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Foreword

The paper extends the previous results of the authors on quantitative stability for chance constrained programming in two directions: it gives verifiable sufficient conditions for Lipschitz property and it indicates the possibility of using the results in connection with a stochastic load dispatch model. The research was carried out in the frame of the IIASA Contracted Study "Parametric Optimization and its Applications."

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

Distribution sensitivity for a chance constrained model of optimal load dispatch

Werner Römisch and Rüdiger Schultz*

Abstract:

Using results from parametric optimization we derive for chance constrained stochastic programs (quantitative) stability properties for (locally) optimal values and sets of (local) minimizers when the underlying probability distribution is subjected to perturbations. Emphasis is placed on verifiable sufficient conditions for the constraint-set-mapping to fulfill a Lipschitz property which is essential for the stability results. Both convex and non-convex problems are investigated.
We present an optimal-load-dispatch model with considering the demand as a random vector and putting the equilibrium between total generation and demand as a probabilistic constraint. Since in optimal load dispatch the information on the probability distribution of the demand is often incomplete, we discuss consequences of our general results for the stability of optimal generation costs and optimal generation policies.

Key words: Parametric optimization, chance constrained stochastic programming, sensitivity analysis, optimal load dispatch.

1. About the Load Dispatch Model

The problem of optimal load dispatch consists of allocating amounts of electric power to generation units such that the total generation costs are minimal while an electric power

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demand is met and certain additional constraints are satisfied. Our purpose is to obtain an optimal production policy for an energy production system consisting of thermal power stations, pumped storage plants and an energy contract for a time period up to one day with a discretization into hourly or half-hourly intervals. Unit commitment and network questions are excluded.

Of course, there is plenty of literature on optimal load dispatch reflecting work beginning with models much more comprehensive than the one presented here and ending with adapted solution procedures and computer codes to find optmal schedules (cf. [9],[10],[11] ,[19],[32],[34]).

Disregarding a quadratic term in the objective and one nonlinear constraint our model is a linear one. From practical viewpoint, however, an incorporation of further nonlinearities would improve the reflection of the reality. Such nonlinearities, if not being too curious, even not destroyed the basis of our distribution sensitivity analysis.

Nevertheless, we prefered to keep the model linear wherever it is possible, since we wanted to have a practicable model also from numerical viewpoint. Due to the number of time discretization intervals, we will face a large-scaled problem already for a comparatively low number of generating units.

A special feature of our model is that we take into account the randomness of the electrical power demand. The equilibrium between total generation and demand is modeled as a probabilistic (or chance) constraint, thus obtaining a high reliability for the equilibrium to hold when the demand is considered as a random vector. Since in practice in general the probability distribution of this random vector is not completely available, the question arises whether our model is a proper one in the sense that optimal solutions behave stable under perturbations of the probability distribution of the demand. For this reason in Section 2 we study qualitative and quantitative aspects of solution stability in chance constrained programming where the entire probability distribution is considered as a parameter the optimization problem depends on. We use a parametric programming framework and we are aiming at comprehensible and verifiable sufficient conditions for stability.

Let K and M denote the number of thermal power stations and pumped storage plants, respectively, the system comprises and N be the number of subintervals in the discretization of the time period. The (unknown) levels of production in the thermal power stations and the pumped storage plants are y_r^i (i=1,...,K; r=1,...,N), s_r^j (j=1,...,M; r=1,...,N) (generation mode) and w_r^j (j=1,...,M; r=1,...,N) (pumping mode). By z_r (r=1,...,N) we denote the (unknown) amounts for energy purchased or sold according to the contract.
The total generation costs are given by the fuel costs of the thermal power stations (which are assumed to be a strictly convex quadratic function of the generated power, cf. [31], [32]) plus the costs (respectively takings) according to the energy constract (which are a linear function of the power). Concerning pumped storage plants we remark that sometimes ([9],[10]) the stock in the upper dam is evaluated by a certain function such that another term enters the objective, which reflects the costs and takings, respectively, according to the change of stock caused by the operation of the plant.
In our model, however, we do not pursue this, and hence the objective becomes

$$y^T Hy + h^T y + g^T z \qquad (1.1)$$

where $y \in \mathbb{R}^{KN}$, $z \in \mathbb{R}^N$, $H \in L(\mathbb{R}^{KN},\mathbb{R}^{KN})$ is positive definite and diagonal, $h \in \mathbb{R}^{KN}$ and $g \in \mathbb{R}^N$.
According to the discretization of the time period we have a demand vector d (of dimension N) which is understood as a random vector with distribution $\mu \in \mathcal{P}(\mathbb{R}^N)$ - the set of all Borel probability measures on $\mathbb{R}^N$. Claiming that a generation (y,s,w,z) fulfills the demand with probability $p_o \in (0,1)$ then means that

$$\mu(\{d \in \mathbb{R}^N: \sum_{i=1}^{K} y_r^i + \sum_{j=1}^{M} (s_r^j - w_r^j) + z_r \geq d_r, r=1,\dots,N\}) \geq p_o. \quad (1.2)$$

