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Frankowska, H. and Quincampoix, M.

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Hélène Frankowska
Marc Quincampoix

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IIASA

International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria

Telephone: (0 22 36) 715 21*0 □ Telex: 079 137 iiasa a □ Telefax: (0 22 36) 71313

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Hélène Frankowska
Marc Quincampoix

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CEREMADE
University of Paris-Dauphine
Place du Maréchal de Lattre de Tassigny
F-75775 Paris cedex 16

IIASA

International Institute for Applied Systems Analysis □ A-2361 Laxenburg □ Austria

Telephone: (0 22 36) 715 21*0 □ Telex: 079 137 iiasa a □ Telefax: (0 22 36) 71313

FOREWORD

The authors investigate a differential inclusion whose solutions have to remain in a given closed set. The viability kernel is the set of the initial conditions starting at which, there exist solutions to the differential inclusion remaining in this closed set. In this paper, the authors provide an algorithm which determine this set and they apply it to some concrete examples.

Alexander B. Kurzhanski
Chairman
System and Decision Science Program

Contents

Viability kernels of differential inclusions with constraints: Algorithm and applications

Hélène Frankowska & Marc Quincampoix

1 Introduction

Let us consider a differential inclusion with constraints:

$$\begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & \forall\, t \geq 0,\ x(t) \in K \end{cases}$$

where F is a set valued map and K a closed subset of a finite dimensional vector space X.

Recall that the *contingent cone* to K at x is the set:

$$T_K(x) := \{\ v \in X \mid \liminf_{h \longrightarrow 0+} d(x + hv, K)/h = 0\ \}$$

Under adequate assumptions, the Viability Theorem of Haddad states that, for all $x_0 \in K$, there exists at least one K-viable solution (i.e. satisfying $ii)$) to the differential inclusion $i)$ starting at x_0, if and only if $F(x) \cap T_K(x) \neq \emptyset$ for any $x \in K$ (see [13], [4], [3], [8], [2], [1]). In this case, K is called a viability domain.

Of course, generally, K is not a viability domain, and we have to solve the inclusion in subsets of K and to determine all the initial conditions providing solutions viable in K. Let us denote by $\mathrm{Viab}_F(K)$, the *viability kernel of* K namely the largest *closed* viability domain contained in K. This set (possibly empty) exists if F is lipschitzean[1] (or even upper semicontinuous) with

[1] Let us recall that the set valued map F is k-lipschitzean if and only if:

$$\forall\, (x, y),\ \ F(x) \subset F(y) + k\|x - y\|B.$$

where B denotes the closed unit ball.

nonempty convex compact values and with linear growth (see for instance [1]). In this paper, the set valued map F is assumed to be such.

In the control theory literature, viability domains are said to be *controlled invariant sets* and many control problems are known to depend on the properties of the maximal controlled invariant subset of a fixed set. For example, the construction of zero dynamics (see [6]) is the construction of the viability kernel of the zero locus of the output map (see [3]).

The related problem of controlled invariance for distributions has played a fundamental role in nonlinear geometric control, for example in the solution of the problem of disturbance decoupling (see [19], [14], [20] and [16] and also [18] for an earlier nonlinear interpretation of transmission zeroes).

Our main aim is to provide a constructive algorithm allowing the computation of the viability kernel when K is assumed to be only closed. We would like to recall that such algorithms were found in several particular cases:

- The V^* algorithm allows one to compute the viability kernel of a closed set defined by a linear equality constraint for control systems (cf [9], [24], [25]) and of course closely parallels the Silverman algorithm [23].

- A "local" viability (*viz*, the zero dynamics) algorithm was extended to nonlinear systems (see [7], [17], [8] for systems which are affine in the control and [15] for partial results in the polynomial non affine case). See also [16] for more detailed discussion and examples. The connections between viability and zero dynamics are described in [3].

- In the case of *descriptor systems*, in a finite number of iterations the viability kernel is obtained, when K is a subspace of a finite dimensional vector space (cf [5]).

Let us notice that there exist algorithms providing viability domains (but not the viability kernel) in particular cases (see [2], [11] and [5]). Our algorithm will be applied to concrete examples and a numerical treatment will be suggested in particular cases.

The authors are indebted to Chris Byrnes for his suggestions and comments.

2 The viability kernel algorithm

2.1 Assumptions

In all this paper, X denotes a finite dimensional vector space, $K \subset X$ is a closed subset. We impose the following assumptions[2] on the set-valued map F from X into itself:

$$(1) \quad \left\{ \begin{array}{l} F \text{ is a } k \text{ -lipshitzean set valued map with nonempty convex} \\ \quad \text{compact values, with linear growth and satisfying the} \\ \quad \text{boundedness condition } M := \sup_{x \in K} \sup_{y \in F(x)} \|y\| < \infty \end{array} \right.$$

Recall that F has a linear growth if there exists $c > 0$ such that:

$$\forall\, x \in X, \;\; F(x) \subset c(1 + \|x\|)B.$$

The problem is to determine the viability kernel, i.e. to find all initial conditions $x_0 \in K$ such that there exists at least one absolutely continuous solution $x(\cdot)$ starting at x_0 to the following differential inclusion:

$$(2) \qquad x'(t) \in F(x(t)) \text{ almost everywhere in } [0, \infty[$$

viable in K in the sense that $x(t) \in K$, for all $t \geq 0$.

