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ON THE INVERTIBILITY OF STORAGE SYSTEMS[1]

S. Muratori[2], C. Piccardi[3]and S. Rinaldi[3]

Abstract

The invertibility of single-input single output storage systems (network of reservoirs) is considered in this paper. The analysis shows that cascade and feedback connections of invertible subsystems give rise to invertible systems, and that parallel connections are invertible provided that the network is not too diversified topologically and that the reservoirs have comparable dynamics. These results often allow one to ascertain the invertibility of a complex storage system by direct inspection of a graph.

Key words: Invertibility, storage, reservoir networks, linear systems, transfer function, positive systems.

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Foreword

The formulation of environmentally sound policies for water resources management requires development of methods necessary to predict the consequences of various activities on the environment. The IIASA *Water Resources Project* (WAT) addresses this issue in several ways, one of them is the development of a Decision Support System for Large River Basins. Its objective is to elaborate a set of models and PC-AT interactive software package capable of analysing problems that may arise in developing hydropower and irrigation systems, land use management and agricultural activity. One of such problems, known as invertibility of storage systems, is that of reconstructing inflow data (e.g. rainfall) from recorded outflow data.

This paper by Simona Muratori, Carlo Piccardi and Sergio Rinaldi from the Politecnico di Milano, Italy, deals with the invertibility of single-input single-output storage systems (network of reservoirs). The analysis shows that cascade and feedback connections of invertible subsystems give rise to invertible systems, and that parallel connections are invertible provided that the network is not too diversified topologically and that the reservoirs have comparable dynamics. These results often allow one to ascertain the invertibility of a complex storage system by direct inspection of a graph.

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INTRODUCTION

Among the many inverse problems one may conceive in analyzing storage systems, perhaps the most simple one is that of reconstructing the unknown upstream flows of a hydrological system from recorded downstream flows. Obviously, this is related to the invertibility (i.e. the possibility of computing inputs from outputs) of a dynamical system. Although the problem has been frequently mentioned, in particular by hydrologists, as a very important problem (see, for example, Sagar et al. [4] with reference to aquifers) and possibly solved heuristically by practitioners in specific fields, general theoretical results are not yet available, even for the most simple storage systems. For this reason we discuss in this paper the invertibility of a quite general class of storage systems, namely networks of linear reservoirs (see Moran [2]). Obviously, these models describe either natural or regulated interconnected reservoirs used for power production, irrigation, flood control, urban drainage, and recreation. But they are also used, with an empirical (black-box) approach (see, for example, the well known Nash model [3]), to predict downstream floods from upstream flows or rainfall, even if there are no real reservoirs in the river basin. Finally, they are also implicitly used when the partial differential equation of an aquifer is numerically solved, because each cell of the spatial discretization is indeed equivalent to a linear reservoir. The invertibility of networks of linear reservoirs is therefore the key feature for reconstructing the daily inflows due to snowmelt into a series of alpine reservoirs used for power production given the production schedule of the last reservoir, or for estimating rainfall from recorded flood data, or for computing the boundary conditions of an aquifer from measured outflows.

In the following we first define the class of systems we are dealing with and we point out their basic properties. Then, we discuss some very simple but general results concerning the invertibility of storage systems composed by the cascade or by the feedback of two networks of reservoirs and we show, by means of an example, how these results can be easily applied. Finally, we consider acyclic networks and we prove the following interesting, although qualitative, result: the invertibility of the network is guaranteed if all the reservoirs have roughly the same dynamics and all the paths from input to output go through almost the same number of reservoirs. If, on the contrary, the network is quite diversified, it might be non-invertible.

STORAGE SYSTEMS

Storage systems can be defined in a very broad sense. For example, distributed parameter systems like aquifer, clouds, and lakes are certainly storage systems. Nevertheless, we restrict ourselves in this paper to a lumped class of storage systems, namely single-input single-output networks of linear reservoirs [2].

