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Working Paper

An Observation Theory for Distributed-Parameter Systems

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WP-90-79
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Foreword

This paper introduces a series of problems on state estimation for parabolic systems on the basis of measurements generated by sensors in the presence of unknown but bounded disturbances. Observability issues, guaranteed filtering schemes for distributed processes and their relation to similar stochastic problems are discussed. The respective problems arise from applied motivations that come, particularly, from ecological and technological issues.

Keywords.
Observers; observability; sensors; state estimation; distributed parameter systems.

An Observation Theory for Distributed-Parameter Systems

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Introduction

This paper deals with the problem of *state estimation* for parabolic systems on the basis of *observations* generated by *sensors*. The issues treated here are the *observability problem* (what types of sensors ensure observability?) and the construction of *observers* for systems subjected to disturbances (in the inputs, in the boundary values and in the measurements). It is indicated that for finite-dimensional measurement outputs the observability property may be ensured through nonstationary ("*scanning*") observations (a respective *duality relation* for problems of *control* is also given). In the state estimation problem the approach discussed here is related to a deterministic model of uncertainty with disturbances taken to be *unknown but bounded*. This approach (also known as the process of *"guaranteed estimation"*) leads to an observer in the form of an evolution equation with set-valued solutions and particularly, in the case of geometric constraints on the unknowns, to an estimator in the form of a *partial differential inclusion*. The set-valued estimate for a finite dimensional projection of the state of the system may then be reached through ***optimization problems for multiple integrals***. An alternative solution may be achieved through stochastic filtering approximations when the set-valued estimate is given through the integration of appropriate ***stochastic filtering equations*** with variable variance terms.

1. The Guaranteed Estimation Problem

In a bounded domain Ω of an n-dimensional Euclidean space consider a distributed field described as the solution to the mixed problem

$$\frac{\partial u(x,t)}{\partial t} = Au(\cdot,t) + f(x,t), \tag{1.1}$$

$$t \in T = (0,\theta), x \in \Omega \subset R^n, Q = \Omega \times T,$$

$$u(x,0) = u_o(x),$$

$$\frac{\partial u(\xi,t)}{\partial n_A} + c(\xi)u(\xi,t) = v(\xi,t), \ \ \xi \in \partial\Omega, \Sigma = \partial\Omega \times T. \tag{1.2}$$

Here $\partial\Omega$ is a piecewise-smooth boundary of Ω,

$$A = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x)\frac{\partial}{\partial x_j}) - a(x)$$

is a symmetric elliptic operator with given coefficients $a_{ij}(x), a(x)$ that satisfies almost everywhere in Ω the condition of coercitivity

$$v\sum_{i=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\ (x)\xi_i\xi_j,$$

$$v = \text{const } > 0,$$

and

$$a_{ij}(x), a(x) \in L_\infty(\Omega), c(\xi) \in L_\infty(\partial\Omega)\,,$$

$$\frac{\partial u(\xi,t)}{\partial n_A} = \sum_{i,j=1}^{n} a_{ij}\ (\xi)\frac{\partial u(\xi,t)}{\partial x_j}\cos\,(n_A(\xi), x_i),\ \xi \in \partial\Omega,$$

where $\cos\,(n_A(\xi), x_i) = i$-th direction cosine of n_A, n_A being the normal at point $\xi \in \partial\Omega$ exterior to Ω; $L_\infty(\Omega), L_\infty(\partial\Omega)$ are spaces of measurable functions that are defined on Ω and $\partial\Omega$ respectively and essentially bounded.

Assuming $f(\cdot,\cdot) \in L_2(Q)$, $u_o(\cdot) \in L_2(\Omega)$, $v(\cdot,\cdot) \in L_2(\Sigma)$ we will consider $u(x,t)$ to be a generalized solution (Sobolev, 1982; Ladyzhenskaya and others, 1968; Lions, 1968) from the Banach space $V_2^{1,0}(Q)$, consisting of all elements of $H^{1,0}(Q)$, that are continuous in t in the norm of $L_2(\Omega)$, with the norm

$$| u |= \max_{0\leq t\leq\theta} \| u(\cdot,t) \|_{L_2(\Omega)} + \| u(\cdot,\cdot) \|_{H^{1,0}(Q)}\,.$$

The symbols $L_2(\Omega), L_2(Q), L_2(\Sigma)$ stand for the spaces of function square integrable on Ω, Q, Σ respectively.

We will further use the following notations for the Sobolev spaces (Sobolev, 1982; Ladyzhenskaya and others, 1968; Lions, 1968):

$$H_l(\Omega) = \{\varphi \mid \varphi \in L_2(\Omega), \frac{\partial \varphi}{\partial x_i} \in L_2(\Omega), \ldots, D^\alpha \varphi \in L_2(\Omega), \forall \alpha,$$

$$\mid \alpha \mid \leq l, \alpha = \{\alpha_1, \ldots, \alpha_n\}, \mid \alpha \mid = \alpha_1 + \ldots + \alpha_n\},$$

$$H_0^l(\Omega) = \{\varphi \mid \varphi \in H^l(\Omega),\ D^\alpha \varphi = 0 \text{ on } \partial\Omega, \mid \alpha \mid \leq l-1\},$$

$$H^{l,0}(Q) = \{\varphi \mid \varphi \in L_2(Q), \frac{\partial \varphi}{\partial x_i} \in L_2(Q), \ldots, D^\alpha \varphi \in L_2(Q),$$

$$\forall \alpha, \mid \alpha \mid \leq l\},$$

$$H_0^{1,0}(Q) = \{\varphi \mid \varphi \in H^{1,0}(Q), \varphi \mid_\Sigma = 0\},$$

$$H^{l,1}(Q) = \{\varphi \mid \frac{\partial \varphi}{\partial t} \in L_2(Q), \varphi \in H^{l,0}(Q)\},$$

$$H_0^{l,1}(Q) = \{\varphi \mid \varphi \in H^{l,1}(Q), \varphi \mid_\Sigma = 0\}, l = 1, 2.$$

Thus the initial boundary value problem (1.1), (1.2) is treated as the following identity

$$\int_0^{\theta_1} \int_\Omega \Big(-u(x,t) \frac{\partial \varphi(x,t)}{\partial t} + \sum_{i,j=1}^n a_{ij}(x) \frac{\partial u(x,t)}{\partial x_i} \frac{\partial \varphi(x,t)}{\partial x_j} + \tag{1.3}$$

$$+ (a(x)u(x,t) - f(x,t))\varphi(x,t)\Big) dx dt +$$

$$+ \int_0^{\theta_1} \int_{\partial\Omega} (c(\xi)u(\xi,t) - v(\xi,t))\varphi(\xi,t) d\xi dt = \int_\Omega u_0(x)\varphi(x,0)dx - \int_\Omega u(x,\theta_1)\varphi(x,\theta_1)dx,$$

for any $\varphi(x,t) \in H^{1,1}(Q)$ and almost all θ_1 from $[0,\theta]$.

It is further assumed that the *input function* $f(x,t)$, *the boundary condition* $v(\xi,t)$ and *the initial distribution* $u_0(x)$ are taken to be *unknown* in advance. However, it is presumed that they satisfy some *preassigned constraints* which will be specified below.

It is supposed that all the available dynamic information on the solution $u(x,t)$ of the problem (1.1)-(1.2) is given through a finite-dimensional *measurement equation*

$$y(t) = \mathbf{G}(t)u(\cdot,t) + \eta(t),\ t \in [\varepsilon, \theta] = T_\varepsilon, \tag{1.4}$$

where $y(t)$ is a measurement data, $y(t) \in R^m$, $y(\cdot) \in L_2^m(T_\varepsilon)$; $\mathbf{G}(t)$ is a linear (nonstationary) observation operator (a "sensor") with its range in R^m ; $\eta(t)$ is the measurement "noise"; ε is a given positive parameter which defines the interval of observations. The operator (the "sensor") $\mathbf{G}(t)$ describes the structure of the observations.

We will suppose that the restriction on the uncertainties $u_o(\cdot), f(\cdot,\cdot), v(\cdot,\cdot), \eta(\cdot)$ can in general be described as

$$w(\cdot) \in \mathbf{W}, \tag{1.5}$$

$$w(\cdot) = \{u_0(\cdot), f(\cdot,\cdot), v(\cdot,\cdot), \eta(\cdot)\}$$

with $\mathbf{W}$ being a given convex set in $L_2(\Omega) \times L_2(Q) \times L_2(\Sigma) \times L_2^m(T_\varepsilon)$.

The guaranteed estimation problem is to estimate the solution $u(x,\theta)$ at instant θ - the terminal point for a trajectory $u(\cdot,t)$ with values in the Hilbert space $L_2(\Omega)$, continuous in t on the interval $[0,\theta]$ - on the basis of the measurement data $y(t)(t \in T_\varepsilon)$ and the available information (1.5) on the uncertainties $f(x,t), u_0(x), v(\xi,t), \eta(t)$.

The estimation problem (1.1) - (1.5) is *a deterministic inverse problem* (Tikhonov, Arsenin, 1979; Lavrentiev and others, 1980) that, in general, obviously has a *nonunique* solution. This leads us to the following (Kurzhanski, 1977)

Definition 1.1. The informational domain $U(\theta, y(\cdot))$ of states $u(x,\theta)$ of system (1.1), (1.2) that are consistent with measurement data $y(t)$ of (1.4) and with restrictions (1.5), is the set of all those functions $u(x,\theta)$ for each of which there exists a quadruple $\omega^*(\cdot) = \{u_0^*(\cdot), f^*(\cdot),$ $v^*(\cdot,\cdot), \xi^*(\cdot)\}$ that satisfies (1.5), and generates a pair $\{u^*(\cdot,\theta),\ y^*(t)\}$ (due to (1.1), (1.2), (1.4)) that satisfies the equalities $u^*(x,\theta) = u(x,\theta), y^*(t) = y(t), t \in T_\varepsilon$.

The linearity of the system (1.1), (1.2), (1.4) and the convexity of $\mathbf{W}$ imply that the domain $U(\theta, y(\cdot))$ is a convex subset of the space $L_2(\Omega)$, that always includes the unknown actual state $u(x,\theta)$.

The estimation problem is to specify the set $U(\theta, y(\cdot))$ and its evolution in time.

Remark 1-a. The domain $U(\theta, y(\cdot))$ may be described by means of its *support function* (Kurzhanski, 1977):

$$\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))) = \sup\{< \varphi(\cdot), u(\cdot, \theta) > \mid u(\cdot, \theta) \in U(\theta, y(\cdot))\}$$

for any element $\varphi(\cdot)$ of the set $\Phi \subseteq L_2(\Omega)$ that defines the generalized solution to the problem (1.1), (1.2) at the instant θ.

Here and below the symbols $< (\cdot), (\cdot) >$ and $\| (\cdot) \|$ stand for the standard scalar product and norm in the respective Hilbert space H which will be clearly specified from the context (in the more complicated cases we will mark the latter by subscripts).

In the sequel, we will pursue the solution to this problem for some specific types of sensors $\mathbf{G}(t)$ and constraints (1.5).

2. Sensors

An observation operator $\mathbf{G}$ ("a sensor") could in general be defined as a map

$$y(\cdot) = \mathbf{G}u(\cdot, \cdot)$$

from $V_2^{1,0}(Q)$ into $L_2^m(T_\varepsilon)$. Particularly, the map $\mathbf{G}u(\cdot, \cdot)$ could be defined through a nonstationary operator $\mathbf{G}(t)$ $(\mathbf{G} = \mathbf{G}(\cdot))$:

$$\mathbf{G}(t)u(\cdot, t) = y(t)$$

from $L_2^m(\Omega)$ into R^m with continuous, piecewise continuous or measuralbe realizations $y(t)$, $t \in T_\varepsilon$, as indicated in (1.4).

Some typical examples of observation operators are as follows

A. Spatially averaged observations:

$$\mathbf{G}(t)u(\cdot, t) = \int_\Omega h\,(x, t)u(x, t)dx, \tag{2.1}$$

with $h(x, t) \in L_2(Q)$ given.

B. A special subclass of observation operators $\mathbf{G}(t)$ *of type A:*

$$\mathbf{G}(t)u(\cdot, t) = \int_\Omega \chi(x, \bar{x}(t))u(x, t)dx, \tag{2.2}$$

where

$$\chi(x, \bar{x}(t)) = \beta(t)\, \delta(x \mid Q_{h(t)}(\bar{x}(t)) \cap \Omega), \;\; \delta(x|S) = \begin{cases} 1, & \text{if } x \in S, \\ 0, & \text{if } x \bar{\in} S, \end{cases}$$

$Q_{h(t)}(\bar{x}(t))$ is the Euclidean neighborhood (in R^n) of radius $h(t)$ of point $\bar{x}(t)$; $\bar{x}(t)$ is a trajectory in the domain Ω; the function $\beta(t) \in L_2(T_\varepsilon)$ is given.

The output of the operator (2.2) is the *spatial average* of the quantity $u(x,t)$ over the sensing region $Q_{h(t)}(\bar{x}(t))$, if $\beta^{-1}(t)$ is the volume of the later, taken along the measurement trajectory $\bar{x}(t)$.

C. Pointwise (stationary or dynamic) observations:

$$\mathbf{G}(t)u(\cdot, t) = \text{ col}\,[u(x^1(t), t), \ldots, u(x^m(t), t)], \tag{2.3}$$

where the measurements are taken at some spatial points or along specified measurement trajectories $x^i(t)$ in the domain Ω. It is clear that this type of sensors requires a corresponding smoothness of the solution $u(x,t)$ to the problem (1.1), (1.2) which is supposed to be assumed below (for example, we will assume that $u(x,t) \in H^{2,1}(\Omega \times T_\varepsilon)$ under $n \leq 3$, see (Ladyzhenskaya and others, 1968; Lions, 1968)).

The mapping $\mathbf{G}(t)$ should be applied throughout the interval T_ε, so that the pointwise sensor would be well-defined.

D. Time averaged (discrete-time) observations:

$$\mathbf{G}(t)u(\cdot, t) = \mathbf{G}(t_i)u(\cdot, \cdot), \; t \in [t_i, t_{i+1}), \; i = 1, \ldots, k,$$

$$\varepsilon = t_1 < \ldots < t_i < \ldots t_k < t_{k+1} = \theta,$$

$$\mathbf{G}(t_i)u(\cdot, \cdot) = \frac{1}{\tau_*} \int\limits_{t_i - \tau_*}^{t_i} \text{ col}\,[u(x^1, t), \ldots, u(x^m, t)]dt,$$

where the measurement data are quantities of the solution $u(x,t)$, taken at spatial points $x^j, j = 1, \ldots, m$ and time averaged over intervals $[t_i - \tau_*, t_i]$ $(i = 1, \ldots, k)$, τ_* is given (sufficiently small).

E. The observation operator may also be a combination of all of the above types of measurements.

As it is clear from the above, *the outputs* of *the sensors* introduced here are all *finite-dimensional* whereas *the system* under observation is *infinite-dimensional.*

In this paper we focus on spatially averaged and dynamic pointwise observations.

Before introducing the notations and definitions and giving the respective proofs, let us turn at first to the finite-dimensional case.

3. Observability in Finite Dimensions

As it is well known, a time-variant finite dimensional system

$$\dot{x} = A(t)x, \tag{3.1}$$

$$y = G(t)x,$$

$$\tau \leq t \leq \theta, x \in R^n, y \in R^m \tag{3.2}$$

is said to be *observable* on the interval $[\tau, \theta]$ once condition $y(t) \equiv 0, t \in [\tau, \theta]$, implies $x(\theta) = 0$, (or, in other words, if two different states $x^{(1)}(\theta) \neq x^{(2)}(\theta)$ generate two different measurements $y^{(1)}(t) \neq y^{(2)}(t)$).

The necessary and sufficient condition for observability is that the symmetric matrix

$$W(\tau, \theta) = \int_{\tau}^{\theta} S'(t, \theta) G'(t) G(t) S(t, \theta) dt$$

would be positive definite:

$$(l, W(\tau, \theta)l) \geq \alpha \parallel l \parallel^2, \forall l \in R^n \tag{3.3}$$

for some $\alpha > 0$ (Krasovski, 1968), symbol $(\cdot, \cdot)$ stands for the scalar product in R^n.

Here $S(t, \theta)$ is the matrix solution to the equation

$$\frac{\partial S(t, \theta)}{\partial t} = A(t) S(t, \theta), \quad S(\theta, \theta) = I_n,$$

where I_n is an identity matrix.

Another formulation for the necessary and sufficient condition of observability (in finite dimensions) may be specified in terms of respective "informational domains".

Consider the system (3.1) subjected to an observation

$$y(t) = G(t)x(t) + \eta(t), \ \tau \leq t \leq \theta \tag{3.4}$$

with an unknown but bounded error $\eta(t)$, so that

$$< \eta(\cdot), \eta(\cdot) > \leq 1, \tag{3.5}$$

with no bounds whatever on the vectors $x(\tau)$ or $x(\theta)$ being presumed.

The ***informational domain*** $X(\theta)$ for system (3.1), (3.4), (3.5) will be defined here as the cross-section at time $t = \theta$ of the bundle of trajectories $\{x(t)\}$ consistent with system (3.1), (3.4) and also with the constraint

$$\int_{\tau}^{\theta} (y(t) - G(t)x(t))'(y(t) - G(t)x(t))dt \leq 1.$$

In our case, by substituting $x(t) = S(t,\theta)x(\theta)$, we may observe that $X(\theta)$ is an *ellipsoid* in R^n defined by the inequality

$$(x, W(\tau,\theta)x) - 2(p,x) + c^2 \leq 1,$$

where

$$p' = + \int_{\tau}^{\theta} y'(t)G(t)S(t,\theta)dt,$$

$$c^2 = \int_{\tau}^{\theta} y'(t)y(t)dt.$$

It is clear that $X(\theta)$ is bounded for any measurement $y(t)$ if and only if $\det W(\tau,\theta) \neq 0$ which is equivalent to (3.3). Therefore the following assertion is true.

Lemma 3.1 The ***informational set*** $X(\theta)$ *(for the problem (3.1) (3.4) (3.5)) is bounded for any* ***measurement*** $y(t)$ ***if and only if*** *the system (3.1), (3.2)* ***is observable*** *on the* ***interval*** $[\tau,\theta]$.

With $\det W(\tau,\theta) \neq 0$ the *support function* for the set $X(\theta)$ can be calculated as follows

$$\rho(l \mid X(\theta)) = \sup\{(l,x) \mid x \in X(\theta)\} = (l, W^{-1}(\tau,\theta)p) + (1-h^2)^{\frac{1}{2}}(l, W^{-1}(\tau,\theta)l)^{\frac{1}{2}},$$

$$h^2 = c^2 - (p, W^{-1}(\tau,\theta)p).$$

It is possible to check that

$$0 \leq h^2 \leq 1.$$

It follows from Lemma 3.1 that the property of $X(\theta)$ being bounded could as well be taken as the *definition* of observability for system (3.1), (3.2).