2.2 Definition of viability kernels

Let us recall the definition of viability domains and kernels (see [2], [1], [4]).

Definition 2.1 *A set A is called a viability domain of F if and only if:*

$$\forall\, x \in A, \;\; F(x) \cap T_A(x) \neq \emptyset$$

The viability kernel $\mathrm{Viab}_F(K)$ *of a closed set K is the largest closed viability domain of F contained in K.*

If F is upper semicontinuous with nonempty compact convex values and linear growth, thanks to Haddad's viability theorem (see [13], [2], [4], [1]) a closed set K is viability domain if and only if starting from any point of K, there exists at least one solution to (2) viable in K (i.e. which remains in the set K).

[2]If K is compact, then the boundedness condition $M < \infty$ is obviously deduced from the lipschitzeanity of F.

2.3 Description of the algorithm

We can divide the boundary of K into three disjoint subsets (see [21]):

$$(3)\qquad \begin{cases} K^i &:= \{\ x \in \partial K \mid F(x) \subset D_K(x)\ \} \\ K^e &:= \{\ x \in \partial K \mid F(x) \subset X \backslash T_K(x)\ \} \\ K^b &:= \{\ x \in \partial K \mid F(x) \cap T_{\partial K}(x) \neq \emptyset\ \} \end{cases}$$

Where $D_K(x)$ denotes the ***Dubovitsky-Miliutin*** tangent cone to K at x defined by:

$$D_K(x) := \{\ v \in\ X \mid \exists\, \alpha > 0,\ x+]0,\alpha[(v+\alpha B) \subset K\ \}.$$

This partition enables us to express the Haddad's Viability Theorem in the following way:

Proposition 2.2 *A nonempty closed set K is a viability domain of F if and only if the set K^e is empty.*

Furthermore, if $K^e \neq \emptyset$ then, $\mathrm{Viab}_F(K) \cap K^e = \emptyset$.

The second statement holds because the viability kernel is a closed set.

There is a "natural" algorithm (see [2], [1], [5]) defined by the following subsequence:

$$(4)\qquad K_0 := K,\ K_1 := \overline{K \backslash K^e},\ \ldots\ ,K_{n+1} := \overline{K_n \backslash K_n^e}.$$

where $\overline{C}$ denotes the closure of a subset $C \subset X$. In some particular cases, this sequence may converge (see, for instance, [5]), but, generally, it is not the case. In fact, it is easy to notice that this sequence is constant ($= K$) as soon as:

$$K = \overline{Int(K)}$$

The idea of our algorithm is to subtract to K not only K^e, but an open neighbourhood of K^e. In fact, since $\mathrm{Viab}_F(K)$ is closed, for any $x_0 \in\ K^e$, there exists a real $\varepsilon^0_{x_0} > 0$ such that:

$$\mathrm{Viab}_F(K) \cap B(x_0, \varepsilon^0_{x_0}) = \emptyset,$$

where $B(x_0, \varepsilon^0_{x_0})$ is the *closed* ball of center x_0 and radius $\varepsilon^0_{x_0}$ and $\overset{\circ}{B}(x_0, \varepsilon^0_{x_0})$, the open one. A sequence of closed subsets of K can be defined in the following way:

$$
(5) \qquad \begin{cases} K_0 := K \\ K_1 := K_0 \backslash \bigcup_{x_0 \in K_0^e} \overset{\circ}{B}(x_0, \varepsilon_{x_0}^0) \\ \text{where } \ B(x_0, \varepsilon_{x_0}^0) \cap \mathrm{Viab}_F(K) = \emptyset \\ \dots \\ K_{n+1} := K_n \backslash \bigcup_{x_0 \in K_n^e} \overset{\circ}{B}(x_0, \varepsilon_{x_0}^n) \\ \text{where } \ B(x_0, \varepsilon_{x_0}^n) \cap \mathrm{Viab}_F(K) = \emptyset \\ \dots \end{cases}
$$

Of course, such sequence depends on the choice of $\varepsilon_{x_0}^n$. Also, since we do not know in advance the set $\mathrm{Viab}_F(K)$, we have to find a procedure which allows to determine $\varepsilon_{x_0}^n$ from the knowledge of K_n and F for all $n \geq 0$. Below, we suggest a particular choice of $\varepsilon_{x_0}^n$ which leads to the viability kernel.

Proposition 2.3 *Consider a sequence of closed subsets K_n, $n \geq 0$ satisfying (5). Set $K_\infty := \bigcap_{n>1} K_n$. Then,*

$$
\begin{array}{l} \mathrm{Viab}_F(K) \subset K_\infty \subset \dots \subset K_{n+1} \subset K_n \subset \dots \subset K_1 \subset K \\ \text{and} \\ \mathrm{Viab}_F(K) = \mathrm{Viab}_F(K_i) \ \text{ for } \ i \geq 1 \end{array}
$$

Proof — Since $K_1 \subset K$, we deduce that $\mathrm{Viab}_F(K_1) \subset \mathrm{Viab}_F(K)$. On the other hand, for all $x_0 \in K^e$, we have $\mathrm{Viab}_F(K) \cap \overset{\circ}{B}(x_0, \varepsilon_{x_0}^0) = \emptyset$. So $\mathrm{Viab}_F(K) \subset K_1$. This and the induction argument end the proof. □

2.4 Approximations

In this section, for each n and for each $x_0 \in K_n^e$, we compute numbers $\varepsilon_{x_0}^n$ depending only on K_n and F. We accomplish this task thanks to Filippov's theorem (see [24]) and so, we build an algorithm. In section 2.5, we shall prove that this algorithm converges to the viability kernel.

