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Foreword

The paper is concerned with an approach to solutions of bargaining problems, i.e. with a rule by which participants of a nonantagonistic game select from the set of all feasible outcomes some "fair" outcome. A rather diverse class of games is considered, and the selection in a concrete game is specified by the class chosen for consideration. Some partial ordering, associated with "contributions of the participants to the game", is given on every class of games, and only monotonic in respect to this ordering solutions are considered. To choose from these solutions a single one it is offered to require the maximum incentive of the participant with the maximum "value of his contribution " but within the limits of monotonicity. The paper contains concrete examples.

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THE MAXIMUM INCENTIVE SOLUTIONS IN BARGAINING PROBLEMS

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1.INTRODUCTION

The present paper deals with solutions of bargaining problems, that is, rules by which an arbitrator or participants of a nonantogonistic game select from the set of all feasible outcomes (or payoffs) some "fair" outcome, which is normally a result of a compromise.

The approach to this problem was first taken in the basic paper Nash [6]. At present in the literature much attention is devoted to this branch of game theory. The survey of many results and a bibliografy are contained e.g. in Roth [9].

The approach which we treat here bases on the following. At first we consider a rather diverse class of games. In particular, this class may be "not too broad". Herewith the selection in a concrete game is specified by the class chosen for consideration, and extending or narrowing the class one may come to a new rule of the selection. (Into each class taken separetly the independence of irrelevant alternatives holds under our axioms).

Secondly (and it seems to be the most essential circumstance) some partial ordering is given on every class of games, and we imply that this ordering is assosiated with "contributions of the participants into the game" (see below for examples). The chosen solution depends on the ordering and, in particular, is monotonic in respect to it.

Monotonic in similiar sense solutions have been investigated before (see, for example, proportional solutions in Isbell [1], Kalai [2], Myerson [3], Roth [9]), but from quite different standpoints. We have noted, that the classes of games considered in the paper might be rather narrow. This firstly ensures, that there is no conflict between the requirement of monotonocity and other natural requirements (e.g. Pareto optimality). Secondly this allows to take account of some prior information about the "interrelationships between the participants" (see below for details).

Under our preliminary axioms an admissable monotonic solution is not unique, and the problem arises to choose a single one. Being another specific feature of our approach, the rule of such choice requires the maximum incentive (or stimulation) of the participant with the maximum "value of his contribution" but within the limits of postulated axioms, in particular, within the limits of monotonocity condition. The latter leads to nontriviality of the solution.

In order to elucidate all this we consider the following simple two-person game.

The Income Allocation problem. Let two participants take part in a business, and we are able to measure their "contributions" into it. The contributions (as well as the participants) can be understood in a very broad sense. For example the contributions may be levels of investments of real individuals or influence characteristics of some factors in a production process (e.g. the labour productivity, the capital etc).

Let number s_i be the value of a contribution (or simply a contribution) of the i-th participant, $s=(s_1,s_2)$ and $R(s)$ (a "production function") be the global income for the vector of contributions s. We are interested to know what parts of the income must be put down to every participant.

Let g_i be a share of the i-th participant. It is clear that in a general case these shares have to depend on the vector of contributions, i.e. $g_i=g_i(s)$. Thus

$$g_1(s) + g_2(s) = R(s).$$

The problem consists in the choice of a vector function $g=(g_1,g_2)$. Each game is assosiated with a vector s, and the class of games may be identified with the set $S=\{s:s_1\geqslant 0,\ s_2\geqslant 0\}$. For any s we have the set of feasible outcomes

$$A(s)= \{ v=(v_1,v_1) : v_1+v_2=R(s) \}$$

It is convenient to elucidate some results of the paper by this model. Assume $R(s_1,s_2)$ be symmetric and nondecreasing in every argument. Then it is natural to assume that

$$g_i(a,a)=R(a,a)/2 \tag{0.1}$$

(the case of equal contributions), and $g_i(s_1,s_2)$ does not decrease at least in s_i. We call this property (may be in a too high flown manner for such a simple model) the Incentive (or Stimulation) property.

In many situations it is natural also to think that the income of any participant must not decline as the contribution of the other one rises, so to say "the rich player must not overwhelm the poor one". In other words, we require that $g_i(s_1,s_2)$ does not decrease in s_j, $j \neq i$. In this case we shall speak about the Nondiminution (or Nonpressing) property.

