Thus, in the neo-classical spirit, a utility maximization problem for pure exchange can be formulated in terms of the progress function Z as follows

$$\text{Maximize } Z(X + dX, M - PdX) \text{ s.t. } PdX = \text{Constant} \tag{50}$$

where X symbolizes the (vector) set of goods. (A good, here is anything that can be exchanged, produced or consumed). Nevertheless for the progress function Z, such an extremum principle cannot be consistently applied, inasmuch as the economic unit is not a maximizer (by assumption) but merely a satisficer. The progress function Z introduced above nevertheless is related to the neo-classical utility function. To show both the similarities and differences it is convenient to begin by considering the simplest possible case, the pure exchange economy where nothing is either produced or consumed.

Consider a hypothetical exchange of an amount **x** of some vector of goods **X** for an amount of money m. The connection between Z and the classical utility function $U(\mathbf{x})$ is straightforward, viz.

$$U(x) = Z(X + x, M - m) - Z(X, M) \tag{51}$$

If only one commodity in amount x, is involved, and P is the price per unit

$$m = Px \tag{52}$$

From the above it is clear that the general form of the utility-function U must be $U = U(X,M,x)$. Evidently this is consistent with the proofs in the literature demonstrating the non-existence of a utility-function with the simpler form $U = U(x)$ [e.g. Kornai 73]. The foregoing argument clearly supports the introduction of the progress function as a more fundamental quantity. The non-transitivity of preference-ordering for U is clear from the above relation. As the economic agent has only imperfect information and bounded rationality (BR), it only *knows* enough to decline any exchange transaction with negative U. (Bargaining strategies leading to lesser or greater gains are still possible, however). Nevertheless the choice is not governed by any principle of $U(\mathbf{x})$ or $Z(\mathbf{X})$ maximization.

Depreciation

One flaw in the picture as presented so far is that physical goods (and, for that matter, even stored information) are subject to aging, deterioration, erosion, corrosion, loss and wear — in short, depreciation. These phenomena are not reflected in (45), as it stands, because (45) is a bookkeeping relationship for truly **conserved** quantities.

Let us return to the information representation by differentiating (40) and substituting (45), dropping the superscript α for convenience:

$$\frac{dH_i}{dt} = b_i J_i + b_i S_i + N_i \frac{db_i}{dt} \tag{53}$$

Now we can interpret the first term as an information flow, and the second term as an information source. The third term contains something new: if b_i stands for the information content of a unit quantity of the commodity X_i, then the third term represents a change in embodied information per unit time that is **not** the result of economic processes. To the extent that it occurs, it is a consequence of the second law of thermodynamics (increasing entropy).

In other words, the third term of (49) reflects the increase of entropy and *disorder* due to physical processes. If we were to repeat the derivation of (47) and (48) from scratch, using the information representation (i.e. substituting (40) into (18) and then carrying out the indicated operations, then an additional term will appear at the end, reflecting the change in information content of the commodity stock of the EU. This additional term, representing spontaneous wealth depreciation D is as follows:

$$D = \sum_i D_i = -\sum_i V_i N_i \frac{db_i}{dt} \frac{1}{T} \tag{54}$$

By defining D_i with the negative sign convention, it can be seen that

$$D_i \geq 0 \tag{55}$$

for all i. The depreciation rate (more familiar) is just $\frac{D_i}{b_i N_i}$. The complete time dependent equation, in simplified verbal form, can be written as follows:

$$\frac{dZ}{dt} = [\textit{trade surplus}] + [\textit{production surplus}] + [\textit{financial in/outflow}] - [\textit{depreciation}] \tag{56}$$

It is implicit in the above, but should be stated explicitly, that the numerical values of depreciation rates and other system parameters that determine the dynamics of the system are not determined by any of the economic characteristics initially assumed. (On the other hand, the economic behavior of any real system is very much affected by these parameters).

A final point worth emphasis is the following: The Z function is a consequence of economic irreversibility. It would be absolutely non-decreasing (in the absence of financial flows) but for the metabolic needs of humans and animals, and depreciation. Both are due to the intervention of the second law of thermodynamics, which is a direct consequence of irreversibility in the physical domain.

Concluding Remarks

Three concluding comments suggest themselves. In the first place, it is very tempting to try to define a *progress function* for the economic system ES as a whole. However, as noted already, BR is not applicable to the economic system as a whole, whence it is not possible to derive either an integrating factor nor a progress function for an ES. It is worth emphasizing yet again that, whereas most axioms and theorems of neoclassical economics deal with an ES, we have restricted ourselves initially to individual economic units (EU's) and pairwise transactions between EU's.

Our exclusive concern with individual EU's and pairwise transactions explains why we do not need to use utility maximization, and why bounded rationality (BR) is sufficient. On the other hand, it seems plausible that – for an EU, where it is definable – one could develop a variational principle for optimization purposes. The basic idea of constrained maximization has already been applied often in economics, with a variety of objective functions often chosen rather arbitrarily. We have already noted that Z is a maximum when the EU is in a Pareto-optimum state, i.e. a state such that further economic activity will not improve its condition. It would seem, therefore, that it is Z that should be maximized. This being so, we can also consider maximizing the sum over all EU's in the ES, viz.

$$Z = \sum_{\alpha} Z^{\alpha} \tag{57}$$

Absent BR for the system as a whole, one cannot prove that Z for the system as a whole actually tends toward such a maximum. However, it is not implausible that a government might reasonably adopt the objective of maximizing Z. It is also quite plausible that the individual EU's within the ES would agree (if consulted) to such a maximization policy for the ES as a whole. We intend to discuss some implications of this in a future paper.

The third and final comment is that there is, indeed, a close analogy between economics and thermodynamics. In fact, the foregoing derivation proceeds in detail along the same lines as Carathéodory's axiomatic development of thermodynamics. The basis of the analogy is that irreversibility plays a key role in each case. It is also true that function T in our derivation is *like* the temperature; the non-decreasing function Z in our derivation as *like* entropy, the product TZ is *like* enthalpy, wealth W is *like* heat, and so on [Martinás 89]. We freely admit having referred to the analogous arguments in thermodynamics to help us see our way. But the analogy was only a guide. The economic derivations as we have presented them above are rigorous: they stand on their own.

Actually, there may be a deeper connection between thermodynamics and economics than we have claimed. The fact the Z plays the same role in our theory as entropy does in classical thermodynamics is not coincidental. Irreversibility is the key in both cases, as we have noted earlier.

It is interesting that a number of physicists from Szilard and Brillouin on have argued that information **is** negative entropy (or **negentropy**, in Brillouin's language)[12]. We, on the other hand, have argued that the function which plays the same role as entropy in our formulation of economics **is** information. The logical circle appears to be closed.

[12] See, for instance [Szilard 29; Brillouin 53; Jaynes 57, 57a].

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SELF-ORGANIZATION OF MARKETS & THE APPROACH TO EQUILIBRIUM

by Robert U. Ayres[1] and Katalin Martinás[2]

Background

The existence of a static general equilibrium (Pareto-optimal) state for a pure exchange economy was conjectured by Walras [Walras 1874, 1954]. It is assumed that each consumer's consumption of commodity X_i is determined by a demand function depending only on the price P_i

$$X_i = D_i(P_i) \quad (1)$$

The problem is to find an equilibrium price P_i for the i^{th} commodity such that total demand does not exceed total supply (viability constraint[3]), viz.

$$X_i(t) \leq W_i \quad (2)$$

and no consumer spends more than he/she earns (budget constraint). The budgetary constraint can be regarded as a property of the individual consumer's demand function (Walras' law).The existence of such a static equilibrium was first proved rigorously for several models in the 1930's by Wald. Simpler and more general proofs, using Kakutani's fixed-point theorem, were given later by McKenzie [McKenzie 54] and by Arrow and Debreu [Arrow & Debreu 54].

For the dynamic case, Walras suggested a hypothetical price adjustment process known as **tâtonnement** (*groping*), which has been described in the following way:

> "Suppose, as Walras did, a set of prices arbitrarily given; then supply may exceed demand on some markets and fall below on others (unless the initial set is in fact the equilibrium set, there must be at least one case of each, by Walras' law[4]). Suppose the markets are considered in some definite order. On the first market, adjust the price so that supply and demand are equal, given all other prices; this will normally require raising the price if demand initially exceeded supply, decreasing it in the opposite case. The change in the first price will change supply and demand in all other

[1] Carnegie-Mellon University, Pittsburgh, PA.,USA, and the International Institute of Applied Systems Analysis (IIASA), Laxenburg, Austria.

[2] Roland Eotvos University, Budapest, Hungary.

[3] It is a fundamental axiom in economics and in life that consumption of physical goods cannot exceed available supply. Koopmans has called this "the impossibility of the land of Cockaigne" [Koopmans 51].