Proposition 2.4 *Let $x_0 \in K^e$ and $\varepsilon := d(F(x_0), T_K(x_0))$. Define*

$$
(6) \qquad t_{max} := \sup\{\, t > 0 \mid (x_0 +]0, t](F(x_0) + \frac{\varepsilon}{2}B)) \cap K = \emptyset \,\}
$$

and set

$$
(7) \qquad t_{x_0} := \min\{\, t_{max}, \ \tfrac{\varepsilon}{2kM} \,\}, \quad \varepsilon_{x_0}^0 := \tfrac{\varepsilon t_{x_0}}{8e^{kt_{x_0}}}
$$

Then,

$$\mathrm{Viab}_F(K) \cap B(x_0, \varepsilon^0_{x_0}) = \emptyset$$

and furthermore,

$$\forall\, y_0 \in\ B(x_0, \varepsilon^0_{x_0}),\ \forall\, y(\cdot) \in\ S(y_0),\ d(y(t_{x_0}), K) \geq \frac{\varepsilon t_{x_0}}{8}$$

In order to prove this proposition, we need two results concerning the distance of a solution starting at $x_0 \in K^e$ from the set K. Let us denote by $S(x_0)$ the set of solutions to (2) starting at x_0 and defined on $[0, \infty[$.

Lemma 2.5 *Let x_0 belong to K^e. If there exist α, $\bar{t} > 0$ such that*

$$(x_0 +]0, \bar{t}\,](F(x_0) + \alpha B)) \cap K = \emptyset$$

then,

$$\forall\, 0 \leq t \leq \min(\bar{t},\ \tfrac{\alpha}{kM}),\quad \forall\, x(\cdot) \in\ S(x_0),\quad d(x(t), K) \geq t\alpha/2$$

Proof — By assumptions, we have:

$$\forall\, 0 < t < \bar{t},\quad K \cap (x_0 + t(F(x_0) + \alpha B)) = \emptyset$$

Hence $(x_0 + t(F(x_0) + \frac{\alpha}{2}B)) \cap (K + \frac{\alpha t}{2}B) = \emptyset$ and therefore

$$d(x_0 + t(F(x_0) + \frac{\alpha}{2}B), K) \geq \frac{\alpha}{2}t$$

On the other hand, for all $t \geq 0$ and for any solution to (2) starting at x_0, we have

$$x(t) = x_0 + \int_0^t x'(s)ds \in\ x_0 + MtB \tag{8}$$

where $x'(s) \in\ F(x(s))$.
Since the set valued map F is $k-$lipschitzean, for almost all $s \geq 0$, $x'(s) \in F(x_0) + k\|x(s) - x_0\|B$, and thanks to (8), we have for almost every $0 \leq s \leq t$, $x'(s) \in\ F(x_0) + ksMB$ and, consequently, by integrating this inclusion and using (8):

$$x(t) \in\ x_0 + F(x_0)t + \frac{1}{2}kt^2MB.$$

Finally $x(t) \in x_0 + t(F(x_0) + \frac{\alpha}{2}B)$ as soon as $\frac{1}{2}ktM \leq \frac{\alpha}{2}$. This is proving that $d(x(t), K) \geq t\alpha/2$. **Q.E.D.**

If a solution $x(\cdot)$ behaves as in the claim of Lemma 2.5, it is the case for at least one solution of $S(y_0)$, for any y_0 near x_0. We shall show this thanks to Filippov's Theorem (see [24] or, for instance [4], chapter 10): Let us state the version of Filippov's Theorem, we shall use here:

$$(9) \qquad \left\{ \begin{array}{l} \forall\, \delta > 0,\ \forall\, y_0 \in B(x_0, \delta),\ \forall\, y(\cdot) \in S(y_0),\ \exists\, x(\cdot) \in S(x_0), \\ \qquad \text{such that } \forall\, t \geq 0,\quad \|x(t) - y(t)\| \leq e^{kt}\delta \end{array} \right.$$

Lemma 2.6 *Let x_0 belong to K and $T > 0$. If for any $x(\cdot) \in S(x_0)$, we have $d(x(T), K) \geq \alpha T/2$, then*

$$\forall\, y_0 \in B(x_0, \tfrac{\alpha T}{4e^{kT}}),\quad \forall\, y(\cdot) \in S(y_0),\quad d(y(T), K) \geq \tfrac{\alpha T}{4}$$

Proof — It is an application of Filippov's Theorem (9) when $\delta := \frac{\alpha T}{4e^{kT}}$. Let y_0 belong to $x_0 + \delta B$. For any $y(\cdot) \in S(y_0)$, there exists $x(\cdot)$ starting at x_0 solution to (2) such that

$$\forall\, t \leq T,\quad \|x(t) - y(t)\| \leq e^{kt}\delta \leq e^{kT}\delta.$$

