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Foreword

The author shows how to model imprecision in the decision maker's judgements, within a Bayesian context. He provides axioms leading to work with families of value functions, if the problem is under certainty, or probability distributions and utility functions, if the problem is under uncertainty, or probability distributions and utility functions, if the problem is under uncertainty. Some solution concepts are suggested. On the whole, he provides a robust decision theory based on a set of axioms embodying coherence, and essentially implies carrying out a family of coherent decision analyses.

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1 Introduction

The foundations of the Bayesian approach to decision making, see French (1986) e.g., require precision in the decision maker's (DM) judgements (preferences, risk attitudes, beliefs) which mean precision in the numerical inputs (values, utilities, probabilities) to the analysis. The assessment of these inputs imply the encoding of the DM's judgemental inputs in parameters and the association of an evaluation function with each alternative. In practice, the DM may not be ready or able to provide the information necessary to locate the parameters precisely. Instead, he may only give some constraints on his judgements: we have to work with a family of values or probabilities and utilities. This clashes with the Bayesian foundations. We shall call this decision making under partial information.

We give axiomatic foundations for this case, according to this principle: the Bayesian approach assumes that comparative judgements follow a weak order (plus some conditions); this leads us to the codification of judgemental inputs in parameters with some uniqueness properties. We shall see that quasi orders regulating comparative judgements (plus some similar conditions) correspond to sets of feasible parameters and, consequently, to constraints of our kind. We start by modelling general preferences under certainty. Then, we analyse belief models. We conclude by analysing preferences under uncertainty. A parametric model is introduced to analyse problems under certainty and under uncertainty in parallel. The proper solution solution concepts are then suggested. An example illustrates the ideas. We study only finite alternative, finite state problems.

Several authors, see Rios (1975) and the references in Nau (1989), have proposed to work with families of utilities and probabilities to account for imprecision in judgements. However, the general emphasis has been on quantitative properties of upper and lower probabilities and/or utilities. Our emphasis will be in (axiomatic) qualitative characterisations.

Let (A, R), (A, S) be binary relations. We say that:

Definition 1

- *(A, R) is a quasi order if it is reflexive and transitive.*
- *(A, R) is a weak order if it is reflexive, transitive and complete.*
- *(A, R) is a strict partial order if it is irreflexive and transitive.*

Definition 2 *S is a covering of R if $a, b \in A$, $aRb \Longrightarrow aSb$. We write it $R \subseteq S$.*

From now on, our binary relations shall be written $(A, \preceq)$. We shall say that

$$a \prec b \text{ if } a \preceq b, \neg(b \preceq a),$$

$$a \sim b \text{ if } a \preceq b, b \preceq a.$$

2 Decision making under certainty

Suppose a DM has to choose among a finite set $A = \{a_1, \ldots, a_m\}$ of alternatives. We model the DM's preferences by a relation $\preceq$ interpreted as follows: let $a_i, a_j \in A$, '$a_i \preceq a_j$' means 'a_i is at most as preferred as a_j'. Our initial aim is to represent it by means of a value function v, see French (1986). Imprecision in the DM's preferences lead us to represent $\preceq$ by a family of value functions.

Roubens and Vincke (1985) give a one way representation of quasi orders. Roberts (1979) presents a characterisation of strict partial orders. We could give a representation of a quasi order by reduction to a strict partial order, but that would lead us to order alternatives in a weak Pareto sense. We provide a characterisation of quasi orders, which will lead us to a Pareto order, as in the rest of the cases we study.

The result is

Proposition 1 *Let A be a finite set and $\preceq$ a relation on A. $(A, \preceq)$ is a quasi order iff there are r real functions $v_1, \ldots, v_r$ such that*

$$a_i \preceq a_j \Longleftrightarrow v_l(a_i) \leq v_l(a_j), l = 1, \ldots, r$$

for $a_i, a_j \in A$, for some $r \in \mathbb{N}$.

$\Leftarrow$ Clearly, $(A, \preceq)$ is reflexive and transitive.
$\Rightarrow$ For each $a_i \in A$, define

$$v_i(a_j) = \begin{cases} 1 & \text{if } a_i \preceq a_j \\ 0 & \text{otherwise} \end{cases}$$

Suppose that $a_i \preceq a_j$. If $a_k \preceq a_i$ then $a_k \preceq a_j$, thus $v_k(a_i) = v_k(a_j)$. If $\neg(a_k \preceq a_i)$, then $v_k(a_i) = 0 \leq v_k(a_j)$. Thus, $v_k(a_i) \leq v_k(a_j), \forall k$.