[4] Walras' law states that the vector product of prices *P* and excess demand *E* is always equal to zero, even when general equilibrium has not established. See footnote 3.

> markets. Repeat the process with the second and subsequent markets. At the end of one round the last market will be in equilibrium, but none of the others need be because the adjustments on subsequent markets will destroy the equilibrium achieved on any one. However, Walras argued, the supply and demand functions for any given commodity will be more affected by the changes in its own price than by changes in other prices; hence after one round the markets should be more nearly in equilibrium than they were to begin with, and with successive rounds the supply and demand on each market will tend to equality" [Arrow & Hahn 71, pp.4-5].

Such a process was described mathematically, and its stability was investigated in the early 1960's, e.g. [Uzawa 62; Morishima 62; Hahn 62; Hahn & Negishi 62; Arrow & Hahn 71]. The tâtonnement approach basically involves a succession of convergent estimates of the **excess demand** *E* (starting from a general condition of shortage[5]) as a function of price. It assumes that the price of any commodity in an economic system follows a path determined by a differential equation of the form

$$\frac{dP_i}{dt} = E_i[P_i(t)] \tag{3}$$

where the final equilibrium state is defined by $E^i = 0$ for every commodity in the system as a whole. However the condition of the market can only be determined by an all-knowing *super-auctioneer* who calls out prices and receives offers[6]. There can be no actual exchange transactions, however, until the market-clearing price has been determined. Hence there is no operational mechanism for the dynamical approach to equilibrium in the Walrasian paradigm. Thus, though the Walrasian model is **decentralized** in the sense that consumers make independent decisions without knowing anything except the price, it is not **implementable** in real markets.

The next step forward was taken by Smale [Smale 76]. In brief, Smale was able to prove the existence of viable price paths corresponding to a sequence of possible exchanges (reflecting transaction costs) between any starting point and the final market-clearing equilibrium state. He also showed that his exchange adjustment process — once begun — will not stop unless forced to by market

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[5] The alternative possibility of starting from a condition of excess supply is ruled out by the assumption of *free disposal* of any hypothetical excess. Of course this is equivalent to assuming free disposal of wastes and pollutants, which is increasingly untenable. The possibility of storage of goods from one period to the next is also neglected.In neo-classical demand theory the individual consumer is assumed to have a continuous (class C^2) utility function $U(X)$ defined on an *n*-vector of goods *X* and to be constrained to buy at given prices from a fixed money income *M*. The *n* goods of the model are assumed to compromise the consumer's universe, so that he spends all his income on those goods. For instance, in Smale's approach an *economic state* means a set of data characterizing the economy at a given time. — and in a pure exchange economy a state will consist of an allocation of the resources, or equivalently the set of goods of each agent and a price system [Smale 76]. So the actual behavior of economic agents is assumed to be **as if** they solved the following classical optimization problem:

$$\text{Maximize } U(X) \text{ s.t. } PX = M$$

The X_o belonging to the set of solutions of the above classical optimization problem, is the demand. The excess demand **E** is then definable by aggregating the amounts that are demanded by all consumers and subtracting the total sum of current production and available supplies. If E_i is the excess demand for the i^{th} good, the definition of equilibrium in the market is as follows:

$$PE = 0, \quad E \leq 0, \quad P \geq 0$$

This implies that either excess demand is zero or price is zero (Walras' Law). Every good with excess supply is a *free good* under the classic prescription.

[6] A computerized system of this sort has been postulated for the stock market, although not yet implemented.

conditions and that it will converge to the Walrasian static equilibrium. However, Smale's model still assumes that all decision-makers possess perfect knowledge of all prices and consumption levels in the market. Thus, Smale's model is implementable, in principle, by an omnipotent central planner but at the sacrifice of the decentralization property.

There is a large literature on (central) planning, in the sense of large-scale constrained maximization. For a real economy this could involve tens of thousands or millions of units, with a correspondingly large number of equations to solve. Hence, during the heyday of mathematical research on planning methodologies, the idea of *informational decentralization* was explored, especially by Hurwicz. In some approaches to planning the central authority provides price guidance to the individual EU's, using quantity production/exchange information as the basis for its computations. Alternatively, the central planner may utilize transactional price data as an input, and provide guidance to the EU's in terms of quantitative production targets. There are numerous schemes for accomplishing this massive information collection/reduction task in several disjoint stages, each of which can be independent of the other; see, for instance, [Arrow & Hurwicz 60]; [Hurwicz 69]; [Heal 73]; [Heal 86]. It should be emphasized that the term *decentralization* in this literature does not imply the absence of planning *per se*.

Aubin has addressed the problem in a different way with his *viability theory* [e.g. Aubin 81]. He has shown that viable price-adjustment trajectories exist, subject to an assumption considerably weaker than Smale's, namely that economic agents know only the prices paid in all transactions, but not the consumption levels of all other agents. Each agent is assumed to be guided by an individualized demand function which depends on knowledge of the (unique) market price at each point in time, but not on the consumption of other agents. Effectively, for the α^{th} consumer and the i^{th} commodity

$$\frac{dX_{\alpha i}}{dt} = D_{\alpha i}\left[X_{\alpha i}(t), P_i(t)\right] \tag{4}$$

where $P_i(t)$ is the market price of the i^{th} commodity. The individual demand functions are assumed to obey a dynamical version of the Walras law (budget constraint). Effectively, the consumer's total expenditure during each infinitesimal time period is non-increasing. Such a price trajectory must also satisfy the supply (viability) constraint (2). Aubin has shown that the instantaneous collective Walras law, which is designed for permanently balancing the budget, provides also price systems embodying enough information to guarantee that the viability constraint is also satisfied in a decentralized way.

Aubin's model is therefore implementable in real markets, in the sense that actual exchange transactions are possible along the path. Moreover, a much greater degree of true decentralization of decision-making is allowed. It does not assume optimization on the part of consumers; *satisficing* (bounded rationality) is sufficient. Nor does it assume a supervisory auctioneer — or the equivalent — to actively match buyers and sellers. It does, however, presume some *invisible hand* mechanism that receives and disseminates price information about all transactions as they occur, thus ensuring the existence of a unique *market* price for each good at each moment in time.

In the discussion of the existence market equilibrium, the existence of a set E_i of excess demand vectors associated with each commodity price P_i is generally accepted as a precondition. This implies the existence of a unique point-mapping $D_i(P_i)$. The adjustment process implied by neo-classical theories represents an arbitrary and unrealistic assumption concerning the behavior of economic agents. However there is no fundamental theory concerning the behavior of economic decision makers away from the equilibrium state; there are only theories of equilibrium behavior.

Thus, there is no neo-classical theory to explain the self-organization of markets. Such a theory requires a model of interactions based on strictly binary (pair-wise) transactions among economic units acting on the basis of bounded rationality and lacking any information about other EU's except for the prices at which they have previously exchanged goods with other EU's. The approach to static equilibrium in a self-organizing pure exchange market must be shown to occur *without* any centralized authority such as an auctioneer (still less a central planner) who receives and processes information about all transactions on a current basis. Only such a model can be the basis for extension to the more general case where production and economic growth (or decline) are also considered.

A Self-Organizing Model of the Market

In the model of equilibration to be described hereafter, there is no auctioneer and prices are determined only by bargaining between individual buyers and sellers. Each agent knows only its own economic state[7] at all times, including worth for all goods and services available in the market. (The problem of valuation is discussed below). Uniqueness of commodity prices away from equilibrium is not assumed, and would not normally occur, because actual exchange prices are known only to the parties. It will be shown that a convergent price adjustment process nevertheless exists in the static, pure exchange case. It leads to a unique final equilibrium price applicable to all economic units.

To summarize the following discussion briefly in advance, it argued that an integrable *progress function* Z exists for each economic unit [Ayres & Martinás 90]. (A consumer is an economic unit). This function has the property that it never decreases in any pure voluntary exchange transaction of infinitesimal magnitude. In other words, $dZ \geq 0$. This property permits us to characterize the final equilibrium state for each economic unit as the state of maximum Z, which corresponds to the condition $dZ = 0$. Convergence is easily demonstrated by simple arguments, not only for the pure exchange case, but also for more general situations. Before addressing the dynamical problem of equilibration, *per se*, we need to introduce the progress function formally.