Alongside with the incentive property this only means that $g(s)$ does not decrease in respect to the standard partial vector ordering $\geqslant$ on S. We call this property Monotonocity one.

Now we accept the principle requiring to give the maximum share of the income to the participant with the maximum contribution but only within the limits of monotonocity and (0.1). Strictly speaking it means the following.

Let $\mathcal{G}$ be the class of all monotonic and satisfying (0.1) vector functions on S. We choose as the solution such the function $g^*=(g^*_1, g_2^*)$, that (0.1) holds, and as $s_1 \geqslant s_2$

$$g^*_1(s) = \sup_{g \in \mathcal{G}} g_1(s). \tag{0.2}$$

(The case $s_1<s_2$ is treated analogously.) If we prove that g^* itself belongs to $\mathcal{G}$ (it is true, though not quite obvious), then (0.2) may be considered as a natural "claimant" to be the solution. We call (0.2) MI(Maximum Incentive)-solution.

This model was investigated in detail in Rotar' and Smirnov [8], where the concrete MI-solution was obtained (see also sec. 3). A similiar approach was used in Katyshev and Rotar'[4], concerning a mutual insurance model. In both these cases the concrete forms of MI-solutions turned out to be not quite trivial. The results from [8] and [4] may serve as examples of applying general results from Rotar'[7], where the notion of MI-solution was defined. The existence of MI-solution was proved in [7] for two-person games, and the multidimensoinal case was treated under burdensome conditions. They were essentially facilitated in Kalashnikov and Rotar' [3].

This paper is mainly devoted to generalization of results from [3]. The assertions given below seem to have a rather completed form. To treat three-person games we shall need also some improvement and generalization of the two-person result. In our view this generalization is interesting in itself too. We also shall give a very brief review of some other results on the subject under discussion.

In Sec.1 the general framework is described; Sec. 2 deals with two-person games. In order to illustrate results of Sec.2 we formulate in Sec.3 some assertions concerning the income allocation problem . Sec. 4-6 are devoted to three-person games. To avoid cumbersome formulas we shall not consider games with more participants. The translation of three person scheme to the k-person case does not meet essential difficulties.

1.THE BASIC FRAMEWORK

Henceforward $S=\{s\}$ be a class of n-person games of arbitrary nature, and for every game s a set of all outcomes $A(s) \subset R^n_+$ is given. Note that the same set of outcomes may be assosiated with different games, as it takes place, for example, in the income allocation problem. We denote points from A(s) by $v=(v_1,\dots,v_n)$.

Set $V= \bigcup_{s \in S} A(s)$.

Let a partial ordering $\succeq$ be given on S.

Assume also that for every game "the rule of priority" is known, namely a breakdown of class S into subclasses S_p is specified, where $p = (p_1,\dots,p_n)$ is one of permutations of $(1,\dots,n)$. We imply that, if $s \in S_p$, then the "contribution into the game s" of the participant with number p_1 is not less then that with number p_2 and so on. Let

$$S_0 = \bigcap_p C_p.$$

The solution for the class S is such a map $h:S\to V$, that

$$h(s) \in A(s) \text{ for all } s \in S.$$

We shall also write $h(s)= (h_1(s),\dots,h_n(s))$, implying that $h_i(s)$ is an income (or utility) of the i-th participant.

Let as before be the usual vector ordering in R^n, and $\Pi(A)$ be the set of all Pareto optimal points from A in respect to . (In particular, $\Pi(A) \subseteq A$).Set $\Pi(s) = \Pi(A(s))$.

Let $\mathcal{H}$ be the class of all solutions with the following properties.

Property 1: Pareto optimality: $h(s) \in \Pi(s)$ for all $s \in S$.

Property 2: Monotonocity: If $s' \succeq s$, then $h(s') \geqslant h(s)$.

Property 3: Priority: If $s \in S_p$, then

$$h_{p_1}(s) \geqslant h_{p_2}(s) \geqslant \dots \geqslant h_{p_n}(s)$$

In particular, if $s \in S_0$, then

$$h_{p_1}(s) = h_{p_2}(s) = \dots = h_{p_n}(s)$$

Of course class $\mathcal{H}$ may be empty or may contain more than one element.