Suppose $\neg(a_i \preceq a_j)$. Then $v_i(a_j) = 0 < 1 = v_i(a_i)$. Thus, $\neg(v_k(a_i) \leq v_k(a_j), \forall k)$.
□

Observe that given $v_1, \ldots, v_r$ representing $\preceq$ we have:

Corollary 1 *For $a_i, a_j \in A$*

$$a_i \preceq a_j \iff \sum_{l=1}^{r} \lambda_l v_l(a_i) \leq \sum_{l=1}^{r} \lambda_l v_l(a_j), \forall \lambda_1, \ldots, \lambda_r : \lambda_l \geq 0, \sum_{l=1}^{r} \lambda_l = 1$$

This result might look innocuous at first sight. However, we easily find that with our choice of $v_1, \ldots, v_r$:

Proposition 2 *Let $(A, \preceq_*)$ be a weak order such that $\preceq_*$ is a covering of $\preceq$. There is $\lambda \geq 0$ such that $\sum_{l=1}^{r} \lambda_l = 1$ and*

$$a_i \preceq_* a_j \iff \sum_{l=1}^{r} \lambda_l v_l(a_i) \leq \sum_{l=1}^{r} \lambda_l v_l(a_j)$$

Therefore, we have a procedure to generate the weak orders covering a quasi order.

3 Decision making under uncertainty

The DM has to choose among the set A of alternatives. The consequences of each alternative are not known with certainty. Nature adopts one of the states of $\Theta = \{1, \ldots, k\}$. We associate the consequence a_{ij} to the alternative a_i when the state is j. a_{ij} belongs to $\mathcal{C} = \{c_1, \ldots, c_s\}$, the finite space of consequences. We study first the problem of modelling beliefs and, then, that of modelling preferences.

3.1 Modelling beliefs

Let Θ be the set of states of Nature. The DM expresses his beliefs over $\mathcal{A}$, the algebra of subsets of Θ. They are modelled in terms of a relation $\preceq_\ell$ on $\mathcal{A}$ interpreted as follows: '$B \preceq_\ell C$' for $B, C \in \mathcal{A}$ means that 'B is at most as likely to occur as C'. We attempt to represent it by a probability distribution. However, we cannot expect the judgemental inputs to be as precise as demanded by the Bayesian foundations. We can only hope, at best, to delimit possible values for a DM's subjective probability to a small set. Given this, it is natural, perhaps, that theories of upper and lower probabilities have developed as in Koopman (1940), Smith (1961), Good (1962), Dempster (1967) or Suppes (1974). See also Walley and Fine (1979), Shafer (1976) and Nau (1989). In general, their emphasis is on quantitative properties of lower and upper probabilities, though Suppes gives a qualitative treatment of the problem, using semiorders, and Nau provides an axiomatic in terms of betting behaviour. In our opinion, the emphasis should be on imprecision in beliefs leading us to work with a family of probabilities. That these probabilities have upper and lower probabilities is interesting, but no more.

We justify this approach axiomatically, characterising probability quasi orders by systems of inequalities. Scott (1964) uses systems of inequalities to represent probability weak orders; Fishburn (1969, 1985) uses them to represent probability semiorders and interval orders. Our result weakens Scott's axioms, giving a simple and elegant qualitative treatment of probability quasi orders. Giron and Rios (1980) provide a characterisation in a more general setting.

We shall need the following property (where χ_B represents the characteristic function of the subset $B \in \mathcal{A}$):

Definition 3 (Property P) *Let $\Theta, \mathcal{A}, \preceq_\ell$ be as above. $(\mathcal{A}, \preceq_\ell)$ satisfies property P if for all integers $l, r \geq 1$ and for all $B_1, C_1, \ldots, B_{l+1}, C_{l+1} \in \mathcal{A}$, such that $B_i \preceq_\ell C_i, i = 1, \ldots, l$ and*

$$\chi_{B_1} + \cdots + \chi_{B_l} + r\chi_{B_{l+1}} = \chi_{C_1} + \cdots + \chi_{C_l} + r\chi_{C_{l+1}}$$

then $C_{l+1} \preceq_\ell B_{l+1}$.