The basic component of the system is therefore the linear reservoir described by

$$\dot{x}(t) = u(t) - ax(t) - bx(t) \tag{1a}$$

$$y(t) = ax(t) \tag{1b}$$

where t is time, x is storage, u is the inflow rate, $y = ax$ is the outflow rate, and bx is the consumption rate (evaporation losses and supply to consumptive users in a real water reservoir). Thus, the transfer function of the reservoir

$$G(s) = \frac{\mu}{1+sT} \qquad \mu = \frac{a}{a+b} \qquad T = \frac{1}{a+b}$$

is characterized by a gain μ smaller than 1 if there is consumption and by a unitary gain in the opposite case. A second important element in a network is the branching point that models the physical bifurcation of a canal with flow rate y into n canals with flow rates $u_i = d_i y, 0 < d_i < 1$, $\sum d_i = 1$. Of course, if one of the canals supplies a consumptive user the branches appearing in the model will be only $(n-1)$ and the corresponding $\sum d_i$ will be smaller than 1. Moreover, one should notice that the d_i's can easily be incorporated in the gains of the reservoirs of the network.

Reservoir networks are in general acyclic, as in the classical cases of irrigation systems and surface runoff through a series of natural lakes, but they can also be cyclic, like in chemical process control and in wastewater treatment systems with recycling or in controlled urban drainage networks where the outflow of an upstream reservoir might depend upon the storage of a downstream reservoir.

Finally, it is important to remark that physically meaningful storage networks are stable as well as all their subnetworks. This trivial property can be taken into account, as shown in the next section, when analyzing the invertibility of complex networks.

INVERTIBILITY OF CASCADE AND FEEDBACK NETWORKS

As is well known a single-input single-output linear system is invertible if and only if all the zeros of its transfer function are stable (i.e. have strictly negative real part). Thus, the invertibility of a reservoir network can be ascertained by explicitly computing its transfer function $G(s) = p(s)/q(s)$ and by applying the Routh test (or any other equivalent test) to the polynomial $p(s)$. Nevertheless, in order to be more effective one can take advantage of the structure of the system by using the following two very simple properties.

Theorem 1. A reservoir network composed by the cascade of two subnetworks is invertible if and only if both subnetworks are invertible. □

Theorem 2. A cyclic reservoir network composed by a forward and a feedback subnetwork is invertible if and only if the forward subnetwork is invertible. □

The first property holds because the subnetworks have only stable poles which, therefore, cannot cancel unstable zeros. Similarly, the second property holds because the unstable zeros of the closed loop system coincide with the unstable zeros of the forward network (the feedback network being stable).

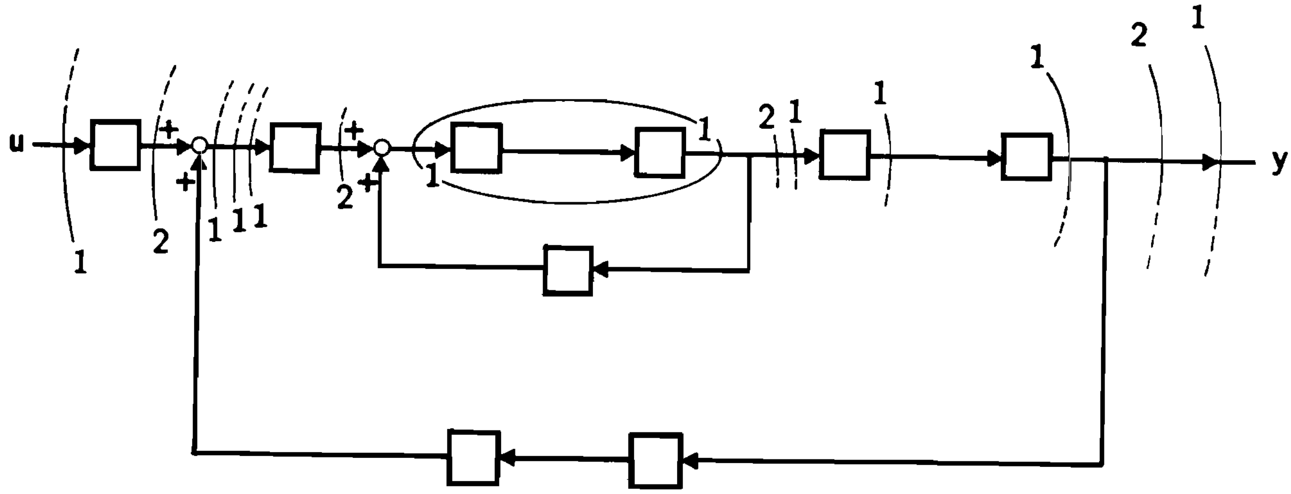

Figure 1: An invertible network of reservoirs.

Although the two above properties are very general, they are quite powerful for analyzing real networks of reservoirs. For example, the system illustrated in Figure 1 can be proven to be invertible by inspection, by successively applying Theorem 1 and 2 in the following order 1, 2, 1, 1, 1, 2, 1, as indicated in the figure.