Let $D = \{ v : v_1=\dots=v_n \geqslant 0 \}$. The map

$$\bar{h}(s) = \Pi(s) \cap D. \tag{1.1}$$

may be the simplest example of a solution from $\mathcal{H}$. (To be sure one can consider $\bar{h}$, if the right side of (1.1) is not empty. It is obvious that $\bar{h}$ posesses properties 1,3 and, as is shown in Sec.4, under rather mild conditions $\bar{h}$ posesses property 2.) We call $\bar{h}$ an evening out solution. It is too primitive and, as a rule, cannot be satisfactory. Below we consider the solution opposed in some sense to $\bar{h}$.

2. THE TWO-PLAYER CASE

Let n=2.

Definition 1. The map h^* is called MI-solution, if $h^* \in \mathcal{H}$, and for every p and all $s \in S_p$

$$h^*_{p_1}(s) = \sup_{h \in \mathcal{H}} h_{p_1} \tag{2.1}$$

In our view the solution h^* seems to be natural in many instances. On the other hand we should note that the choice of such a solution would be the reflection of a logical but extreme position. The solutions $\bar{h}$ and h^* are the extreme ones, and ensuring, for example, "social stability" or a more favorable "psychological atmosphere" in the game we may come to the adoption of a solution intermediate between $\bar{h}$ and h^*. The choice of this intermediate solution must apparently be based on the special features of a particular case. Our aim is to state the bounds on this choice.

Before the following proposition it is appropriate to note, that the existense of MI-solution is not quite obvious, because it is not quite obvious that the map defined in (2.1) belongs to $\mathcal{H}$.

Condition A. For all $s \in S$ the set $\Pi(s)$ is compact.

Theorem 1. Let condition A hold, and class $\mathcal{H}$ be not empty. Then MI-solution exists and is unique.

We slightly generalize this assertion. Let $z: S \to R^1$ and $\mathcal{H}^z$ be the class of all solutions $h \in \mathcal{H}$ and such that $h_i(s) \leqslant z(s)$ for i=1,2. One may interpreted z as a maximum "allowed income".

We call the map h^{*z} MI-solution in respect to $\mathcal{H}^z$, if $h^{*z} \in \mathcal{H}^z$, and for every p and all $s \in S_p$

$$h^{*z}_{p_1}(s) = \sup_{h \in \mathcal{H}^z} h_{p_1}(s)$$

Theorem 1′. Let condition A hold, and $\mathcal{H}^z$ be not empty. Then MI-solution in respect to $\mathcal{H}^z$ exists and is unique.

Proof of theorem 1'. It suffices to consider classS_{12} . Since $\mathcal{H}^z$ is not empty, the set

$$Q(s) = \{ v: v = h(s) \text{ for some } h \in \mathcal{H}^z \}$$

is not empty either (we omit the upper index z for simplicity). By property 1 $Q(s) \subseteq \Pi(s)$. Let $\bar{Q}$ be the closure of Q. By condition A

$$\bar{Q}(s) \subseteq \Pi(s). \tag{2.2}$$

Let $h^*(s)$ be the point from $\bar{Q}(s)$ with the maximum first coordinate. Since $\Pi(s)$ is bounded, (2.2) causes the existence of such a point. It is unique because of Pareto-optimality of points from $\Pi(s)$. Finally (2.2) implies that $h^*(s) \in \Pi(s) \subseteq A(s)$.

We shall prove that $h^* \in \mathcal{H}^z$. If $s \in S_{12}$, then $v_1 \geqslant v_2$ for all $v \in Q(s)$. Consequently $h^*_1(s) \geqslant h^*_2(s)$, and property 3 holds. From (2.2) property 1 also follows. It is also obvious that $h^*_1(s) \leqslant z(s)$ for all s. Thus it remains to prove monotonocity of map h^* in respect to $\succeq$ on S_{12}.

Let $s' \succeq s$. Assume that

$$h^*_2(s') < h^*_2(s).$$

By construction for all $h \in \mathcal{H}$

$$h_2(s) \geqslant h^*_2(s), \tag{2.3}$$

because otherwise point h(s) would not be Pareto optimal. For any $\varepsilon > 0$ there exists such $h^\varepsilon \in \mathcal{H}$ that

$$h^\varepsilon_2(s') \leqslant h^*_2(s') + \varepsilon.$$

Setting $\varepsilon = (h^*_2(s) - h^*_2(s'))/2$ and using (2.3) we get

$$h^\varepsilon_2(s') \leqslant [h_2(s) + h^*_2(s')]/2 < h^*_2(s) \leqslant h^\varepsilon_2(s)$$

It is not possible, because $h^\varepsilon \in \mathcal{H}$. Analogously one can prove monotonocity of h_1^*. The theorem is proved.