In addition to this probabilistic constraint we take into account conditions which characterize the operation of the different plants:

$$\underline{a}_1 \leq y \leq \bar{a}_1,\ 0 \leq s \leq \bar{a}_2,\ 0 \leq w \leq \bar{a}_3,\ \underline{a}_4 \leq z \leq \bar{a}_4; \quad (1.3)$$

$$S_j^{oo} - S_j^o \leq \sum_{r=1}^{\tau} (s_r^j - \eta_j w_r^j) \leq S_j^{oo} \ (j=1,\dots,M;\ \tau=1,\dots,N); \quad (1.4)$$

$$\sum_{r=1}^{N} (s_r^j - \eta_j w_r^j) = b_{1j} \ (j=1,\dots,M),\ \sum_{r=1}^{N} z_r = b_2. \quad (1.5)$$

Restrictions for the power output are modeled in (1.3). The inequalities (1.4) reflect the balance between generation and pumping (measured in energy) in the pumped storage plants, S_j^{oo} and S_j^o denote the initial respectively maximal stocks (in energy) in the upper dam. For each pumped storage plant we assume that the maximal stock (in water) of the upper dam equals that of the lower dam and that no additional in- or outflow occurs. We then put the pumping efficiency, denoted η_j, as the quotient of the energy that is gained when letting the full content of the upper dam go down and the energy that is needed when pumping the full content of the lower dam upward. A further refinement of the model is possible if the pumping efficiency is not put as a constant but as a function of the actual stock in the upper dam (cf. [16]). The equations (1.5) are balances over the whole time period for the pumped storage plants and according to the energy contract, respectively. The model can be supplemented by further linear (non-probabilistic) constraints, for instance those reflecting fuel quotas in the thermal power stations.

Due to the practical background (generation costs in each thermal plant are strictly monotonically increasing) the function $y^T H y + h^T y$ is strictly monotonically increasing in each component of y with respect to the corresponding one-dimensional projection of the KN - dimensional interval $[\underline{a}_1, \bar{a}_1]$.

A special feature of the above model is that different variables have been introduced for the pumping and the generation modes in the pumped storage plants. For this reason there should be additional constraints to exclude situations where for some $j \in \{1,...,M\}$ and $r \in \{1,...,N\}$ both $s_r^j > 0$ and $w_r^j > 0$. However, such constraints can be omitted which might be seen as follows:
Let (y,s,w,z) be an optimal solution to the problem given by (1.1) - (1.5) and let there be $j \in \{1,...,M\}$ and $r \in \{1,...,N\}$ such that $s_r^j > 0$ and $w_r^j > 0$. According to whether $s_r^j - \eta_j w_r^j \geq 0$ or $s_r^j - \eta_j w_r^j < 0$ we construct a point $(\bar{y},\bar{s},\bar{w},\bar{z})$ which differs from (y,s,w,z) only in the components $\bar{s}_r^j$ and $\bar{w}_r^j$. We put

$$\bar{s}_r^j := s_r^j - \eta_j w_r^j, \ \bar{w}_r^j := 0 \quad \text{if } s_r^j - \eta_j w_r^j \geq 0 \text{ and}$$
$$\bar{s}_r^j := 0, \ \bar{w}_r^j := -\frac{1}{\eta_j} s_r^j + w_r^j \quad \text{else.}$$

In both situations we then have

$$\begin{aligned} \bar{s}_r^j - \eta_j \bar{w}_r^j &= s_r^j - \eta_j w_r^j \quad \text{and} \\ \bar{s}_r^j - \bar{w}_r^j &> s_r^j - w_r^j \ . \end{aligned} \qquad (1.6)$$

From this we conclude that $(\bar{y},\bar{s},\bar{w},\bar{z})$ fulfills (1.2) - (1.5). Furthermore, the objective values for (y,s,w,z) and $(\bar{y},\bar{s},\bar{w},\bar{z})$ are the same. hence, if (y,s,w,z) is optimal so is $(\bar{y},\bar{s},\bar{w},\bar{z})$, and the latter point can be obtained from the former one very easily.
In the case $y = \bar{y} \neq \underline{a}_1$ the argument can be extended: Consider one component, say $\bar{y}_r^i$, of $\bar{y}$ in which $\bar{y}$ differs from $\underline{a}_1$. Then there exists $\varepsilon > 0$ such that the point $(\tilde{y},\tilde{s},\tilde{w},\tilde{z})$ whose components coincide with those of $(\bar{y},\bar{s},\bar{w},\bar{z})$ with the exception of $\bar{y}_r^i$ where we put $\bar{y}_r^i - \varepsilon$ instead fulfills constraint (1.2) (note that (1.6) holds) and - of course - the remaining constraints. Due to strict monotonicity, however, the objective value of $(\tilde{y},\tilde{s},\tilde{w},\tilde{z})$ is less than that of (y,s,w,z). Hence (y,s,w,z) cannot have been optimal.