The equation in property P simply states that each $i \in \Theta$ appears the same number of times in the sets $B_1, \ldots, B_l, B_{l+1}, \overset{r}{\ldots}, B_{l+1}$ than in the sets $C_1, \ldots, C_l, C_{l+1}, \overset{r}{\ldots}, C_{l+1}$. We can give a betting interpretation of property P: suppose the DM receives a prize of utility 1 for each B_i or C_i that obtains, $i = 1, \ldots, l$, and a prize of utility r for obtaining B_{l+1} or C_{l+1}. Then, if $B_i \preceq_\ell C_i$ for $i = 1, \ldots, l$, it should be $C_{l+1} \preceq_\ell B_{l+1}$.

The main result of this section is

Proposition 3 *Let $\Theta, \mathcal{A}, \preceq_\ell$ be as above. Then:*

- *P1. $\emptyset \prec_\ell \Theta, \emptyset \preceq_\ell B, \forall B \in \mathcal{A}$,*
- *P2. Property P,*

are equivalent to the existence of a (nonempty, finite) family $\mathcal{P}$ of probabilities p such that

$$B \preceq_\ell C \iff p(B) \leq p(C), \forall p \in \mathcal{P}.$$

$\Leftarrow$ Immediate.

$\Rightarrow$ We first see that there is a weak order $(\mathcal{A}, \preceq_\ell')$, such that $\preceq_\ell'$ is a covering of $\preceq_\ell$, which may be represented by a probability distribution. The proof is based in similar results mentioned above. The assertion is true if there is a probability distribution p such that

$$B \preceq_\ell C \Longrightarrow B \preceq_\ell' C \iff p(B) \leq p(C).$$

Given p, we define $p_i = p(i)$, so we can write $p = (p_i)_{i=1}^k$. Associate an inequality

$$p(B) \leq p(C)$$

with each comparison $B \preceq_\ell C$. Associate the inequality

$$0 < p(\Theta)$$

with the comparison $\emptyset \prec_l \Theta$. Then, we can rewrite the system as

$$\left\{ \begin{array}{l} \{pb_r \quad \geq \quad 0\}_r \\ pb_* > 0 \end{array} \right\} S$$

where the b's are vectors whose elements are in {-1, 0, 1} and b_* is a vector of 1's. The assertion is true iff S is consistent. Suppose S is inconsistent, then (see, e.g., Rockafellar, 1970, p. 198) there are numbers $\mu_r \geq 0$ and $\mu_* > 0$ such that

$$\sum \mu_r b_r + \mu_* b_* = 0.$$

As the b's are vectors of rational components, the μ's may be taken rationals, and, consequently, integers. Also, at least one of the μ_r's has to be positive. As each inequality is related to a comparison $B_r \preceq_\ell C_r$, we deduce that, for some $l \geq 1$,

$$\mu_1 \chi_{B_1} + \dots + \mu_l \chi_{B_l} + \mu_* \chi_\emptyset = \mu_1 \chi_{C_1} + \dots + \mu_l \chi_{C_l} + \mu_* \chi_\Theta.$$

By property P, $\Theta \preceq_\ell \emptyset$, which is a contradiction. So S is consistent: there is a weak order $(\mathcal{A}, \preceq_\ell')$, such that $\preceq_\ell'$ is a covering of $\preceq_\ell$, which may be represented by a probability distribution.

Let us call $(\mathcal{A}, \preceq_p)$ to the the weak order associated with p. Let $\mathcal{P} = \{p : \preceq_\ell \subseteq \preceq_p\}$. We know that $\mathcal{P} \neq \emptyset$. Let $\preceq^* = \bigcap_{p \in \mathcal{P}} \preceq_p$; clearly, $\preceq_\ell \subseteq \preceq^*$. Suppose $B, C \in \mathcal{A}$ are such that $B \neq C$, $B \preceq^* C$: $\forall p \in \mathcal{P}$, it is $B \preceq_p C$. Thus, writing

$$p(B) \leq p(C)$$

as

$$pb \geq 0,$$

with $b \in \{-1, 0, 1\}$, this inequality is a consequence of the system

$$\left. \begin{cases} \{pb_r & \geq \ 0\}_r \\ pb_* \geq 0 \end{cases} \right\} S'$$

(Observe that S' adds only to S the equation

$$pb_* = 0.$$

When this happens, $p_i = 0, \forall i$, $pb \geq 0, \forall b$).