Theorem 1 essentially generalizes the corresponding theorem from [7], though the proofs are similiar.

We should also compare our axioms with some well known ones. The question has been disscussed in [7], and so we shall only note the following. Assume for simplicity that

$$s' \succeq s \Rightarrow A(s) \leq A(s') \qquad (2.4)$$

It is a natural assumption; in Sec.4-6 we shall use it. It is easy to see that monotonocity together with (2.4) and Pareto optimality implies the independence of irrelevant alternatives. The reverse is not generally true, and, in particular, the Nash solution may be not monotonic (an example see e.g. in [7]).

As to the independence of equivalent utility representations, this property may be redundant for our scheme, because S may not contain sets derived from one to another by a linear transformation. But even in the opposite case one may construct the class for which there is no solution possessing the latter property and properties 1 through 3. Therefore we need another rule to distinguish the unique solution in $\mathcal{H}$.

Now let us turn to the proportional solutions. We shall follow Roth [9], where, in particular, class M of all games with freely disposable utility is considered (one can see the accurate definition in Sec.4), and a number of axioms is discussed: independence of common scale changes, strong individual rationality and decomposability. The latter property seems to be the most important.

It was shown that the fullfillment of these axioms implies monotonocity, and , as is proved in Kallai [2], if these axioms are fullfilled on M, then a solution may be only proportional, i.e. the result of the selection is the point of the intersection of a ray starting from the origin with the boundary of the set of outcomes. In other words we deal with a solution similiar to $\bar{h}$.

Such solution cannot be satisfactory in cases which we discuss here. On the other hand the opportunity to restrict ourselves to a narrow class of games allows to choose a more resoursefull solution, for example h^*.

It should be noted that the above reasoning should not be taken as a criticism of Nash or other schemes. Our aim is only to discuss some differences and to emphasize that one of the basic differences is that we each time choose a rather diverse and maybe narrow class of games which is also partially ordered.

We already have noted that MI-solution in a concrete problem might be not trivial. To illustrate this, we discuss some results from [8] concerning

3. MI-SOLUTION IN THE INCOME ALLOCATION PROBLEM

For simplicity we sligtly change the denotations of the Introduction. Henceforward we shall write x in place of s_1 and y in place of s_2. Set $u(x,y) = g_1(x,y)$. Since

$$g_2(x,y) = R(x,y) - u(x,y), \tag{3.1}$$

it suffices to deal with function u. Let $R(x,y)$ be a symmetric, nondecreasing in all arguments and twice differentiable function. It was shown in [8], that in this case the functions g_1^*, g_2^* are smooth. Hence in view of (3.1) and the properties 1-3 we must consider on the set $B=\{(x,y): x \geqslant y\}$ such functions $u(x,y)$, that

$$u(x,x)=R(x,x)/2, \tag{3.2}$$

$$0 \leqslant \frac{\partial u(x,y)}{\partial x} \leqslant R_1(x,y) := \frac{\partial R(x,y)}{\partial x}, \tag{3.3}$$

$$0 \leqslant \frac{\partial u(x,y)}{\partial y} \leqslant R_2(x,y) := \frac{\partial R(x,y)}{\partial y} \ . \tag{3.4}$$

Thus MI-solution is the function

$$u^*(x,y) = \sup_{u \in \mathcal{U}} u(x,y). \tag{3.5}$$

where $\mathcal{U}$ is the class of all functions defined on B and satisfying "boundary" condition (3.2) and conditions on the derivatives (3.3),(3.4). The problem of seeking for this function seems to be interesting in pure mathematical sense too. Firstly we elucidate the following.

Let x>y. Together with the point z=(x,y) we consider the points $\underline{z}=(y,y)$ and $\bar{z}=(x,x)$. Let us transit from the point $\underline{z}$ to the point z (the first participant increases his contribution and the second one does not do it). It might seem that MI-solution requires to give the whole arising increment of the income to the first player, that is to choose the solution (see also Fig.1)

$$u^+(x,y) := R(x,y) - R(y,y)/2$$

(the symbolism will be clear later). Solution u^+ may be, however, nonmonotonic. Really, let us transit now from z to $\bar{z}$ (the second player increases his contribution up to the value of the first player's contribution). The payoff of the first player must become equal to the right side of (3.2), but it may turn out that $u^+(x,y) > u(x,x)$, i.e.. u^+ does not belong to $\mathcal{U}$. Thus for the vector of contributions (x,y) the first player's income must not exceed $u^-(x,y) := R(x,x)/2$.