From the formal point of view our model can be expressed as

$$\min \{f(x): x \in X_0,\ \mu(\{d \in \mathbb{R}^N: Ax \geq d\}) \geq p_0\} \quad \text{or}$$

$$\min \{f(x): x \in X_0,\ F_\mu(Ax) \geq p_0\} \tag{1.7}$$

where $x = (y,s,w,z) \in \mathbb{R}^m$ with $m := N(K+2M+1)$, $f(x)$ is defined by (1.1), $X_0 \subset \mathbb{R}^m$ is the bounded convex polyhedron given by (1.3) - (1.5), $A \in L(\mathbb{R}^m, \mathbb{R}^N)$ is a suitable matrix, μ is the probability distribution of the (random) demand and F_μ its distribution function.

2. Sensitivity Analysis

Let us consider the following general chance constrained model

$$\min\{f(x): x \in \mathbb{R}^m,\ \mu(\{z \in \mathbb{R}^s: x \in X(z)\}) \geq p_0\} \tag{2.1}$$

where f is a real-valued function defined on $\mathbb{R}^m$, X is a set-valued mapping from $\mathbb{R}^s$ into $\mathbb{R}^m$, $p_0 \in (0,1)$ is a prescribed probability level and μ is a probability distribution on $\mathbb{R}^s$. For basic results on chance constrained problems consult [13],[36] and the references therein.
We are going to study the behaviour of (2.1) with respect to (small) perturbations of the probability distribution μ. Our approach relies on stability results for parametric optimization problems with parameters varying in metric spaces (see [15] for quantitative and [1],[25] for qualitative aspects). As parameter space we consider the space $\mathcal{P}(\mathbb{R}^s)$ of all Borel probability measures on $\mathbb{R}^s$ equipped with a suitable metric. We are aiming at (quantitative) continuity properties for the mappings assigning to each parameter the (local) optimal value and the set of (local) minimizers, respectively.
Because of its central place in the convergence theory for probability measures it seems appropriate to study stability with respect to the topology of weak convergence on $\mathcal{P}(\mathbb{R}^s)$. This has been done in the analysis carried out in [14] (using the results of [25]) and in [35]. An example in [28] indicates that stability of (2.1) with respect to the topology of weak

convergence cannot be expected in general without additional smoothness assumptions on the measure μ. It turned out in [29], [27] and [28] that the so-called $\mathcal{B}$-discrepancy

$$\alpha_{\mathcal{B}}(\mu,\nu):= \sup\{|\mu(B)-\nu(B)| : B\in\mathcal{B}\}\ (\mu,\ \nu\in\mathcal{P}(\mathbb{R}^s)),\qquad(2.2)$$

where $\mathcal{B}$ is a proper subclass of Borel sets in $\mathbb{R}^s$, is a suitable metric on $\mathcal{P}(\mathbb{R}^s)$ for the sensitivity analysis of (2.1). In the following, $\mathcal{B}$ will be chosen such that $\alpha_{\mathcal{B}}$ forms a metric on $\mathcal{P}(\mathbb{R}^s)$ (i.e. $\mathcal{B}$ is a determining class [8]) and that it contains all the pre-images $X^-(x) := \{z\in\mathbb{R}^s : x\in X(z)\}$ $(x\in\mathbb{R}^m)$. We also refer to [5] where sensitivity of optimal solutions to chance constrained problems involving parameter-dependent distributions is investigated by an approach via the implicit function theorem (cf. [7]). Stability in chance constrained programming is studied also in [30] and [33]. Whereas the results of [30] are relevant for approximation schemes, [33] deals with a statistical approach.

Next we introduce some basic concepts and notations which are used throughout. For $\nu\in\mathcal{P}(\mathbb{R}^s)$ we denote by F_ν the distribution function of ν and set for $p\in[0,1]$
$C_p(\nu) := \{x\in\mathbb{R}^m : \nu(X^-(x)) \geq p\}$, hence problem (2.1) becomes $\min\{f(x) : x\in C_{p_o}(\mu)\}$. Given $V\subseteq\mathbb{R}^m$ and $\nu\in\mathcal{P}(\mathbb{R}^s)$ we denote

$$\varphi_V(\nu) := \inf\{f(x) : x\in C_{p_o}(\nu)\cap \mathrm{cl}\ V\} \quad\text{and}$$

$$\Psi_V(\nu) := \{x\in C_{p_o}(\nu)\cap \mathrm{cl}\ V : f(x) = \varphi_V(\nu)\},$$

where we employ the abbreviation cl for closure. Following [25],[15] we call a nonempty subset M of $\mathbb{R}^m$ a complete local minimizing set (CLM set) for (2.1) with respect to Q if Q is an open subset of $\mathbb{R}^m$ such that $Q\supset M$ and $M = \Psi_Q(\mu)$. Later on we will briefly say that $\Psi_Q(\mu)$ is a CLM set for (2.1) which means that the set in question is a CLM set for (2.1) with respect to Q. Examples for CLM sets are the set of global minimizers (which we shall denote by $\Psi(\mu)$ and, accordingly, the global optimal value by $\varphi(\mu)$) or strict local minimizing points.
We call a multifunction Γ from a metric space (T,d) to $\mathbb{R}^m$