While being of no special significance in the finite-dimensional case, this "alternative" definition proves, as we shall see, to be useful in infinite dimensions (see also Remark 4-b in the sequel).

Remark 3-a. The equivalence of the property of observability for (3.1), (3.2) and of the boundedness of $X(\theta)$ for (3.1), (3.4), (3.5) is true with the bounds on $\eta(t)$ being taken not only in the form of (3.5) but also for any constraint of type

$$\eta(\cdot) \in Q(\cdot),\ (\eta(\cdot) \equiv \eta(t), \tau \leq t \leq \theta),$$

provided the set $Q(\cdot) = \{q(\cdot)\}$ of functions $q(\cdot)$ is such that

$$\sigma_\varepsilon^{(p)}(0) \subset Q(\cdot) \subset \sigma_r^{(2)}(0),$$

for some $\varepsilon > 0, p \in [2, \infty]$ and for r sufficiently large. Here $\sigma_\alpha^{(p)}(0)$ is a ball of radius α in the space $L_p[\tau, \theta]$.

Prior to the treatment of the infinite dimensional case, however, let us deal with the dual *controllability problem* (in finite dimensions). Although this problem is well known, in the sense that the observability of system (3.1), (3.2) is equivalent to the controllability of system

$$\dot{s} = -sA(t) + w(t)G(t), \quad \tau \leq t \leq \theta \tag{3.6}$$

(the ability to steer $s(t)$ from $s(\theta) = 0$ to any preassigned state $s(\tau) = s$ by a selection of $w(t)$, s being a vector-row), let us formulate the controllability property also in some alternative terms that would be dual to the property that the set $X(\theta)$ should be bounded.

Once $X(\theta)$ is defined for the observed system (3.1), (3.4), (3.5), what would be its equivalent for the controlled system (3.6)?

Calculating the support function $\rho(\ell \mid X(\theta))$ we notice that

$$y(t) = G(t)S(t, \theta)x(\theta) + \eta(t), \quad < \eta(\cdot),\ \eta(\cdot) > \leq 1.$$

From here it follows

$$\rho(l \mid X(\theta)) = \inf\{< w(\cdot), y(\cdot) > + \| w(\cdot) \| \mid w(\cdot) \in W(l)\},$$

where $W(l)$ consists of all the functions $w(\cdot)$ of $L_2^m(\tau, \theta)$ that satisfy

$$\int_{\tau}^{\theta} w'(t)G(t)S(t,\theta)dt = l.$$

Since $w(\cdot) \in W(l)$ implies $-w(\cdot) \in W(-l)$, we observe that the *diameter* of $X(\theta) = X(\theta, y(\cdot))$ (i.e. the diameter of the smallest ball that contains $X(\theta)$) is given by

$$d(X(\theta, y(\cdot))) = \sup_{\|l\| \le 1} \{ \rho(l \mid X(\theta, y(\cdot))) + \rho(-l \mid X(\theta, y(\cdot)))\} =$$

$$= \sup_{\|l\| \le 1} \{ \max \{(l, x) \mid x \in X(\theta, y(\cdot))\} - \min \{(l, x) \mid x \in X(\theta, y(\cdot))\}\}.$$

This yields

$$d(X(\theta, y(\cdot))) = \max_{\|l\| \le 1} \{ \inf \{< w(\cdot), y(\cdot) > + \| w(\cdot) \| \mid w(\cdot) \in W(l)\} + \tag{3.7}$$

$$+\inf\{- < w(\cdot), y(\cdot) > + \| w(\cdot) \| \mid w(\cdot) \in W(l)\}\} \le 2 \max_{\|l\| \le 1} \inf \{\| w(\cdot) \| \mid w(\cdot) \in W(l)\}.$$

Since, obviously,

$$d(X(\theta, \{0\})) = 2 \max_{\|l\| \le 1} \inf \{\| w(\cdot) \| \mid w(\cdot) \in W(l)\},$$

formula (3.7) implies

$$d(X(\theta, y(\cdot))) \le d(X(\theta, \{0\}))$$

for any $y(\cdot)$ generated by system (3.1), (3.2).

As a consequence we come to the following propositions.

Lemma 3.2. The set $X(\theta, y(\cdot))$ is bounded for any $y(t)$ if and only if $X(\theta, \{0\})$ is bounded.

Lemma 3.3. The set $X(\theta, \{0\})$ is bounded if and only if the minimum norm ($\| w_l^0(\cdot) \| = \min$) controls $w_l^0(\cdot)$ for the two-point boundary-value problem

$$\dot{s} = -sA(t) + w(t)G(t), \quad s(\theta) = 0, \quad s(\tau) = l,$$

are bounded in the norm $\| w(\cdot) \|$ *uniformly over all* $l : \| l \| \leq 1$.

The latter property is obviously true if and only if again $| W(\tau, \theta) | \neq 0$. Hence rather than checking that $| W(\tau, \theta) | \neq 0$, it may sometimes be simpler to check that the domain $X(\theta, \{0\})$ is bounded.

Further on we propagate this scheme to parabolic systems. Among the early solutions to the observability problem in infinite dimensions is the one given in (Krasovski, Kurzhanski, 1966).

4. Observability in Infinite Dimensions

In this paragraph we will substitute (1.2) by the boundary-value problem

$$u(\xi, t) \equiv 0, (\xi, t) \in \Sigma. \tag{4.1}$$

Consider the initial boundary value problem (1.1), (4.1) assuming that the input $f(x, t) \equiv 0$ and that the initial state $u_0(x)$ is unconstrained. Moreover, suppose that the measurement $y(t)$ is exact so that we may write

$$u_0(x) \in L_2(\Omega), \quad f(x, t) \equiv 0, \quad x \in \Omega, \ t \in T, \tag{4.2}$$

$$\eta(t) \equiv 0, \quad t \in T_\varepsilon. \tag{4.3}$$

Let us start with a traditional notion:

Definition 4.1. We will say that the system (1.1), (1.4), (4.1), (4.2), (4.3) is observable with sensor $\mathbf{G}(t)$ *if the measurement* $y(t) \equiv 0, \ t \in T_\varepsilon$, *yields* $u(x, \theta) \equiv 0$.

Definition 4.1 is equivalent to the fact that in the absence of errors ($\eta(\cdot) \equiv 0$) the linear mapping

$$\mathbf{T}u(\cdot, \theta) \ = \ y(\cdot)$$

is such that $\mathrm{Ker}\mathbf{T} = \{0\}$.

From this definition it obviously follows that two different states $u^{(1)}(x, \theta) \neq u^{(2)}(x, \theta)$ yield two different measurements $y^{(1)}(t) \neq y^{(2)}(t)$, $t \in T_\varepsilon$. However, definition 4.1 is nonconstructive, whereas the main issue here is to *reconstruct* the state $u(x, \theta)$ from the measurement $y(t)$. We will therefore introduce another definition:

Definition 4.2 We will say that the system (1.1), (1.4), (4.1)-(4.3), is strongly observable with sensor $\mathbf{G}(t)$ *if the informational domain* $U(\theta, y(\cdot))$ *for the estimation problem (1.1), (4.1), (4.2), (1.4) under unknown but bounded error* $\eta(t)$,

$$< \eta(\cdot), \eta(\cdot) >_{L_2(T_\epsilon)} \leq 1 \tag{4.4}$$

is a bounded set in $L_2(\Omega)$, *whatever is the measurement* $y(\cdot)$.

Remark 4-a. The inequality (4.4) for error $\eta(t)$ can be replaced by any restriction of the type

$$\| \eta(\cdot) \|_B \leq 1,$$

where B is some Banach space (see also Remark 3-a), particularly with $B = C(T_\epsilon)$ or $L_\infty(T_\epsilon)$.

It is clear that Definition 4.2 implies Definition 4.1. Indeed, suppose Def. 4.2 holds but Def. 4.1 is false. Then $\mathrm{Ker}\mathbf{T} \neq \{0\}$ and there exists such an element $u^*(\cdot, \theta) \neq 0$, that $\mathbf{T}\alpha u^*(\cdot, \theta) \equiv 0$ for any $\alpha \in R$. Taking the informational domain $U(\theta, \{0\})$, we now observe that it consists of all the states $u(\theta, \cdot)$ that satisfy the equation

$$\mathbf{T}u(\theta, \cdot) = -\eta(\cdot), \text{ under } < \eta(\cdot), \eta(\cdot) > \leq 1. \tag{4.5}$$

Clearly, with $u(\cdot, \theta) = \alpha u^*(\cdot, \theta), \eta(\cdot) = \eta^*(\cdot) = 0$ we have $y^*(\cdot) = \alpha \mathbf{T} u^*(\cdot, \theta) + \eta^*(\cdot) = 0$ for any α. With $u^*(\cdot, \theta) \neq 0$ and α arbitrary this indicates that $U(\theta, \{0\})$ is unbounded in $L_2(\Omega)$.

To compare the "sizes" of various bounded domains $U(\theta, y(\cdot))$, we need the notion of an appropriate "diameter" for these sets.

The diameter of $U(\theta, y(\cdot))$ is defined as

$$d(U(\theta, y(\cdot))) = \sup\{\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))) + \rho(-\varphi(\cdot) \mid U(\theta, y(\cdot))) \mid \| \varphi(\cdot) \| \leq 1, \varphi \in \Phi\}.$$

Similar to the finite dimensional case it is possible to prove that

$$d(U(\theta, y(\cdot))) \leq d(U(\theta, \{0\})),$$

whatever is the measurement generated due to the system (1.1), (4.1), (4.2), (1.4), (4.4) (the nature of the restriction (4.4) does not affect this result). This can be summarized in

Lemma 4.1. The system (1.1), (4.1), (1.4), (4.2), (4.3) is strongly observable if and only if the set $U(\theta, \{0\})$ *for the estimation problem (1.1), (4.1), (1.4), (4.2), (4.4) is bounded.*

We will further use the latter Lemma to investigate the property of strong observability for different types of sensors $\mathbf{G}(t)$. This property however may turn to be a rather strong requirement on $\mathbf{G}(t)$. It seems reasonable, therefore, to introduce a weaker notion.

Let $\lambda_i, \ \ \omega_i(\cdot) \ \ (i = 1,2,3,\ldots)$ be the sequence of eigenvalues and respective eigenfunctions for the problem

$$A\omega_i(\cdot) = -\lambda_i\omega_i(\cdot), \ \omega_i(\cdot) \in H_0^1(\Omega), \tag{4.6}$$

$$< \omega_i(\cdot), \omega_j(\cdot) >= \delta_{ij},$$

so that

$$\lambda_{i+1} > \lambda_i; \lambda_i \to +\infty, \ \ i \to +\infty \ ; \ \ \delta_{ij} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Let $X_r(\Omega) = \mathrm{Span}\{\omega_{i_j}(\cdot)\}_{j=1}^r$ stands for an r-dimensional linear subspace generated by $\omega_{i_j}(\cdot), j = 1,\ldots,r$ and $U_r(\theta, y(\cdot))$ for the orthogonal projection of $U(\theta, y(\cdot))$ on $X_r(\Omega)$, so that

$$U_r(\theta, y(\cdot)) = \bigcup\{\sum_{j=1}^{r} \beta_j(y(\cdot), u(\cdot))\omega_{i_j}(\cdot)\}$$

over all the values $\beta_j(y(\cdot), u(\cdot))$ that satisfy

$$\beta_j(y(\cdot), u(\cdot)) =< u(\cdot), \omega_{i_j}(\cdot) >, \ \ u(\cdot) \in U(\theta, y(\cdot)).$$

Definition 4.3 We will say that the system (1.1), (4.1) - (4.3), (1.4) is weakly observable with sensor $\mathbf{G}(t)$ *if the projection* $U_r(\theta, y(\cdot))$ *of the set* $U(\theta, y(\cdot))$ *of Definition 4.2 on any finite-dimensional subspace* $X_r(\Omega) = \mathrm{Span}\{w_{i_j}(\cdot)\}_{j=1}^r$ *is bounded, whatever is the measurement* $y(\cdot)$.

Def. 4.3 then again implies Def. 4.1, since Ker $\{\mathbf{T}\} \neq \{0\}$ leads to the existence of an element $u^*(\cdot) \neq 0, \mathbf{T}u^*(\cdot) = 0$, and as the system $\{\omega_i(\cdot)\}_{i=1}^{\infty}$ is complete, to the existence of an element $\omega_{i_*}(\cdot) \in \{\omega_i(\cdot)\}_{i=1}^{\infty}$ such that $\alpha\beta_{i_*}^*(0, u^*(\cdot)) =< \alpha u^*(\cdot), \omega_{i_*}(\cdot) >\neq 0$, $\forall\alpha \in R$. This indicates that both the "line" $\alpha u^*(\cdot) \in U(\theta, \{0\}), \forall\alpha$, and its projection $\alpha\beta_{i_*}^*(0, u^*(\cdot))\omega_{i_*}(\cdot) \in U_1(\theta, \{0\}), \forall\alpha$, are unbounded.

It is also clear that Definition 4.2 implies Definition 4.3.

Remark 4-b. The definitions of the above could also be interpreted as follows: given a unit ball $\sigma_1(0)$ in B, the system (1.1), (1.4), (4.1) - (4.3) is *strongly observable*, once the preimage U of $\sigma_1(0)$ due to the mapping

$$TU = \sigma_1(0)$$

is bounded in $L_2(\Omega)$. The latter system is ***weakly observable*** if any finite-dimensional projection U_r of the set U is bounded. The given definitions are thus clearly related to the invertibility properties of the mapping T.

The forthcoming examples demonstrate that the definitions of the above are nonredundant.

5. Examples

Example 1. Consider a one-dimensional heat equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1, \quad t \in T, \tag{5.1}$$

$$u(t,0) = u(t,1) = 0, \quad u(x,0) = u_0(x)$$

under a stationary pointwise observation operator (with measurement at point $x = \bar{x}$)

$$y(t) = u(\bar{x},t) + \eta(t), \quad t \in T_\epsilon. \tag{5.2}$$

It is well-known that the eigenvalues and the (orthonormalized) eigenfunctions for problem (5.1) are given by

$$\lambda_k = -(\pi k)^2, \omega_k(x) = \sqrt{2}\,\mathrm{Sin}\;\pi k x, \quad k = 1, 2, \dots .$$

Expanding the output of system (5.1), (5.2) in a series of exponents we come to

$$y(t) = \sqrt{2}\sum_{k=1}^{\infty} e^{-(\pi k)^2 t}\, u_{0k}\omega_k(\bar{x}) + \eta(t), \quad t \in T_\epsilon, \tag{5.3}$$

where

$$u_{0k} = \sqrt{2}\int_0^1 u(x,0)\,\mathrm{Sin}\;\pi k x\; dx.$$

Due to Lemma 4.1 we will restrict ourselves to the case of $y(t) \equiv 0, t \in T_\varepsilon$.

As it follows from the Müntz-Szácz type theorems (Luxemburg, Korevaar, 1971; Fattorini. Russell, 1974) the distance d_k between an arbitrary function $e^{-(\pi k)^2 t}$ and the closed span $L^k = \text{Span}\{e^{-(\pi i)^2 t} \mid i = 1, 2, \ldots, i \neq k\}$ when taken in the space $B = C[\varepsilon, \theta]$ or $L_p(T_\varepsilon)(p \geq 1)$ is non-zero so that

$$d_k = \inf\{\| e^{-(\pi k)^2 t} - v(t) \|_B \mid v(\cdot) \in L^k\} \neq 0, \quad k = 1, 2, \ldots \ . \tag{5.4}$$

Assume that a solution $u(x, t)$ of the problem (5.1) does satisfy the observation equation (5.2) under $y(t) \equiv 0$ and under the constraint

$$\| \eta(\cdot) \|_B \leq 1.$$

Then, for any integer k we have

$$\sqrt{2} \, \| u_{0k} \text{ Sin } \pi \ k\bar{x} \ e^{-(\pi k)^2 t} + \sum_{\substack{j=1 \\ j \neq k}}^{\infty} u_{0j} \text{ Sin } \pi j \bar{x} \ e^{-(\pi j)^2 t} \|_B \leq 1. \tag{5.5}$$

Taking into account (5.4) we obtain for an arbitrary coefficient $u_{0k} \neq 0$ and an irrational $\bar{x}$ the chain of inequalities

$$| u_{0k} | \cdot | \text{ Sin } \pi \ k\bar{x} \ | \cdot d_k \leq | u_{0k} | \cdot | \text{ Sin } \pi \ k\bar{x} \ | \cdot \| e^{-(\pi k)^2 t} - \sum_{\substack{j=1 \\ j \neq k}}^{\infty} e^{-(\pi j)^2 t} \alpha_j \|_B \leq \frac{1}{\sqrt{2}},$$

where

$$\alpha_j = -\frac{u_{0j} \text{ Sin } \pi \ j\bar{x}}{u_{0k} \text{ Sin } \pi k \ \bar{x}}.$$

This leads to estimates

$$| u_{0k} | \leq \frac{1}{\sqrt{2} d_k \mid \text{Sin } \pi k \ \bar{x} \mid} \quad \text{for any } k = 1, 2, \ldots . \tag{5.6}$$

The boundedness of u_{ok} clearly implies the same property for $e^{-(\pi k)^2 \theta} u_{0k}$. The system (5.1), (5.2) will thus be weakly observable at an initial instant of time as well as at time θ if and only if the coordinate for the location point of the sensor is an irrational number (Sin $\pi \ k\bar{x} \neq 0$ for any $k = 1, 2, \ldots$).

Moreover if $\bar{x}$ is an irrational number of a special type such that the series $\sum_{k=1}^{\infty} e^{-2(\pi k)^2 \theta}/$ $(\mid \text{Sin } \pi \ k\bar{x} \mid d_k)^2$ does converge, then the system (5.1), (5.2) will be strongly observable. The measure of the points of the latter type on the interval [0,1] is equal to 1. This follows from asymptotic estimates for the values of d_k (Luxemburg, Korevaar, 1971; Fattorini, Russell, 1974). For instance, this occurs if one substitutes the point $\bar{x}$ in (5.2) for an arbitrary number of "constant type" (Sakawa, 1975), for example

$$\bar{x} = a + b\sqrt{c},$$

where a, b are arbitrary rational numbers, c is a positive integer which is not a square, and all these are such that $\bar{x} \in (0,1)$.

Remark here that due to (5.3), under $\text{Sin}\pi k\bar{x} = 0$, $\bar{x}$ being rational, the coefficient u_{0k} will be unobservable and as it further follows from (5.5), the system (5.1), (5.2) will not be even weakly observable.