By Farkas' lemma, see Rockafellar (1970), there are nonnegative numbers μ_r, μ_* such that

$$\sum_r \mu_r b_r + \mu_* b_* = b$$

As $b \neq 0$, at least one of the μ's has to be positive. As the b's are vectors with rational components, the μ's may be chosen rational. Thus, there are nonnegative integers β_r, β_*, β, with β and at least one of the other β's positive such that

$$\sum_r \beta_r b_r + \beta_* b_* = \beta b.$$

This implies that there are $B_i, C_i \in A, i = 1, \ldots, l$, with $l \geq 1$, such that $B_i \preceq_\ell C_i, \forall i$ and

$$\chi_{B_1} + \cdots + \chi_{B_l} + \beta \chi_C = \chi_{C_1} + \cdots + \chi_{C_l} + \beta \chi_B$$

By property P, $B \preceq_\ell C$. That is, $\preceq^* \subseteq \preceq_\ell$.

The $\preceq_p$'s are weak orders in $\mathcal{A}$, which is finite. Thus, we may choose $p^1, \ldots, p^t$ such that

$$\preceq_\ell = \bigcap_{i=1}^{t} \preceq_{p^i}.$$

□

Clearly, $(\mathcal{A}, \preceq_\ell)$ is a quasi order. Again, we have:

Corollary 2 *Let $\mathcal{P} = \{p^1, \ldots, p^t\}, t \geq 1$, be a family of probabilities characterising $\preceq_\ell$ on $\mathcal{A}$. Then*

$$B \preceq_\ell C \Longleftrightarrow \sum_{i=1}^{t} \lambda_i p^i(B) \leq \sum_{i=1}^{t} \lambda_i p^i(C), \forall \lambda_1, ..., \lambda_t : \lambda_i \geq 0, \sum_{i=1}^{t} \lambda_i = 1.$$

Note that if there is no judgement gathered concerning events A and B, then there is $p \in [\mathcal{P}]$ such that $p(A) = p(B)$, as is frequently assumed. Observe also that if $\emptyset \prec_\ell D$, there is $p \in \mathcal{P}$ such that $p(D) > 0$, a fact which we use in proposition 4.

To end this section, we study conditional probabilities. Non unicity of $\mathcal{P}$ leads to problems while defining probabilities of events conditional on different events. However, in our context, it is enough to deal with probabilities conditional on the same event. $B|D$ represents the event B given D. The result, of simple proof, is

Proposition 4 *Let $\Theta, \mathcal{A}, \preceq_\ell$ be as before. Suppose $(\mathcal{A}, \preceq_\ell)$ verifies P1, P2 and*

- *P3. For any B, C, $D \in \mathcal{A}$, such that $\emptyset \prec_\ell D$, $B|D \preceq_\ell C|D$ iff $B \cap D \preceq_\ell C \cap D$.*

Then, there is a family $\mathcal{P}$ of probabilities such that for any $B, C, D \in \mathcal{A}$, with $\emptyset \prec_\ell D$,

$$B|D \preceq_\ell C|D \iff p(B|D) \leq p(C|D), \forall p \in \mathcal{P} : p(D) > 0$$

where $p(B|D) = p(B \cap D)/p(D)$.

Therefore, Bayes' theorem holds in this case.

3.2 Modelling preferences under uncertainty

The DM may express his preferences over the set $\mathcal{P}_0(\mathcal{C})$ of probability measures over $\mathcal{C}$, the (finite) space of consequences. We model the DM's preferences over

$\mathcal{P}_0(\mathcal{C})$ with the relation $\preceq_*$, interpreted as follows: '$p \preceq_* q$' means 'p is at most as preferred as q'. We attempt to represent it via expected utility of a utility function. Imprecision in the DM's preferences leads us to the representation

$$p \preceq_* q \iff E(u,p) \leq E(u,q), \forall u \in \mathcal{U} \tag{1}$$

where $\mathcal{U}$ is a family of utility functions.