When the market is not in equilibrium there is no unique *market price*. On the contrary, prices negotiated between individual buyers and sellers may differ considerably from each other. (This phenomenon is easily observable in real marketplaces). Nevertheless, as the number of exchanges becomes very large, the price fluctuations gradually decrease and prices approach limiting values. It is this convergent process, which can occur **without** any centralized information gathering and dissemination mechanism, that should probably be termed *the invisible hand.* It is what we mean, today, when we speak of *self-organization.*

The Progress Function Z for Economic Units

In a previous paper [Ayres & Martinás, op cit.], the authors have derived a computable *progress function* Z and an associated integrating function T (which can be interpreted as liquidity). These two functions are definable for each economic unit (EU) depending only on the observable **extensive** variables (e.g. money and goods) that define its economic state (see footnote 5). The *meta-economic* assumptions needed to derive the progress function and the associated liquidity function, are the following:

[7] The economic state of an economic unit (EU) is defined by observables, such as stocks of money, fixed capital, inventory and consumption goods, potential output of labor and capital per unit time (productivity), current levels of consumption of goods and economic services – and environmental services – per unit time, stock of economically useful knowledge (*know-how*) and in the case of individuals, physical health and well-being and life expectancy.

I. Standard microeconomic assumptions, viz.

(i) economic processes consist of trade and production/consumption;

(ii) an economic unit is an entity capable of independent *rational* decisions with respect to whether or not to undertake economic processes, based on current information; an economic unit can be a firm, a family unit, or an individual worker/consumer;

(iii) the state of an economic unit is characterized by the quantity of goods (including capital stock, inventory, and consumer goods) which appear in economic processes, and by its money stock); and

(iv) the system has no memory of previous transactions (Markovian processes).

II. Non-standard microeconomic assumptions are as follows:

(i) economic units do not optimize, but make decisions under bounded rationality (BR); in a spontaneous economic process no economic agent voluntarily undertakes any economic activity resulting in decrease of its wealth;

(ii) money is a universal equivalent, for all units, and

(iii) there is no unique *market* price (except in the static equilibrium limit); all transactions are pairwise exchanges of goods for money, and the price of the goods exchanged is determined in pairwise bargaining.

In summary, the derivation requires three key non-standard assumptions: (i) that money — a universal medium of exchange — is defined exogenously, (ii) that EU's make decisions based on *bounded rationality* (BR) in the sense of Simon [Simon 55, 59, 82] — so-called perfect rationality is not required — and (iii) there is no unique market price. It is important to emphasize that BR is a sufficient condition to guarantee the irreversibility of voluntary exchange transactions. Irreversibility of exchange transactions, in turn, is a sufficient condition for the existence of the function Z and the liquidity function T.

An important feature of the progress function is that it does not depend on the market price of any commodity. Indeed, except for the assumed existence of a universal medium of exchange (money) no market price in the usual sense is presumed to exist. The progress function for an economic unit generally can be written as,

$$Z = Z(X_1, X_2, \cdots, X_n, M) \tag{5}$$

It is a consequence of irreversibility, in the sense noted above, that

$$dZ \geq 0 \tag{6}$$

Dynamics of Pure Exchange

To compare the different approaches it is useful to begin with a concrete illustration of an exchange market. Smale offers, as an example, the *mineral bourse* or a large mineral show. This kind of market lasts two or three days; initially buyers, dealers, traders (many agents are all three) bring mineral specimens of some worth and/or money. When the show starts, minerals are traded, bought and sold, more quickly at first, and one sees price equilibrium reached in the afternoon of the last day as prices stabilize and exchange slows to a halt. The exact equilibrium depends on factors such as which agents first encounter each other [Smale 76 p. 212].

Here a process is a path (over time) in the space of states of an economy. Smale has given the exchange axioms for the process in his model of the approach to the equilibrium state [Smale 76]. The exchange axioms for the process assert that:

a. the total resources of the economy are constant (i.e.there is no production);
b. exchange takes place at current prices (there is only one price for each good, at a given time)
c. an exchange increases satisfaction of the participating agents;
d. some exchange will take place provided that it is possible consistent with a, b, and c above.

Our case differs from Smale's in one main respect only. Whereas the neo-classical theory assumes a unique price for all transactions of the same type at a given time (axiom b), we can dispense with this assumption. We merely assume that a price is determined by *pairwise* bargaining between the parties for each transaction, subject to (c).

To prove convergence of the process to equilibrium, recall that Z is increased for both parties in every transaction (a consequence of (6)). Transactions will cease when Z is maximum, i.e. when for all possible transactions $dZ < 0$ at least for one of the parties (Pareto-optimum). Thus transactions can occur if, and do occur only if, prior to the transaction, the worth of the commodity to the buyer (V_i^b) is greater than (or equal to) the agreed price, which is in turn greater than or equal to the worth of the commodity to the seller (V_i^s), that is

$$V_i^b > p_i > V_i^s \qquad (7)$$

From (10), it is easy to prove that, in the final equilibrium state, the worth V for each commodity is identical for all economic units, since $V_i^b = p_i = V_i^s$ is the condition under which the exchanges must stop.

The price at which an exchange takes place may be anywhere between the bottom and the top of the allowed range $V_i^b - V_i^s$ depending on the bargaining strategies of the parties. Assuming the actual quantity exchanged is infinitesimal, the result of the exchange is as follows: (i) Z increases for both parties as a consequence of (6); (ii) the worth (reservation price) V_i of the i^{th} good shifts infinitesimally up for the seller (who now has less) and down for the buyer (who now has more) and, (iii) the liquidity T shifts infinitesimally up for the seller (who receives money) and down for the buyer (who spends money).

The next time the same two parties meet to bargain, the range of possible prices is accordingly reduced. The increase of Z and the convergence of T values are both measures of this narrowing of the *window* of possible transactions. If only infinitesimal exchanges are permitted, convergence to the final state is obvious though we have not excluded the possibility that it might take an infinite amount of time. (It is interesting to note the similarity — and the difference — between the process considered here and the Walrasian **tâtonnement** described above).

For the more realistic situations of finite exchanges, convergence is still assured by the following simple device. Let the quantity exchanged in each transaction be reduced monotonically. (Many rules are possible: for instance, the quantity offered for sale by an EU could be reduced each time in proportion to the percentage change — normally an increase — in the worth of that commodity resulting from the last sale). After a transaction, there are now two possibilities. The first possibility is that each EU is closer to equilibrium than before, in which case the Z value for each is larger. In this case the buyer will continue to buy, and the seller will continue to sell. The second possibility is that the transaction was *too big* resulting in an overshoot. The buyer (who inadvertently bought too much) will observe that his worth has fallen to the point where he will now become a seller if another EU is ready to offer a higher price, and vice versa. The possibility of indefinite oscillation is precluded by the rule of declining quantities.

The non-negativity of dZ (equation (6)) evidently does **not** completely define the path to equilibrium. It does, however, constrain the direction of any voluntary process. The actual quantity of goods exchanged and the rate of (decentralized) price evolution will depend on other characteristics of the economic units, including their pricing strategies and rules for fixing quantities offered or sought in any one exchange transaction. The path also depends on the operation of the marketplace itself, especially the rate at which *encounters* between potential buyers and sellers occurs.

If the process proceeds to a global optimum, then $V_i^\alpha = P_i$ for all α (i.e. for all buyers and sellers) and $T^b = T^s$, i.e. all the EU's have the same final T. This is a criterion for global optimum. It was shown in our previous paper [Ayres & Martinás 90a] that liquidities of different EU's do not normally approach equality, except (i) in the special case where all EU's are indistinguishable (i.e. they have exactly the same T,V and Z functions) and (ii) at global optimum. In all other cases the decentralized exchange process proceeds only to a Pareto optimum.

Simulations of the Approach to Equilibrium

In the usual Markovian approach the *flow* of goods between sellers and buyers, J can be written

$$J_{sb} = f(X_s, X_b, V_s, V_b) \tag{8}$$

and, in principle, the price P is determined by the worth

$$P = P(V_s, V_b) \tag{9}$$

We have emphasized that economic interactions occur only in a pairwise fashion. However, we now wish to focus on a single economic unit (EU). Thus we designate one EU in a market, i.e. surrounded (as it were) by an undifferentiated *cloud* of other economic units with which it interacts, one at a time. The flow, in this approximation is between the designated unit and all others. Now the flow is only a function of V for the designated unit and P, the market price, i.e the price at which exchanges between the designated unit and the *cloud* occurs. In this approximation the designated EU cannot influence the market price: it is a *price-taker.*

In principle f could be a continuous function, but in reality transactions are discontinuous and finite. This case reduces to the continuous case in the limit of very small, very frequent exchanges. Thus, the discontinuous case is the more general one. Since the flow J disappears when $P = V$ (for the designated EU), it follows that

$$J = L(V - P) \tag{10}$$

where V and P are vectors and L is a matrix. By bounded rationality BR (and normal sign convention)

$$\sum_i J_i(V_i - P_i) \geq 0 \tag{11}$$

Thus L is a positive definite matrix.