Clearly, $d(y(T), K) + \|x(T) - y(T)\| \geq d(x(T), K)$ and therefore

$$d(y(T), K) + e^{kT}\delta \geq d(x(T), K) \geq \frac{T\alpha}{2}.$$

Since $e^{kT}\delta = \frac{T\alpha}{4}$, we proved that $d(y(T), K) \geq \frac{T\alpha}{4}$. **Q.E.D.**

Thanks to lemmas 2.5 and 2.6, we shall determinate a radius $\varepsilon^0_{x_0}$ such that $\mathrm{Viab}_F(K) \cap B(x_0, \varepsilon^0_{x_0}) = \emptyset$, and consequently, we shall define the first step of our algorithm:

Proof of proposition 2.4 — Let us consider $x_0 \in K^e$, then $\varepsilon := d(F(x_0), T_K(x_0)) > 0$ hence

$$(F(x_0) + \frac{\varepsilon}{2}B) \cap T_K(x_0) = \emptyset.$$

Since $F(x_0) + \frac{\varepsilon}{2}B$ is compact, by the very definition of the contingent cone, we can find a positive t satisfying:

$$(10) \qquad (x_0 +]0, t](F(x_0) + \frac{\varepsilon}{2}B)) \cap K = \emptyset$$

We have defined t_{max}, the largest t (possibly equal to $+\infty$) satisfying (10). Thanks to lemma 2.5, we know that:

$$\forall\, 0 \le t \le t_{x_0} := \min(t_{max}, \tfrac{\varepsilon}{2kM}), \quad \forall\, x(\cdot) \in S(x_0), \quad d(x(t), K) \ge \tfrac{t\varepsilon}{4}$$

From lemma 2.6, we deduce that

$$\mathrm{Viab}_F(K) \cap B(x_0, \frac{\varepsilon t_{x_0}}{8e^{kt_{x_0}}}) = \emptyset$$

This is ending the proof of proposition 2.4. **Q.E.D.**

Now, we have defined for each $x_0 \in K^\epsilon$, a positive number $\varepsilon^0_{x_0}$ and consequently the set K_1 by using (5). Clearly, K_1 is a closed subset of K. This and the induction argument allow us to define a decreasing sequence of closed sets.
Set $K_\infty := \bigcap_{n\ge 1} K_n$. Observe that if K is compact, then $K_\infty = \emptyset$, if and only if for some $N \ge 1$, $K_N = \emptyset$.
If K have some additional regularity properties at x_0, then the number $\varepsilon^0_{x_0} > 0$ satisfying $\mathrm{Viab}_F(K) \cap \overset{\circ}{B}(x_0, \varepsilon^0_{x_0}) = \emptyset$ can be estimated from the distance between $F(x_0)$ and $T_K(x_0)$.

Corollary 2.7 *Let x_0 belong to K^ϵ. Set $\varepsilon := d(F(x_0), T_K(x_0))$. If $K \subset x_0 + T_K(x_0)$ then,*

$$\mathrm{Viab}_F(K) \cap \overset{\circ}{B}(x_0, \frac{\varepsilon^2}{16kM} e^{-\frac{\varepsilon}{2M}}) = \emptyset \tag{11}$$

Proof — It is an obvious consequence of proposition 2.4, if we notice that $t_{max} = \infty$. **Q.E.D.**

Remark — If the set K is convex, then, for any $x \in K$, K is contained in $x + T_K(x)$ and corollary 2.7 can be applied. □

In the next section, we shall prove that the sequence K_n converges to the viability kernel of K.

2.5 Convergence of the algorithm

In section 2.3, we have shown that algorithms defined by formula (5) lead to the inclusion $\mathrm{Viab}_F(K) \subset \bigcap_{n>1} K_n$. In section 2.4, we have suggested, thanks to proposition 2.4, one possible choice of numbers $\varepsilon^n_{x_0}$ satisfying requirements of (5), namely $B(x_0, \varepsilon^n_{x_0}) \cap \mathrm{Viab}_F(K) = \emptyset$ for all $x_0 \in K^e_n$. Now we check that our algorithm converges to the viability kernel, i.e. that $\mathrm{Viab}_F(K) = \bigcap_{n>1} K_n$.

Theorem 2.8 *Let K be a closed set and K_∞ be defined as in section 2.4. Then,*

$$\mathrm{Viab}_F(K) = K_\infty$$

Proof — By proposition 2.3 and the choice of $\varepsilon^n_{x_0}$, $\mathrm{Viab}_F(K) \subset K_\infty$. Let us assume, for a moment, that K_∞ is not a viability domain, namely $K^e_\infty \neq \emptyset$. Pick x in K^e_∞ and set $\bar{\varepsilon} := d(F(x), T_{K_\infty}(x)) > 0$. Let us define the following *finite* number:

$$\bar{t}_{max} := \sup\{\, t \in\ [0,1] \mid (x+]0,t](F(x) + \frac{\bar{\varepsilon}}{2}B)) \cap K_\infty = \emptyset \,\} > 0$$