closed at $t_o \in T$ if $t_k \to t_o$, $x_k \to x_o$, $x_k \in \Gamma(t_k)$ $(k \in \mathbb{N})$ imply $x_o \in \Gamma(t_o)$, Γ is said to be upper semicontinuous (usc) at $t_o \in T$ if for any open set $G \supset \Gamma(t_o)$ there exists a neighbourhood U of t_o such that $\Gamma(t) \subset G$ whenever $t \in U$, and Γ is said to be pseudo-Lipschitzian at $(x_o, t_o) \in \Gamma(t_o) \times T$ (cf. [26]) if there are neighbourhoods U and V of t_o and x_o, respectively, and a constant $L > 0$ such that

$$\Gamma(t) \cap V \subseteq \Gamma(\tilde{t}) + Ld(t,\tilde{t})B_m \quad \text{whenever } t,\tilde{t} \in U,$$

where B_m is the closed unit ball in $\mathbb{R}^m$. For $x_o \in \mathbb{R}^m$ and $\varepsilon > 0$ we denote $B(x_o, \varepsilon) := \{x \in \mathbb{R}^m : \|x - x_o\| \leq \varepsilon\}$ (thus $B_m = B(0,1)$), where $\|.\|$ is the Euclidean norm on $\mathbb{R}^m$.

The following theorem asserts in a fairly general frame sensitivity properties for solutions of a parametric chance constrained problem. The proof which relies on stability results for abstract parametric programming problems obtained by D. Klatte in [15] can be found in [27] (Theorem 5.4).

<u>Theorem 2.1:</u>
Let in (2.1) $\mu \in \mathcal{P}(\mathbb{R}^s)$, $p_o \in (0,1)$ and $\{X^-(x) : x \in \mathbb{R}^m\} \subseteq \mathcal{B}$. Let further X be a closed multifunction and f be locally Lipschitzian. Assume that there exists a bounded open set $V \subset \mathbb{R}^m$ such that $\Psi_V(\mu)$ is a CLM set for (2.1). Let the multifunction $p \mapsto C_p(\mu)$ be pseudo-Lipschitzian at each $(x_o, p_o) \in \Psi_V(\mu) \times \{p_o\}$.
Then Ψ_V is usc at μ with respect to the metric $\alpha_{\mathcal{B}}$ on $\mathcal{P}(\mathbb{R}^s)$ and there exist constants $L > 0$ and $\delta > 0$ such that

$\Psi_V(\nu)$ is a CLM set for (2.1) and
$|\varphi_V(\mu) - \varphi_V(\nu)| \leq L\alpha_{\mathcal{B}}(\mu, \nu)$ whenever $\alpha_{\mathcal{B}}(\mu, \nu) < \delta$.

Under more restrictive assumptions it is possible to quantify also the upper semicontinuity of the solution set mapping.

<u>Theorem 2.2:</u>
Let μ, p_o, X, $\mathcal{B}$ and f be as in Theorem 2.1. Let further $|.|_*$ be a (non-trivial) semi-norm on $\mathbb{R}^m$.
Assume that there exist $x_o \in C_{p_o}(\mu)$ and constants $c > 0$, $\varrho > 0$

and $q \geq 1$ such that for all $x \in C_{p_o}(\mu) \cap B(x_o, \rho)$ we have

$$f(x) \geq f(x_o) + c|x-x_o|_*^q. \qquad (2.3)$$

Further, let the multifunction $p \mapsto C_p(\mu)$ be pseudo-Lipschitzian at (x_o, p_o).
Then there exist $\varepsilon \in (0,q]$, $L > 0$ and $\delta > 0$ such that with V taken as the open ball in $\mathbb{R}^m$ around x_o with radius ε the set $\Psi_V(\nu)$ is a CLM set for (2.1) and

$$|x-x_o|_*^q \leq L\,\alpha_{\mathfrak{B}}(\mu, \nu) \quad \text{for all } x \in \Psi_V(\nu),$$

whenever $\alpha_{\mathfrak{B}}(\mu, \nu) < \delta$, $\nu \in \mathcal{P}(\mathbb{R}^s)$.

To prove the above theorem one proceeds in principal as in [27] (Theorem 5.4), i.e. first derive continuity properties for the constraint set mapping $\nu \mapsto C_{p_o}(\nu)$ at μ (with respect to the distance $\alpha_{\mathfrak{B}}$ on $\mathcal{P}(\mathbb{R}^s)$) and then apply a quantitative stability result for parametric programs which is a slight relaxation of a theorem due to D. Klatte [15] and quoted as Theorem 2.6 in [27] . The relaxation concerns condition (2.3) where, compared to [15] and [27], we use a semi-norm rather than a norm. A direct inspection of the proof given in [15] then shows the validity of the result.