We further proceed by introducing a class of *dynamic* pointwise operators ("scanning observers") that ensure a strong observability for (5.1), (5.2) and such that in the case of a one-dimensional heat equation it would be possible to construct a broad class of appropriate measurement trajectories *explicitly*.

Consider the observation equation

$$y(t) = y(\bar{x}(t), t) + \eta(t), \quad t \in T_\varepsilon. \tag{5.7}$$

For any value θ we will consider a class of dynamic pointwise observation operators under measurement trajectories of the following type

$$\bar{x}(t) = \begin{cases} k(t-\varepsilon), & \varepsilon \leq t \leq \theta_k, \\ 1, & \theta_k \leq t \leq \theta, \end{cases} \tag{5.8}$$

where $\theta_k = k^{-1} + \varepsilon$.

The above class is nonempty if $k \geq 1/(\theta - \varepsilon)$.

Indeed, modifying the classical maximum principle for the solution to the mixed problem (5.1) for the region $\{(x,t) \mid 0 \leq x \leq \bar{x}(t), t \in T_\varepsilon\}$ one can obtain the estimate

$$\max\{| u(x,\theta) | \mid x \in [0,1]\} \leq \max\{ | u(\bar{x}(t),t) |,\ t \in [\varepsilon,\theta]\}. \tag{5.9}$$

The latter estimate yields strong observability of the system, (5.1), (5.7), (5.8) under

$$| \eta(t) | \leq 1, \ \ t \in T_\varepsilon.$$

It is important to stress that the set of continuous curves (5.8) is *stable* with respect to possible perturbations in the space $C[\varepsilon,\theta]$, and it may be extended to the set of *all continuous curves* defined on the interval T_ε with values running through the whole spatial interval $[0,1]$.

Applying Green's formula to (5.1) and taking into account estimate (5.9) one may obtain strong observability for the system (5.1), (5.7), (5.8) under restriction on $\eta(t)$ taken in the space $L_2(T_\varepsilon)$. A theorem in Section 7 will point out that the transition to nonstationary observation operators may ensure obseravability also in the general case.

Example 2. Consider the heat equation in a rectangle

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x_1^2} + \frac{\partial^2 u(x,t)}{\partial x_2^2}, \ \ x' = (x_1,x_2), \ \ t \in T, \tag{5.10}$$

$$\Omega = \{x \mid 0 < x_1 < 1,\ 0 < x_2 < \frac{1}{a}\},$$

$$u(x,t) \mid_\Sigma = 0$$

with the observation equation

$$y(t) = u(\bar{x}(t),t),\ t \in T_\varepsilon. \tag{5.11}$$

For this example $\{\lambda_k\}_{k=1}^\infty = \{\bar{\lambda}_{lm}\}_{l,m=1}^\infty$, $\{w_k(x)\}_{k=1}^\infty = \{\bar{w}_{lm}(x)\}_{l,m=1}^\infty$, where $\bar{\lambda}_{lm} = \pi^2(l^2 + a^2m^2)$, $\bar{w}_{lm}(x) = 2\sin \pi l x_1 \cdot \sin\ \pi a m x_2$, $l,m = 1,2,\dots$.

It is known that the series $\sum_{i=1}^\infty \lambda_k^{-1}$ diverges. Therefore, in this case, all of the values d_{lm} taken for the exponents $\{e^{-\bar{\lambda}_{lm}t}\}$ and defined similar to the values d_k of (5.4) are equal to zero, due to (Luxemburg, Karevaar, 1971; Fattorini, Russell, 1974). Hence there does not exist any stationary observation operator with one dimensional output that can ensure the system (5.10) to be either strongly or even weakly observable under $B = C[\varepsilon,\theta], L_p(T_\varepsilon), p \geq 1$.

The introduction of dynamic pointwise measurements allows to construct the measurement trajectory so that the system (5.10), (5.11) would be strongly and, therefore also weakly observable. The corresponding class of measurement trajecotries is, in general, unstable with respect to possible perturbations. The way out here can be found in increasing the spatial dimension of the measurements.

For example, instead of the pointwise measurements we may consider a "zone" sensor (El Jai and Pritchard, 1988):

$$y(x,t) = \begin{cases} u(x,t), & x \in \Omega_\delta^*(\bar{x}(t)), t \in T_\varepsilon, \\ 0, & x \bar{\in} \Omega_\delta^*(\bar{x}(t)), \end{cases}$$

where the measurements are taken at each instant t over the domain $\Omega_\delta^*(\bar{x}(t)) = \{x \mid x \in \Omega, \| x - \bar{x}(t) \|_{R^2} \leq \delta\}$, $\delta > 0$.

It is clear that if $\bar{x}(t)$ is a trajectory that ensures the system (5.10), (5.11) to be strongly observable under $B = C[\varepsilon, \theta]$, the system (5.10), (5.12) will be also strongly observable. Moreover, this property will be stable with respect to perturbations of the curve $\bar{x}(t)$.

Remark 5-a The latter was an example of an observable system, where $\mathbf{G}(t)$ is a "zone" sensor and $x \in R^2$. Here the measurement is therefore infinite-dimensional. Further in Section 7 it will be shown that observability could be attained for the same system with a pointwise observation along a scanning trajectory $\bar{x}(t) = (\bar{x}_1(t), \bar{x}_2(t))'$, where $\bar{x}_1(t) \equiv \bar{x}_1^*$ is a given point and $\bar{x}_2(t)$ is constructed along the lines of example 1.

6. Duality in Infinite Dimensions

Let us now formulate the problems of control that are dual to those of observation as given in Section 4.

Assume $\mathbf{T}_0, \mathbf{S}(\cdot)$ to denote the linear bounded maps

$$\mathbf{T}_0 u_0(\cdot) = y(\cdot),\ \mathbf{S}(t) u_0(\cdot) = u(\cdot, t),\ t \in T$$

so that $U(\theta, y(\cdot)) = \mathbf{S}(\theta) U(0, y(\cdot))$, $\mathbf{T}_0 = \mathbf{G}\mathbf{S}(\cdot), U(\theta, y(\cdot)) \subset H(\Omega)$, $\mathbf{S}(t)$ is continuous in t.

Here the respective mappings are defined as

$$\mathbf{T}_0\ :\ H(\Omega)\ \rightarrow\ H_1(T_\varepsilon),$$

$$\mathbf{S}(t) : H(\Omega) \to H(\Omega),\ t \in T,$$

$$\mathbf{S}(\cdot) : H(\Omega) \to H_2(Q),$$

$$\mathbf{G} : H_2(Q) \to H_1(T_\varepsilon),$$

where H, H_1, H_2 are Hilbert spaces. In particular, when dealing with the problems of Sections 1,2 we may put $H = L_2, H_1 = L_2, H_2 = H^{1,0}$.

The set $U(\theta, \{0\})$ of states $u(\cdot, \theta)$ consistent with system

$$\mathbf{T}_0 u_0(\cdot) = y(\cdot) - \eta(\cdot),\ \ < \eta(\cdot), \eta(\cdot) > \leq 1$$

will have the following support function

$$\rho(\varphi(\cdot) \mid U(\theta, \{0\})) = \sup\{< \varphi(\cdot), u(\cdot) > \mid u(\cdot) \in U(\theta, \{0\})\} = \rho(\mathbf{S}^*(\theta)\varphi(\cdot) \mid U(0, \{0\})) = f(\varphi(\cdot)),$$

$$f(\varphi(\cdot)) = \inf\{< \lambda(\cdot), \lambda(\cdot) >^{1/2} \mid \mathbf{T}_0^* \lambda(\cdot) = \mathbf{S}^*(\theta)\varphi(\cdot)\}. \tag{6.1}$$

Here according to (Rockafellar, 1970), one should also allow the value $f(\varphi(\cdot)) = +\infty$.

In order that the primal system

$$\frac{\partial u(x,t)}{\partial t} = Au(\cdot, t),\ (x,t) \in Q,\ y(t) = \mathbf{G}u(\cdot, t),\ t \in T_\varepsilon, \tag{6.2}$$

$$u(x, 0) = u_0(x),\ u(\xi, t) \equiv 0, (\xi, t) \in \Sigma$$

would be ***strongly observable*** it is necessary and sufficient that the function $\rho(\varphi(\cdot) \mid U(\theta, \{0\}))$ would be bounded uniformly in $\varphi(\cdot) \in \Xi(0)$,

$$\Xi(0) = \{\varphi(\cdot) : < \varphi(\cdot), \varphi(\cdot) > \leq 1\}.$$

This means that the minimum-norm solution $\lambda_\varphi^0(\cdot)$ to problem (6.1) should be bounded uniformly in $\varphi(\cdot) \in \Xi(0)$. From the properties of Hilbert space it follows that

$$\lambda^0_\varphi(\cdot) = \mathbf{T}_0 w^0_\varphi(\cdot),$$

where

$$\mathbf{T}^*_0\mathbf{T}_0 w^0_\varphi(\cdot) = \mathbf{S}^*(\theta)\varphi(\cdot), w^0_\varphi(\cdot) \in L_2(\Omega)$$

and the uniform boundedness does hold if and only if there exists a constant $\gamma > 0$ that ensures

$$< w(\cdot), \mathbf{T}^*_0\mathbf{T}_0 w(\cdot) > \geq \gamma < \mathbf{S}(\theta)w(\cdot), \mathbf{S}(\theta)w(\cdot) >, w(\cdot) \in L_2(\Omega). \tag{6.3}$$

Then obviously $(\mathbf{T}^*_0\mathbf{T}_0)^{-1}$ exists, so that $w^0_\varphi(\cdot) = (\mathbf{T}^*_0\mathbf{T}_0)^{-1}\mathbf{S}^*(\theta)\varphi(\cdot)$ and

$$< \lambda^0_\varphi(\cdot), \lambda^0_\varphi(\cdot) > = < \varphi(\cdot), \mathrm{S}(\theta)(\mathrm{T}^*_0\mathrm{T}_0)^{-1}\mathrm{S}^*(\theta)\varphi(\cdot) > \leq \| \mathrm{S}(\theta)(\mathrm{T}^*_0\mathrm{T}_0)^{-1} \mathrm{S}^*(\theta) \|,$$

whenever $\varphi(\cdot) \in \Xi(0)$.

Problem (6.1) may be interpreted as a *control problem* for the system

$$\frac{\partial v(\cdot,t)}{\partial t} = -Av(\cdot,t) + \mathbf{G}^*\lambda(\cdot), (x,t) \in Q, \; v(\xi,t) \equiv 0, \; (\xi,t) \in \Sigma, \tag{6.4}$$

$$v(\cdot,\theta) = 0, \;\; v(\cdot,0) = \mathbf{S}^*(\theta)\varphi(\cdot), \tag{6.5}$$

where the control $\lambda(t)$ is to be selected so as to solve a two-point boundary value problem ($v(\cdot,\theta) = \{0\}$, $v(\cdot,0) = \mathbf{S}^*(\theta)\varphi(\cdot)$) with minimum-norm.

Definition 6.1. We will say that the system (6.4) is strongly controllable if the two point boundary-value problem (6.5) is solvable for any $\varphi(\cdot) \in L_2(\Omega)$ and if the minimum-norm solution $\lambda^0_\varphi(\cdot)$ to (6.5) is bounded uniformly in $\varphi(\cdot) \in \Xi(0)$.

The property of *strong observability* is thus equivalent to the one that the *minimum-norm solution $\lambda^0_\varphi(\cdot)$ to the control problem (6.4), (6.5) would be bounded uniformly in $\varphi(\cdot) \in \Xi(0)$. The latter is precisely the property of strong controllability* for system (6.4), (6.5). As indicated in Section 5 the class of such systems is nonvoid for dim $x = 1$. However, as we shall see in the sequel, this property does hold for parabolic systems with dim of $x \geq 2$ only if the sensors A, B, C are described by a *nonstationary* operator $\mathbf{G}(t)$. (Particularly if $\mathbf{G}^*\lambda(\cdot) = f(\cdot), f(t) =$

$\lambda(t)\delta(x - x^*(t)), t \in T_\varepsilon$; $f(t) \equiv 0$, $t \in [0, \varepsilon)$ is a *dynamic actuator* along a certain continuous or piecewise continuous spatial curve $x^*(t)$. The existence of a curve $x^*(t)$ that would ensure strong controllability will be proved in Section 7).

Specifying equation (6.4) we remind that according to the definition of adjoint operators we observe that operator $\mathbf{G}^*$ maps $H_1(T_\varepsilon)$ into the dual space for $H_2(Q)$ (particularly, $L_2(T_\varepsilon)$ into the dual space for $H^{1,0}(Q)$ for the specific problems of Sections 1 and 2). More explicitly, taking the sensors A, B of Section 2 and calculating the respective relation $\mathbf{G}^*\lambda(\cdot) = f(\cdot, \cdot)$, we have:

A. $f(x,t) = h(x,t)\lambda(t)$, $t \in T_\varepsilon$; $f(t) \equiv 0, t \in [0, \varepsilon)$,

B. $f(x,t) = \beta(t)\lambda(t)\delta(x \mid Q_{h(t)}(\bar{x}(t)) \cap \Omega)$, $t \in T_\varepsilon$; $f(t) \equiv 0, t \in [0, \varepsilon)$,

so that here $f(x,t) \in L_2(Q)$.

A separate issue arises for case C where $\mathbf{G}u(\cdot, \cdot)$ is a mapping from either $C(Q)$ or $H^{2,1}(Q)$ (for $n \leq 3$) into $L_2(T_\varepsilon)$ so that $(m = 1)$

$$f(x,t) = \delta(x - \bar{x}(t))\lambda(t),\ t \in T_\varepsilon;\ f(x,t) \equiv 0,\ t \in [0, \varepsilon)$$

should be interpreted along the conventional lines of the theories of Sobolev spaces and generalized functions (Sobolev, 1982; Ladyzhenskaya and others, 1968; Lions, 1968).

Theorem 6.1. The property of strong observability for system (6.2) is equivalent to the property of strong controllability of system (6.4), (6.5). (The uniform boundedness of the minimum-norm solution $\lambda^0_\varphi(\cdot)$ to (6.4), (6.5) over all $\varphi(\cdot) \in \Xi(0)$).

If we now refer to the property of weak observability then obviously, for any finite-dimensional subspace $X_r(\Omega)$ the projection $U_r(\theta, \{0\})$ on $X_r(\Omega)$ will be bounded if and only if the function $\rho(\varphi(\cdot) \mid U(\{0\}))$ will be bounded uniformly in $\varphi(\cdot) \in \Xi^{(r)}(0)$ where $\Xi^{(r)}(0) = \{\varphi(\cdot) : \varphi(\cdot) \in X_r(\Omega), < \varphi(\cdot), \varphi(\cdot) > \leq 1\}$ (for $\varphi(\cdot) \in X_r(\Omega)$ clearly $< \varphi(\cdot), \varphi(\cdot) > = < \varphi(\cdot), \varphi(\cdot) >_r = < \varphi_r(\cdot), \varphi_r(\cdot) >$, where $\varphi_r(\cdot)$ is the projection of $\varphi(\cdot)$ on $X_r(\Omega)$ and $< \cdot, \cdot >_r$ is the scalar product in $X_r(\Omega)$).

For a given $\varphi(\cdot) \in X_r(\Omega)$ and a given $\mu > 0$ the problem

$$< \lambda(\cdot), \lambda(\cdot) > \leq \mu^2,\ \ \mathbf{T}_0^*\lambda(\cdot) = \mathbf{S}^*(\theta)\varphi(\cdot),$$

will be solvable if and only if the inequality

$$\mu \parallel \mathbf{T}_0 w(\cdot) \parallel - < w(\cdot), \mathbf{S}^*(\theta)\varphi(\cdot) > \geq 0$$

does hold for any $w(\cdot) \in L_2(\Omega)$. In order that problem (6.1) would be solvable uniformly in $\varphi(\cdot) \in \Xi^{(r)}(0)$, it is necessary and sufficient that there would exist a number $\mu_r > 0$ such that

$$\mu_r \parallel \mathbf{T}_0 w(\cdot) \parallel \geq < \mathbf{S}(\theta)w(\cdot), \mathbf{S}(\theta)w(\cdot) >_r^{1/2}$$

or in other words, that

$$\mu_r < w(\cdot), \mathbf{T}_0^* \mathbf{T}_0 w(\cdot) > \geq < w(\cdot), \mathbf{S}^*(\theta)\mathbf{S}(\theta)w(\cdot) >_r, \tag{6.6}$$

whatever is $w(\cdot)$ that belongs to $\in L_2(\Omega)$.

Lemma 6.1. In order that system (6.2) would be weakly observable it is necessary and sufficient that for any finite-dimensional subspace $X_r(\Omega)$ there would exist a number $\mu_r > 0$, such that (6.6) would be true.

(Note that strong observability yields the existence of a number μ that *does not depend on* r).

The dual property of *weak controllability* for system (6.4), (6.5) now sounds as follows

Definition 6.2. The system (6.4), (6.5) is said to be weakly controllable if for any finite dimensional subspace $X_r(\Omega) \subset L_2(\Omega)$ the minimum norm solution $\lambda_\varphi^0(\cdot)$ to problem (6.4), (6.5) is bounded uniformly in $\varphi(\cdot) \in \Xi^{(r)}(0)$.

Lemma 6.2. In order that (6.4), (6.5) would be weakly controllable, it is necessary and sufficient that for any given $X_r(\Omega)$ the relation (6.6) would hold for some $\mu_r > 0$.

Since both strong and weak observability imply that Ker $\mathbf{T}_0 = \{0\}$, we will now demonstrate that the latter property is equivalent to the property of *ε-controllability* of the dual system.

Definition 6.3. The system (6.4) is said to be ε-controllable if for any $\varphi(\cdot) \in L_2(\Omega)$ and any $\varepsilon > 0$ there exists a number $\mu_{\varphi\varepsilon} > 0$ such that the problem

$$\mathbf{T}_0^* \lambda(\cdot) \in \mathbf{S}^*(\theta)\varphi(\cdot) + \varepsilon \Xi(0), \ \parallel \lambda(\cdot) \parallel \leq \mu \tag{6.7}$$

is solvable for $\mu \geq \mu_{\varphi\varepsilon}$.

Lemma 6.3. The system (6.4), (6.5) is ε-controllable iff Ker $\mathbf{T}_0 = \{0\}$.