We shall state, thanks to a technical lemma given below, that:

$$(12) \qquad \begin{cases} \exists\, N > 0, \text{ such that } \forall\, n > N, \\ \exists\, x_n \in\ K_n \cap (x + [0, \frac{1}{2}\bar{t}_{max}](F(x) + (\bar{\varepsilon}/2)B)) \\ \text{satisfying } K_n \cap (x_n+]0, \frac{1}{2}\bar{t}_{max}](F(x) + \frac{\bar{\varepsilon}}{2}B)) = \emptyset \end{cases}$$

For this aim, we need the following well known result:

Lemma 2.9 *Let C be a convex closed cone*[3] *and H be a compact subset of X. If C does not contain any whole line, then there exists $y \in\ H$ such that:*

$$(y + C) \cap H = \{\, y \,\}$$

Proof — The proof results from the lemma of Zorn. Let us define the following relation for the points of H:

$$a \leq b \iff b \in\ a + C$$

[3] Recall that a subset $C \subset X$ is called cone with the vertex at 0 if and only if:

$$\forall\, \lambda > 0,\ \forall\, x \in\ C,\ \lambda x \in\ C.$$

We can easily check that, if C does not contain whole lines and since it is convex, then this relation is an order. We shall prove that all subset P of H *which is totally ordered* has a majorant.

Clearly, for any $a \in P$, $(a + C) \cap H \neq \emptyset$. Since these sets are nonempty compact and are included one in the other, we deduce that:

$$\bigcap_{a \in P} ((a + C) \cap H) \neq \emptyset$$

Let b belong to $\bigcap_{a \in P}(a + C) \cap H$. Obviously, b is larger than any element of P for the relation $\leq$. According to Zorn's lemma, there exists a maximal element $y \in H$ namely, if $z \in H$ is different from y, then $y \notin z + C$. Hence, $(y + C) \cap H = \{ y \}$. **Q.E.D.**

Thanks to Lemma 2.9, we can achieve the proof of Theorem 2.8:

Since $x \in K^{\varepsilon}_{\infty}$, $F(x)$ does not contain 0 and so does $F(x) + (\bar{\varepsilon}/2)B$. Consequently, the convex closed cone $C := \mathbb{R}_+(F(x) + \frac{\bar{\varepsilon}}{2}B)$ does not contain any whole line and by setting $H := K_n \cap (x + [0, \bar{t}_{max}](F(x) + \frac{\bar{\varepsilon}}{2}B))$, we can assert, thanks to Lemma 2.9:

$$\exists\, x_n \in H, \text{ such that } (x_n + C) \cap H = \{ x_n \}$$

On the other hand, by the very definition of K_∞ and the choice of x, the bounded sequence $(x_n)_n$ converges to x. Hence for all n large enough,

$$x_n \in x + [0, \frac{1}{2}\bar{t}_{max}](F(x) + \frac{\bar{\varepsilon}}{2}B).$$

Thus,

$$\begin{array}{l} K_n \cap (x_n + [0, \frac{1}{2}\bar{t}_{max}](F(x) + \frac{\bar{\varepsilon}}{2}B)) \subset \\ K_n \cap (x + [0, \bar{t}_{max}](F(x) + \frac{\bar{\varepsilon}}{2}B)) \cap (x_n + C) = (x_n + C) \cap H = \{ x_n \}. \end{array}$$

This is proving (12) and clearly $x_n \in \partial K_n$. For n large enough, as F is lipschitzean, we have $F(x_n) \subset F(x) + \frac{\bar{\varepsilon}}{4}B$, hence:

$$\forall\, t \in]0, \tfrac{1}{2}\bar{t}_{max}], \quad (x_n + t(F(x_n) + \tfrac{\bar{\varepsilon}}{4}B)) \cap (K_n + \tfrac{t\bar{\varepsilon}}{4}B) = \emptyset.$$

Consequently, for any $t < \frac{1}{2}\bar{t}_{max}$,

$$d(x_n + t(F(x_n) + \frac{\bar{\varepsilon}}{4}B), K_n) \geq \frac{\bar{\varepsilon}t}{4}.$$

Thus $d(F(x_n), T_{K_n}(x_n)) \geq \bar{\varepsilon}/2$, and since $0 \in T_{K_n}(x_n)$, we have also $d(F(x_n), T_{K_n}(x_n)) \leq M$. Let us denote by $\bar{t} := \min\{ \frac{\bar{t}_{max}}{2}, \frac{\bar{\varepsilon}}{2kM} \}$.
If $t^n_{x_n}$ is defined by (7) for the set K_n, then

$$\frac{\bar{t}}{2} \leq \min\{ \frac{1}{2}\bar{t}_{max}, \frac{\bar{\varepsilon}}{4kM} \} \leq t^n_{x_n} \leq \frac{1}{2k}.$$

Since the function $\sigma \mapsto \sigma e^{-k\sigma}$ is increasing for $\sigma \in [0, \frac{1}{k}]$, we can assert thanks to the definition of $\varepsilon^n_{x_n}$ (see (7) in proposition 2.4):

$$\frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}} \leq \varepsilon^n_{x_n}$$

By the very definition of K_{n+1}:

$$K_{n+1} \cap \mathring{B}\,(x_n, \frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}}) = \emptyset$$

Let us notice that $\frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}}$ does not depend on n. Consequently, since x belongs to K_{n+1}, the two following contradictory statements would hold:

$$\begin{cases} i) & \|x_n - x\| \geq \frac{\bar{\varepsilon}\bar{t}}{32} e^{-k\frac{\bar{t}}{2}} \\ ii) & \lim_{n \longrightarrow \infty} x_n = x \end{cases}$$

Q.E.D

The algorithm (5) provides closed sets K_n which contain the viability domain. Consequently, if there exists N such that K_N is compact then we can deduce:

- For all $n \geq N$, the set K_n is compact.
- The set $\mathrm{Viab}_F(K)$ is compact.