Aumman (1962) and Fishburn (1982) provide sufficient conditions for a a one way representation. White (1972) gives conditions to represent an order in $\mathbf{R}^n$ by a finite family of linear value functions, which may be translated to this context.

The following simple result establishes the necessity of the main conditions we shall use to get the representation.

Proposition 5 *Suppose that $(\mathcal{P}_0(\mathcal{C}), \preceq_*)$ admits the representation (1). Then,*

- *B1. $(\mathcal{P}_0(\mathcal{C}), \preceq_*)$ is a quasi order.*
- *B2. For $\alpha \in (0,1), p \preceq_* q \iff \alpha p + (1-\alpha) r \preceq_* \alpha q + (1-\alpha) r$.*
- *B3. $\alpha p + (1-\alpha) r \preceq_* \alpha q + (1-\alpha) s, \forall \alpha \in (0,1] \Longrightarrow r \preceq_* s$.*

As a first step, we shall look for a one way utility representation as Aumman's or Fishburn's. We substitute their continuity condition by a nontriviality condition (B4): thus, *continuity is not essential for a one way utility representation.*

Proposition 6 *Let $\mathcal{C}, \mathcal{P}_0(\mathcal{C}), \preceq_*$ be as before. Suppose that $\preceq_*$ satisfies B1, B2. Then if:*

- *B4. $\exists p', q' \in \mathcal{P}_0(\mathcal{C}) : \neg(p' \preceq_* q')$,*

there is a real function u on $\mathcal{C}$ such that

$$p \preceq_* q \Longrightarrow E(u,p) \leq E(u,q).$$

Our proof will be based up to a point on Fishburn's (1982) proof, which in turn develops Aumman's ideas. Let us call

$$E = \{p - q : q \preceq_* p\}.$$

We first prove that:

1. E is convex.

Let $y, z \in E$. There are p, q, r, s such that $q \preceq_* p$, $r \preceq_* s$, $y = p - q, z = s - r$. Applying B2, we have

$$\alpha q + (1-\alpha) r \preceq_* \alpha p + (1-\alpha) r \preceq_* \alpha p + (1-\alpha) s$$

for $\alpha \in (0,1)$. Thus,

$$\alpha(p-q) + (1-\alpha)(s-r) = \alpha y + (1-\alpha) z \in E.$$

Extending the above argument by induction we may prove that

2. $\alpha_j > 0, \sum \alpha_j = 1, q_j \preceq_* p_j \Longrightarrow \sum \alpha_j q_j \preceq_* \sum \alpha_j p_j$. Then:

3. $\alpha_j > 0, \sum \alpha_j = 1, q_j \preceq_* p_j, j = 1, \ldots, n-1, \sum \alpha_j q_j = \sum \alpha_j p_j \Longrightarrow p_n \preceq_* q_n$.

Because of 2, we only have to prove the result for $n = 2$. If $\alpha \in (0,1)$, $\alpha q_1 + (1-\alpha) q_2 = \alpha p_1 + (1-\alpha) p_2$ and $q_1 \preceq_* p_1$, we have : $\alpha q_1 + (1-\alpha) p_2 \preceq_* \alpha p_1 + (1-\alpha) p_2 = \alpha q_1 + (1-\alpha) q_2$, so that $p_2 \preceq_* q_2$.

Let S be the minimal convex cone containing E. From 3, we get:

4. $p, q \in \mathcal{P}_0(\mathcal{C}), p - q \in S \Longrightarrow q \preceq_* p$.

We have that

$$p - q = \sum \alpha_n (p_n - q_n)$$

with $\alpha_n > 0, q_n \preceq_* p_n$. Then,

$$\sum (\alpha_n / (\sum \alpha_n + 1)) q_n + (1/(\sum \alpha_n + 1)) p =$$

$$\sum (\alpha_n / (\sum \alpha_n + 1)) p_n + (1/(\sum \alpha_n + 1)) q.$$

As $q_n \preceq_* p_n$, applying 3, we get $q \preceq_* p$. Hence we may write,

5. $p, q \in \mathcal{P}_0(\mathcal{C}), p - q \in E \iff q \preceq_* p$.