Having said this, it is clear that L depends on the details of the transactional process. There is no other general restriction on L. However there are basically three different cases that are interesting to consider. The first case is rather hypothetical but worth describing: it is conceivable that the designated EU (knowing its own V and Z functions, and considering a range of possible prices P) attempts to reach the Pareto optimum by maximizing its Z-function in a single transaction. This program involves solving the following equations:

$$V(X+J, M-JP) = P \tag{12}$$

It is important to note that the *price-taker* assumption is not essential here. Pairwise Pareto-optimum can be determined through the same scheme. Each of the two EU's solves equation (12) for its individual V function, for all possible *market* prices. For each price equation (12) yields a corresponding supply (demand) J. If the exchange is between the α^{th} and β^{th} economic units and if the commodity has a material nature, so that it obeys the general conservation of mass law, then

$$J_i^{\alpha\beta} = -J_i^{\beta\alpha} \tag{13}$$

In this case, the system is completely determined except for the dynamics of the approach to local equilibrium (or steady-state). In particular, the prices are determined from equations (10) and (13). Since the V function is derivable from the Z function, the latter in effect contains almost all the information needed to characterize the dynamics. The only missing information is the frequency and sequence of pairwise encounters between EU's. These data would have to be provided exogenously for any given system.

The next case to be considered may be called the *quasi-linear* approach. Here we assume simple proportionality between demand and the difference between internal worth and exchange price. In this case the L-matrix does not depend on price. Now the flow J can be written as:

$$J_i^{\alpha\beta} = \sum_j L_{ij}^{\alpha}\left(V_j^{\alpha} - P_j^{\alpha\beta}\right) \tag{14}$$

If the off-diagonal elements of matrix L are zero, then one get a simpler formula

$$P_i^{\alpha\beta} = \frac{L_i^{\alpha} V_i^{\alpha} + L_i^{\beta} V_i^{\beta}}{L_i^{\alpha} + L_i^{\beta}} \tag{15}$$

Inserting it into (14) one gets

$$J_i^{\alpha\beta} = L_i^{\alpha\beta}\left(V_i^\alpha - V_i^\beta\right) \tag{16}$$

where

$$L_i^{\alpha\beta} = \frac{L_i^\alpha * L_i^\beta}{L_i^\alpha + L_i^\beta} \tag{17}$$

The rate of change of the quantity of goods owned by the αth EU is defined by the demand-supply vector

$$\frac{dX_i^\alpha}{dt} = \sum_\beta J_i^{\alpha\beta} \tag{18}$$

Assuming that changes in the X-vector are small compared to the absolute magnitude of X, one can carry out a Taylor-series expansion for Z around its steady-state average (or equilibrium) value. When this is carried out to second order, one obtains a differential equation defining the time dependence of the system as follows:

$$\frac{dX_i^\alpha}{dt} = \sum_{\beta jk} L_{ij}^{\alpha\beta}\left[\frac{\partial(T^\alpha \partial Z^\alpha)}{\partial X_j^\alpha \partial X_k^\alpha}\delta X_k^\alpha - \frac{\partial(T^\beta \partial Z^\beta)}{\partial X_i^\beta \partial X_k^\beta}\delta X_k^\beta\right] \tag{19}$$

The usual stability analyses can be applied[8]. In this case, note that the dynamics of the system are completely defined by the Z-function and the L-matrix.

The third case differs from the first in that the path to steady-state or equilibrium is assumed to require a number of transactions, such that the price changes in any one exchange are small relative to the total change. Ditto the quantity exchanged at any one time is relatively small in the same sense. This case differs from the second case in two ways. First, the system is not restricted to small changes. Second, the system is not endogenously determined; that is, prices and quantities exchanged are determined by factors outside the system. In this case the rate of approach to steady-state or equilibrium is strongly affected by external constraints.

To simulate the behavior of an economic system we have to define the agents (market, producers, (factories), consumers (humans, workers), their possible interactions, and their initial stocks (endowments) of goods and money. We must also specify their decision rules, beyond the simple requirement of bounded rationality. Agents can then be programmed to determine whether to do or not to undertake the possible activities. We also need further data and assumptions (e.g. parametric values) to fully specify the worth functions and the liquidity functions for a multi-good system (equations (8) and (9)).

The explicit form of one simple form of Z is as follows

[8] If the Taylor expansion is taken to higher order, one can also derive equations with non-linear characteristics, such as self-organization, analogous to the Belousov-Zhabotinski reactions, or the so-called *Brusselator* and *Oregonator* systems.

$$Z = \sum_i \frac{V_i X_i}{T} + \frac{M}{T} \tag{20}$$

where X_i is the i^{th} extensive variable (commodity stock), M is the money stock, V_i is the internal worth of the i^{th} good, and T is the liquidity of the EU [Ayres & Martinás, op cit]. For the sake of concreteness, we also note here the explicit forms for V_i and T, namely

$$V_i = \left[\frac{g_i M}{\sum_i g_i X_i}\right]\left[\ln\left(\frac{M}{X_i}\right) + c_i\right] \tag{21}$$

and

$$T = \frac{M}{\sum_i g_i X_i} \tag{22}$$

Substituting (8) and (9) into (7) yields the following convenient form:

$$Z = \sum_i g_i X_i \ln\left(\frac{M}{k_i X_i}\right) \tag{23}$$

where the g_i and k_i are constants be determined[9]. Figure 1 shows the relationships graphically for a single EU and a single good X. Z is plotted in Fig 1a as a function of X for various endowments of money M, and in Fig 1b as a function of M for various endowments of X. T and V are plotted similarly in Figs 1c through 1f. Note that the logarithmic form has the property that if X becomes large enough, V and Z become negative. (See, for instance, Figure 1d).

Another possible form for the Z function arises from the following choice for T, viz.

$$T = \sqrt{\frac{M}{\sum_i g_i X_i}} \tag{24}$$

This leads to the following form for Z,

[9] It has been emphasized several times that the explicit forms shown here are not unique. The only requirement on T was that it be homogeneous to the order zero, while Z must be homogeneous to the first order. Let Y be the simplest choice, namely $Y = \frac{M}{X}$, where X is the weighted sum over goods. Then every function T of Y with positive first derivative with respect to Y and negative second derivative with respect to M (e.g. Y^q for $0 < q \leq 1$) is also possible. There are an infinite number of such functions. We can assume that economic units that are distinguishable in some way, are also characterized by different forms of T (and Z).

$$Z = \tfrac{1}{2}\sqrt{M * \sum_i g_i X_i} \tag{25}$$

and the worth V becomes

$$V_i = \frac{g_i M}{\sum_i g_i X_i} \tag{26}$$

Figure 2 displays exactly comparable plots, for the same set of values of M and X.

The interaction between two EU's, exchanging a single good for money is shown graphically in Figures 3,4 and 5. In Figure 3 we exhibit the relationship between supply/demand and price for two different definitions of supply. In both cases price is assumed to be exogenously determined by an *auctioneer* and supply/demand is computed for a number of different values of the original stock of the good, X. In the first (3a), the quantity of the good X that will be sold (i.e. the supply) or bought (i.e. the demand) is calculated by postulating a single transaction in which the equilibrium point is determined by the *indifference* condition that Z is the same before and after the transaction. It seems to confirm the classical assumption that supply increases monotonically with price.

However Figure 3b shows the results of a different calculation, with (perhaps) greater claim to realism: it assumes a sequence of exchange transactions limited to one unit of the good X per transaction, such that the sequence terminates only when Z is a maximum (i.e. the internal worth equals the price). In this second case it is interesting to note that, for large initial endowments and high enough prices, the supply actually declines slightly beyond a maximum. This is a theoretical prediction of the model. It reflects the fact that the internal worth of the remaining stock of the good increases sharply as the cash income of the EU (based on sales) grows [10].

Figure 4 exhibits one version of the well-known ***Edgeworth Box***. There is still only one good X, the total quantity of which is 200 units. Starting from **any** initial allocation of the good and money, the final equilibrium lies on the so-called *contract curve*, which lies near the diagonal of the box. In the case of *indistinguishable* EU's (having the same form of Z function), the equilibrium states must be exactly on the diagonal. However, for purposes of illustration, we have chosen the more interesting case where one EU has the logarithmic form of Z-function (equation (23)), while the other has the square root form (equation (25)). Figures 4a-4d show the *fine-structure* of the Box.

Figure 5 still refers to the Edgeworth Box, but for a different situation. Here the EU's are indistinguishable, with logarithmic Z-functions. The total quantity of the good X is still 200. It displays a variety of paths to equilibrium, for two different cases. In the first case (upper left hand corner), the path is plotted on the assumption that only a single unit of the good can be exchanged in one transaction. We have assumed a general price determination *rule* of the form $P = aV_1 + (1-a)V_2$. Results are plotted for a range of values of a, from .001 to .032. In the second case (lower right hand side), a similar price rule is assumed, with the added complexity that exchange transactions begin with quantities of 20 units, and continue at that level until no further exchange of that magnitude was possible. The next attempt was at 10 units, and so on with successively smaller quantities being exchanged.