INVERTIBILITY OF ACYCLIC NETWORKS

Theorem 1 implies, in particular, that the connection of n reservoirs in cascade is invertible. The same property holds for parallel connections, as pointed out by the following theorem that somehow recalls studies on minimum phase of ladder networks (S.P. Chan [1]).

Theorem 3. The parallel connection of n reservoirs is invertible (and the zeros and poles of its transfer function $G^{(n)}(s)$ are alternate). □

Proof.

First of all, notice that the property holds for $n = 2$, since

$$G^{(2)}(s) = \frac{\alpha_1}{s+p_1} + \frac{\alpha_2}{s+p_2} = \alpha^{(2)}\frac{s+z^{(2)}}{(s+p_1)(s+p_2)}$$

with

$$z^{(2)} = \frac{\alpha_2}{\alpha_1+\alpha_2}p_1 + \frac{\alpha_1}{\alpha_1+\alpha_2}p_2$$

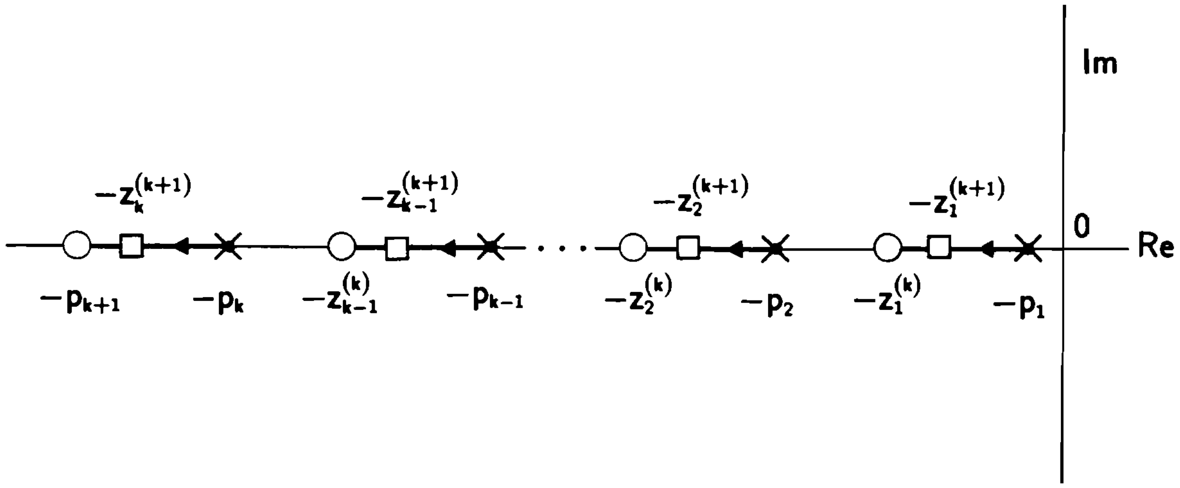

Figure 2: The root locus of the open-loop transfer function $G^{(k)}(s)/H_{k+1}(s)$.

Moreover, the property holds for $k+1$ if it holds for k. In fact, notice that the zeros of the transfer function $G(s) + H(s)$ of the parallel connection of two systems are the poles of the closed loop system which has $G(s)$ as forward transfer function and $1/H(s)$ as feedback transfer function. Thus, the zeros of the parallel connection of

$$G^{(k)}(s) = \sum_{i=1}^{k} \frac{\alpha_i}{s+p_i} = \alpha^{(k)} \frac{\Pi_{i=1}^{k-1}(s+z_i^{(k)})}{\Pi_{i=1}^{k}(s+p_i)}$$

with the $(k+1)$-st reservoir $H_{k+1}(s) = \frac{\alpha_{k+1}}{s+p_{k+1}}$ can be found by drawing the root locus (Truxal [5]) of the open-loop transfer function $G^{(k)}(s)/H_{k+1}(s)$ which has k poles $(-p_1, -p_2, ..., -p_k)$ and k zeros $(-z_1^{(k)}, -z_2^{(k)}, ..., -z_{k-1}^{(k)}, -p_{k+1})$ alternate on the negative real axis. Thus, the root locus (see Figure 2) goes from each pole to the closest zero on its left and the zeros of $G^{(k+1)}(s)$ are therefore alternate with its poles. □

Since cascade and parallel connections of linear reservoirs are invertible, one could naturally conjecture that any acyclic reservoir network is invertible. Unfortunately, this is false, as can be shown by means of relatively simple examples (see below). Nevertheless, a weaker form of the conjecture holds as indicated by the following theorem.