It is obvious now that (see also Fig.1)

$$u^*(x,y) \leqslant \tilde{u}(x,y) := \min \{ u^+(x,y), u^-(x,y)\}. \tag{3.6}$$

It follows from (3.6), that, if $\tilde{u}$ satisfies (3.3)-(3.4), then it is MI-solution. However we shall see that it is not always true.

Firstly we consider the case when really $u^* = \tilde{u}$. At the start let

$$R(x,y) = R(x+y). \qquad (3.7)$$

It is easy to calculate that in this case the following holds. If for all $t>0$ the function $R(t)$ is concave from below, ($R''(t) \geqslant 0$ for all t) then $u^+ \leqslant u^-$, the function u^+satisfies (3.3),(3.4) and $u^* = u^+$. If $R(t)$ is concave from above ($R''(t) \leqslant 0$ for all t), then $u^+ \geqslant u^-$ and $u^* = u^-$.

We turn to the general case. Let

$$L = \{\ z=(x,y) \in B:\ R(x,x) + R(y,y) = 2R(x,y)\ \}$$

We assume also that L is continious and decreasing curve in B, i.e. one can write that $L=\{(x,y):\ x \geqslant y,\ y = \varphi(x)\}$, where φ is a continious and decreasing function.

Let $R_{12}(x,y)= \partial^2 R(x,y)/\partial x \partial y$, the curve

$$M = \{\ z \in B:\ \ R_{12}(z)=0\ \},$$

and, as above, $M = \{(x,y) \in B:\ y = \ \psi(x)\}$, where ψ is also continious and decreasing.

Let ψ^{-1}, φ^{-1} be the corresponding inverse functions. Set

$M_1=\{\ (x,y) \in B:\ x \leqslant \psi^{-1}(y)\}$, $\qquad M_2=\{\ (x,y) \in B:\ y > \psi(x)\}$,

$L_1=\{\ (x,y) \in B:\ x \leqslant \varphi^{-1}(y)\}$, $\qquad L_2=\{\ (x,y) \in B:\ y > \varphi(x)\}$,

(see also Fig.2), and

$$\tilde{u}(z) = \begin{cases} u^+(z), \text{ if } & z \in L_1, \\ u^-(z), \text{ if } & z \in L_2. \end{cases}$$

Proposition 1. Let $R_{12}(z) > 0$, if $z \in M_1$; and $R_{12}(z) < 0$, if $z \in M_2$. Then $u^* = \tilde{u}$.

We consider now an opposite in some sense case, when MI-solution does not coincide with $\tilde{u}$. Let x_0 be a solution of the equation $x=\psi(x)$. It is not difficulte to calculate that this solution is unique and $(x_0,x_0) \in \mathbf{M}$.

Proposition 2. Let $R_{12}(z) < 0$, if $z \in \mathbf{M}_1$; and $R_{12}(z) > 0$, if $z \in \mathbf{M}_2$. Then

$$u^*(x,y) = u^-(x,y), \text{ if } x \leqslant x_0; \text{ and}$$

$$u^*(x,y) = u^+(x,y), \text{ if } y \geqslant x_0.$$

If $(x,y) \in \mathbf{M}$, then

$$u^*(x,y) = (1/2)R(x_0,x_0) + \int_{(x_0,x_0)}^{(x,y)} R_1(a,b)da ,$$

where we integrate along the curve $\mathbf{M}$.

If $x_0 \leqslant x \leqslant \psi^{-1}(y)$, then $u^*(x,y) = u^*(x, \psi(x))$;
and if x_0 $y \geqslant \psi(x)$, then $u^*(x,y) = R(x,y) - u^*(\psi^{-1}(y),y)$.

Note that in the both cases (proposition 1 and proposition 2) the solution is the result of the corresponding integration of the function $R_1(a,b)$. The ways of integration are shown in Fig.2 and in Fig.3 correspondingly. In the first case the curve L play the role of a "separating curve", in the second case the curve $\mathbf{M}$ play the role of a turnpyke.