[10] As a possible real world example, in the early decades of the industrial revolution factory owners often complained that hired workers from peasant backgrounds often worked only long enough to earn enough cash for some particular purpose, at which point they would quit their jobs and return to their villages. This was distressing to factory owners, who could not count on a stable labor supply. A similar phenomenon is observed in parts of Africa today (where it is known as *target farming*). During the early 1970's when the price of petroleum increased 5-fold virtually overnight, some producers (in particular, Iran) began to consider cutting back on production.

Simulation of an ES with Production

To simulate an ES including production as well as exchange processes we introduce a production vector. It describes what happens in the i^{th} firm in an elementary step. In general, in the i^{th} firm $R(i,k)$ units of the k^{th} good are produced $(R(i,k) > 0)$ or consumed $(R(i,k) < 0)$. One explicit class of EU's consists of individual humans (workers/consumers). We assume that they eat, work, and collect durable goods with finite useful lifetimes. They sell their labor, if the price in the labor market is higher at the margin then the internal worth of leisure time. They buy food, until a specified maximal food consumption level (fc) is reached. The quantity of durable goods (personal wealth) increases only by accumulation, but there is a spontaneous depreciation of 10% per period $(dn = 0.9n)$.

External assumptions:

1. *Money*: the liquidity of each EU (?) is assumed to be constant in the market (another possibility is the bank system).

2. *Lifetime*: the lifetime of all goods is ten cycles, with a depreciation rate of 10% per cycle.

3. *Labor*: labor is purchased from worker/consumers by firms for one period at a time

4. *Inexhaustible resources*: (for the sake of simplicity), except for food (agriculture), which is assumed to be based on a finite supply of land. Land can be bought and sold on the market.

5. *No explicit capital*: Except for industrial goods (noted above) and land, there is no explicit capital (again, for the sake of simplicity).

We have introduced four generic EU's in this simulation. They are as follows: (i) a trading company which buys and sells all goods from/to other EU's; (ii) a farm, which buys durable goods and labor from the trader and sells food to the trader; (iii) a manufacturer, which buys agricultural raw materials and labor from the trader and sells durable goods to the trader; (iv) a worker/consumer, who buys food and goods from the trader, and sells labor to the trader.

Results of the simulation are displayed in Figures 6, 7, 8 and 9. In Figure 6a-6d the time dependence (up to 200 cycles) of the Z-functions for the worker/consumers and the two production firms (excluding the Trader) are displayed for various fixed values of the liquidity T, ranging from 0.5 to 400. The results show generally increasing levels over time (cycles) as liquidity increases to 300; but surprisingly, the ES is unstable and fails to grow for $T = 400$. (This curious T-dependence behavior is shown more explicitly in Figures 7,8). Figure 7a-7d displays production and stocks of the two types of goods (food and durables) respectively. The same type of behavior occurs.

Figures 8 and 9 display the same variables (goods output and stocks) at the end of 500 cycles, as an explicit function of liquidity. Figure 8a-8d assumes 100 worker/consumers in the system, while Figure 9a-9d assumes 30 worker/consumers. Otherwise the two cases are the same. The results are obviously qualitatively similar.

It is clear that liquidity T dominates the functioning of the model ES as a whole. In cases where the liquidity is very low or very high, the model economy simply does not function, while there is a middle region characterized by prosperity generally increasing with liquidity. This result is counter-intuitive, but quite robust. It is in direct contradiction to the standard result of classical general equilibrium theory which asserts that the quantity of money in the system is not important. On the other hand, it is consistent (in a general sense) with the main hypothesis of monetary theory. The latter has achieved fairly wide acceptance as a predictor of economic growth.

We cannot yet assert that the observed behavior is characteristic of real economies, since we cannot really explain it. However, the result is sufficiently remarkable to suggest that economic theorists should focus much more attention to elucidating the underlying relationships between liquidity and economic growth.

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FIGURE 1. Logarithmic form of Z.

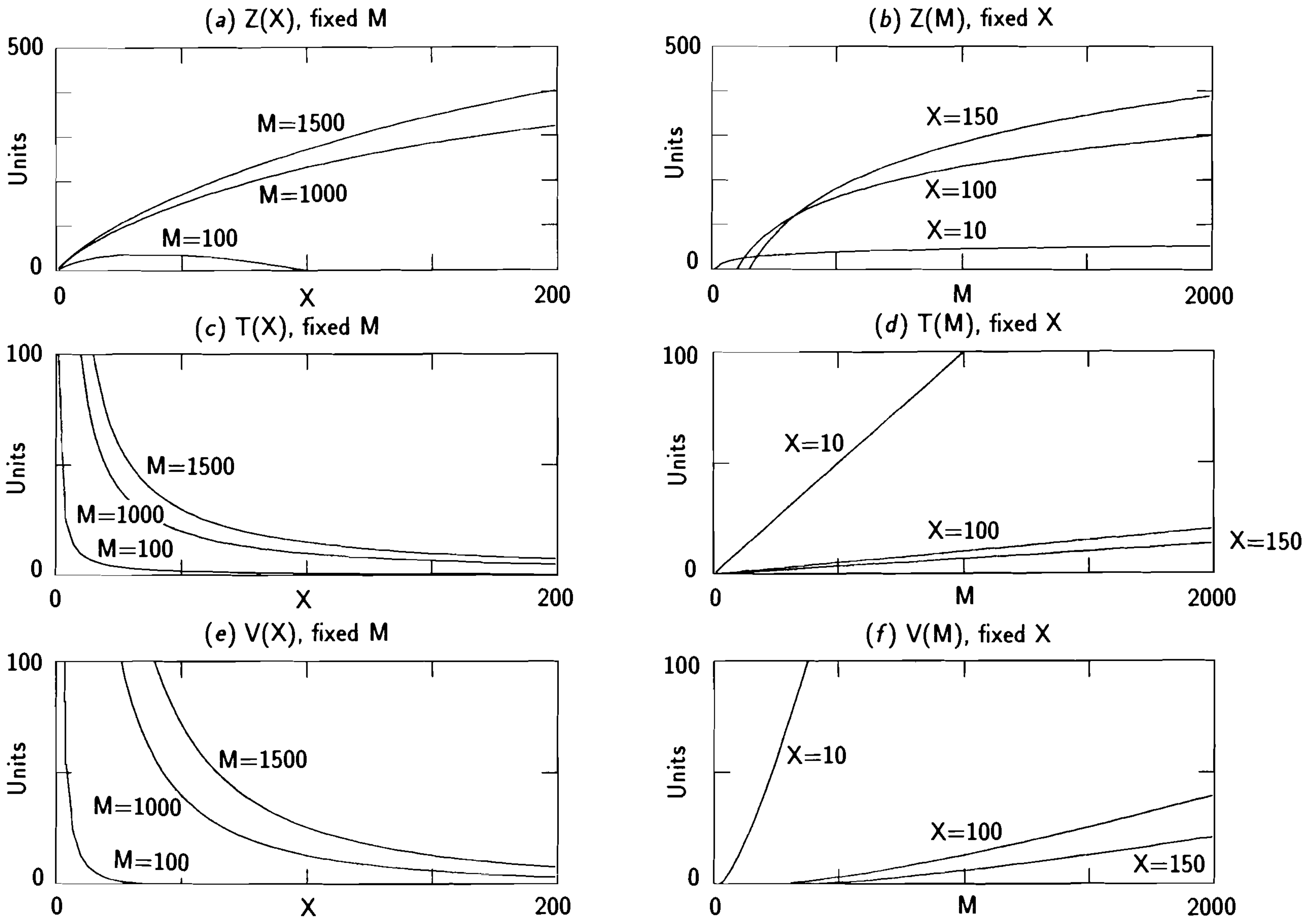


FIGURE 2. Square root form of Z.