Theorem 4. If the reservoirs contained in each input-output path of an acyclic network differ at most by two, then the network is invertible provided the time constants of the reservoirs are not too different. □

Proof.

The proof of this property is very simple. Indeed, assume that all reservoirs have the same time constant and that each input-output path contains $k, k+1$, or $k+2$ reservoirs. Thus, the transfer function of the network is

$$G(s) = \frac{A}{(s+p)^k} + \frac{B}{(s+p)^{k+1}} + \frac{C}{(s+p)^{k+2}} =$$

$$= \frac{As^2 + (2Ap + B)s + (Ap^2 + Bp + C)}{(s+p)^{k+2}}$$

and A, B and C are nonnegative and at least one of them is positive. Obviously, the zeros of $G(s)$ (if any) have negative real part if $ABC = 0$. But the same is true if $ABC > 0$ because the coefficients of the second order polynomial are positive in such a case. Since the system is not completely reachable and observable when all the reservoirs have the same dynamics, any small, but generic, perturbation of the time constants of the reservoirs will give rise to a completely reachable and observable system with a higher number of zeros and poles. Nevertheless, two of such zeros are, by continuity, close to the two roots of the above second order polynomial and have therefore negative real parts, while the others are close to the eigenvalues of the non reachable and/or nonobservable part of the system. But such eigenvalues, the network being acyclic, are the eigenvalues of some of the reservoirs and are therefore negative. Thus, all zeros have negative real parts provided the time constants of the reservoirs are not too different. □

Theorem 4 says that storage systems with "low diversity" are invertible, where "diversity" is a combination of the topological diversity of the network and of the dynamic diversity of the reservoirs. Of course, if the network has a very low topological diversity it might easily be that the system is invertible even if the reservoirs have highly differentiated dynamics. For example, the parallel connection of a single reservoir with the cascade of two reservoirs is structurally invertible (easy to check) as well as the parallel connection of n reservoirs (Theorem 3).

We now show, by means of two simple examples, that the converse of Theorem 4 (namely "high diversity implies non invertibility") is also true.

Example 1.

Consider the parallel connection of two subnetworks, the first being the cascade of three reservoirs

$$G_1(s) = \frac{\mu_1}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

and the second being a single reservoir

$$G_2(s) = \frac{\mu_2}{(1+sT_4)}$$

In this system the reservoirs contained in each input-output path differ by two. Thus, the condition on the topological diversity required by Theorem 4 is satisfied. Nevertheless, let us assume that the time constants $T_1, ..., T_4$ are different. As already said, the zeros of $G(s) = G_1(s)+G_2(s)$ are the poles of the closed loop system which has $G_1(s)$ as forward transfer function and $1/G_2(s)$ as feedback transfer function. By applying the standard root locus technique to the open-loop transfer function

$$\frac{G_1(s)}{G_2(s)} = \frac{\mu_1}{\mu_2}\frac{1+sT_4}{(1+sT_1)(1+sT_2)(1+sT_3)}$$

one can easily discuss the location of the zeros of $G(s)$ in the complex plane, and the conclusion is that the system is non-invertible provided that

$$\frac{1}{T_4} > \sum_{i=1}^{3}\frac{1}{T_i}$$

and that the gain μ_2 of the single reservoir is sufficiently smaller than the gain μ_1 of the cascade of reservoirs. This means that if the reservoirs belonging to the two subnetworks are highly diversified in their dynamics and the flows through the two paths are suitably tuned, the system is non-invertible.

Example 2.

Consider the parallel connection of two subnetworks, the first being the cascade of four reservoirs and the second being a simple reservoir and assume that all the reservoirs have the same time constant. Thus, the condition on the topological diversity of the network required by Theorem 4 is not satisfied while the condition on dynamical diversity is strictly satisfied. Since the zeros of $G(s) = G_1(s) + G_2(s)$ are the poles of the feedback system with open-loop transfer function

$$\frac{G_1(s)}{G_2(s)} = \frac{\mu_1/\mu_2}{(1+sT)^3}$$

the system is not invertible if $\mu_1/\mu_2 > 8$. Once again, this shows that if the two conditions required by Theorem 4 are not satisfied and the flows are suitable the system is non-invertible.