Once (6.7) is solvable, we obviously have Ker $\mathbf{T}_0 = \{0\}$. Indeed, if $\mathbf{T}_0 w^*(\cdot) = \{0\}$ for some $w^*(\cdot) \neq 0$ and if $\lambda^*(\cdot)$ is a solution to (6.7), then one should have

$$0 =< \mathbf{T}_0 w^*(\cdot), \lambda^*(\cdot) > \leq < w^*(\cdot), \mathbf{S}^*(\theta)\varphi(\cdot) > +\varepsilon \parallel w^*(\cdot) \parallel .$$

However, one could always chose $\varphi(\cdot)$, ε so that $< w^*(\cdot),\ \mathbf{S}^*(\theta)\varphi(\cdot) > \leq -\frac{1}{2} \parallel w^*(\cdot) \parallel$, $\varepsilon < 1/2$. The previous inequality will then turn to be false.

On the opposite, suppose Ker $\mathbf{T}_0 = \{0\}$. Let us prove that (6.7) is solvable. The following part of the proof gives a constructive estimate for $\mu = \mu_{\varphi\varepsilon}$.

Pressuming

$$h(\cdot) = \mathbf{S}^*(\theta)\varphi(\cdot),\ h(\cdot) \in H(\Omega) = L_2(\Omega),$$

we observe that $h(\cdot)$ allows an expansion along the complete system of orthonormalized functions $\{\omega_i(\cdot)\}_{i=1}^{\infty}$, so that

$$h(\cdot) = \sum_{i=1}^{\infty} \alpha_i \omega_i(\cdot),$$

and for a given $\varepsilon > 0$ we may find $r = r(\varepsilon) > 0$ that yields $h(\cdot) = h_r(\cdot) + h_r^{\perp}(\cdot)$, where

$$h_r(\cdot) = \sum_{i=1}^{r} \alpha_i \omega_i(\cdot),\ h_r^{\perp}(\cdot) = \sum_{i=r+1}^{\infty} \alpha_i \omega_i(\cdot),$$

$$\parallel h_r^{\perp}(\cdot) \parallel^2 = \sum_{i=r+1}^{\infty} \alpha_i^2 \leq \varepsilon^2/4.$$

It now suffices to prove the solvability of the inclusion

$$\mathbf{T}_0^* \lambda(\cdot) \in h_r(\cdot) + \delta\Xi(0),\ \delta = \frac{\varepsilon}{2},$$

where $h_r(\cdot)$ is a finite-dimensional element that depends on parameters

$$\alpha_i,\ i = 1, \ldots, r(\varepsilon)\ (h_r(\cdot) \in X_r(\Omega)).$$

For $h_r(\cdot) \in L_2(\Omega)$, $h_r(\cdot) \neq \{0\}$, $\mu > 0$ the problem (6.7) is solvable iff (Kurzhanski, 1977)

$$\mu \parallel \mathbf{T}_0 w(\cdot) \parallel + \delta \parallel w(\cdot) \parallel + < w(\cdot), h_r(\cdot) > \geq 0 \tag{6.8}$$

holds for any $w(\cdot) \in L_2(\Omega)$. We will prove the existence of a number $\mu = \mu_{\varphi\epsilon}$ that depends on $h_r(\cdot), \delta$ and ensures (6.8) to be true for any $w(\cdot)$.

Instead of (6.8) we may consider the condition that the inequality

$$\mu \parallel \mathbf{T}_0 w(\cdot) \parallel + \delta \parallel w(\cdot) \parallel \geq 1 \tag{6.9}$$

should be true for any $w(\cdot)$ such that

$$< w(\cdot), h_r(\cdot) >= -1. \tag{6.10}$$

Obviously the latter are equivalent to (6.8). Without loss of generality we may also assume $< h_r(\cdot), h_r(\cdot) > = 1$ (as the equation (6.7) is linear in $\lambda(\cdot)$).

Further on we come to

$$\mu < w(\cdot), \mathbf{T}_0^* \mathbf{T}_0 w(\cdot) >^{1/2} + \delta < w(\cdot), w(\cdot) >^{1/2} \geq$$

$$\geq < w(\cdot), (\mu^2 \mathbf{T}_0^* \mathbf{T}_0 + \delta^2 I)\, w(\cdot) >^{1/2} .$$

Therefore, in order to ensure (6.9), (6.10) for any $w(\cdot)$, we may first secure

$$\chi^0 = \min \{< w(\cdot), \mathbf{K}_{\mu\delta} w(\cdot) >^{1/2} \mid < w(\cdot), h_r(\cdot) > = -1\} \geq 1, \tag{6.11}$$

where $\mathbf{K}_{\mu\delta} = \mu^2 \mathbf{T}_0^* \mathbf{T}_0 + \delta^2 I$ is an invertible map with bounded inverse $\mathbf{K}_{\mu\delta}^{-1}$. A direct calculation of (6.11) by Hilbert space techniques gives

$$\chi^0 =< h_r(\cdot), \mathbf{K}_{\mu\delta}^{-1} h_r(\cdot) >^{-\frac{1}{2}} .$$

Therefore the problem (6.9), (6.10) is solvable once

$$(\chi^0)^{-2} =< h_r(\cdot),\ \mathbf{K}_{\mu\delta}^{-1}\ h_r(\cdot) > \leq 1. \tag{6.12}$$

The latter relation is obviously ensured if

$$\max_{\|z(\cdot)\|=1,\ z(\cdot) \in X_r(\Omega)} < z(\cdot),\ \mathbf{K}_{\mu\delta}^{-1} z(\cdot) > \ \leq 1.$$

However, we have

$$(\max_{||z(\cdot)||=1,\, z(\cdot)\in X_r(\Omega)} < z(\cdot),\ \mathbf{K}_{\mu\delta}^{-1} z(\cdot) >)^{-1} = \min_{||z(\cdot)||=1,\, z(\cdot)\in X_r(\Omega)} < z(\cdot),\ \mathbf{K}_{\mu\delta} z(\cdot) > .$$

Therefore (6.12) will be ensured if

$$\min_{||z(\cdot)||=1, z(\cdot)\in X_r(\Omega)} < z(\cdot), \mathbf{K}_{\mu\delta} z(\cdot) > \ \geq 1$$

or, in more detail, if

$$\mu^2\gamma + \delta^2 \ \geq 1, \tag{6.13}$$

where

$$\gamma = \min_{||z(\cdot)||=1,\, z(\cdot)\in X_r(\Omega)} < z(\cdot), \mathbf{T}_0^*\mathbf{T}_0 z(\cdot) > \ \ > 0 \tag{6.14}$$

(since Ker $\mathbf{T}_0 = \{0\}$ and $X_r(\Omega)$ is finite-dimensional).

Inequality (6.13) yields $\mu \geq \gamma^{-1/2}(1-\delta^2)^{1/2}$. We thus come to

Lemma 6.4 For the solvability of (6.7) it suffices to select $\mu > \gamma^{-\frac{1}{2}}$, *where* γ *is given by (6.14) with dimension* $r = r(\varepsilon)$ *of* $X_r(\Omega)$ *being dependent on* ε.

We will now prove the property of observability under scanning observers starting with pointwise sensors.

7. Observability Under Pointwise Dynamic Observations.

The examples of Section 5 give us a hint as to how to prove the existence of a measurement trajectory $\bar{x}(t)$ that would ensure observability for the system (6.2) where

$$y(t) = \mathbf{G}\,(t)u(\cdot,t) = u(\bar{x}(t),t),\ \ t \in T_\varepsilon. \tag{7.1}$$

We further assume that system (6.2) under $u_0(x) \in L_2(\Omega)$ is such (either classical on T_ε or $u(\cdot,\cdot) \in H^{2,1}(\Omega \times T_\varepsilon)$ with $n \leq 3$) that its arbitrary solution is a continuous function on $[\varepsilon,\theta]$ satisfying *the maximum principle* (Ladyzhenskaya and others, 1968):

$$\max\{|\ u(x,t')\ ||\ x \in \bar{\Omega}\} \ \geq |u(x,t'')|,\ \forall x \in \bar{\Omega},\quad t'' > t' \geq \varepsilon. \tag{7.2}$$

As it was demonstrated earlier in Section 4, the system (6.2), (7.1) will be strongly observable if the informational domain $U(\theta, \{0\})$ for the system (6.2) under "noisy" observation

$$y(t) = u(\bar{x}(t), t) + \eta(t), \quad \| \eta(\cdot) \|_{C[\varepsilon,\theta]} \leq 1 \tag{7.3}$$

with unknown but bounded "noise" $\eta(\cdot)$ will be bounded (see Remark 4-a). We therefore have to prove the existence of a measurement trajectory $\bar{x}(t)$ that would ensure this property. We will start to seek for the function $\bar{x}(t)$ in the class $X[\varepsilon, \theta]$ of piecewise-continuous functions on the interval $[\varepsilon, \theta]$.

Let U_ε stand for the set of all the solutions to the initial boundary value problem (6.2) generated by all the possible functions $u_0(x)$, with $U_\varepsilon[t]$ standing for the crossection of U_ε at instant t. Since the set

$$U_\varepsilon \subseteq C(\bar{\Omega} \times [\varepsilon, \theta]),$$

and since the space $C(\bar{\Omega} \times [\varepsilon, \theta])$ is separable, it is possible for any $\gamma > 0$ to indicate a countable γ-net for U_ε

$$U_\varepsilon^\gamma = \{u_i(\cdot, \cdot)\}_{i=1}^{\infty}, \quad u_i(\cdot, \cdot) \in U_\varepsilon.$$

Any crossection $U_\varepsilon^\gamma[t]$ at instant t of the γ-net U_ε^γ will hence be a γ-net in $U_\varepsilon[t]$.

In other words, for any element $u^*(\cdot, \cdot) \in U_\varepsilon$ there exists an integer $i = i_*$ such that

$$\| u^*(\cdot, \cdot) - u_{i_*}(\cdot, \cdot) \|_{C(\bar{\Omega} \times [\varepsilon,\theta])} \leq \gamma.$$

This yields

$$\| u^*(\cdot, t) - u_{i_*}(\cdot, t) \|_{C(\bar{\Omega})} \leq \gamma, \quad \forall t \in T_\varepsilon.$$

We will now indicate a possible measurement trajectory $\bar{x}(t)$ that would ensure the set $U(\theta, \{0\})$ to be bounded.

Consider a monotone sequence of points $t_i, i = 1, 2, \ldots$ such that

$$\varepsilon < t_1 < t_2 < \ldots < t_i < t_{i+1} < \ldots < \theta.$$

Clearly there exists a limit

$$\lim_{i\to\infty} t_i = a \leq \theta.$$

Denote $x^{(i)}$ to be the lexicographic minimum for the set $X^{(i)}$, where

$$X^{(i)} = \arg \{\max \mid u_i(x, t_i) \mid \ \mid x \in \bar{\Omega}\}.$$

The function $\bar{x}(t)$ will now be constructed in the form of a *spline-function*

$$\bar{x}(t) \equiv x^*(t), \quad t \in [\varepsilon, \theta]$$

such that

$$x^*(t_i) = x^{(i)}, i = 1, 2, \ldots$$

with $x^*(t)$ being continuous for $t \in [\varepsilon, a)$, $t \in [a, \theta]$.

Clearly $x^*(t)$ is continuous at all the points $t \in [\varepsilon, \theta]$, except for point $t = a$. Therefore, $x^*(\cdot) \in X[\varepsilon, \theta]$. Let us show that this function satisfies the necessary requirements.

Take any element $\bar{u}(\cdot) \in U(\theta, \{0\})$ generated by a solution $\bar{u}(x,t)$ to (6.2) and (7.3), $y(t) \equiv 0$, so that $\bar{u}(x,\theta) = \bar{u}(x)$. For a given $\gamma > 0$ select an element $u_k(\cdot,\cdot) \in U_\varepsilon^\gamma$ so that

$$\| \bar{u}(\cdot,\cdot) - u_k(\cdot,\cdot) \|_{C(\bar{\Omega}\times[\varepsilon,\theta])} \leq \gamma. \tag{7.4}$$

Then, due to (7.3), taking $y(t) \equiv 0$, we have

$$\| u_k(x^*(t), t) \|_{C[\varepsilon,\theta]} \leq 1 + \gamma.$$

The latter inequality indicates, in particular, that

$$u_k(x, \theta) \in U_\gamma(\theta, \{0\}), \tag{7.5}$$

where $U_\gamma(\theta, y(\cdot))$ is the informational domain for problem (6.2), (7.3) with constraint

$$\| \eta(\cdot) \|_{C[\varepsilon,\theta]} \leq 1 + \gamma,$$

so that $U_0(\theta, y(\cdot)) = U(\theta, y(\cdot))$.

Applying the maximum principle (7.2), we now come to the relations

$$| u_k(x,\theta) | \leq \max_{x \in \bar{\Omega}} | u_k(x, t_k) | = | u_k(x^{(k)}, t_k) | =$$

$$= | u_k(x^*(t_k), t_k) | \leq 1 + \gamma$$

for any $x \in \bar{\Omega}$. The later inequality, taken together with (7.4), gives us the final estimate:

$$| \bar{u}(x,\theta) | \leq 1 + 2\gamma, \quad \forall x \in \bar{\Omega}. \tag{7.6}$$

The bound (7.6) is uniform in all $u(\cdot, \theta) \in U(\theta, \{0\})$, so that

$$U(\theta, \{0\}) \subseteq \Xi^c_{1+2\gamma}(0)$$

which proves *strong observability* under the pointwise observation $\mathbf{G}(t)u(\cdot, t) = u(\bar{x}(t), t)$ generated by the trajectory $\bar{x}(t) = x^*(t)$. The symbol $\Xi^c_r(0)$ stands for the ball

$$\Xi^c_r(0) = \{u(\cdot) : \| u(\cdot) \|_{C(\bar{\Omega})} \leq r\}.$$

Theorem 7.1 There exists a pointwise observation trajectory $\bar{x}(t)$ (a "scanning observer") selected in the class $X[\varepsilon, \theta]$ of piecewise-continuous functions with a finite number of discontinuities that ensures strong observability for the system (1.1), (4.1), (4.2), (7.3).

Remark 7-a

(i) From the proof of Theorem 7.1 it follows that the function $\bar{x}(t)$ could also be selected as *piecewise-constant*, so that

$$\bar{x}(t) \equiv x_i(t_i), \text{ if } t_i \leq t < t_{i+1},$$

$$(i = 1, 2, \ldots),$$

$$\bar{x}(t) \equiv 0, \text{ if } \varepsilon \leq t < t_1, \quad a \leq t \leq \theta.$$

Function $\bar{x}(t)$ is *measurable*, it has but a *countable set of discontinuities* at points t_i, a.

(ii) The result of Theorem 7.1 does not depend on the dimension of the space variable x and on the stationarity of the elliptic operator A.

(iii) The property of strong observability is *unstable* with respect to pertubation of the function $\bar{x}(t)$ (the measurement curve) when taken in the metric of $C[\varepsilon,\theta]$ or $L_p[\varepsilon,\theta], p > 0$.

(iv) Since the solution $u(x,t)$ is continuous in $\{x,t\}$ ($x \in \bar{\Omega},\ t \in T_\varepsilon$) while $\bar{x}(t) \in \bar{\Omega}$ is measureable and bounded, the superposition $y(t) = u(\bar{x}(t),t)$ will be measurable and bounded and therefore Lesbesgue-integrable on $[\varepsilon,\theta]$ (Sansone, 1949).

Example 3. Consider again the system (5.10), (5.11)

The techniques applied in *the example 1* allow us to obtain the estimate

$$\| \tilde{u}(x_1,x_2,\theta) \|_{L_2(\Omega)} \leq M \max_{\substack{x_2 \in [0,a^{-1}] \\ t \in [\varepsilon,\theta]}} | \tilde{u}(\bar{x}_1,x_2,t) | \tag{7.7}$$

for an arbitrary solution to (5.10). Here M is a constant, $\bar{x}_1$ is an irrational number of "constant type", $u(x,t) = \tilde{u}(x_1,x_2,t)$.

Indeed, put

$$\max_{\substack{x_2 \in [0,a^{-1}] \\ t \in [\varepsilon,\theta]}} | \tilde{u}(\bar{x}_1,x_2,t) | = c.$$

Then we have

$$\| 2 \sum_{l,m=1}^{\infty} e^{-\bar{\lambda}_{lm}t} u_{0lm} \sin \pi l \bar{x}_1 \sin \pi a m x_2 \|_{C([0,a^{-1}]\times[\varepsilon,\theta])} \leq c,$$

where

$$u_{0lm} = 2 \int_0^1 \int_0^{1/a} u(x,0) \sin \pi l x_1 \sin \pi a m x_2 dx_2 dx_1.$$

The latter inequality yields

$$| \sum_{m=1}^{\infty} 4 (\sum_{l=1}^{\infty} e^{-\bar{\lambda}_{lm}t} \sin \pi l \bar{x}_1 u_{0lm})^2 | \leq c^2 \cdot a^{-1}, \forall t \in [\varepsilon,\theta],$$

from where it follows

$$\max_{t\in[\varepsilon,\theta]} \mid 2\sum_{l=1}^{\infty} e^{-\bar{\lambda}_{lm}t} \sin \pi l \bar{x}_1 u_{0lm} \mid \le c(1/a)^{1/2}$$

for any integer $m = 1, 2, \ldots$.

Since the series $\sum_{l=1}^{\infty} 1/\lambda_{lm}$ converges, one can obtain (along the lines of (5.4) - (5.6)) the following sequence of estimates for the values u_{0lm} under an arbitrary irrational $\bar{x}_1$:

$$\mid u_{0lm} \mid \le \frac{c}{2\sqrt{a} \mid \text{ Sin } \pi l \bar{x}_1 \mid d_{lm}} \, , \, l, m = 1, 2, \ldots, \tag{7.8}$$

where

$$d_{lm} = \text{ inf } \{\| \, e^{-\bar{\lambda}_{lm}t} - v(t) \, \|_{C[\varepsilon,\theta]} \mid v(\cdot) \in L^{lm}\} \ne 0,$$

$$L^{lm} = \text{ Span } \{e^{-\bar{\lambda}_{im}t} \mid i = 1, 2, \ldots, i \ne l\}.$$

As in *the example 1* the latter leads to (7.7).

The estimate (7.7) gives an idea as to how to construct a dynamic pointwise observation operator in the form

$$\mathbf{G}(t)u(\cdot, t) = u(\bar{x}(t), t) = \tilde{u}(\bar{x}_1, \bar{x}_2(t), t) \tag{7.9}$$

that ensures system (5.10), (5.11) to be strongly observable under $B = C(T_\varepsilon)$.