Remark 2.3:

The above results may also be viewed as stability results with respect to perturbations of μ in the space $\mathcal{P}(\mathbb{R}^s)$ equipped with the topology of weak convergence if $\mathfrak{B}$ is a μ-uniformity class of Borel sets in $\mathbb{R}^s$. Recall that $\mathfrak{B}$ is a μ-uniformity class if $\alpha_{\mathfrak{B}}(\mu_n, \mu) \to 0$ holds for every sequence (μ_n) converging weakly to μ ([2]). If $\mathfrak{B}$ is a subclass of $\mathfrak{B}_C := \{B \subset \mathbb{R}^s$: B is convex and Borel$\}$, the following result is known (Theorem 2.11 in [2]): $\mathfrak{B}$ is a μ-uniformity class if $\mu(\partial B) = 0$ for all $B \in \mathfrak{B}$ (here ∂B denotes the topological boundary of B). Hence, the class $\mathfrak{B}_R := \{\emptyset, (-\infty, z] : z \in \mathbb{R}^s\}$ is a μ-uniformity class if the distribution function F_μ (of μ) is continuous on $\mathbb{R}^s$, and $\mathfrak{B}_C$ is a μ-uniformity class if μ has a density (with respect to Lebesgue measure on $\mathbb{R}^s$). We note that

$d_K(\mu,\nu) := \alpha_{\mathcal{B}_R}(\mu,\nu) = \sup_{z\in\mathbb{R}^s} |F_\mu(z)-F_\nu(z)|$ is the so-called Kolmogorov distance on $\mathcal{P}(\mathbb{R}^s)$.

We now reveal conditions on the measure μ and on the multifunction X to have the mapping $p \longmapsto C_p(\mu)$ pseudo-Lipschitzian at some point $(x_o,p_o) \in \mathbb{R}^m \times [0,1]$, thus arriving at stability results which are specifications of the Theorems 2.1 and 2.2.

The first part of our analysis concerns the special case where the sets $C_p(\mu)$ ($p \in [0,1]$) are convex.

We say that $\mu \in \mathcal{P}(\mathbb{R}^s)$ belongs to the class $\mathcal{M}_r$ ($r \in [-\infty,+\infty)$) if for all $\lambda \in [0,1]$ and all Borel sets B_1, $B_2 \subset \mathbb{R}^s$,

$$\mu(\lambda B_1+(1-\lambda)B_2) \geq \{\lambda[\mu(B_1)]^r + (1-\lambda)[\mu(B_2)]^r\}^{1/r}. \quad (2.4)$$

Here $\lambda B_1 + (1-\lambda)B_2 := \{\lambda b_1+(1-\lambda)b_2 \colon b_i \in B_i,\ i=1,2\}$. In the case $r = 0$ and $r = -\infty$ the right-hand side of (2.4) is interpreted by continuity as $[\mu(B_1)]^\lambda [\mu(B_2)]^{1-\lambda}$ and $\min\{\mu(B_1),\mu(B_2)\}$, respectively. The classes $\mathcal{M}_r$ have been introduced and studied in [3], [17], [22]. Clearly, we have $\mathcal{M}_{r_1} \supseteq \mathcal{M}_{r_2}$, $-\infty \leq r_1 \leq r_2 < +\infty$. Measures belonging to $\mathcal{M}_0$ ($\mathcal{M}_{-\infty}$) are called logarithmic concave (quasi-concave). $\mathcal{M}_0$ was first and extensively studied by Prékopa [17], [18].

It is known (cf. e.g. Theorem 1 in [22]) that μ belongs to $\mathcal{M}_r$ ($r \in [-\infty,0]$) if μ has a density f_μ and $f_\mu^{r/(1-rs)}$ is convex ($-\infty \leq r < 0$), $\log f_\mu$ is concave ($r=0$).

It is well-known that the (non-degenerate) multivariate normal, the multivariate beta, Dirichlet and Wishart, a special multivariate gamma, and the multivariate Pareto, t and F distributions (cf. [12]) belong to $\mathcal{M}_r$ for some $r \leq 0$ (see [3], [17], [18], [20]).

For convex chance constraints we now have the following corollary to Theorem 2.1.

<u>Corollary 2.4:</u>

Assume that in (2.1) $\mu \in \mathcal{M}_r$ for some $r \in (-\infty,0]$, $p_o \in (0,1)$, X has closed convex graph and f is locally Lipschitzian.

Let $\{X^-(x)\colon x \in \mathbb{R}^m\} \subseteq \mathcal{B} \subseteq \mathcal{B}_C$ and $\bigcup_{z\in\mathbb{R}^s} X(z)$ be bounded. Assume

that there exists $\bar{x} \in \mathbb{R}^m$ such that $\mu(X^-(\bar{x})) > p_0$ (Slater condition).
Then Ψ is usc at μ with respect to $\alpha_{\mathcal{B}}$ on $\mathcal{P}(\mathbb{R}^s)$ and there exist constants $L > 0$ and $\delta > 0$ such that $\Psi(\nu) \neq \emptyset$ and $|\varphi(\mu) - \varphi(\nu)| \leq L\alpha_{\mathcal{B}}(\mu, \nu)$ whenever $\alpha_{\mathcal{B}}(\mu, \nu) < \delta$, $\nu \in \mathcal{P}(\mathbb{R}^s)$.