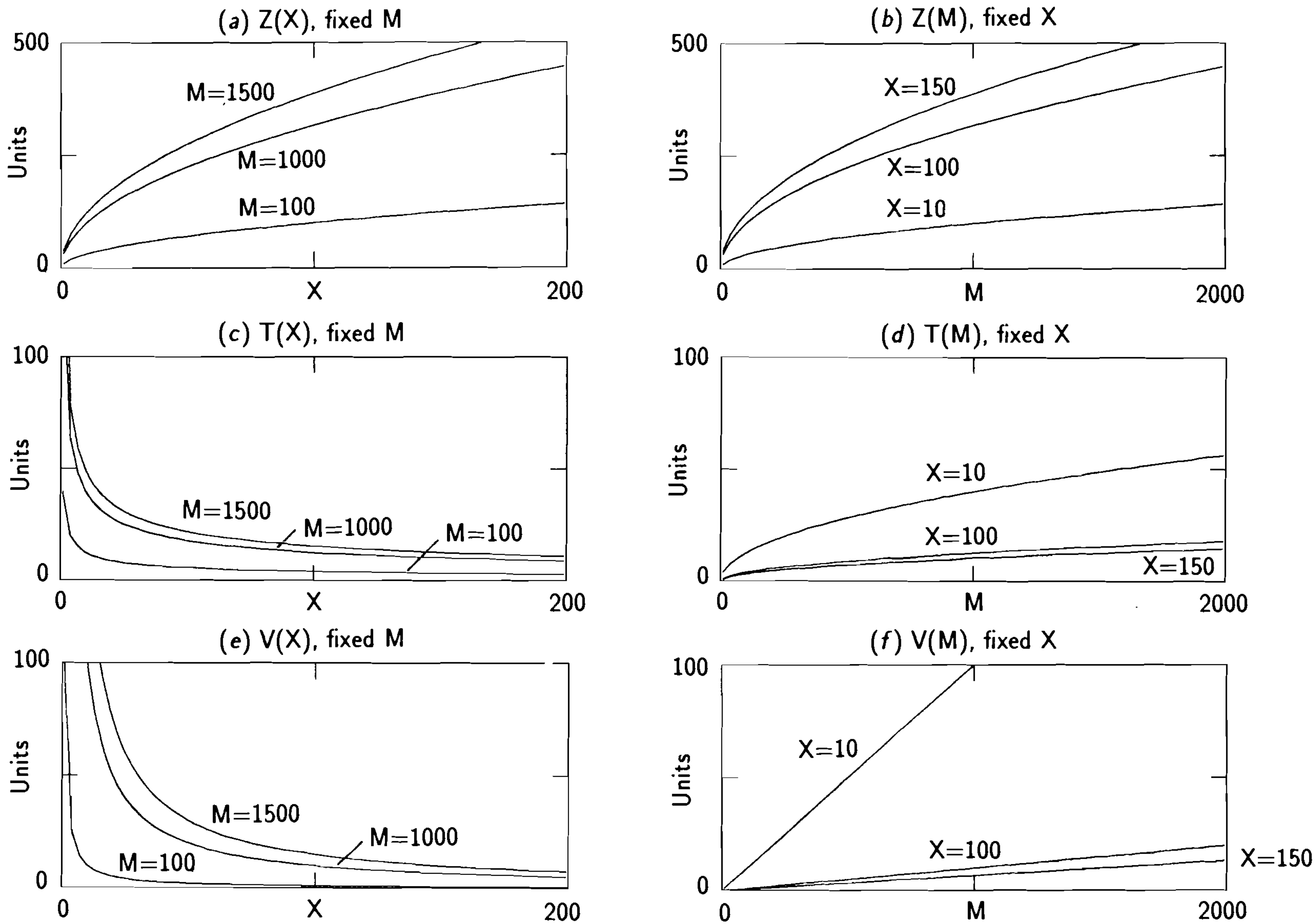


FIGURE 3. Supply/demand at given prices, fixed X, M = 1200.

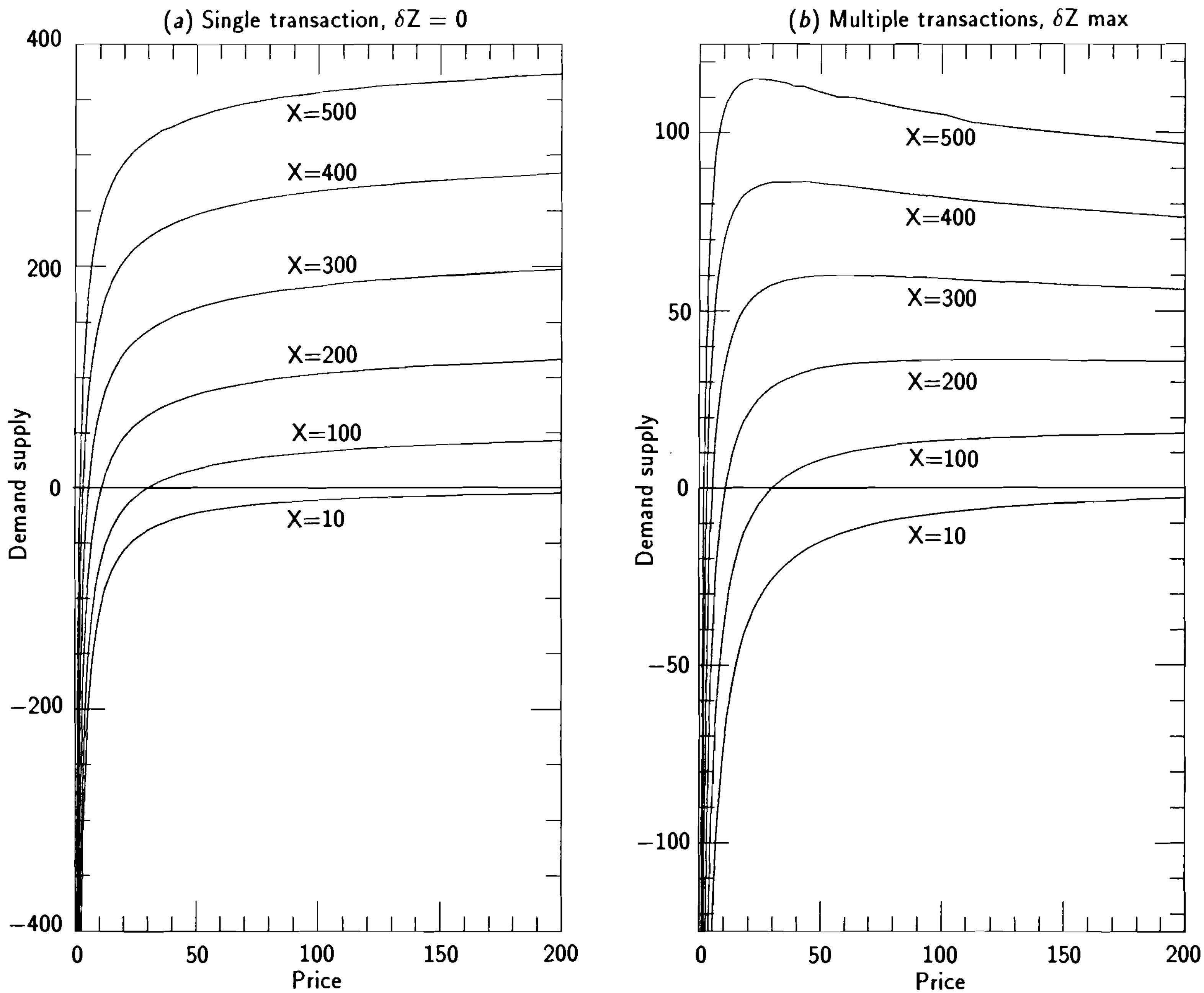


FIGURE 4. Structure of Edgeworth box.