Let coordinate $x_1(t)$ of the measurement trajectory $\bar{x}'(t) = (\bar{x}_1(t), \bar{x}_2(t))$ be fixed so

$$\bar{x}_1(t) = \bar{x}_1, \; t \in T_\varepsilon,$$

where $\bar{x}_1$ is an irrational number of a "constant type" (see Section 5). The problem is to find the function $\bar{x}_2(t)$ for the second coordinate so that the domain $U(\theta, \{0\})$ for the system (5.10) under the measurements

$$0 = \tilde{u}(\bar{x}_1, \bar{x}_2(t), t) + \eta(t), \; t \in T_\varepsilon$$

and the constraints

$$\mid \eta(t) \mid \le 1, \; t \in T_\varepsilon \tag{7.10}$$

would be bounded in $L_2(\Omega)$.

Let $U_\varepsilon^\gamma \subset U_\varepsilon$ be a countable γ-net in $C(\bar{\Omega} \times T_\varepsilon)$ for the set of all the possible solutions to the problem (5.10) taken on the time interval T_ε so that

$$U_\varepsilon^\gamma = \{u_i(x,t)\}_{i=1}^{\infty}, \;\; u_i(\cdot,\cdot) = \tilde{u}_i(\cdot,\cdot,\cdot) \in C(\bar{\Omega} \times T_\varepsilon).$$

Denote by $(x_2^{(i)}, t_i)$ an arbitrary solution to the optimization problem

$$\mid \tilde{u}(\bar{x}_1, x_2, t) \mid \to \max$$

$$x_2 \in [0, a^{-1}],$$

$$t \in T_\varepsilon.$$

Suppose at the beginning that all of the instants t_i are different. In this case, an arbitrary curve $\bar{x}_2(t) = x^*(t)$, piecewise-continuous on $[\varepsilon, \theta]$, and such that

$$x^*(t_i) = x_2^{(i)}, \; i = 1, 2, 3, \ldots$$

ensures strong observability of the system (5.10), (7.9), (7.10).

Indeed, taking any element $\tilde{u}^*(x_1, x_2, \theta) \in U(\theta, \{0\})$ generated due to (5.10) and selecting $\tilde{u}_{i_*}(x_1, x_2, t)$ as an element of the γ-net U_ε^γ we observe

$$\mid \tilde{u}^*(x_1, x_2, t) - \tilde{u}_{i_*}(x_1, x_2, t) \mid \leq \gamma, \; x_1 \in [0,1], \; x_2 \in [0, a^{-1}], \; t \in T_\varepsilon. \qquad (7.11)$$

The estimate (7.7) applied for $\tilde{u}_{i_*}(x_1, x_2, t)$ then leads to

$$\| \tilde{u}^*(x_1, x_2, \theta) \|_{L_2(\Omega)} \leq \| \tilde{u}_{i_*}(x_1, x_2, \theta) \|_{L_2(\Omega)} + \gamma a^{-\frac{1}{2}} \leq$$

$$\leq M \mid \tilde{u}_{i_*}(\bar{x}_1, \bar{x}_2^{(i_*)}, t_{i_*}) \mid + \gamma a^{-\frac{1}{2}} \leq M(\mid \tilde{u}^*(\bar{x}_1, x^*(t_{i_*}), t_{i_*}) \mid + \gamma) + \gamma a^{-\frac{1}{2}} \leq$$

$$\leq M(1+\gamma) + \gamma a^{-\frac{1}{2}}.$$

In the case of coinciding points t_i it is again possible to obtain the same property of strong observability. Instead of the values t_i we may take some other values close to those but such that all the new t_i's will be different. The necessary property then follows from the countability of the pairs $\{x_2^{(i)}, t_i\}$ and the continuity of the solutions to the system (5.10).

8. Observability under Spatially Averaged Observations.

Consider the Dirichlet problem

$$\frac{\partial u(x,t)}{\partial t} = Au(\cdot,t), \;\; x \in \Omega, \; t \in T, \tag{8.1}$$

$$u(x,0) = 0, \quad u(\xi,t) = 0, \quad \xi \in \partial\Omega$$

and the measurement equation (1.4) under a spatialy averaged observation operator $\mathbf{G}$ of the type B

$$y(t) = \beta(t) \int\limits_{\Omega} \delta(x \mid Q_{h(t)}(\bar{x}(t)) \cap \Omega) u(x,t) dx + \eta(t), \; t \in T_\varepsilon. \tag{8.2}$$

The observability problem for such a sensor is to specify a curve $\bar{x}(t)$, a neighborhood $Q_{h(t)}(\bar{x}(t))$ of radius $h(t)$ and with a volume $\beta^{-1}(t)$ so that system (8.1), (8.2) would be either strongly or weakly observable.

It is known that for an arbitrary generalized solution $u(x,t) \in V_2^{1,0}(Q)$ to problem (8.1) the following estimate does hold (Ladyzhenskaya and others, 1968 , p. 193):

$$\operatorname*{vrai\,max}_{Q_\epsilon} \mid u(x,t) \mid \leq L(\varepsilon) \parallel u(\cdot,\cdot) \parallel_{L_2(Q)}, \tag{8.3}$$

where $Q_\varepsilon = \Omega \times (\varepsilon, \theta)$, $L(\varepsilon)$ is a positive function.

Moreover $u(x,t)$ satisfies the *generalized maximum principle* (Ladyzhenskaya and others, 1968)

$$\operatorname*{vrai\,max}_{x \in \Omega} \mid u(x,t') \mid \geq c \operatorname*{vrai\,max}_{x \in \Omega} \mid u(x,t'') \mid, \; t'' \geq t' \geq \varepsilon, \tag{8.4}$$

$$c = \text{const.}$$

Let U be the set of all generalized solutions taken for the time-interval T. Since

$$U \subset V_2^{1,0}(Q)$$

it is possible to indicate for U a countable γ-net U_g^γ ($\gamma > 0, \gamma$ given) so that

$$U_g^\gamma = \{u_i(\cdot,\cdot)\}_{i=1}^{\infty}, \quad u_i(\cdot,\cdot) \in U.$$

Hence for any solution $u(x,t)$ there exists an element (solution) $u_i(x,t)$ such that

$$\max_{t\in T} \parallel u(\cdot,t) - u_i(\cdot,t) \parallel_{L_2(\Omega)} \leq \gamma, \tag{8.5}$$

$$\parallel u(\cdot,\cdot) - u_i(\cdot,\cdot) \parallel_{H^{1,0}(Q)} \leq \gamma.$$

Consider again an arbitrary monotone sequence of points $\{t_i\}_{i=t}^{\infty}$ such that

$$\varepsilon < t_1 < \ldots < t_i < \ldots < \theta$$

and

$$a = \lim_{i\to\infty} t_i, \; a \leq \theta.$$

Due to the properties of Lebesque points for each (square integrable) element $u_i(\cdot,t)$ of U_g^γ there exists a point $x^{(i)} \in$ int Ω such that for some neighborhood $Q_{h_i}(x^{(i)})$ of the latter the following estimate is true

$$\mid \operatorname*{vrai\,max}_{x\in\Omega} \mid u_i(x,t_i) \mid - \psi_i \int\limits_{Q_{h_i}(x^{(i)})} \beta_i u_i(x,t_i)dx \mid \leq \nu, \tag{8.6}$$

where

$$\psi_i = \begin{cases} 1, & \text{if} \;\; \text{vrai}\max_{x\in\Omega} |u_i(x,t_i)| = \text{vrai}\max_{x\in\Omega} u(x,t_i), \\ -1, & \text{if} \;\; \text{vrai}\min_{x\in\Omega} |u_i(x,t_i)| = -\,\text{vrai}\min_{x\in\Omega} u(x,t_i); \end{cases}$$

ν is positive (given in advance); β_i^{-1} and h_i are the volume and the radius of the ball $Q_{h_i}(x^{(i)})$ ($Q_{h_i}(x^{(i)}) \subset \Omega$) respectively; $i = 1,2,3,\ldots$.
Thus we obtain a sequence $\{t_i, x^{(i)}, h_i, \beta_i\}_{i=1}^{\infty}$ that allows to construct spline-functions $\bar{x}(t) = x^*(t)$, $h(t) = h^*(t)$, $\beta(t) = \beta^*(t)$ such that

$$x^*(t_i) = x^{(i)}, \; h^*(t_i) = h_i, \; \beta^*(t_i) = \beta_i \; (i = 1,2,3,\ldots), \; Q_{h^*(t)}(x^*(t)) \subset \Omega.$$

Let us show that the weight function

$$\chi(x, x^*(t)) = \beta^*(t)\delta(x \mid Q_{h^*(t)}(x^*(t))),\ x \in \Omega,\ t \in T_\varepsilon$$

generated by the above parameters ensures strong observability for the system (8.1), (8.2) under

$$\max \{\mid \eta(t) \mid \mid t \in T_\varepsilon\} \ \leq\ 1. \tag{8.7}$$

Consider any element $u^*(\cdot) \in U(\theta, \{0\})$ generated by a solution $u^*(x,t)$ so that $u^*(x,\theta) = u^*(x)$. Select $u_{i_*}(\cdot,\cdot) \in U_g^\gamma$ such that estimates (8.5) do hold.

Note that for $i = 1, 2, 3, \ldots$

$$\operatorname*{vrai\,min}_{x\in\Omega} v(x)\ \leq \int\limits_{Q_{h_i}(x^{(i)})} \beta_i v(x)dx\ \leq\ \operatorname*{vrai\,max}_{x\in\Omega}\ v(x),\ \forall v(\cdot) \in L_\infty(\Omega).$$

Therefore one can obtain

$$\mid \int\limits_\Omega \chi^*(x, x^*(t_{i_*}))(u_{i_*}(x, t_{i_*}) - u^*(x, t_{i_*}))dx \mid \leq\ \operatorname*{vrai\,max}_{x\in\Omega} \mid u_{i_*}(x, t_{i_*}) - u^*(x, t_{i_*}) \mid .$$

Then, due to (8.3)

$$\mid \int\limits_\Omega \chi^*(x, x^*(t_{i_*}))u_{i_*}(x, t_{i_*})dx \mid \leq \mid \int\limits_\Omega \chi^*(x, x^*(t_{i_*}))u^*(x, t_{i_*})dx \mid + L(\varepsilon)\gamma. \tag{8.8}$$

Due to the generalized maximum principle (8.4) and also (8.3), (8.5) - (8.8) we come to the estimate

$$\operatorname*{vrai\,max}_{x\in\Omega} \mid u^*(x,\theta) \mid \leq\ c \operatorname*{vrai\,max}_{x\in\Omega} \mid u^*(x, t_{i_*}) \mid \leq$$

$$\leq c(\operatorname*{vrai\,max}_{x\in\Omega} \mid u_{i_*}(x, t_{i_*}) \mid + L(\varepsilon)\gamma) \leq c(1 + 2L(\varepsilon)\gamma + \nu).$$

Theorem 8.1. There exists a spatially averaged nonstationary observation operator (a "scanning" sensor) of type (2.2) that ensures strong observability for the system (8.1), (8.2), (8.7). The respective weight function $\chi^*(x, x^*(t))$ may be chosen continuous excluding the only point of T_ε.

9. The Informational Domain: An Ellipsoidal Case

Assume the set $\mathbf{W}$ to be defined by a quadratic inequality

$$\mathbf{W} = \{(u_0(\cdot),\, f(\cdot,\cdot),\, v(\cdot,\cdot),\, \eta(\cdot)) \mid < u_0(\cdot) - \bar{u}_0(\cdot),\, \mathbf{I}_0(u_0(\cdot) - \bar{u}_0(\cdot)) > +$$

$$+ < f(\cdot,\cdot) - \bar{f}(\cdot,\cdot),\, \mathbf{I}_1(f(\cdot,\cdot) - \bar{f}(\cdot,\cdot)) > + < v(\cdot,\cdot) - \bar{v}(\cdot,\cdot), \mathbf{I}_2(v(\cdot,\cdot) - \bar{v}(\cdot,\cdot)) > +$$

$$+ < \eta(\cdot) - \bar{\eta}(\cdot), \mathbf{N}(\eta(\cdot) - \bar{\eta}(\cdot)) > \leq 1\}, \tag{9.1}$$

where the operators I_i and the scalar products in the respective Hilbert spaces $L_2(\Omega), L_2(Q), L_2(\Sigma)$ are defined as

$$< \varphi_1(\cdot),\, \mathbf{I}_0\varphi_2(\cdot) > = \int_\Omega \varphi_1(x)m(x)\varphi_2(x)dx,$$

$$< \psi_1(\cdot,\cdot),\, \mathbf{I}_1\psi_2(\cdot,\cdot) > = \int_Q \psi_1(x,t)k(x,t)\psi_2(x,t)\, dx\, dt,$$

$$< v_1(\cdot,\cdot),\, \mathbf{I}_2\, v_2(\cdot,\cdot) > = \int_\Sigma v_1(\xi,t)n(\xi,t)v_2(\xi,t)d\xi dt,$$

$$< \eta_1(\cdot),\, \mathbf{N}\eta_2(\cdot) > = \int_\epsilon^\theta \eta_1'(t)N(t)\eta(t)\, dt,$$

with continuous functions $m(x)$, $k(x,t)$, $n(\xi,t)$ and the symmetric matrix $N(t)$ being given in advance and such that

$$\min_{\substack{x\in\bar{\Omega},t\in[\epsilon,\theta]\\ \xi\in\partial\Omega}} \{m(x), k(x,t), n(\xi,t)\} > 0,\ \min_{t\in[\epsilon,\theta]} l'N(t)l \geq c \parallel l \parallel,$$

$$c = \text{const} > 0, \text{for any } l \in R^m.$$

The set $\mathbf{W}$ is convex and weakly compact in the Hilbert space $\hat{H} = L_2(\Omega) \times L_2(Q) \times L_2(\Sigma) \times L_2^m(T_\epsilon)$. Therefore the respective informational domain $U(\theta, y(\cdot))$ will be convex and weakly compact in $L_2(\Omega)$.

It is well-known that the solution to the problem (1.1), (1.2) allows a unique representation as

$$u(\cdot,t) = \mathbf{S}_0(t)u_0(\cdot) + \mathbf{S}_1(t)f(\cdot,\cdot) + \mathbf{S}_2(t)v(\cdot,\cdot), \tag{9.2}$$

where the operator $\mathbf{S}_0(t)$ coinsides with $\mathbf{S}(t)$ from Section 6,

$$\mathbf{S}_0(t) \;:\; L_2(\Omega) \to L_2(\Omega),\; \mathbf{S}_0(t)u_0 = \sum_{i=1}^{\infty} e^{-\lambda_i t} < u_0(\cdot),\, \omega_i(\cdot) > \omega_i(x),$$

$$\mathbf{S}_1(t) \;:\; L_2(Q) \to L_2(\Omega),\; \mathbf{S}_1(t)f(\cdot,\cdot) = \sum_{i=1}^{\infty} \int_0^t e^{-\lambda_i(t-\tau)} < f(\cdot,\tau), \omega_i(\cdot) > d\tau\omega_i(x),$$

$$\mathbf{S}_2(t) \;:\; L_2(\Sigma) \to L_2(\Omega),\; \mathbf{S}_2(t)v(\cdot,\cdot) = \sum_{i=1}^{\infty} \int_0^t e^{-\lambda_i(t-\tau)} < v(\cdot,\tau), \omega_i(\cdot) > d\tau\omega_i(x),$$

and $\{\lambda_i\}_{i=1}^{\infty}$, $\{\omega_i(x)\}_{i=1}^{\infty}$ are here the eigenvalues and the eigenfunctions for the elliptic operator A under the homogeneous boundary condition of type (1.2).

For simplicity we will restrict ourselves below to the case of the observation operators A, B. In the case of pointwise operator C adjout operators should be interpreted along the conventional lines of the theory of respective Sobolev spaces.

Due to (9.2) the measurement equation (1.4) could be written as

$$y(t) = \mathbf{G}(t)\,\mathbf{S}_0(t)\,u_0(\cdot) + \mathbf{G}(t)\,\mathbf{S}_1(t)f(\cdot,\cdot) + \mathbf{G}(t)\mathbf{S}_2(t)\,v(\cdot,\cdot) + \eta(t),\; t \in T_\varepsilon, \tag{9.3}$$

$$\mathbf{G}(t) \;:\; L_2(\Omega) \to R^m,\; \mathbf{G}(\cdot) \;:\; L_2(Q) \to L_2^m(T_\varepsilon).$$

Therefore the informational domain $U(\theta, y(\cdot))$ is the reachable set at time θ for the system (9.2) under constraints (9.3) and (9.1).

Theorem 9.1. The informational domain $U(\theta, y(\cdot))$ for the estimation problem (1.1), (1.2), (1.4), (9.1) is an ellipsoid in the space $L_2(\Omega)$:

$$U(\theta, y(\cdot)) = \{u(x) \,|< u(\cdot) - u^{0*}(\cdot,\theta),\, \hat{\mathbf{P}}^{-1}(\theta)(u(\cdot) - u^{0*}(\cdot,\theta)) > \le 1 - h^2(\theta)\}, \tag{9.4}$$

where

$$\hat{\mathbf{P}}(\theta) = \mathbf{P}(\theta) - \mathbf{B}(\theta),$$

$$\mathbf{P}(\theta) : L_2(\Omega) \to L_2(\Omega),\ \mathbf{P}(\theta)\varphi(\cdot) = \sum_{i=0}^{2} \mathbf{S}_i(\theta)\mathbf{I}_i^{-1}\mathbf{S}_i^*(\theta)\varphi(\cdot), \tag{9.5}$$

$$\mathbf{B}(\theta) : L_2(\Omega) \to L_2(\Omega),\ \mathbf{B}(\theta)\varphi(\cdot) = \mathbf{F}^*(\theta)\mathbf{L}^{-1}\mathbf{F}(\theta)\varphi(\cdot), \tag{9.6}$$

$$u^{0*}(\cdot,\theta) = \bar{u}(\cdot,\theta) + u^0(\cdot,\theta),\ u^0(\cdot,\theta) = \mathbf{F}^*(\theta)\hat{y}(\cdot), \tag{9.7}$$

$$\bar{u}(\cdot,t) = \mathbf{S}_0(t)\bar{u}_0(\cdot) + \mathbf{S}_1(t)\bar{f}(\cdot,\cdot) + \mathbf{S}_2(t)\bar{v}(\cdot,\cdot),\ \ t \in T,$$

$$\bar{y}(t) = y(t) - \mathbf{G}(t)\bar{u}(\cdot,t) - \bar{\eta}(t),\ \ t \in T_\varepsilon,$$

$$\mathbf{F}(\theta) : L^2(\Omega) \to L_2^m(T_\varepsilon),\ \mathbf{F}(\theta)\varphi(\cdot) = \sum_{i=0}^{2} \mathbf{G}(\cdot)\mathbf{S}_i(\cdot)\mathbf{I}_i^{-1}\ \mathbf{S}_i^*(\theta)\varphi(\cdot), \tag{9.8}$$

$$\mathbf{K} : L_2^m(T_\varepsilon) \to L_2^m(T_\varepsilon),\ \mathbf{K}\lambda(\cdot) = \sum_{i=0}^{2} \mathbf{G}(\cdot)\mathbf{S}_i(\cdot)\mathbf{I}_i^{-1}\mathbf{S}_i^*(\cdot)\mathbf{G}^*(\cdot)\lambda(\cdot),$$

$$\hat{y}(\cdot) = \mathbf{L}^{-1}\bar{y}(\cdot),\ \mathbf{L} : L_2^m(T_\varepsilon) \to L_2^m(T_\varepsilon),\ \mathbf{L} = \mathbf{N}^{-1} + \mathbf{K},$$

$$h^2(\theta) = < \hat{y}(\cdot), \bar{y}(\cdot) > . \tag{9.9}$$

Proof. The brief scheme of the proof of Theorem 9.1 can be done as follows.