Proof:

Since $\mathcal{M}_0 \subseteq \mathcal{M}_r$ for each $r \in (-\infty, 0)$, we assume w.l.o.g. that $r \in (-\infty, 0)$ and write (2.1) in the equivalent form

$$\min\{f(x) : x \in \mathbb{R}^m,\ [\mu(X^-(x))]^r \leq p_0^r\}. \tag{2.5}$$

Since the constraint set of (2.5) is closed (see [27]) and bounded (according to the assumptions), we have that the set of global minimizers $\Psi(\mu)$ to (2.5) is nonempty and that the assumptions in Theorem 2.1 concerning the CLM set may be fulfilled with a bounded open set $V \supset \bigcup_{z \in \mathbb{R}^s} X(z)$ (hence the mappings Ψ and Ψ_V, φ and φ_V coincide).
We define the function $g(x) := [\mu(X^-(x))]^r$ from $\mathbb{R}^m$ to $(-\infty, \infty]$ and have for all $x_1, x_2 \in \mathbb{R}^m$ and $\lambda \in [0,1]$ that

$$\begin{aligned} g(\lambda x_1 + (1-\lambda)x_2) &= [\mu(X^-(\lambda x_1 + (1-\lambda)x_2))]^r \\ &\leq [\mu(\lambda X^-(x_1) + (1-\lambda)X^-(x_2))]^r \\ &\leq \lambda[\mu(X^-(x_1))]^r + (1-\lambda)[\mu(X^-(x_2))]^r. \end{aligned}$$

(Here we used in the first inequality that X has convex graph, and in the second that (2.4) is valid.)
Hence g is convex and the multifunction Γ (from $\mathbb{R}$ to $\mathbb{R}^m$) defined by $\Gamma(t) := \{x \in \mathbb{R}^m : g(x) \leq t\}$ $(t \in \mathbb{R})$ has closed convex graph. Due to Theorem 2 in [23], Γ is pseudo-Lipschitzian at each (x_0, t_0) with $x_0 \in \Gamma(t_0)$ and t_0 belonging to the interior of $\{t \in \mathbb{R} : \Gamma(t) \neq \emptyset\}$. Since $g(\bar{x}) < p_0^r$, p_0^r is an interior point of $\{t \in \mathbb{R} : \Gamma(t) \neq \emptyset\}$. Therefore, Γ is pseudo-Lipschitzian at (x_0, p_0^r) for each $x_0 \in \Gamma(p_0^r)$. In view of $C_p(\mu) = \Gamma(p^r)$, this means that there exist positive constants L, δ and a neighbourhood V of $x_0 \in C_{p_0}(\mu)$ such that

$$C_p(\mu) \cap V \subseteq C_{\tilde{p}}(\mu) + L|p^r - \tilde{p}^r| B_m$$

whenever $p^r, \tilde{p}^r \in B(p_o^r, \delta)$. Since the function $\xi \mapsto \xi^r$ is locally Lipschitzian for positive ξ, we obtain that the multifunction $p \mapsto C_p(\mu)$ is pseudo-Lipschitzian at each $(x_o, p_o) \in C_{p_o}(\mu) \times \{p_o\}$. The assertion now follows from Theorem 2.1. □

The above corollary entends results obtained by Salinetti ([30], Corollary 3.2.2) and Wang ([35], Theorem 6).
We remark that the Lipschitz modulus L in Corollary 2.4 can be estimated above provided that δ (which restricts $\alpha_{\mathfrak{B}_{\mathcal{X}}}(\mu, \nu)$) is sufficiently small. According to [15] such a bound for L is given by $L_f(L_C+1)$ where L_f is the (local) Lipschitz modulus for f and L_C the modulus we have for $p \mapsto C_p(\mu)$ since it is pseudo-Lipschitzian (cf. the proofs of Prop. 5.3 and Th. 5.4 in [27]). Starting from results of e.g. Robinson ([23], Theorem 2) or Psheničnyi ([21], Theorem 1.2, p. 100) a further estimation of L_C is possible. This would exploit the uniform compactness of the sets $C_p(\mu)$ $(p \in (0,1))$ and explicit knowledge of the Slater point $\bar{x}$.