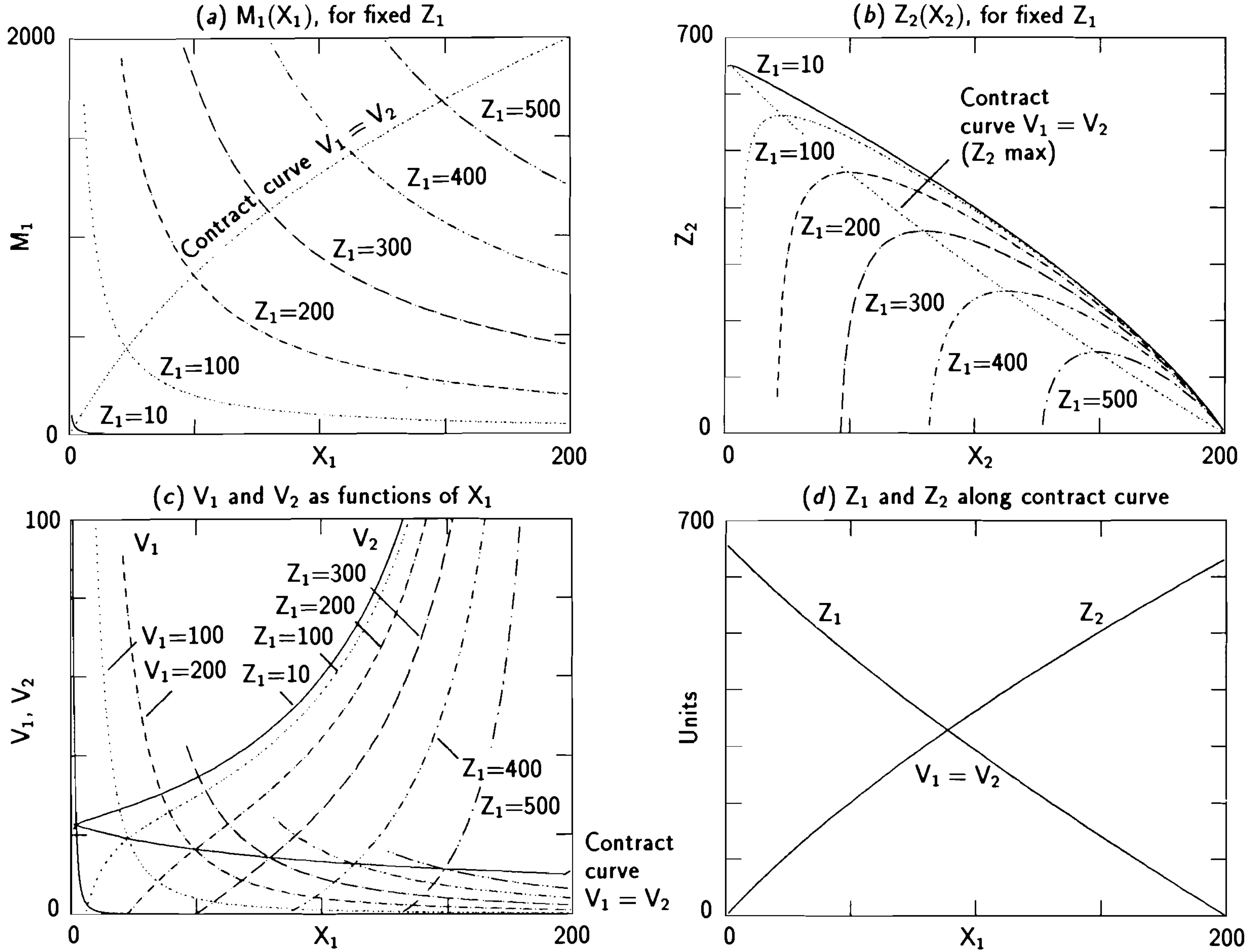
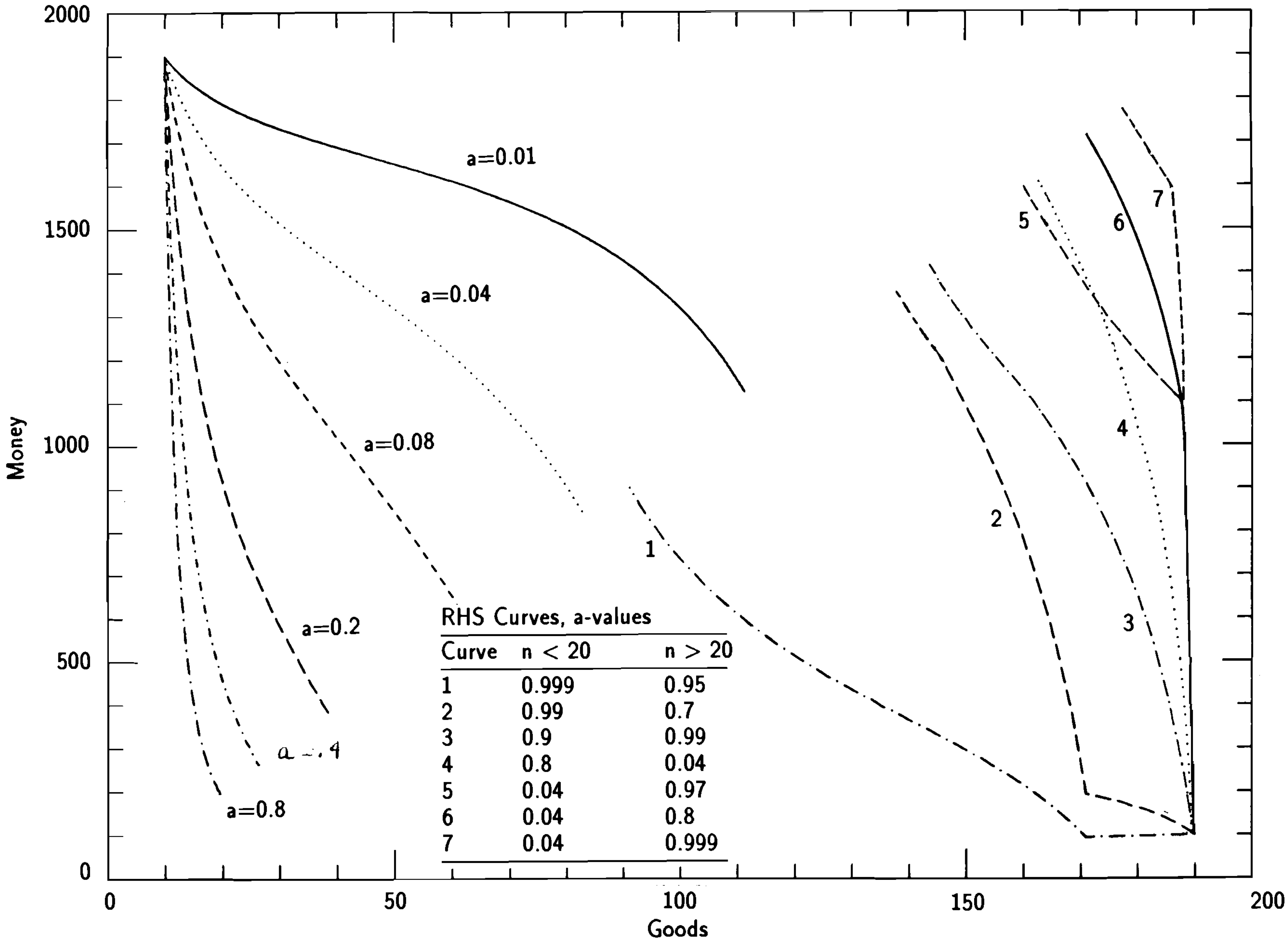


FIGURE 5. Edgeworth box: Approach to equilibrium for various price rules.



RHS Curves, a-values

Curve	n < 20	n > 20
1	0.999	0.95
2	0.99	0.7
3	0.9	0.99
4	0.8	0.04
5	0.04	0.97
6	0.04	0.8
7	0.04	0.999

FIGURE 6. Z-functions over time for fixed liquidities.

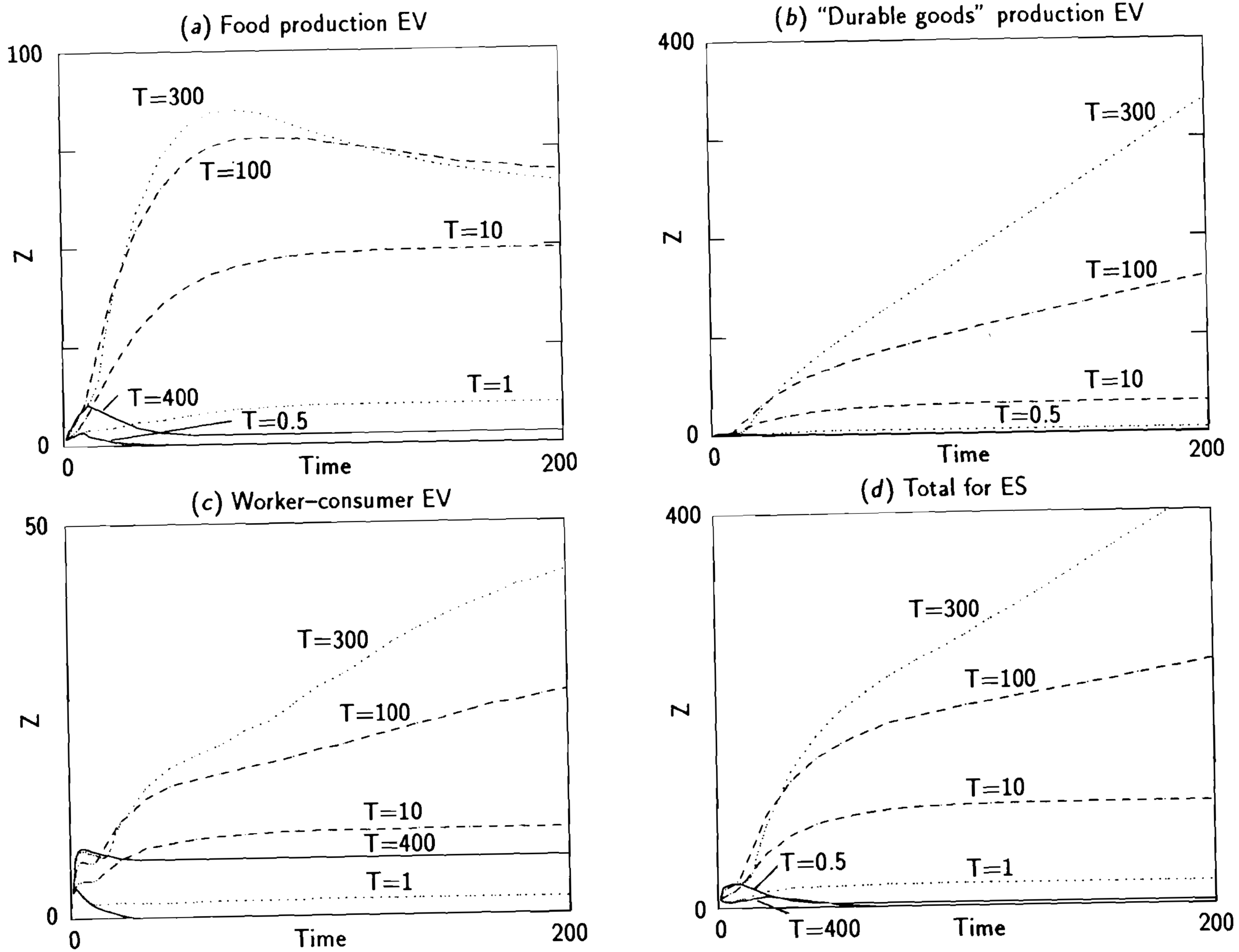


FIGURE 7. Production and stocks overtime for fixed liquidities.