Due to criterion of the consistency of the system of inequalities (Kurzhanski, 1977) the set of the operator equations (9.2), (9.3) is consistent with constraint (9.1) iff the inequality

$$\max_{\mathbf{W}} L(u_0(\cdot), f(\cdot,\cdot), v(\cdot,\cdot), \eta(\cdot), \lambda(\cdot), \varphi(\cdot)) - < u(\cdot,\theta), \varphi(\cdot) > \geq 0 \tag{9.10}$$

does hold for any $\lambda(\cdot) \in L_2^m(\mathbf{T}_\varepsilon)$, $\varphi(\cdot) \in \mathbf{\Phi}$, where

$$L(u_0(\cdot), f(\cdot,\cdot), v(\cdot,\cdot), \eta(\cdot), \lambda(\cdot), \varphi(\cdot)) = < \lambda(\cdot), y(\cdot) > +$$

$$+ < \mathbf{S}_0^*(\theta)\varphi(\cdot) - \mathbf{S}_0^*(\cdot)\mathbf{G}^*(\cdot)\lambda(\cdot), u_0(\cdot) > + < \mathbf{S}_1^*(\theta)\varphi(\cdot) - \mathbf{S}_1^*(\cdot)\mathbf{G}^*(\cdot)\lambda(\cdot), f(\cdot,\cdot) > +$$

$$+ < \mathbf{S}_2^*(\theta)\varphi(\cdot) - \mathbf{S}_2^*(\cdot)\mathbf{G}^*(\cdot)\lambda(\cdot),\ v(\cdot,\cdot) > - < \lambda(\cdot), \eta(\cdot) > .$$

Calculating the maximum in (9.10), after a number of transformations we come to the formula of support function for the set $U(\theta, y(\cdot))$:

$$\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))) = \inf_{\lambda(\cdot) \in L_2^m(T_\epsilon)} \{< \lambda(\cdot), \bar{y}(\cdot) > + < \varphi(\cdot), \bar{u}(\cdot, \theta) > + \tag{9.11}$$

$$+(< \lambda(\cdot), \mathbf{N}^{-1}\lambda(\cdot) > +$$

$$+ < \lambda(\cdot), \mathbf{K}\lambda(\cdot) > -2 < \lambda(\cdot), \mathbf{F}(\theta)\varphi(\cdot) > + < \varphi(\cdot), \mathbf{P}(\theta)\varphi(\cdot) >)^{1/2}\},$$

for an arbitrary element $\varphi(\cdot) \in \mathbf{\Phi}$.

The calculation of the infimum in the latter relation leads to Theorem 9.1.

From above it follows that a consequence of Theorem 9.1 is that $\mathbf{F}(\theta)$ and $\mathbf{B}(\theta)$ are integral operators and that $h^2(\theta) \in [0, 1]$.

Lemma 9.1. The support function $\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))) = \sup\{< \varphi(\cdot), u(\cdot, \theta) > \mid u(\cdot, \theta) \in U(\theta, y(\cdot))\}$ is given by

$$\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))) = (1 - h^2(\theta))^{1/2} < \varphi(\cdot), \hat{\mathbf{P}}(\theta)\varphi(\cdot) >^{1/2} + < \varphi(\cdot), u^{0*}(\cdot, \theta) >$$

being defined for any element $\varphi(\cdot) \in \mathbf{\Phi} \subseteq L_2(\Omega)$.

A specific question that arises here is how to describe the *best* and the *worst* measurements $y(\cdot)$ which could be defined as such that the domain $U(\theta, y(\cdot))$ would be either the "smallest" or the "largest" possible. Observing that operator $\hat{\mathbf{P}}(\theta)$ does not depend upon $y(\cdot)$, one may reduce the problem to finding the measurements $y(\cdot)$ for which the parameter $h^2(\theta)$ would be equal either to 1 (the case when $U(\theta, y(\cdot))$ is a singleton) or to zero (this gives the "largest" $U(\theta, y(\cdot))$ with respect to the inclusion).

The answer to the problem is given by the following two propositions:

Lemma 9.2. The "worst case" measurement $y(t) = \bar{y}(t), t \in T_\epsilon$ is the one generated by the set $\bar{w}(\cdot) = \{\bar{u}_0(\cdot),\ \bar{f}(\cdot, \cdot),\ \bar{v}(\cdot, \cdot),\ \bar{\eta}(\cdot)\}$ due to equations (1.1), (1.2), (1.4). This ensures the existence for any feasible $y(t)$ of an element $\varphi^*(\cdot, y(\cdot)) \in L_2(\Omega)$ such that

$$\rho(\varphi(\cdot) \mid U(\theta, \bar{y}(\cdot))) \geq <\varphi(\cdot), \varphi^*(\cdot, y(\cdot))> + \rho(\varphi(\cdot) \mid U(\theta, y(\cdot))), \forall \varphi(\cdot) \in \Phi.$$

In other words the "worst case" $\bar{y}(\cdot)$ is such that for any other $y(\cdot)$ the domain $U(\theta, y(\cdot))$ could be shifted (by $\varphi^*(\cdot, y(\cdot))$) so that it would lie entirely within $U(\theta, \bar{y}(\cdot))$.

An example of the best measurement where $U(\theta, y(\cdot))$ reduces to a singleton could be as follows.

Suppose that the initial value $u_0(x)$ is the only uncertainty in the system (1.1), (1.2) and that the inputs for $f(x,t)$ and $v(\xi,t)$ are given and such that $f(x,t) \equiv \bar{f}(x,t)$, $v(\xi,t) = \bar{v}(\xi,t)$. Therefore we can put

$$w(\cdot) = \{u_0(\cdot), \bar{f}(x,t), \bar{v}(\xi,t), \eta(t)\}$$

and denote

$$Y = \{y(\cdot) \mid y(t) = \mathbf{G}(t)\, \mathbf{S}_0(t) u_0(\cdot),\ t \in T_\varepsilon,\ u_0(\cdot) \in L_2(\Omega)\}.$$

An arbitrary element of $L_2^m(T_\varepsilon)$ could be represented as

$$y(\cdot) = y_Y(\cdot) + y(\cdot)^\perp,$$

where

$$y_{Y(\cdot)} \in Y \text{ and } < y_Y(\cdot), \mathbf{N} y^\perp(\cdot) >= 0.$$

Lemma 9.3 Assume that the available observation $y(\cdot) = \tilde{y}(\cdot)$ *is such that*

$$< \tilde{y}^{*\perp}(\cdot), \mathbf{N}\tilde{y}^{*\perp}(\cdot) >= 1,$$

where

$$\tilde{y}^*(\cdot) = \tilde{y}(\cdot) - \mathbf{G}(\cdot)\mathbf{S}_1(\cdot)\bar{f}(\cdot,\cdot) - \mathbf{G}(\cdot)\mathbf{S}_2\bar{v}(\cdot,\cdot).$$

Then the set $U(\theta, \tilde{y}(\cdot))$ *is a singleton.*

In other words, here the whole "resource" assigned to the error $\eta(t)$ is completely "spent" on producing $\tilde{y}^{*\perp}(\cdot)$ which is orthogonal to Y.

Remark 9-a. Assume now that the set $\mathbf{W}$ is unbounded with respect to the initial value $u_0(\cdot)$ and that

$$\mathbf{W} = \{(u_0(\cdot), f(\cdot,\cdot), v(\cdot,\cdot), \eta(\cdot)) \mid u_0(\cdot) \in L_2(\Omega), < f(\cdot,\cdot)) - \bar{f}(\cdot,\cdot), \mathbf{I}_1(f(\cdot,\cdot) - \bar{f}(\cdot,\cdot)) > +$$

$$+ < v(\cdot,\cdot) - \bar{v}(\cdot,\cdot), \mathbf{I}_2(v(\cdot,\cdot) - \bar{v}(\cdot,\cdot)) > + < \eta(\cdot) - \bar{\eta}(\cdot), \mathbf{N}(\eta(\cdot) - \bar{\eta}(\cdot)) > \leq 1\}. \qquad (9.12)$$

Under constraint (9.12) the informational domain $U(\theta, y(\cdot))$ is a convex, but in general, a non-closed unbounded set in $L_2(\Omega)$. Nevertheless, the relations given by Theorem 9.1 allow to derive some formulae for its approximating.

10. Evolution Equations: The Ellipsoidal Case

In this section we consider the dynamic guaranteed estimation problem (1.1), (1.2), (1.4), (9.1) with dependence on the measurement interval. From the theorem 9.1 it follows that the domain $U(\theta, y(\cdot))$ can be completely described by its parameters $h^2(\theta)$, $\mathbf{P}(\theta), \mathbf{B}(\theta)$, $u^{0*}(\cdot, \theta)$. We therefore proceed to specify the evolution of these parameters in time.

Denote $q(x,t,\theta)$ and $b(x,y,\theta)$ to be the kernels of the operators $F(\theta)$ and $B(\theta)$ respectively, so that

$$\mathbf{F}(\theta)\varphi(\cdot) = \int_\Omega q(x,t,\theta)\varphi(x)dx,$$

$$\mathbf{B}(\theta)\varphi(\cdot) = \int_\Omega b(x,y,\theta)\varphi(y)dy.$$

Then

$$q(x,t,\theta) = \sum_{i=1}^{\infty} e^{-\lambda_i\theta}\Big(\int_\Omega \Big(\sum_{j=1}^{\infty} e^{-\lambda_j t}(\mathbf{G}(t)\omega_j(\cdot))\omega_i(z)\Big)\, m^{-1}(z)\omega_i(z)dz\omega_i(x) + \qquad (10.1)$$

$$+ \sum_{i=1}^{\infty}\Big(\int_{Q_\epsilon}\Big(\Big(\sum_{j=1}^{\infty} e^{-\lambda_j(t-\tau)}(\mathbf{G}(t)\omega_j(\cdot))\omega_j(z)\Big)e^{-\lambda_i(\theta-\tau)}k^{-1}(z,\tau)\omega_i(z)\Big)dzd\tau\Big)\omega_i(x)$$

$$+ \sum_{i=1}^{\infty}\Big(\int_{\Sigma_\epsilon}\Big(\Big(\sum_{j=1}^{\infty} e^{-\lambda_j(t-\tau)}(\mathbf{G}(t)\omega_j(\cdot))\omega_j(\xi)\Big)e^{-\lambda_i(\theta-\tau)}n^{-1}(\xi,\tau)\omega_i(\xi)\Big)d\xi d\tau\Big)\omega_i(x),$$

$$Q_\varepsilon = \Omega \times T_\varepsilon,\ \Sigma_\varepsilon = \partial\Omega \times T_\varepsilon,$$

$$b(x,y,\theta) = \int_\varepsilon^\theta q'(x,t,\theta)\hat{q}(y,t,\theta)dt, \tag{10.2}$$

$$u^{0*}(x,\theta) = \bar{u}(x,\theta) + \int_\varepsilon^\theta \hat{q}'(x,t,\theta)\bar{y}(t)dt, \tag{10.3}$$

where the function $\hat{q}(x,t,\theta)$ is a unique solution to the following integral equation

$$N^{-1}(t)\hat{q}(x,t,\theta) + \int_\varepsilon^\theta \hat{K}(t,\tau)\hat{q}(x,\tau,\theta)d\tau = q(x,t,\theta), \tag{10.4}$$

$\hat{K}(t,\tau)$ is a non-negative kernel of the operator $\mathbf{K}$.

Using the Schwarz inequality and the equivalence (Sobolev, 1982; Ladyzhenskaya and others, 1968; Lions, 1968) of the usual norm in the Sobolev space $H^1(\Omega)$ and the norm (for simplicity we can put $a(x) > 0, c(\xi) \neq 0$)

$$d(\varphi(\cdot)) = (\int_\Omega (\sum_{i,j=1}^n a_{ij}(x)\frac{\partial\varphi(x)}{\partial x_i}\frac{\partial\varphi(x)}{\partial x_j} + a(x)\varphi^2(x))dx + \int_{\partial\Omega} c(\xi)\varphi^2(\xi)d\xi)^{1/2},$$

one can observe

$$u^{0*}(x,\theta) \in H^{1,0}(\Omega \times (\varepsilon,\Theta)),$$

$$b(x,y,\theta) \in H^{1,1,0}(\Omega \times \Omega \times (\varepsilon,\Theta))$$

for arbitrary interval (ε,Θ) (where the parameters used in (1.1), (1.4), (9.1) can be defined), $\Theta > \varepsilon$, where

$$H^{1,1,0}(\Omega \times \Omega \times (\varepsilon,\Theta)) = \{\varphi \mid \varphi, \frac{\partial\varphi}{\partial x_i}, \frac{\partial\varphi}{\partial y_i} \in L_2(\Omega \times \Omega \times (\varepsilon,\Theta))\}.$$

Formulae (10.2), (10.3) lead us to the following system of partial differential equations for functions $u^{0*}(x,\theta)$ and $b(x,y,\theta)$:

$$\frac{\partial u^{0*}(x,\theta)}{\partial\theta} = Au^{0*}(\cdot,\theta) + \chi_\varepsilon(\theta)(y(\theta) - \mathbf{G}(\theta)u^{0*}(\cdot,\theta) - \bar{\eta}(\theta))'N(\theta)\times \tag{10.5}$$

$$\times(q(x,\theta,\theta)-\mathbf{G}(\theta)b(x,\cdot,\theta))+\bar{f}(x,\theta),\;\; x\in\Omega,\; \theta\in(0,\Theta),$$

$$\frac{\partial u^{0*}(\xi,\theta)}{\partial n_A}+c(\xi)u^{0*}(\xi,\theta)=\bar{v}(\xi,\theta),\;\; \xi\in\partial\Omega,$$

$$u^{0*}(x,0)=\bar{u}_0(x),$$

$$\frac{\partial b(x,y,\theta)}{\partial\theta}=\hat{A}b(\cdot,\cdot,\theta)+\chi_\varepsilon(\theta)(q(x,\theta,\theta)-\mathbf{G}(\theta)b(x,\cdot,\theta))'\times \tag{10.6}$$

$$\times N(\theta)(q(y,\theta,\theta)-\mathbf{G}(\theta)b(\cdot,y,\theta)),\; x,y\in\Omega,\; \theta\in(0,\Theta),$$

$$b(x,y,0)=0,\frac{\partial b(\xi,y,\theta)}{\partial n_A}+c(\xi)b(\xi,y,\theta)=0,$$

$$\frac{\partial b(x,\xi,\theta)}{\partial n_A}+c(\xi)b(x,\xi,\theta)=0,\quad \xi\in\partial\Omega,$$

where

$$\hat{A}=\sum_{i,j=1}^{n}\frac{\partial}{\partial x_i}(a_{ij}(y)\frac{\partial}{\partial y_i})+\frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial y_j})-a(x)-a(y),$$

$$\chi_\varepsilon(\theta)=\begin{cases}0, & 0<\theta<\varepsilon,\\ 1, & \varepsilon\le\theta<\Theta.\end{cases}$$

We give here the brief formal scheme for the derivaiton of equations (10.5) (which may be strictly justified on the basis of Galerkin's method).

Differentiating formally the relation (9.7) for $u^0(\cdot,\theta)$ with respect to θ one can obtain

$$\frac{\partial u^0(\cdot,\theta)}{\partial\theta}=\frac{\partial}{\partial\theta}(\int_\varepsilon^\theta \hat{y}\prime(t,\theta)q(x,t,\theta)dt)=$$

$$=\hat{y}\prime(\theta,\theta)N^{-1}(\theta)\hat{q}(x,\theta,\theta)+\int_\varepsilon^\theta \hat{y}\prime(t,\theta)N^{-1}(t)\frac{\partial\hat{q}(x,t,\theta)}{\partial\theta}dt+$$

$$+\int_\varepsilon^\theta\int_\varepsilon^\theta \hat{y}\prime(t,\theta)\hat{K}(t,\tau)\frac{\partial\hat{q}(x,\tau,\theta)}{\partial\theta}d\tau dt+\int_\varepsilon^\theta\frac{\partial\hat{y}\prime(t,\theta)}{\partial\theta}N^{-1}(t)\hat{q}(x,t,\theta)dt+$$

$$+\int_{\varepsilon}^{\theta}\int_{\varepsilon}^{\theta}\frac{\partial\hat{y}\prime(t,\theta)}{\partial\theta}\hat{K}(t,\tau)\hat{q}(x,\tau,\theta)d\tau dt+\hat{y}\prime(\theta,\theta)\int_{\varepsilon}^{\theta}\hat{K}(\theta,\tau)\hat{q}(x,\tau,\theta)d\tau+$$

$$+\int_{\varepsilon}^{\theta}\hat{y}(t,\theta)\hat{K}(t,\theta)\hat{q}(x,\theta,\theta)dt,$$

where

$$N^{-1}(t)\hat{y}(t,\theta)+\int_{\varepsilon}^{\theta}\hat{K}(t,\tau)\hat{y}(\tau,\theta)d\tau=\bar{y}(t). \tag{10.7}$$

Then taking into account that $\hat{K}(\theta,t)$ is a kernel of the integral operator $\mathbf{F}(\theta)\mathbf{G}^*(\theta)$, formula (10.3) and

$$\hat{y}(\theta,\theta)=N(\theta)(\bar{y}(\theta)-\mathbf{G}(\theta)u^0(\cdot,\theta)), \tag{10.8}$$

$$\hat{q}(x,\theta,\theta)=N(\theta)(q(x,\theta,\theta)-\mathbf{G}(\theta)b(x,\cdot,\theta)), \tag{10.9}$$

we come to the mixed problem (10.5).