CONCLUDING REMARKS

The invertibility (minimum phase) of linear storage systems (networks of linear reservoirs) has been investigated in this paper. The conclusion is that these systems are very often invertible because their main component (i.e. the single reservoir) is such and invertibility cannot be lost through cascade and feedback connections (Theorems 1 and 2). Moreover, parallel connections of reservoirs give rise to invertible systems if the network is not too diversified topologically and the reservoirs are not too different in their dynamics.

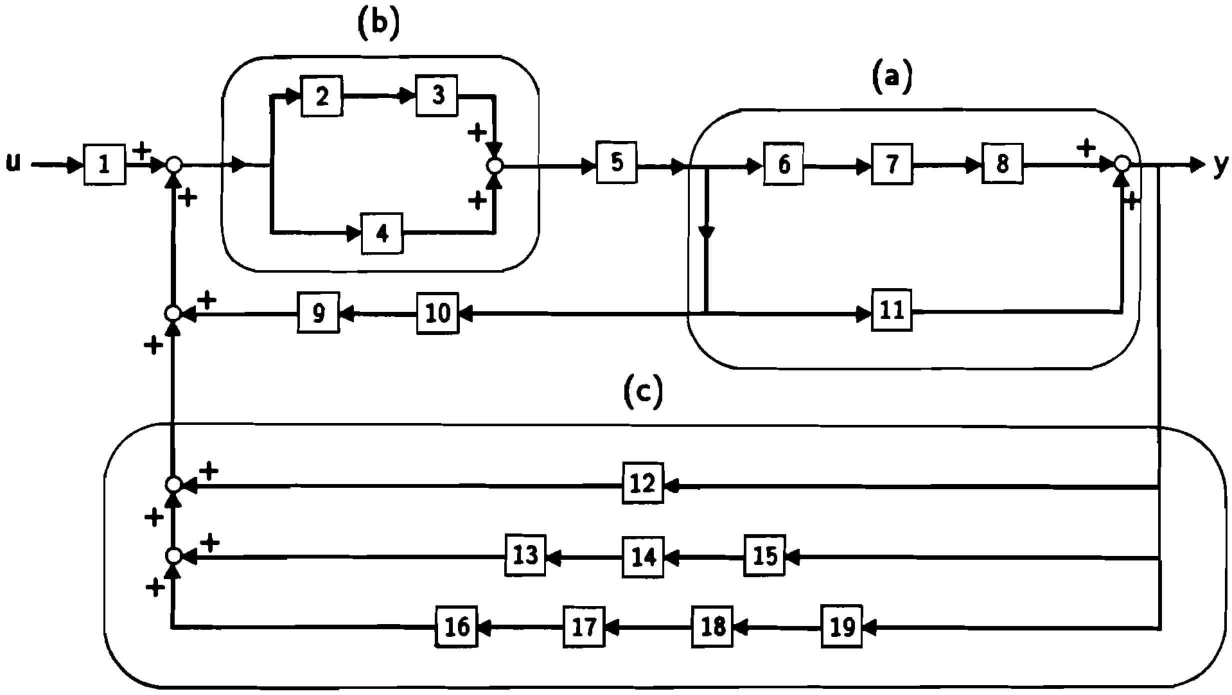

Figure 3: A network of reservoirs with parallel connections (a), (b), and (c).

The properties pointed out in the paper often allows one to ascertain the invertibility of a complex storage system by direct inspection of its graph. For example, the network represented in Figure 3 is certainly invertible if the time constant of reservoir 11 is greater than the time constant of at least one of the reservoirs 6, 7, and 8. In fact, under this assumption, the parallel connection indicated by (a) in Figure 3 is invertible (see Example 1), while the parallel connection indicated by (b) is structurally invertible, as pointed out in the paper. Thus, in view of Theorems 1 and 2 the network is invertible even if the subnetwork indicated by (c) is not. In fact, the system is composed by cascade connections of invertible networks and by feedback connections the forward components of which are invertible.

The results presented in this paper are certainly significant in the analysis of hydrological systems, where the problem of reconstructing missing input data from output recorded data is of definite importance. They are also of interest in the design of feed-forward adaptive storage control schemes, since the minimum phase of the system is a necessary condition. We also hope that these results can be extended to distributed storage systems, in particular to aquifers, where the identification of the boundary conditions is a problem of paramount importance.

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