In the next section, we shall make this algorithm more precise when F and K are regular enough.

3 The convex case

In all this section, the set K is compact and convex, the set valued map F is k-lipschitzean with nonempty convex compact values and with linear growth. Furthermore F is *assumed to be convex* (i.e. its graph is convex). In this case, we know that $\mathrm{Viab}_F(K)$ is convex (see [5]) and we can apply corollary 2.7 at least at the first step.

We shall modify our algorithm in such way that for any $n > 0$, the subset K_n is convex.

3.1 Description of the algorithm in the convex case

We define the following sequence:

$$(13) \qquad \begin{cases} \widetilde{K_0} := K \\ \widetilde{K_1} := \mathrm{co}(K \backslash \bigcup_{x_0 \in K^c} \overset{\circ}{B}(x_0, \varepsilon^0_{x_0})) \\ \widetilde{K_2} := \mathrm{co}(\widetilde{K_1} \backslash \bigcup_{x_0 \in \widetilde{K}_1^c} \overset{\circ}{B}(x_0, \varepsilon^1_{x_0})) \\ \dots \\ \widetilde{K}_\infty := \bigcap_{n=1}^\infty \widetilde{K}_n \end{cases}$$

where $\mathrm{co}(A)$ denotes the convex closure of the closed subset A and $\varepsilon^i_{x_0}$ are defined as in the previous section and are associated to $\widetilde{K}_i$.

It is obvious that $\widetilde{K}_\infty$ is closed and convex and that it contains the viability kernel of K thanks to the results of section 2.

3.2 Convergence of the modified algorithm in the convex case

Lemma 3.1 *Let K be compact and convex, F be a convex set-valued map satisfying[4] (1). Then K is a viability domain of F if and only if for any x which is an extremal point[5] of K, we have: $T_K(x) \cap F(x) \neq \emptyset$.*

[4]Let us notice that the boundedness condition is automatically satisfied because K is compact.

[5]Recall that a point x of a convex set K is an extremal point whenever:

Proof — We only need to check that, if for any extremal point x of K, we have $F(x) \cap T_K(x) \neq \emptyset$, the same property holds for any point of K. Let be $x \in K$. Then, there exist extremal points $x_1, x_2, \ldots x_p$ such that $x \in \mathrm{co}(\{ x_i, i = 1, 2 \ldots, p \})$. Consider $\lambda_i \geq 0$ such that:

$$\sum_{i=1}^{p} \lambda_i = 1, \ x = \sum_{i=1}^{p} \lambda_i x_i$$

Let $v_i \in T_K(x_i) \cap F(x_i)$. Then $\sum_{i=1}^{p} \lambda_i v_i \in F(x)$ because F is convex, and $\sum_{i=1}^{p} \lambda_i v_i \in T_K(x)$ because $x \mapsto T_K(x)$ has a convex graph (see [5]). Consequently $F(x) \cap T_K(x) \neq \emptyset$. **Q.E.D.**

Now, we can prove the convergence of the algorithm (13).

Theorem 3.2 *Let K be compact and convex, F be a convex set-valued map. Then*

$$\mathrm{Viab}_F(K) = \widetilde{K}_\infty$$

Proof — We know that $\mathrm{Viab}_F(K) \subset \widetilde{K}_\infty$. Assume for a moment that $\widetilde{K}_\infty$ is not a viability domain. It means, by lemma 3.1, that there exists an extremal point x of $\widetilde{K}_\infty$ such that: $F(x) \cap T_{\widetilde{K}_\infty}(x) = \emptyset$. We denote by $\varepsilon := d(F(x), T_{\widetilde{K}_\infty}(x))$ and by $C := \mathbb{R}_+(F(x) + \frac{1}{2}\varepsilon B)$. For $\widetilde{K}_\infty$, we define t_x as in proposition 2.4.

Thanks to the lemma 2.9, there exists $x_n \in \widetilde{K}_n \cap (x + C)$ such that:

$$(x_n + C) \cap \widetilde{K}_n = \{ x_n \}$$

It is clear that the sequence x_n converges to x. Let $t^n_{x_n}$, $\varepsilon^n_{x_n}$ be defined as in proposition 2.4 applied to $\widetilde{K}_n$. Then the sequence $t^n_{x_n}$ satisfies, for n large enough:

$$\frac{\varepsilon}{8kM} \leq t^n_{x_n} \leq \frac{1}{2k} \tag{14}$$

In fact, since $0 \in T_{\widetilde{K}_n}(x_n)$, for all n large enough,

$$\frac{\varepsilon}{4} \leq d(F(x_n), T_{\widetilde{K}_n}(x_n)) \leq M \tag{15}$$

If there exist $\lambda \in [0,1]$, $(y,z) \in K \times K$ such that $y \neq z$ and $x = \lambda y + (1-\lambda) z$, then necessarily $\lambda = 0$ or $\lambda = 1$.