Then, due to B4, $S \neq \mathbf{R}^s$: as S is a convex cone (see Rockafellar 1970, p. 99) there is $u \neq 0$ such that $x \in S \Longrightarrow ux \geq 0$. Let us call $u(c_i) = u_i, i = 1, \ldots, s$. We have:

$$q \preceq_* p \Longrightarrow p - q \in E \subset S \Longrightarrow uq \leq up \Longrightarrow \sum u(c_i) q_i \leq \sum u(c_i) p_i \Longrightarrow$$

$$E(u,q) \leq E(u,p).$$

□

To get the two way representation we need to restore continuity, through axiom B3. Aumman (1964) provides a related result of more complicated proof and under stronger conditions.

Proposition 7 *Let $\mathcal{C}, \mathcal{P}_0(\mathcal{C}), \preceq_*$ be as before. Suppose that $\preceq_*$ satisfies B1-B3. Then, there is a family $\mathcal{U}$ of real functions u such that*

$$p \preceq_* q \iff E(u,p) \leq E(u,q), \forall u \in \mathcal{U}.$$

Let E, S be as in proposition 6. $cl(E)$ is convex (and closed). Let $u \in cl(E)$. There is $u_0 \in E$ such that

$$\alpha u_0 + (1-\alpha)u \in E, \forall \alpha \in (0,1].$$

Define $u_n = \alpha_n u_0 + (1-\alpha_n)u$, with $1 \geq \alpha_n \downarrow 0$. We have $u_n = p_n - q_n$ with $q_n \preceq_* p_n$. Then, there are convergent subsequences, say $p_{n_k} \xrightarrow{k} p^*$, $q_{n_k} \xrightarrow{k} q^*$. Consequently,

$$p_{n_k} - q_{n_k} = u_{n_k} \xrightarrow{k} u$$

$$p_{n_k} - q_{n_k} \xrightarrow{k} p^* - q^*,$$

that is, $u = p^* - q^*$. Hence, as $u_0 = p_0 - q_0$, with $q_0 \preceq_* p_0$,

$$\alpha(p_0 - q_0) + (1-\alpha)(p^* - q^*) \in E, \forall \alpha \in (0,1]$$

therefore

$$\alpha q_0 + (1-\alpha)q^* \preceq_* \alpha p_0 + (1-\alpha)p^*, \forall \alpha \in (0,1].$$

By B3, $q^* \preceq_* p^*$ and $u \in E$. Thus, E is closed. Consequently, S is a closed, convex set (cone) and it can be characterised by means of its support hyperplanes: there is a family $\mathcal{W}$ of vectors ω such that

$$x \in S \iff \omega x \geq 0, \forall \omega \in \mathcal{W}.$$

Then

$$q \preceq_* p \iff p - q \in E \iff \omega(p-q) \geq 0, \forall \omega \in \mathcal{W} \iff \omega q \leq \omega p, \forall \omega \in \mathcal{W}.$$

Defining $u_\omega(c_i) = \omega_i$ and $\mathcal{U} = \{u_\omega : \omega \in \mathcal{W}\}$ we get

$$q \preceq_* p \iff E(u,q) \leq E(u,p), \forall u \in \mathcal{U}.$$

□

As in the previous models,

Corollary 3 *Let $\mathcal{C}, \mathcal{P}_0(\mathcal{C}), \preceq_*, \mathcal{U}$ be as before. Then*

$$p \preceq_* q \iff E(u,p) \leq E(u,q), \forall u \in [\mathcal{U}].$$

3.3 Modelling judgements under uncertainty

We summarise results concerning belief and preference modelling. Imprecision in the DM's beliefs can be modelled with a family $\mathcal{P}$ of probability distributions p. Consequently, we associate a family $\{p(a_i)\}_{p \in \mathcal{P}}$ of lotteries with each alternative a_i. Alternatives are compared on the basis of the associated lotteries. Imprecision in preferences between lotteries leads to imprecision in preferences between alternatives. We assume a preference relation $\preceq$ on A, represented by '$a_i \preceq a_j$' for $a_i, a_j \in A$ and interpreted as 'a_i is at most as preferred as a_j', given by

$$a_i \preceq a_j \iff p(a_i) \preceq_* p(a_j), \forall p \in \mathcal{P}.$$

Proposition 8 *Let $A, \mathcal{C}, \Theta, \mathcal{A}, \preceq_\ell, \preceq_*$ be as propositions 3 and 7, and $\mathcal{P}$, $\mathcal{U}$, the corresponding families of probabilities and utilities. Let $\{p(a_i)\}_{p \in \mathcal{P}}$ be the set of lotteries associated with each alternative a_i. Let $\preceq$ be defined as above. Then,*

$$a_i \preceq a_j \iff E(u, p(a_i)) \leq E(u, p(a_j)), \forall u \in \mathcal{U}, \forall p \in \mathcal{P}.$$

The proof is simple.