From (10.9) there follows an ordinary differential equation for $h^2(\theta)$

$$\frac{dh^2(\theta)}{d\theta}=(y(\theta)-\mathbf{G}(\theta)u^{0*}(\cdot,\theta)-\bar{\eta}(\theta))'N(\theta)(y(\theta)- \tag{10.10}$$

$$-\mathbf{G}(\theta)u^{0*}(\cdot,\theta)-\bar{\eta}(\theta)),\quad \theta\in(\varepsilon,\Theta),$$

$$h^2(\varepsilon)=0.$$

The operator $\mathbf{P}(\theta)$ does not depend upon any measurements and as it follows from (9.5) it describes the structure of the reachable set of the system (1.1), (1.2), (9.1) in the absence of the measurement equation (and measurement "noise" $\eta(t)$ in (9.1) in particular). The operator $\mathbf{B}(\theta)$ and scalar $h^2(\theta)$ describe the correction of the latter set due to the estimation process.

Theorem 10.1. The evolution in θ of the informational domain $U(\theta,y(\cdot))$ for the estimation problem (1.1), (1.2), (1.4), (9.1) is given by the joint system for the initial boundary value problems (10.5), (10.6), the ordinary differential equation (10.10) and the formula (9.5).

Remark 10-a. The solutions to the initial boundary value problems (10.5) and (10.6) are treated as generalized solutions in the sense of the corresponding integral identities (see (1.3)).

In the case of the integral quadratic constraint of general (operator) type the mixed problem (10.6) should be modified in the form of a respective differential equation for the operator $\mathrm{B}(\theta)$.

11. The Informational Domain: Finite-Dimensional Outputs

In this paragraph we will consider a particular case of the problem (1.1), (1.2), (1.4), (9.1) when the aim is to estimate a *finite-dimensional output* of the system.

We therefore introduce the ***estimation problem in finite-dimensional outputs*** which is to determine the set of all elements

$$z(\cdot) = \mathbf{H}u(\cdot,\theta)$$

that are defined at instant θ being consistent with the system (1.1), (1.2), the measurement data $y(t), t \in T_\varepsilon$ and the constraint (9.1), the linear operator $\mathbf{H}$ being given:

$$\mathbf{H} \, : \, L_2(\Omega) \rightarrow R(\mathbf{H}), \dim R(\mathbf{H}) < \infty.$$

The informational domain $Z(\theta, y(\cdot))$ for the latter problem is the projection of the respective set $U(\theta, y(\cdot))$ on the subspace $R(\mathbf{H})$:

$$Z(\theta, y(\cdot)) = \mathbf{H}U(\theta, y(\cdot)).$$

Therefore, due to Lemma 9.1, we come to

$$\rho(\varphi(\cdot) \mid Z(\theta, y(\cdot))) = (1 - h^2(\theta))^{1/2} < \varphi(\cdot), \hat{\mathbf{P}}(\theta)\varphi(\cdot) >^{1/2} + < \varphi(\cdot), u^{0*}(\cdot,\theta) > \qquad (11.1)$$

for any $\varphi(\cdot) \in R(H)$.

Consider $\mathbf{H}$ to be the operator $\mathbf{\Pi}_r$ of orthogonal projection on an arbitrary subspace $X_r(\Omega)$:

$$\mathbf{\Pi}_r \, : \, L_2(\Omega) \rightarrow X_r(\Omega),$$

$$\mathbf{\Pi}_r v(\cdot) = \omega(\cdot), \omega(x) = \sum_{i=1}^{r} < v(\cdot), \omega_i(\cdot) > \omega_i(x).$$

The respective set $Z(\theta, y(\cdot))$ will then be denoted as $Z_r(\theta, y(\cdot))$.

Assume that the boundary value $v(\xi, t)$ is given:

$$v(\xi,t) \equiv \bar{v}(\xi,t), (\xi,t) \in \Sigma\,, \tag{11.2}$$

and that the operators $\mathbf{I}_i (i = 0, 1)$ in the constraint (9.1) are identities.

Along the scheme of (9.2) - (9.11), (10.1) - (10.4) one can obtain the following formulae for the parameters of $Z_r(\theta, y(\cdot))$ which are all further marked by a lower index "r" and which are represented through the parameters and functions specified in (9.5) - (9.9) and (10.1) - (10.4):

$$\hat{\mathbf{P}}_r(\theta) = \mathbf{\Pi}_r \hat{\mathbf{P}}(\theta)\mathbf{\Pi}_r,\ \mathbf{\Pi}_r^* = \mathbf{\Pi}_r,$$

$$\mathbf{P}_r(\theta) = \sum_{i=1}^{2} \mathbf{\Pi}_r\ \mathbf{S}_i(\theta)\mathbf{S}_i^*(\theta)\mathbf{\Pi}_r\varphi(\cdot), \tag{11.3}$$

$$\mathbf{B}_r(\theta) = \mathbf{\Pi}_r\mathbf{B}(\theta)\mathbf{\Pi}_r, \quad \mathbf{F}_r(\theta) = \mathbf{F}(\theta)\mathbf{\Pi}_r,$$

$$\mathbf{B}_r(\theta)\varphi(\cdot) = \int\limits_{\Omega} b_r(x,y,\theta)\varphi(y)dy,$$

$$z_r^{0*}(\cdot,\theta) = \mathbf{\Pi}_r u^{0*}(\cdot,\theta), \quad \hat{q}_r(\cdot,t,\theta) = \mathbf{\Pi}_r\hat{q}(\cdot,t,\theta),$$

$$h_r^2(\theta) = h^2(\theta), \quad q_r(\cdot,t,\theta) = \mathbf{\Pi}_r q(\cdot,t,\theta).$$

Moreover, (10.8) and (10.9) could be modified for values $\hat{y}_r(\theta,\theta)$, $\hat{q}_r(x,\theta,\theta)$ so as to yield

$$\hat{y}_r(\theta,\theta) = N(\theta)(\bar{y}_r^*(\theta) - \mathbf{G}(\theta)z_r^0(\cdot,\theta)), \tag{11.4}$$

$$\hat{q}_r(x,\theta,\theta) = N(\theta)(q_r^*(x,\theta,\theta) - \mathbf{G}(\theta)b_r(x,\cdot,\theta)). \tag{11.5}$$

Here

$$z^0(\cdot,\theta) = \int\limits_{\varepsilon}^{\theta} \hat{q}_r'(x,t,\theta)\,\bar{y}(t)dt,$$

$$b_r(x,y,\theta) = \int\limits_{\varepsilon}^{\theta} q_r'\ (x,t,\theta)\,\hat{q}_r(x,t,\theta)dt,$$

$$\bar{y}_r^*(\theta) = \bar{y}(\theta) - \bar{y}_r(\theta,\theta), \quad \bar{y}_r(\cdot,\theta) = \mathbf{K}_r\hat{y}(\cdot,\theta),$$

$$q_r^*(x,\theta,\theta) = q_r(x,\theta,\theta) - \bar{q}_r(x,\theta,\theta), \;\; \bar{q}_r(\cdot,\cdot,\theta) = \mathbf{K}_r\hat{q}(\cdot,\cdot,\theta),$$

$$\mathbf{K}_r = \sum_{i=1}^{2} \mathbf{G}(\cdot)\mathbf{\Pi}_r^{\perp}\mathbf{S}_i(\cdot)\mathbf{S}_i^*(\cdot)\mathbf{\Pi}_r^{\perp}\mathbf{G}^*(\cdot),$$

$\mathbf{\Pi}_r^{\perp}$ is the operator of orthogonal projection on $X_r^{\perp}(\Omega)$ $(\mathbf{\Pi}_r^{\perp}v(\cdot) = v(\cdot) - \mathbf{\Pi}_r v(\cdot))$.

On the basis of relations (11.1) - (11.5), similarly the proofs of the theorems 9.1, 10.1 we obtain

Theorem 11.1. The informational domain $Z_r(\theta, y(\cdot))$ is an ellipsoid in the finite-dimensional subspace $X_r(\Omega)$ of the space $L_2(\Omega)$:

$$Z_r(\theta, y(\cdot)) = \{z(x) \mid z(x) \in X_r(\Omega),$$

$$< z(\cdot) - z_r^{0*}(\cdot,\theta),\ \hat{\mathbf{P}}_r^{-1}(\theta)(z(\cdot) - z_r^{0*}(\cdot,\theta)) > \leq 1 - h^2(\theta)\}$$

with support function (11.1).

The evolution in θ of the parameters of the domain $U_r(\theta, y(\cdot))$ under condition (11.2) and identical operators $\mathbf{I}_i (i = 0,1)$ can be described for any r by formula (11.3) and by the following joint system of partial differential equations for initial-boundary value problems in the finite-dimensional subspace $X_r(\Omega)$ and of an ordinary differential equation for the value $h^2(\theta)$:

$$\frac{\partial z_r^{0*}(x,\theta)}{\partial\theta} = Az_r^{0*}(\cdot,\theta) + \chi_\epsilon(\theta)(y_r(\theta) - \mathbf{G}(\theta)z_r^{0*}(\cdot,\theta) - \bar{\eta}(\theta))'N(\theta)\times \tag{11.6}$$

$$\times(q_r^*(x,\theta,\theta) - \mathbf{G}(\theta)b_r(x,\cdot,\theta)) + \bar{f}_r(x,\theta), \;\; x \in \Omega, \;\; \theta \in (0,\Theta),$$

$$\frac{\partial z_r^{0*}(\xi,\theta)}{\partial n_A} + c(\xi)z_r^{0*}(\xi,\theta) = 0, \quad \xi \in \partial\Omega,$$

$$z_r^{0*}(\cdot,0) = \bar{z}_{0r}(\cdot) = \mathbf{\Pi}_r\bar{u}_0(\cdot), \quad \bar{f}_r(\cdot,\theta) = \mathbf{\Pi}_r\bar{f}(\cdot,\theta),$$

$$y_r(\theta) = y(\theta) - \bar{y}_r(\theta),$$

$$\frac{\partial b_r(x,y,\theta)}{\partial\theta} = \hat{A}(\cdot,\cdot,\theta) + \chi_\epsilon(\theta)(q_r^*(x,\theta,\theta) - \mathbf{G}(\theta)b_r(x,\cdot,\theta))'N(\theta)\times \tag{11.7}$$

$$\times\,(q_r^*(y,\theta,\theta)-\mathbf{G}(\theta)b_r(\cdot,y,\theta)),x,y\in\Omega,\ \theta\in(0,\Theta),$$

$$b_r(x,y,0)=0,\frac{\partial b_r(x,\xi,\theta)}{\partial n_A}+c(\xi)\,b(x,\xi,\theta)=0,$$

$$\frac{\partial b_r(\xi,y,\theta)}{\partial n_A}+c(\xi)\,b(\xi,y,\theta)=0,\ \ \xi\in\partial\Omega,$$

$$\frac{dh^2(\theta)}{d\theta}=(y_r(\theta)-\mathbf{G}(\theta)z_r^{0*}(\cdot,\theta)-\bar{\eta}(\theta))'N(\theta)(y_r(\theta)-\mathbf{G}^*(\theta)z_r^{0*}(\cdot,\theta)-\bar{\eta}(\theta)),\tag{11.8}$$

$$\theta\in(\varepsilon,\Theta),\ \ h^2(\varepsilon)=0.$$

The mixed problems (11.6) - (11.7) are finite-dimensional. Therefore, they may be reformulated through a system of ordinary differential equations.

Indeed, put

$$z_r^{0*}(x,\theta)=\sum_{i=1}^{r}z_{ir}^{0*}\ (\theta)\omega_i(x),\ \ z_r^{0*}[\theta]=\mathrm{col}[z_{1r}^{0*}(\theta),\ldots,z_{rr}^{0*}(\theta)],$$

$$b_r(x,y,\theta)=\sum_{i,j=1}^{r}b_{ijr}\ (\theta)\omega_i(x)\omega_j(y),\ \ B_r[\theta]=\{b_{ijr}(\theta)\}\ \text{ is a }\ [r\times r]-\ \text{matrix},$$

$$A_r=\mathrm{diag}\,\{-\lambda_1,\ldots,-\lambda_r\},\ \bar{f}_r[\theta]=\mathrm{col}\,[<\bar{f}(\cdot,\theta),\omega_1(\cdot)>,\ldots,<\bar{f}(\cdot,\theta),\omega_r(\cdot)>],$$

$$\bar{u}_{0r}=\mathrm{col}\,[<\bar{u}_0(\cdot),\omega_1(\cdot)>,\ldots,<\bar{u}_0(\cdot),\omega_r(\cdot)>],$$

$$G_r[\theta]=(\mathbf{G}(\theta)\omega_1(\cdot),\ldots,\mathbf{G}(\theta)\omega_r(\cdot)),$$

$$q_r^*(x,\theta,\theta)=\sum_{i=1}^{r}q_{ir}^*(\theta)\omega_i(x),\ \ Q_r^*[\theta]=\{q_{ir}^*(\theta)\}\ \text{ is a }\ [r\times m]-\text{matrix}.$$

Then the problems (11.6), (11.7) generate the system

$$\frac{dB_r[\theta]}{d\theta}=A_rB_r[\theta]+B_r[\theta]A_r+\chi_\varepsilon(\theta)(Q_r^*[\theta]-G_r[\theta]B_r[\theta])'\times$$

$$\times N(\theta)(Q_r^*[\theta] - G_r[\theta]B_r[\theta]),\ \theta \in (0, \Theta),$$

$$B_r[0] = \mathrm{diag}\{0, \ldots, 0\},$$

$$\frac{dz_r^{0*}[\theta]}{d\theta} = A_r z_r^{0*}[\theta] + \chi_\varepsilon(\theta)(y_r(\theta) - G_r(\theta)z_r^{0*}[\theta] - \bar{\eta}(\theta))' N(\theta) \times$$

$$\times\ (Q_r^*[\theta] - G_r[\theta]B_r[\theta]) + \bar{f}_r[\theta],\quad \theta \in (0, \Theta),$$

$$z_r^{0*}[0] = \bar{u}_{0r},$$

which should be treated together with

$$\frac{dh^2(\theta)}{d\theta} = (y_r(\theta) - G_r[\theta]z_r^{0*}[\theta] - \bar{\eta}(\theta))' N(\theta) \times$$

$$\times\ (y_r(\theta) - G_r[\theta]z_r^{0*}[\theta] - \bar{\eta}(\theta)),$$

$$h^2(\varepsilon) = 0,\ \ \theta \in (\varepsilon, \Theta)$$

and formula (11.3).

Remark 11-a. We have used ***square brackets*** above for the description of *finite-dimensional vectors* obtained through a truncation of respective infinite-dimensional elements. This type of notation will also be used below.

12. The Informational Domain: Instantaneous Constraints

Assume now that the unknown inputs $u_0(\cdot)$, $f(\cdot)$, $\eta(\cdot,\cdot)$ in the system (1.1), (4.1), (1.4) satisfy some preassigned constraints of an instantaneous type, namely

$$u_0(\cdot) \in U_0;\ f(\cdot, t) \in F(t),\ t \in T; \eta(t) \in \Delta(t), t \in T_\varepsilon, \tag{12.1}$$

where U_0 is a given weakly compact convex set in $L_2(\Omega)$; $F(t)$ is a continuous multivalued map from T into the set of convex weakly compact subsets of $L_2(\Omega)$; $\Delta(t)$ is a continuous multivalued map from T_ε into the set conv R^m of convex compact subsets of R^m and int $\Delta(t) \neq \phi$, $t \in T_\varepsilon$;

$\varepsilon < \theta \leq \Theta$. We will also restrict the equation (1.4) to the case of spatially averaged sensors A and B.

Due to formula (9.10), an arbitrary informational domain $U(\theta, y(\cdot))$ for the estimation problem (1.1), (1.2), (1.4), (12.1) is a closed convex and bounded subset of the space $L_2(\Omega)$. Its evolution in time may be described through the techniques of partial differential inclusions in Hilbert space.

The scheme for deriving appropriate inclusions is based on a limit transition along the results obtained for ordinary linear differential systems in (Kurzhanski, Filippova, 1989) for guaranteed estimation problems under instantaneous constraints.

Consider a sequence of infinite-dimensional informational domains $U_{(r)}(\theta, y(\cdot))$ that are the solutions to estimation problem (1.1), (4.1), (1.4), (12.1) under condition

$$\mathbf{G}(t) = \mathbf{G}_r(t), \ \mathbf{G}_r(t)u(\cdot,t) = \mathbf{G}(t)\mathbf{\Pi}_r u(\cdot,t), \tag{12.2}$$

where $\mathbf{\Pi}_r$ stands for the operator of orthogonal projection on an arbitrary subspace $X_r(\Omega)$ generated by first r eigenfunctions for the problem (4.6). We will investigate a limit transition for these with $r \to \infty$. In order to do that we introduce

Condition 12-a. We will say that the measurement output $y^*(t), t \in T_\varepsilon$ satisfies *the regularity condition of the constraint qualification type* if among all of the triplets that generate $y^*(t)$ due to (1.1), (4.1), (1.4), (12.1) there exists a triplet $\{u_0^*(\cdot), f^*(\cdot,\cdot), \eta^*(\cdot)\}$ that ensures

$$\eta^*(t) \in \text{ int } \Delta(t), \ t \in T_\varepsilon.$$

Lemma 12.1. Assume that the measurement output $y^*(t), t \in T_\varepsilon$ satisfies Condition 12-a. Then

$$d(U(\theta, y^*(\cdot)), U_{(r)}(\theta, y^*(\cdot))) \to 0 \textit{ with } r \to \infty. \tag{12.3}$$

Here $d(A_1, A_2)$ stands for the Hausdorff metric (Kuratowski, 1966) for the sets $A_1, A_2 \subset L_2(\Omega)$.

Proof. Due to the given assumptions all of the sets $U_{(r)}(\theta, y(\cdot))$ are nonvoid once r exceeds some value $r^* = r^*(y^*(\cdot))$.

We may split an arbitrary solution $u(x,t)$ to the system (1.1), (4.1) generated by pair $\{u_0(\cdot), f(\cdot,\cdot)\}$ into two terms:

$$u(\cdot,t) = u_r(\cdot,t) + u^r(\cdot,t),$$

So that

$$u_r(\cdot,t) \in X_r(\Omega), \ < u^r(\cdot,t), u_r(\cdot,t) > = \ 0.$$

Here $u_r(\cdot,t), u^r(\cdot,t)$ are solutions to (1.1), (4.1) generated respectively by the pairs

$$\{u_{0r}(\cdot), f_r(\cdot,\cdot)\}, \ \{u_0^r(\cdot), f^r(\cdot,\cdot)\},$$

so that

$$u_{0r}(\cdot) = \mathbf{\Pi}_r u_0(\cdot), \ u_0^r(\cdot) = \mathbf{\Pi}_r^{\perp} u_0(\cdot),$$

$$f_r(\cdot,t) = \mathbf{\Pi}_r f(\cdot,t), \ f^r(\cdot,t) = \mathbf{\Pi}_r^{\perp} f(\cdot,t), \ t \in T.$$

Due to formula (9.2) we have

$$\| u^r(\cdot,t) \|_{L_2(\Omega)}^2 \ \leq c(e^{-2\lambda_r t} \| u_0(\cdot) \|_{L_2(\Omega)}^2 \ + \frac{1}{\lambda_r} \| f(\cdot,\cdot) \|_{L_2(Q)}^2) \ , \ \ \forall t > 0,$$

$$r = 1, 2, \ldots, \quad c = \ \text{const}.$$

Let $\hat{u}(x,\theta)$ be an element of $U(\theta, y^*(\cdot))$ generated by $\{\hat{u}_0(\cdot), \hat{f}(\cdot,\cdot)\}$ together with $\hat{\eta}(t)$ and $\hat{u}(\cdot,t)$ - the respective solution to (1.1), (4.1).