As $\widetilde{K}_n$ is convex, the number t^n_{max} associated to x_n thanks to (6) in proposition 2.4 applied to $\widetilde{K}_n$, is equal to ∞ and thanks to (7) and (15), we proved inequalities (14). This is also proving that there exists $\rho > 0$ such that for all n large enough,

(16) $$\varepsilon^n_{x_n} \geq \rho$$

(thanks to (7), it is possible to choose $\rho := \frac{\varepsilon^2}{256kM\sqrt{e}}$). We have to consider two different cases:

Case 1 $\widetilde{K}_\infty = \{ x \}$.

Assume for a moment that $0 \notin F(x)$ and let $p \in \mathbb{R}^n$ and $\bar{\varepsilon} > 0$ be such that:

$$\inf_{y \in F(x)} < p, y > \geq \bar{\varepsilon}.$$

Pick $y_n \in \widetilde{K}_n$ such that $< p, y_n >= \sup_{y \in \widetilde{K}_n} < p, y >$. Then for all large n, $d(F(y_n), T_{\widetilde{K}_n}(y_n)) \geq \frac{\bar{\varepsilon}}{2}$. Then, the choice of $\varepsilon^n_{y_n}$ and the definition of the algorithm (13) imply that $x \notin \widetilde{K}_{n+1}$ for all large n and we derived a contradiction.

Case 2 $\widetilde{K}_\infty \neq \{ x \}$.

Pick $\delta > 0$ such that for any $0 < \eta < \delta$, $Q := \widetilde{K}_\infty \backslash B(x, \eta) \neq \emptyset$. Fix $0 < \eta < \delta$. Then $\mathrm{co}(Q) \subset \widetilde{K}_\infty$ (because $\widetilde{K}_\infty$ is convex) and since x is extremal, it does not belong to $\mathrm{co}(Q)$. Consequently

$$\mathrm{co}(Q) \cap (x + C) = \emptyset$$

and we can separate these two sets by an hyperplane:

$$\begin{cases} \exists\, p \in X \text{ such that} \\ \inf_{e \in x+C} < p, e > > \sigma(Q, p) := \sup_{q \in \mathrm{co}(Q)} < p, q > \end{cases}$$

It means that the set $x + C$ is contained in the open half-space

$$H := \{ e \mid < p, e > > \sigma(Q, p) \}$$

and $H \cap (\widetilde{K}_\infty \backslash B(x, \eta)) = \emptyset$. Since x is an extremal point of $\widetilde{K}_\infty = \bigcap_{n>1} \widetilde{K}_n$, for any n large enough, $H \cap \widetilde{K}_n \subset B(x, \eta)$. From (16), using that x_n converge to x, we deduce that, if $\eta < \min\{ \frac{\rho}{4}, \delta \}$, for all n large enough,

$$B(x, 2\eta) \subset B(x_n, \varepsilon^n_{x_n})$$

So, according to proposition 2.4, $H \cap \widetilde{K}_n = \emptyset$, for n large enough: a contradiction. **Q.E.D.**

Remark 1 — When the closed set K is convex and the set valued-map F satisfies

$\forall\, \lambda \in [0,1],\ \forall\, (x,y) \in X \times X,\ F(\lambda x + (1-\lambda)y) = \lambda F(x) + (1-\lambda)F(y).$

then algorithms (13) and (5) define the same sets, *i.e.* $\forall\, i,\ \ K_i = \widetilde{K}_i$. □

Remark 2 — From the proof of theorem 3.2, follows that in the algorithm (13), instead of considering all points of $\widetilde{K}_n^e$, we could restrict our attention only to those elements of $\widetilde{K}_n^e$ which are extremal points of $\widetilde{K}_n$. The algorithm would still converge to the viability kernel $\mathrm{Viab}_F(K)$. We shall apply this remark in a concrete example in section 4.2. □

4 Examples

We shall give some examples of computation of the viability kernel.

4.1 Control systems

Consider a metric space Z, a continuous set-valued map U from X into Z with closed nonempty values and let $f : X \times Z \mapsto X$ be a continuous function. Then the control system:

$$x'(t) = f(x(t), u(t)),\quad u(t) \in U(x(t)) \tag{17}$$

can be reduced to the following (equivalent) differential inclusion:

$$x'(t) \in F(x(t)) \text{ where } F(x) := \{\, f(x,u) \mid u \in U(x) \,\}$$

(see [2] and [4] for details).

4.1.1 Linear control systems in the two dimensional space

Let us consider the system in $\mathbb{R}^2$:

$$\left\{ \begin{array}{ccc} x'(t) & = & x(t) + \alpha u_1(t) \\ y'(t) & = & y(t) + \alpha u_2(t) \\ \text{where} & & \\ (u_1(t),\ u_2(t)) & \in & B(0,1) \end{array} \right. \tag{18}$$

Example 1- the constraint set is a ball

$K = B(0,r),\ \ 0 < \alpha \leq r$

Here, clearly $F(x,y) := \{(a,b) \in \mathbb{R}^2 \,|\, (a-x, b-y) \in B(0,\alpha)\}$. It is easy to check that the Lipschitz constant is $k = 1$ and that $M = r + \alpha$. If $(x_0, y_0) \in \partial K$, then we can notice that:

$$\varepsilon(x_0,y_0) := d(F(x_0,y_0), T_K(x_0,y_0)) = r - \alpha$$

It means that $K^e = \partial K$ when $\alpha < r$.