<u>Remark 2.5:</u>
Let, additionally to the assumptions of Corollary 2.4, there exist $x_o \in C_{p_o}(\mu)$ and $c > 0$ such that

$$f(x) \geq f(x_o) + c|x - x_o|_*^q \quad \text{for all } x \in C_{p_o}(\mu), \tag{2.6}$$

where $|.|_*$ is a (non-trivial) semi-norm on $\mathbb{R}^m$. Then, using Theorem 2.2, we arrive at the following quantitative stability result for the global minimizers:
There exist constants $L > 0$ and $\delta > 0$ such that

$$|x - x_o|_*^q \leq L \alpha_{\mathfrak{B}_{\mathcal{X}}}(\mu, \nu) \qquad \text{for all } x \in \Psi(\nu)$$

whenever $\alpha_{\mathfrak{B}_{\mathcal{X}}}(\mu, \nu) < \delta$, $\nu \in \mathcal{P}(\mathbb{R}^s)$.

We proceed with the non-convex case. Here we assume that the multifunction X is given by

$$X(z) := \{x \in X_o : Ax \geq z\} \quad (z \in \mathbb{R}^s) \tag{2.7}$$

where $X_o \subseteq \mathbb{R}^m$ is a nonempty closed set and $A \in L(\mathbb{R}^m, \mathbb{R}^s)$. Again, sufficient conditions are essential under which the multifunction $p \mapsto C_p(\mu)$ is pseudo-Lipschitzian at certain points (x_o, p_o). From the literature it is known that constraint qualifications are such sufficient conditions (cf.[24], [26]). As an example for results that can be derived in this way we present the following:

Proposition 2.6:

Let the distribution function F_μ of $\mu \in \mathcal{P}(\mathbb{R}^s)$ be locally Lipschitzian, $p_o \in (0,1)$, X_o be a closed set and $x_o \in X_o$ such that $F_\mu(Ax_o) \geq p_o$. In case $F_\mu(Ax_o) = p_o$ let further $\partial F_\mu(Ax_o) \cap N_{X_o}(x_o) = \emptyset$, where ∂ denotes the Clarke generalized gradient of $F_\mu(A\cdot)$ and $N_{X_o}(x_o)$ is the Clarke normal cone to X_o at x_o ([4]).

Then the multifunction $p \mapsto \{x \in X_o : F_\mu(Ax) \geq p\}$ is pseudo-Lipschitzian at (x_o, p_o).

Proof:

Define $\Gamma(p) := \{x : p - F_\mu(Ax) \leq 0, \ (p,x) \in \mathbb{R} \times X_o\}$. According to Theorem 3.2 in [26] the multifunction Γ is pseudo-Lipschitzian at (x_o, p_o) if the following holds:

If there are $y, z \in \mathbb{R}$ such that
$y \geq 0$, $y(p_o - F_\mu(Ax_o)) = 0$ and
$(0,z) \in \{y(x,1) + (\tilde{x},0) : -x \in \partial F_\mu(Ax_o), \ \tilde{x} \in N_{X_o}(x_o)\}$
then $y = z = 0$.

Now assume that in our situation the above did not hold. Then there were $y > 0$, $\bar{x} \in \partial F_\mu(Ax_o)$ and $\tilde{x} \in N_{X_o}(x_o)$ such that $-y\bar{x} + \tilde{x} = 0$. The last identity, however, implies $\bar{x} \in N_{X_o}(x_o)$ which contradicts $\partial F_\mu(Ax_o) \cap N_{X_o}(x_o) = \emptyset$. $\square$

Of course, making use of Proposition 2.6 hinges upon whether one is able to check the constraint qualification $\partial F_\mu(Ax_o) \cap N_{X_o}(x_o) = \emptyset$. In applications this may be a formidable task, especially when exploiting the result in its fullest generality.

Therefore, in the following we establish by an alternative way sufficient conditions which are easier to verify and similar to that given in [29].

Corollary 2.7:

In (2.1) let $\mu \in \mathcal{P}(\mathbb{R}^s)$ have a continuous distribution function F_μ, further let $p_o \in (0,1)$ and the multifunction X be given by (2.7) where the set X_o is convex and closed.
Suppose there exists a bounded open set $V \subset \mathbb{R}^m$ such that $\Psi_V(\mu)$ is a CLM set for (2.1). For each $x_o \in \Psi_V(\mu)$ with $F_\mu(Ax_o) = p_o$ let there exist reals $\varepsilon_o > 0$ and $c > 0$ such that for any $x \in X_o \cap B(x_o, \varepsilon_o)$ there exists $\bar{x} \in X_o$ with the property

$$F_\mu(Ax + tA(\bar{x}-x)) \geq F_\mu(Ax) + ct \quad \text{for all } t \in [0,1]. \tag{2.8}$$

Then Ψ_V is upper semicontinuous at μ with respect to the metric d_k on $\mathcal{P}(\mathbb{R}^s)$ and there exist constants $L > 0, \delta > 0$ such that $\Psi_V(\nu)$ is a CLM set for (2.1) and

$$|\varphi_V(\mu) - \varphi_V(\nu)| \leq L d_K(\mu, \nu)$$

whenever $d_K(\mu, \nu) < \delta$, $\nu \in \mathcal{P}(\mathbb{R}^s)$.