We illustrate propositions 1 and 2 by the particular case (3.7). Let for some $x_0 > 0$ the second derivative $R''(t) < 0$ if $t < 2x_0$, $R''(t)= 0$ if $t=2x_0$, and $R''(t) > 0$ if $t < x_0$. Then

$$L = \{ z \in B : x + y = 2x_0\},$$

and, as is easy to calculate,in this case MI-solution u^* takes the following values:

$$R(2x)/2 \quad \text{if } x < x_0, \qquad R(x+y) - R(2y)/2 \quad \text{if } y > x_0 ;$$

$$(1/2)\, R(2x_0) + R'(2x_0)(x - x_0) \qquad \text{if } x_0 \leqslant x \leqslant \psi^{-1}(y) ;$$

$$R(x+y) - (1/2)\, R(2x_0) + R'(2x_0)(x_0 - y) \qquad \text{if } x_0 \geqslant y \geqslant \psi(x).$$

We see that u^* is linear in the third zone and depends only on the value of the derivative in point $2x_0$.

4. THE THREE-PERSON CASE

For the present we are not able to translate theorem 1 to the k-person case without supplementary conditions. At any rate a literal translation of the proof does not work. We should not analyse details and note only the following. If we gave the preference to the first player, it would not be obvious that there was a monotonic solution, which divides the "remainder of the income" between the second and the third players.

Set $\mathcal{A} = \{A(s); s \in S\}$.

Condition I. Every set from $\mathcal{A}$ is compact.

Condition II Every game from S is a game with freely disposable utility, i.e. for all $A \in \mathcal{A}$

$$A=\{v:\ v \leqslant x \text{ for some } x \in A\} \tag{4.1}$$

Since we deal with Pareto optimal solutions, this condition, in fact, does not restrict generality.

The following condition slightly narrows the class of games under discussion and concerns the part of the boundary of A, which lies outside the coordinate planes.

Namely let us consider a space R^k_+, where k is arbitrary. Let R^k_{+0} $\{v: v_i > 0,\ i=1,\dots,k\}$, $\bar{A}$ be the closure of set A, and $\sigma(A)$ be the boundary of A. We define the set-to-set map $\mathbb{K}$ by the following:

$$\mathbb{K}(A) = \overline{\sigma(A) \cap R^k_{+0}}. \tag{4.2}$$

Now we return to class $\mathcal{A}$ and set $K(s) = \mathbb{K}(A(s))$.

Condition III. For all $s \in S$

$$K(s) = \Pi(s) \neq \emptyset \tag{4.3}$$

Condition A is fullfilled under conditions I,III of course.

At last we consider

Condition IV. If $s' \succeq s$, then $A(s') \supseteq A(s)$.

Theorem 2. If conditions I through IV are fullfilled, then the class $\mathcal{H}$ is not empty, in particular, $\mathcal{H} \ni \bar{h}$ where $\bar{h}$ is the same as in (1.1).

Proof is very simple. Firstly we show that the intersection in (1.1) is not empty. Let $A \in \mathcal{A}$. Since $\mathbb{K}(A)$ is not empty, A contains points from R^3_{+0} and, by condition II, points from D with positive coordinates. Since A is bounded, there is a point $d \in D$ coordinates of which are equal to $\sup\{v_1 : v \in A \cap D, v_1 > 0\}$.

By (4.3), $d \in \Pi(A) \subseteq A$, and the intersection in (1.1) is not empty.

It is obvious that map $\bar{h}$ possesses properties 1,3. Let $s' \succeq s$. By construction either $h(s) \leqslant h(s')$, or $h(s) > h(s')$. The latter is impossible since by condition IV and Pareto optimality of solution $\bar{h}$. The proof is complete.

Now we turn to MI-solutions. Let $\mathcal{H}_i(z) = \{h \in \mathcal{H} : h_i = z\}$, where a map $z: S \to R^1$. This class may be empty of course.

Definition 2. The map h^* is called MI-solution, if $h^* \in \mathcal{H}$, and for every p and all $s \in S_p$

$$h^*_{p_1}(s) = \sup_{h \in \mathcal{H}} h_{p_1}(s),$$

$$h^*_{p_2}(s) = \sup_{h \in \mathcal{H}_{p_1}(h^*_{p_1})} h_{p_2}(s),$$

We shall need one more condition on sets from $\mathcal{A}$. This condition seems not too burdensome, but anyway it is "significant".

For any set A by $\mathbb{P}_i(A;a)$ we denote the projection of the section $\{v \in A: v_i=a\}$ on the subspace generated by the "rest" coordinates.