In this case, ε is constant on the boundary, so thanks to corollary 2.7, we can assert that

$$K_1 = B(0, r - m(r-\alpha)), \text{ where } m(r-\alpha) := \frac{(r-\alpha)^2}{16(r+\alpha)} e^{-\frac{r-\alpha}{2(r+\alpha)}}$$

By induction, we determine the sequence of the algorithm (5):

$$K_n = B(0,r_n) \text{ where } r_n := r_{n-1} - m(r_{n-1} - \alpha)$$

Thanks to this, we obtain the viability kernel:

Proposition 4.1 *If $0 < \alpha \leq r$, then:*

$$\mathrm{Viab}_F(B(0,r)) = B(0,\alpha)$$

Proof — For doing this, it is sufficient to prove that the sequence r_n converges to α. Let us introduce $g(s) = s - m(s-\alpha)$, then $r_{n+1} = g(r_n)$. We check that $r_n \geq 0$ is a decreasing sequence. Thus it converges to some $s \geq 0$. Since the unique positive solution to $g(s) = s$ is $s = \alpha$, we deduce that r_n converges to α. The proof is complete. **Q.E.D.**

We can easily extend this example to the nonlinear case when α is a function depending on the radius, i.e. $\alpha := \beta(x^2 + y^2)$, where $\beta : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a given function. It is easy to check that now, when the viability kernel is nonempty:

$$\mathrm{Viab}_F(B(0,r)) = B(0,\alpha_0)$$

where α_0 is the largest solution (if it exists) to the equation:

$$\beta(\gamma^2) = \gamma, \quad \gamma \in [0,r]$$

Example 2- the constraint set is a square

Now, $K := [-1,1] \times [-1,1]$ and the dynamics is still (18). It is easy to check that $K_e = \{ (-1,1), (-1,-1), (1,1), (1,-1) \}$ and for any $x \in K_e$, $d(F(x), T_K(x)) = \sqrt{2} - \alpha$.

Here K_1 is not convex; consequently, we introduce $\widetilde{K_1}$ and we use the convex-case algorithm.

We obtain $\widetilde{K_\infty} = \mathrm{Viab}_F(K) = B(0,1)$.

Remark — In these two examples, the viability kernels are exactly the sets of equilibrium points. □

4.1.2 A linear control system in the four dimensional space

It is a little different version of example 1-1 in [16] page 298. Consider the following control system in $\mathbb{R}^4$:

$$
(19) \qquad \begin{cases} x_1'(t) &= u_1(t) \\ x_2'(t) &= x_4(t) + x_3(t)u_1(t) \\ x_3'(t) &= -x_3(t) + x_4(t) \\ x_4'(t) &= u_2(t) \\ \text{where} & (u_1(t), u_2(t)) \in [-1,\ 1] \end{cases}
$$

With the following set of constraints:

$$
K := \{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 = x_2 = 0,\ (x_3, x_4) \in \mathbb{R}^2 \}
$$

Here, we shall determine the viability kernel (or the *playable kernel* if we consider the system as a model of a differential game, see [22]). Firstly, we determine K^e: here $\partial K = K$. It is easy to check that $F(x) \cap T_K(x) \neq \emptyset$ if and only if: $x_1 = x_2 = x_4 = 0$ and $x_3 \in \mathbb{R}$. Now we can immediately notice that this set, i.e. $K \backslash K^e$ is a viability domain. This is a case of a "degenerateted" application of the algorithm 4. This example allows to see that it is possible, sometimes, to obtain the viability kernel in a finite number of steps thanks to our algorithm even if K is not compact or if the boundedness condition of (1) is not satisfied.

4.2 "Convex" differential inclusion on a polyhedral convex set: A numerical method

Here $X = \mathbb{R}^n$.

We consider the system (2) and assume that the set of constraints K is a *compact, polyhedral set*, i.e the intersection of p ($\geq n+1$) half-spaces such that their intersection is compact (clearly it is also convex). We assume that K has exactly p faces and N_0 vertices: A_1^0, ..., $A_{N_0}^0$. We denote by $s^0(A_i^0)$ the number of faces of K which contains the vertex A_i^0. Define $\varepsilon_{A_i^0}^i$ by (7) in proposition 2.4 and set

$$\widetilde{K}_1 = \mathrm{co}(K \backslash \bigcup_{A_i^0 \in K^e} B(A_i^0, \varepsilon_{A_i^0}^i)).$$

It is clear that we have a new polyhedral convex compact set which has, no more than $\sum_{i=1}^{N_0} s^0(A_i^0)$ vertices and $p + N_0$ faces. By iterating this algorithm, we have an approximation of $\mathrm{Viab}_F(K)$ by polyhedral sets, and for any step we do a finite number of calculus of the real numbers $\varepsilon_{A_i^0}^i$.

A simple example illustrating this approach is:

$$K = [0,1]^n, \quad F(x) := x + B(0,1)$$

and we obtain:

$$\mathrm{Viab}_F(K) = K \cap B(0,1).$$

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