Then

$$y^*(t) = \mathbf{G}(t)\hat{u}(\cdot,t) + \hat{\eta}(t) = \mathbf{G}_r(t)\hat{u}(\cdot,t) + \hat{\eta}(t) + \eta_r(t), \quad t \in T_\varepsilon, \tag{12.4}$$

where

$$\eta_r(t) = \mathbf{G}(t)\hat{u}(\cdot,t) - \mathbf{G}_r(t)\hat{u}(\cdot,t)$$

and (see 8.3)

$$\| \eta_r(\cdot) \|_{L_\infty^m(T_\varepsilon)} \leq s(r); \ \ s(r) \rightarrow 0, \ r \rightarrow \infty. \tag{12.5}$$

Relations (12.4), (12.5) mean that

$$\hat{u}(\cdot,\theta) \in U_{(r)}^{s(r)}(\theta, y^*(\cdot)), \tag{12.6}$$

where $U_{(r)}^{s(r)}(\theta, y^*(\cdot))$ stands for the informational domain of the estimation problem (1.1), (4.1), (12.1), (12.2) and

$$\eta(t) \in \Delta(t) + s(r)\sigma(0) = \Delta^{s(r)}(t), \quad t \in T_\varepsilon,$$

$\sigma(0)$ is a ball of unit radius in R^m.

Conversely, if an element $\tilde{u}(x, \theta)$ belongs to $U_{(r)}(\theta, y^*(\cdot))$, we can similarly obtain

$$\tilde{u}(\cdot, \theta) \in U^{s(r)}(\theta, y^*(\cdot)), \tag{12.7}$$

where the upper index $s(r)$ means the same as in (12.6).

Noticing that the sets of type $U_{(r)}^{\beta}(\theta, y^*(\cdot))$, $U^{\beta}(\theta, y^*(\cdot))$ are continuous in $\beta (\beta \geq 0)$ under condition 12-a, we observe that inclusions (12.6), (12.7) yield the assertion of Lemma 12.1. The further results follow those of a paper by (Kurzhanski, Filippova, 1989). The results of this paper sound as follows.

Denote $X[t] = X(t, t_0, x^0)$, $X[t_0] = X^0$, to be the solution tube (generated by initial set X^0) to the system

$$\dot{x} \in A(t)x + P(t), \; t \geq t_0$$

$$G(t)x \in Q(t), \; x(t_0) = x^0,$$

$$x^0 \in X^0$$

($A(t), G(t)$ are continuous matrices; $P(t), Q(t)$ are set-valued maps, convex compact valued, continuous in t).

Also denote $X_M[t] = X_M(t, t_0, X^0)$ to be the solution tube (generated by initial set X^0) to the system

$$\dot{x} \in (A(t) - M(t)G(t))x + M(t)Q(t) + P(t),$$

$$X_M[t_0] = X^0.$$

Theorem. The following relation is true

$$\bigcap_{M(\cdot)} X_M[t] = X[t],$$

where the intersection is taken over all continuous matrix valued function $M(t)$ $(T \to R^{m\times n})$.

Returning to the basic problem of this paragraph, consider the sequence of sets $U_r^\sigma(\theta, y(\cdot))$ $(r = 1, 2, \ldots)$ each of which admits the following representation:

$$U_r^\sigma(\theta, y(\cdot)) = \{u(x) | u(x) = \sum_{i=1}^{r} u_{ir}(\theta)\omega_i(x),$$

$$\hat{u}_r[\theta] = \text{ col } [u_{1r}(\theta), \ldots, u_{rr}(\theta)] \in \hat{U}_r^\sigma(\theta, y(\cdot))\},$$

where the sequence $\hat{U}_r^\sigma(\theta, y(\cdot))$ $(r = 1, 2, \ldots)$ comes from the solutions to appropriate finite dimensional guaranteed estimation problems:

$$\frac{d\hat{u}_r[t]}{dt} = A_r\hat{u}_r[t] + \hat{f}_r[t],\ \hat{u}_r[t] \in R^r,\ t \in T, \tag{12.8}$$

$$\hat{u}_r[0] \in \hat{U}_{0r},\ \hat{f}_r[t] \in \hat{F}_r[t],$$

$$y(t) = G_r[t]\hat{u}_r[t] + \eta(t),\ t \in T_\varepsilon,$$

$$\eta(t) \in \Delta^\sigma(t),$$

$$\hat{U}_{0r} = \{\hat{u}_r | \hat{u}_r = \text{ col } [u_{1r}, \ldots, u_{rr}], u_r(x) = \sum_{i=1}^{r} u_{ir}\omega_i(x), u_r(\cdot) \in \Pi_r U_0\},$$

$$\hat{F}_r[t] = \{\hat{f}_r[t] | \hat{f}_r(t) = \text{ col}[f_{1r}(t), \ldots, f_{rr}(t)] \in L_2^r(T),$$

$$f_r(x,t) = \sum_{i=1}^{r} f_{ir}(t)\omega_i(x), f_r(\cdot, t) \in \Pi_r F(t), t \in T\}.$$

Lemma 12.2. Assume the set $U_r^\sigma(\theta, y^*(\cdot))$ to be generated by measurement $y^*(t)$, $t \in T_\varepsilon$, that satisfies condtion 12-a. Then the following representation is true

$$U_r^\sigma(\theta, y^*(\cdot)) = \bigcap\{U_r^\sigma(\theta, M^r(\cdot,\cdot)) | M^r(\cdot,\cdot) \in \mathcal{M}^r(\cdot)\}, \tag{12.9}$$

where

$$U_r^\sigma(\theta, M^r(\cdot,\cdot)) = \bigcup\{u(\cdot,\theta|M^r(\cdot,\cdot))\},$$

over all solutions $u(\cdot,\theta|M^r(\cdot,\cdot))$ (taken at instant θ) to the initial boundary value problem

$$\frac{\partial u(x,t)}{\partial t} = (A - \chi_\varepsilon(t)M^r(x,t)\mathbf{G}_r(t))u(\cdot,t) + f(x,t)+ \tag{12.10}$$

$$+\chi_\varepsilon(t)M^r(x,t)(y^*(t) - \eta(t)),\ \ t \in T,\ \ x \in \Omega,$$

$$u(\cdot,0) = u_0(\cdot) \in \mathbf{\Pi}_r U_0,\ \ f(\cdot,t) \in \mathbf{\Pi}_r F(t)\ (t \in T),\ \eta(t) \in \Delta^\sigma(t)\ (t \in T_\varepsilon),$$

$$\mathcal{M}^r(\cdot) = \{M^r(x,t)|M^r(x,t) = (M_1^r(x,t),\dots,M_m^r(x,t)),$$

$$M_i^r(\cdot,\cdot) \in C([0,\theta]; X_r(\Omega))\}.$$

Problem (12.10) is a finite-dimensional problem in $X_r(\Omega)$ treated in a generalized sense (see (1.3)).

It is also clear that

$$U_{(r)}^\sigma(\theta, y^*(\cdot)) = \{u(x)|\ \mathbf{\Pi}_r u(\cdot) \in U_r^\sigma(\theta, y^*(\cdot))\}.$$

Therefore, the relations (12.9), (12.10) are valid for the domain $U_{(r)}^\sigma(\theta, y^*(\cdot))$ with $U_0, F(t)$ substituted by $\mathbf{\Pi}_r U_0$ and $\mathbf{\Pi}_r F(t)$ respectively.

Let now $u^*(x,t)$ be an arbitrary solution to the initial boundary value problem (12.10) generated by the triplet $\{u_0^*(\cdot), f^*(\cdot,\cdot), \eta^*(\cdot)\}$ with an arbitrary function

$$M(\cdot,\cdot) \in \mathcal{M}(\cdot) = \{M(\cdot,t)|M(\cdot,t) = (M_1(\cdot,t),\dots,M_m(\cdot,t));$$

$$M_i(\cdot,t) \in C([0,\theta]; L_2(\Omega))\}$$

and with $\mathbf{G}(t)$ substituted for $\mathbf{G}_r(t)$.

Using the method of transposition (Lions, 1968) we can observe that this solution does exsit in the space $C([0,\theta]; L_2(\Omega))$ and

$$\max_{t\in[0,\theta]} \| u^*(\cdot,t) \|_{L_2(\Omega)} \leq \beta_1(M(\cdot,\cdot))(\| u_0^*(\cdot) \|_{L_2(\Omega)} + \| f^*(\cdot,\cdot) \|_{L_2(Q)} + \| \eta^*(\cdot) \|_{L_2^m(T_\epsilon)}), \quad (12.11)$$

where $\beta_1(M(\cdot,\cdot))$ depends upon $M_i(\cdot,\cdot)(i = 1,\ldots,m)$ continuously in the norm of $C(T_\epsilon; L_2(\Omega))$. Denote by $u^*_{(r)}(x,t)$ the solution of (12.10) generated by the same triplet as above but with $M^r(x,t)$ taken as a truncation of $M(x,t)$. Then for the difference $\tilde{u}^{(r)}(x,t) = u^*(x,t) - u^*_{(r)}(x,\cdot)$ we obtain the mixed problem

$$\frac{\partial \tilde{u}^{(r)}(x,t)}{\partial t} = (A - \chi_\epsilon(t)M(x,t)\mathrm{G}(t))\tilde{u}^{(r)}(x,t) + \chi_\epsilon(t)(M(x,t)\mathrm{G}(t)- \quad (12.12)$$

$$-M^r(x,t)\mathrm{G}_r(t))u^*_{(r)}(x,t) + \chi_\epsilon(t)(M(x,t) - M^r(x,t))\times$$

$$\times(y^*(t) - \eta^*(t)), t \in T, x \in \Omega,$$

$$\tilde{u}^{(r)}(x,0) = 0, \quad \tilde{u}^{(r)}(x,t)|_{\partial\Omega} = 0.$$

Therefore, due to (12.11) we have

$$\| \tilde{u}^{(r)}(\cdot,\theta) \|_{L_2(\Omega)} \leq \beta_2(r, M(\cdot,\cdot)), \quad (12.13)$$

where $\beta_2(r, M(\cdot,\cdot)) \to 0$ when $r \to \infty$ whatever $M(\cdot,\cdot) \in \mathcal{M}(\cdot)$.

Taking into account the lemmas 12.1, 12.2, the estimate (12.13) and taking $r \to \infty$, $\sigma - 0$ we come to

Theorem 12.1. Once the measurement $y^*(t)(t \in T_\epsilon)$ satisfies condition 12-a, the informational domain $U(\theta, y^*(\cdot))$ for the problem (1.1), (4.1), (1.4), (12.1) may be described as

$$U(\theta, y^*(\cdot)) = \bigcap\{U(\theta, M(\cdot,\cdot)) | M(\cdot,\cdot) \in \mathcal{M}(\cdot)\}, \quad (12.14)$$

where $U(\theta, M(\cdot,\cdot))$ is the cross-section at instant θ of the set of all solutions to the partial differential inclusion

$$\frac{\partial u(x,t)}{\partial t} \in (A - M(x,t)\mathrm{G}(t))u(\cdot,t) + F(t) + M(x,t)(y^*(t) - \Delta(t)), \; x \in \Omega, \; t \in [0,\theta], \quad (12.15)$$

$$u(x,0) \in U_0, \; u(x,t)|_{\partial\Omega} = 0.$$

Remark 12-a. The condition for the measurement data $y^*(t)$, $t \in T_\varepsilon$ in Theorem 12.1 may be repalced by a more general condition. Indeed the statement of the latter theorem (and of Lemmas 12.1, 12.2) will be true under the assumption:

$$\lim_{\delta \to +0} d(U(\theta, y(\cdot)), U^\delta(\theta, y(\cdot))) = 0. \tag{12.16}$$

In particular, (12.16) does hold for an arbitrary measurement $y(t)$, $t \in T_\varepsilon$ if the system (1.1), (4.1), (1.4) is strongly observable.

13. Interrelation Between Guaranteed and Stochastic Estimation

Let $(\tilde{\Omega}, B(\tilde{\Omega}), \mu)$ be a probability space (Curtain, Pritchard, 1978; Sawaragi and others, 1978) with $\tilde{\Omega}$ as a topological space, $B(\tilde{\Omega})$ as the Borel field generated by $\tilde{\Omega}$, and μ as the probability measure on $\tilde{\Omega}$.

Suppose that $\tilde{u}_0(\cdot) \in L_2(\tilde{\Omega}, \mu \; ; \; L_2(\Omega))$ and is Gausian with zero mean and with covariance operator P_0 ; $\tilde{f}(\cdot, t)$ is a Wiener process on $L_2(\Omega)$ with covariance operator $Q(t)$; $\tilde{v}(\cdot, t)$ is a Wiener process on $L_2(\partial\Omega)$ with covariance operator $R(t)$; $\tilde{\xi}(t)$ is a vector valued Wiener process on R^m with covariance matrix $N(t)$.

Instead of the deterministic mixed problem (1.1), (1.2) consider a similar problem for a *stochastic partial differential equation*

$$< d\tilde{u}(\cdot, t), \varphi(\cdot) > + a(\tilde{u}(\cdot, t), \varphi(\cdot))dt = \tag{13.1}$$

$$= < d\tilde{f}(\cdot, t), \varphi(\cdot) > +$$

$$< d\tilde{v}(\cdot, t) - c(\cdot)\, \tilde{u}(\cdot, t), \varphi(\cdot)\, |_{\partial\Omega} >,$$

$$\forall \varphi(\cdot) \in H^1(\Omega),\ t \in T,$$

$$a(u_1(\cdot), u_2(\cdot)) = \int_\Omega \Big(\sum_{i,j=1}^{n} a_{ij}\,(x) \frac{\partial u_1(x)}{\partial x_i} \cdot \frac{\partial u_2(x)}{\partial x_j} +$$

$$+ \ a(x)u_1(x)u_2(x))dx,$$

$$\bar{u}(\cdot,0) = \bar{u}_0(\cdot),$$

where $\bar{u}_0(\cdot) = \tilde{u}_0(\cdot)+u_0(\cdot), \bar{f}(\cdot,t) = \tilde{f}(\cdot,t)+f(\cdot,t), \bar{v}(\cdot,t) = \tilde{v}(\cdot,t)+v(\cdot,t)$, the set $\{u_0(\cdot), f(\cdot,t), v(\cdot,t)\}$ satisfies the restriction (12.1),

$$v(\cdot,t) \subset V(t), \ t \in T, \tag{13.2}$$

where $V(t)$ is a continuous multivalued map from T into the set of convex weakly compact sets of $L_2(\partial\Omega)$ and

$$E[\bar{u}_0(\cdot)] = u_0(\cdot), E[\bar{f}(\cdot,t)] = f(\cdot,t), E[\bar{v}(\cdot,t)] = v(\cdot,t).$$

The last two terms in the right hand part of (13.1) are interpreted as respective Ito integrals.

Suppose that we can observe the process

$$dy(t) = \mathbf{G}(t)\bar{u}(\cdot,t)dt \ + \ d\bar{\xi}(t), t \in T_\varepsilon, \tag{13.3}$$

where $\bar{\xi}(t) = \tilde{\xi}(t) + \eta(t), \eta(\cdot)$ satisfies (12.1),

$$E[\bar{\xi}(t)] = \eta(t).$$

The processes $\tilde{f}(\cdot,t), \tilde{v}(\cdot,t)$ are assumed to be statistically independent and also independent of the initial function $\tilde{u}_0(\cdot)$. The relations (13.1), (13.3) define a conventional *stochastic optimal filtering problem* (Falb, 1967; Bensoussan, 1971). We will denote the respective optimal estimate for this problem as $u^o(\cdot,\theta \mid w(\cdot), \wedge(\cdot))$, where $\wedge(\cdot)$ is the quadruple

$$\wedge(\cdot) = \Big\{P_0, \ Q(t), \ R(t), \ N(t)\Big\}.$$

Follow the lines of (Kurzhanski, 1988) for the informational domain $U(\theta, y(\cdot))$ of the deterministic inverse problem (1.1) - (1.5), (12.1), (13.2) we then have

Theorem 13.1. The following relations are true

$$U(\theta, y(\cdot)) \subseteq \cap \cup \left\{u^0(\cdot, \theta \mid w(\cdot),\ \wedge(\cdot)) \mid w(\cdot) \in \mathbf{W}, \wedge(\cdot)\right\}. \tag{13.4}$$

Therefore the projection of the domain $U(\theta, y(\cdot))$ over a prescribed direction $\varphi(\cdot)$ may now be evaluated as follows

$$- \inf \left\{J(-\varphi(\cdot), \wedge(\cdot)) \mid \wedge(\cdot)\right\} \leq < \varphi(\cdot), u(\cdot, \theta) > \leq$$

$$\leq \inf \left\{J(+\varphi(\cdot), \wedge(\cdot)) \mid \wedge(\cdot)\right\},\ \forall u(\cdot, \theta) \in U(\theta, y(\cdot)),$$

where

$$J(\varphi(\cdot), \wedge(\cdot)) = \tag{13.5}$$

$$= \sup\left\{< \varphi(\cdot), u^o(\cdot, \theta \mid \omega(\cdot), \wedge(\cdot)) > \mid \mathbf{W}\right\}.$$

The nature of the relation of (13.4) is such that the substitution of any element $\wedge(\cdot)$ into $J(\varphi(\cdot), \wedge(\cdot))$ gives a guaranteed estimate of the actual state $u(x, \theta)$.

Remark 13-a. From theorem 13.1 it follows that the support function $\rho(\varphi(\cdot) \mid U(\theta, y(\cdot))$ may be calculated by minimizing a multiple integral of type (13.5) over $\wedge(\cdot)$.

Remark 13-b. A number of important physical processes may well be modeled on the basis of the theory of guaranteed estimation. As an example we indicate the problem of estimating the spatial and temporal distributions of air pollution levels (Omatu and others, 1988) where under natural absence of complete statistical information on the inputs and parameters of the system the given approach may turn to be rather relevant.

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