4 A parametric model

The previous results show how to model imprecision in judgements by families of values or utilities and probabilities. Problems under certainty and under uncertainty can be treated in parallel if we introduce a convenient parametric representation, as we briefly illustrate.

4.1 The certainty case

When $(A, \preceq)$ is a quasi order, $\preceq$ may be modelled by

$$a_i \preceq a_j \iff v(a_i) \leq v(a_j), \forall v \in V.$$

For a given $v \in V$, let $w_i = v(a_i)$ and $w = (w_1, \ldots, w_m)$. Let $S = \{w : w = v(A), v \in V\}$. Typically, S is defined by constraints of the kind $w_i \leq w_j$. We can define the evaluation of the j-th alternative as $\Psi_j(w) = w_j$.

Then, taking into account the results of section 1, we can represent $\preceq$ as follows

$$a_i \preceq a_j \iff \Psi_i(w) \leq \Psi_j(w), \forall w \in S.$$

4.2 The uncertainty case

When $(\mathcal{A}, \preceq_\ell)$ is a probability quasi order, $\preceq_\ell$ can be modelled by a family $\mathcal{P}$ of probabilities p:

$$C \preceq_\ell B \iff p(C) \leq p(B), \forall p \in \mathcal{P}.$$

For a given $p \in \mathcal{P}$, let us call $w_i = p(\theta_i)$, $w^1 = (w_1, \ldots, w_k)$ and $S_1 = \{w^1 : w^1 = p(\Theta), p \in \mathcal{P}\}$. S_1 is defined by constraints of the kind $cw^1 \geq 0$, where the c's are vectors of -1's, 0's and 1's, and a constraint $\sum_{i=1}^{k} w_i > 0$.

Imprecision in preferences has been modelled with a family $\mathcal{U}$ of utility functions. For a given $u \in \mathcal{U}$, call $w_{k+l} = u(c_l), w^2 = (w_{k+1}, \ldots, w_{k+s})$ and $S_2 = \{w^2 = u(\mathcal{C}), u \in \mathcal{U}\}$. Calling $w = (w^1, w^2)$, the evaluation of the j-th alternative takes the form

$$\Psi_j(w) = w^1 B_j w^2$$

where B_j is a matrix of zeroes and ones. Calling $S = S_1 \times S_2$, we have

$$a_i \preceq a_j \iff \Psi_i(w) \leq \Psi_j(w), \forall w \in S,$$

as a consequence of proposition 8.

As we see, we can give the same parametric treatment, deduced from the axioms, to problems under certainty and under uncertainty. We shall use this general parametric model in the following section. An example will be given in the last section.

5 Orders and solution concepts

The main thrust of the parametric model is to order the alternatives in a Pareto sense.

$$a_k \preceq a_j \iff \Psi_k(w) \leq \Psi_j(w), \forall w \in S.$$

The solutions of the problem are the nondominated ones.

Definition 4 *a_j dominates a_k if $a_k \preceq a_j$ and $\neg(a_j \preceq a_k)$. We write it $a_k \prec a_j$.*

Definition 5 *a_j is nondominated if there is no $a_k \in A$ such that $a_j \prec a_k$.*

It may seem appealing to propose as solutions those a_j that maximise $\Psi_j(w)$ for some $w \in S$. They have received several names in the literature, e.g. quasi-Bayesian (Rios, 1975, in a context in which there is uncertainty in the probabilities) and potentially optimal (Hazen, 1986, in an uncertain value or utility functions context). We adopt the latter term.