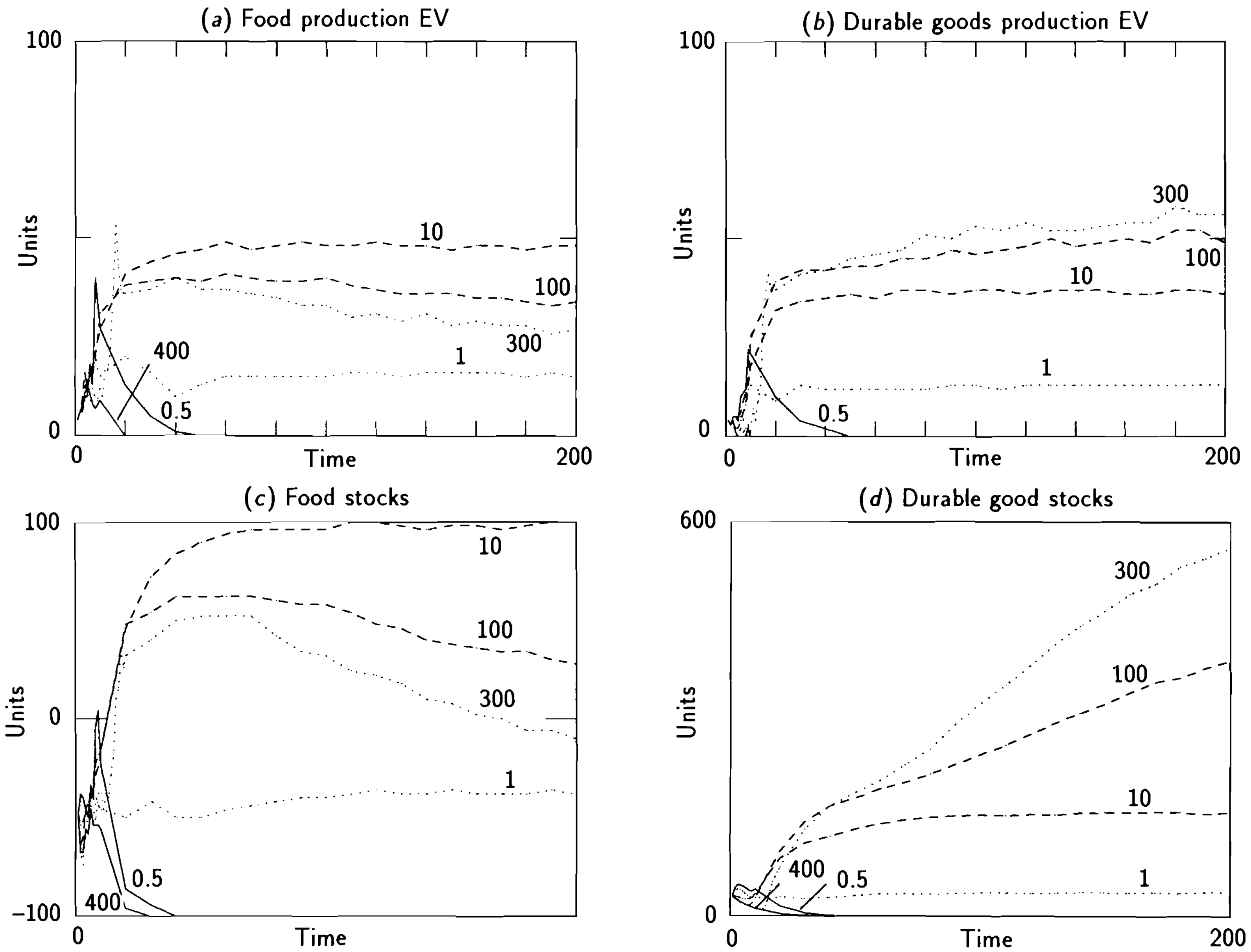


FIGURE 8. State of ES after 500 cycles (100 workers).

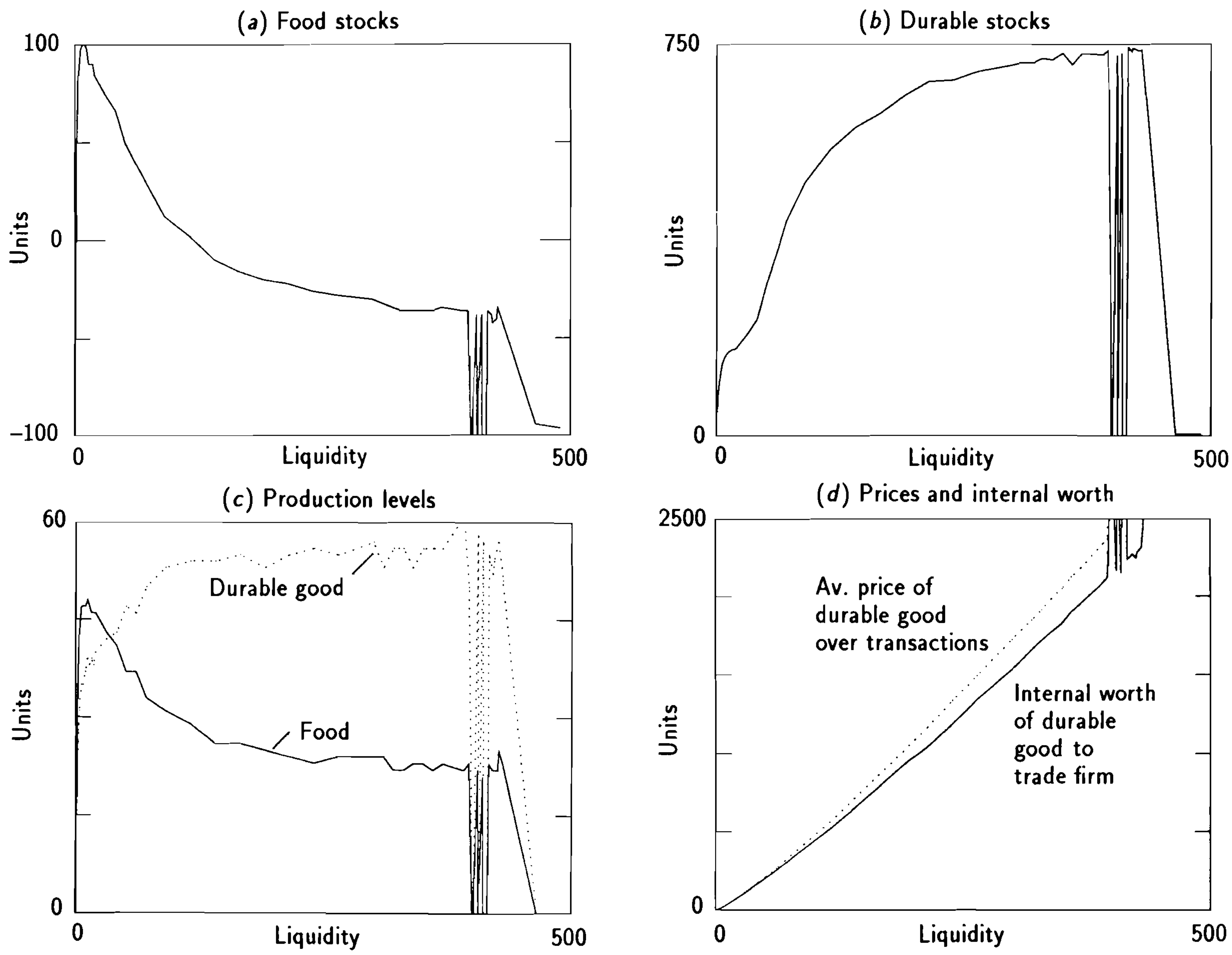


FIGURE 9. State of ES after 500 cycles (30 workers).