For sets A,B from a space R_+^k, we shall write $A \supset^* B$, if $A \supset B$, and $\mathbb{K}(A) \cap \mathbb{K}(B) = \emptyset$.

Condition V. For all $A,A' \in \mathcal{A}$ and numbers $x,y>0$ either

$$\mathbb{P}_i(A;x) \supset^* \mathbb{P}_i(A';y),$$

or

$$\mathbb{P}_i(A;x) \ {}^*\!\subset \mathbb{P}_i(A';y),$$

or

$$\mathbb{P}_i(A;x) = \mathbb{P}_i(A';y).$$

Let, for example, as in the income allocation problem, for any $s \in S$ the set $A(s)= \{v \in R_+^3: v_1+v_2+v_3 \leqslant R(s)\}$, where $R(s)$ is a function. Then condition V is fullfilled.

Theorem 3. Let conditions I through V be fullfilled. Then MI-solution exists and is unique.

5. LEMMAS

We assume henceforward that conditions I through V hold.

Lemma 1. Let $A \in \mathcal{A}$; $i=1,2,3$; $y \geqslant x > 0$; and the sets $\mathbb{P}_i(A;x)$, $\mathbb{P}_i(A;y)$ are not empty. Then

$$\mathbb{P}_i(A;x) \supset^* \mathbb{P}_i(A;y), \tag{5.1}$$

and $\mathbb{P}_i(A;x) = \mathbb{P}_i(A;y)$ iff $x=y$.

The proof is simple and we leave it out.

Lemma 2. Let $h \in \mathcal{H}$; $i=1,2,3$; $s' \succeq s$; and $A=A(s)$, $A'=A(s')$. Then

$$\mathbb{P}_i(A;h_i(s)) \subseteq \mathbb{P}_i(A';h_i(s')) \tag{5.2}$$

Proof. Let, for example, (5.2) does not hold for $i=1$. Then, by condition V

$$\mathbb{P}_1(A';h_1(s')) \ {}^*\!\subset \mathbb{P}_1(A;h_1(s)) \tag{5.3}$$

By (4.3)

$$(h_2(s),\ h_3(s)) \in \mathbb{K}(\mathbb{P}_1(A;h_1(s))),$$

and the same holds under replacement s by s'.(Here the set-to-set map $\mathbb{K}(.)$ is considered on R^2_+). In view of (5.3) and property 1 the latter means that either $h_2(s)>h_2(s')$ or $h_3(s)>h_3(s')$, which contradicts to the proposition that $h \in \mathcal{H}$. The lemma is proved.

For all sets A,B we define

$$\rho(A,B)= \inf_{x \in A,\ y \in B} |x-y|,,$$

$$\rho_1(A,B)= \sup_{x \in B} \rho(x,B),$$

Lemma 3. Let $A \in \mathcal{A}$, and a number x be an interior point of the projection A on the first coordinate axis. Then the function

$$\rho_1(\mathbb{K}(\mathbb{P}_1(A;x+\varepsilon)),\mathbb{K}(\mathbb{P}_1(A;x))) \qquad (5.4)$$

is continious in ε.

Proof. Let x' be the supremum of all points x described above. The set A is bounded, and all Pareto optimal points are limits of sequences of points from $\{\mathbb{K}(\mathbb{P}_1(A;x)),\ x<x'\}$. Therefore for all $x<x'$ sets $\mathbb{K}(\mathbb{P}_1(A;x))$ are not empty. The set $\mathbb{K}(\mathbb{P}_1(A;x'))$ is not empty either, because otherwise condition III would not hold. The latter set contains only one point, namely the origin, because otherwise one would be able to show such a sequence of points from sets $\{\mathbb{K}(\mathbb{P}_1(A;x)),\ x<x'\}$, that the limit of this sequence would not be Pareto optimal.

It follows from the above reasoning, that if $x<x'$, then the intersection $\mathbb{K}(\mathbb{P}_1(A;x))$ with the line $\{(v_2,v_3):\ v_2=v_3 tg\ \varphi\}$, where $0 \leqslant \varphi \leqslant \pi/2$, contains one and only one point. We denote it by $l(x,\varphi)$.

The Pareto optimal and bounded surface $\mathbb{K}(A)$ is continious.