Definition 6 *a_j is potentially optimal (p.o.) if $\Psi_k(w) \leq \Psi_j(w), \forall a_k \in A$, for some $w \in S$.*

In principle, there is no reason to introduce p.o. solutions into a decision analysis under partial information. The DM needs only consider the nondominated solutions. However, if we knew w for sure, we should propose those a_j maximising $\Psi_j(w)$; as we know that $w \in S$, our final solution will be among the potentially optimal alternatives.

Some relations between both concepts can be seen in White (1982). Further relations, taking into account this parametric representation, can be seen in Rios Insua (1990). Those results show that we should look for the nondominated potentially optimal alternatives, what essentially implies carrying out a family of coherent decision analyses and basing conclusions on common grounds, thus embodying coherence.

6 A portfolio selection example

This example is adapted from French (1986). A broker has to decide among six investments $\{a_1, a_2, a_3, a_4, a_5, a_6\}$. Their next year payoffs depend on the future financial market which might have very low (θ_1), low (θ_2), medium (θ_3) or high (θ_4) activity. Each investment pays off after one year. Payoffs are given in the table. The broker must choose one and only one of the investments.

	θ_1	θ_2	θ_3	θ_4
a_1	800	900	1000	1100
a_2	600	1000	1000	1000
a_3	400	800	1000	1200
a_4	600	800	800	1200
a_5	400	600	1100	1200
a_6	800	800	1100	1100

Alternatives are ranked according to their expected utility

$$E(u(a),p) = \sum_{i=1}^{4} p(\theta_i)u(a,\theta_i).$$

Let us call:

$$\begin{aligned}
w_i &= p(\theta_i), i = 1,\ldots,4 \\
w_5 &= u(1200) \\
w_6 &= u(1100) \\
w_7 &= u(1000) \\
w_8 &= u(900) \\
w_9 &= u(800) \\
w_{10} &= u(600) \\
w_{11} &= u(400)
\end{aligned}$$

There are some constraints in the probabilities

$$w_1 + w_2 \geq w_3 + w_4$$

$$w_2 + w_3 \geq w_4$$

$$w_2 \geq w_3$$

$$\sum_{i=1}^{4} w_i = 1$$

(for example, the first one means that the broker believes that a very low to low market is at least as likely as a medium to high market). To ensure monotonicity in the utilities, we add the constraints

$$w_{10} \leq w_9$$

$$w_8 \leq w_7$$

$$w_7 \leq w_6.$$

The bounds of the parameters are:

Par	LB	UB
w_1	.20	.50
w_2	.20	.40
w_3	.10	.40
w_4	.10	.60
w_5	1.0	1.0
w_6	.85	.95
w_7	.80	.90
w_8	.75	.85
w_9	.60	.75
w_{10}	.30	.60
w_{11}	.00	.00

6.1 The model

The uncertainty in this problem is both in the probabilities and the utilities. We have

$$\Psi_j(w) = \sum_{i=1}^{4} w_i w_{ij},$$

where $w_{ij} = u(a_{ij}) \in \{w_5, \ldots, w_{11}\}$. For example,

$$\Psi_1(w) = w_1 w_9 + w_2 w_8 + w_3 w_7 + w_4 w_6$$

which can be written

$$\Psi_1(w) = w^1 B_1 w^2,$$

with

$$w^1 = (w_1, w_2, w_3, w_4)$$
$$w^2 = (w_5, w_6, \ldots, w_{11})$$

and B_1 the matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Similarly, we associate bilinear evaluation functions with the rest of the alternatives. The constraints can be written

$$Dw \leq d.$$

The set of potentially optimal nondominated solutions is

$$\{a_1, a_6, a_2, a_4\}.$$

See Rios Insua and French (1990) for algorithms to compute these solutions.

7 Summary

We have seen a principle allowing us to model imprecision within a Bayesian context: if the Bayesian foundations require that comparative judgements follow a weak order (plus some conditions), comparative judgements regulated by a quasi order (plus some similar conditions) lead to modelling judgements by families of value functions or by families of probability distributions and utility functions. The proper solution concepts are described. In summary, we have a more robust decision theory based on a weaker set of axioms, but embodying coherence, since it essentially implies carrying out a family of coherent decision analyses and basing conclusions on common grounds. The ideas are illustrated with a financial example. Their role in a framework for sensitivity analysis in multiobjective decision making can be seen in French and Rios Insua (1989) and Rios Insua and French (1990).

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