Therefore in R_+^3 points

$$(x+\varepsilon, l(x+\varepsilon,\varphi)) \to (x, l(x,\varphi)) \quad \text{as} \quad \varepsilon \to 0 \tag{5.5}$$

The set

$$C_x = \bigcup_{\varphi} l(x,\varphi) \subset \mathbb{K}(A)=\Pi(A),$$

since all points from C_x are limits of points from the bounded set $\mathbb{K}(A)$. Then

$C_x = \mathbb{K}(\mathbb{P}_1(A;x))$,

since otherwise condition III would not hold. Therefore (5.4) is equal to $\rho_1(\mathbb{K}(\mathbb{P}_1(A;x+\varepsilon)), C_x)$.

It remains to note that the convergence in (5.5) is uniform in φ for the simple reason that a function, continious on a compact, is uniformly continious. The proof is complete.

Set

$$h_1^*(s) = \sup_{h \in \mathcal{H}} h_1(s) \tag{5.6}$$

Lemma 4. Let $s' \succeq s$, $A=A(s)$, $A'=A(s')$, $B = \mathbb{P}_1(A;h_1^*(s))$, $B' = \mathbb{P}_1(A';h_1^*(s'))$. Then

$$B \leq B' \tag{5.7}$$

Proof. Assume that (5.7) does not hold. Then, by condition V,

$$B \supset B', \tag{5.8}$$

and $\tilde{B}' \cap \tilde{B} = \emptyset$, where $\tilde{B}=\mathbb{K}(B)$, $\tilde{B}'=\mathbb{K}(B')$.

Hence, since the sets $\tilde{B}$, $\tilde{B}'$ are closed, there is such $\delta>0$, that

$$\rho(\tilde{B},\tilde{B}') > \delta. \tag{5.9}$$

By (5.6) for every $\varepsilon>0$ there exists such $h^\varepsilon \in \mathcal{H}$ that

$$|h_1^\varepsilon(s')-h_1^*(s')| \leqslant \varepsilon \tag{5.10}$$

Set $B'_\varepsilon = \mathbb{P}_1(A';h_1^\varepsilon(s\ ))$, $\tilde{B}'_\varepsilon= \mathbb{K}(B'_\varepsilon)$. By lemma 3 and (5.10) there is such ε that

$$\rho_1(\tilde{B}'_\varepsilon,\tilde{B}) < \delta/2. \qquad (5.11)$$

Now it is not difficult to realise that (5.11),(5.9) and (5.8) imply the relation

$$\mathbb{P}_1(A';h_1^\varepsilon(s\ ))\ {}^*\!\subset B\ . \qquad (5.12)$$

By construction and lemma 1

$$B \subseteq \mathbb{P}_1(A;h_1^\varepsilon(s)). \qquad (5.13)$$

From (5.12), (5.13) we obtain that

$$\mathbb{P}_1(A';h_1^\varepsilon(s\ )) \subset \mathbb{P}(A;h_1^\varepsilon(s)),$$

which contradicts to lemma 2. Lemma 4 is proved.

6. PROOF OF THEOREM 3

It suffices to consider the class of games $S_1=S_{123} \cup S_{132}$. By theorem 2 the class $\mathcal{H}$ is not empty. Let $h_1^*(s)$ be the same as in (5.6). Following the logic of the proof of theorem 1′ ,it is easy to prove that the map $h_1^*(s)$ is monotonic in respect to $\succeq$. Let now

$$B(s) = \mathbb{P}_1(A(s);h_1^*(s)).$$

By analogy with the beginning of the proof of theorem 1, one easily proves that for sets $\{B(s),\ s\in S\}$ the "two dimensional" variants of conditions I,II,III are fullfilled. By lemma 4, if $s'\succeq s$, then

$$B(s) \subseteq B(s'),$$

i.e condition IV is also fullfilled.

Set $S_{23} = S_{123}$, $S_{32} = S_{132}$, and $z(s)=h_1^*(s)$.

We define class $\mathcal{H}^z$ as in Sec. 2 in respect to the two-person problem, specified by class S_1, the point-to-set mapping $B(s)$, ordering $\succeq$ on S_1, and subclasses S_{23}, S_{32}.

It is clear (see also theorem 2) that class $\mathcal{H}^z$ is not empty. Then by theorm 1′ there exists MI-solution

$$(h_2^*,\ h_3^*\ ) \in \mathcal{H}^z.$$

Set $h^* = (h_1^*, h_2^*, h_3^*)$. It is obvious that the latter map is the one which we seek for. The theorem is proved.

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