Proof:

Once more we apply Theorem 2.1. We merely have to check whether the mapping $p \mapsto C_p(\mu)$ is pseudo-Lipschitzian at each $(x_o, p_o) \in \Psi_V(\mu) \times \{p_o\}$.
Let $x_o \in \Psi_V(\mu)$ and consider at first the case where $F_\mu(Ax_o) > p_o$. Then there exists $\delta_o > 0$ such that $F_\mu(Ax_o) > p_o + \delta_o$ and due to the continuity of F_μ we have $\varepsilon_o > 0$ such that

$$F_\mu(Ax) \geq p_o + \delta_o \qquad \text{for all } x \in B(x_o, \varepsilon_o).$$

Hence $C_p(\mu) \cap B(x_o, \varepsilon_o) \subseteq C_{p+\delta}(\mu)$ for each $p \in (p_o - \delta_o, p_o]$ and each $\delta \in (0, \delta_o)$. Therefore the multifunction $p \mapsto C_p(\mu)$ is pseudo-Lipschitzian at (x_o, p_o). Now let $F_\mu(Ax_o) = p_o$. Take $\varepsilon_o > 0$ and $c > 0$ according to the assumption and define $\delta_o := c$ and $L := c^{-1}$. We will show that

$$C_p(\mu) \cap B(x_o, \varepsilon_o) \subseteq C_{p+\delta}(\mu) + L\delta B_m$$

for each $p \in (p_o - \delta_o, p_o]$ and each $\delta \in (0, \delta_o)$, which yields

the desired pseudo-Lipschitzian property.
Let $p \in (p_o - \delta_o, p_o]$, $\delta \in (0, \delta_o)$ be chosen arbitrarily and consider $x \in C_p(\mu) \cap B(x_o, \varepsilon_o)$. Due to the assumption there exists $\bar{x} \in X_o$ such that (2.8) holds. In view of the convexity of X_o, without loss of generality, it is possible to select this $\bar{x}$ in a way such that we additionally have $\|\bar{x}-x\| \leq 1$.
Consider $y := x + \delta c^{-1}(\bar{x}-x) \in X_o$. Now

$$\|x-y\| \leq L\delta \quad \text{and}$$
$$F_\mu(Ay) = F_\mu(Ax+\delta c^{-1}A(\bar{x}-x)) \geq F_\mu(Ax)+c\delta c^{-1} \geq p+\delta .$$

Hence $y \in C_{p+\delta}(\mu)$ and $x \in C_{p+\delta}(\mu) + L\delta B_m$.
The assertion finally follows from Theorem 2.1. □

Remark 2.8:

If F_μ is continuously differentiable at Ax_o then (2.8) implies the constraint qualification used in Proposition 2.6.

Remark 2.9:

Corollary 2.7 is a generalization of Corollary 2.4 when X is given as in (2.7).
To see this suppose that $\mu \in \mathcal{M}_r$ for some $r \in (-\infty, 0]$ and assume that there exists $\bar{x} \in X_o$ such that $F_\mu(A\bar{x}) > p_o$ (Slater condition). Then the distribution function F_μ is continuous, since μ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^s$ ([3]). Now let $x_o \in X_o$ such that $F_\mu(Ax_o) = p_o$.
There exist $\varepsilon_o > 0$ and $\delta_o > 0$ such that
$0 < p_o - \delta_o \leq F_\mu(Ax) \leq p_o + \delta_o < F_\mu(A\bar{x})$ for all $x \in B(x_o, \varepsilon_o)$.
We are going to show that, with a suitable $c > 0$, condition (2.8) is fulfilled for any $x \in X_o \cap B(x_o, \varepsilon_o)$.
For this, let w.l.o.g. $r < 0$ and define

$$a := p_o - \delta_o > 0 \quad \text{and} \quad b := [F_\mu(A\bar{x})]^r - (p_o + \delta_o)^r < 0.$$

We obtain for arbitrary $t \in [0,1]$:

$$\begin{aligned} F_\mu(Ax+tA(\bar{x}-x))^r &\leq t\, F_\mu(A\bar{x})^r + (1-t)F_\mu(Ax)^r \\ &\leq a^r + t(F_\mu(A\bar{x})^r - (p_o + \delta)^r) = a^r + tb \end{aligned}$$

and therefore

$$F_\mu(Ax+tA(\bar{x}-x)) \geq (a^r+tb)^{1/r}$$
$$\geq a+tr^{-1}ba^{1-r} \quad \text{for all } t\in[0,-a^rb^{-1}].$$

The last inequality holds since the function $g(t):= (a^r+tb)^{1/r}$ is convex for $t\in[0,-a^rb^{-1}]$ and consequently

$$g(t) \geq g(0) + g'(0)t \quad \text{for } t\in[0,-a^rb^{-1}].$$

Taking finally into account that $-a^rb^{-1} > 1$ we obtain (2.8) with $c := r^{-1}ba^{1-r}$.

The following lemma is very useful when verifying the uniform growth condition (2.8). Its proof is essentially based on an idea that has already been developed in [29], Lemma 4.9.

Lemma 2.10:

Let $\mu\in\mathcal{P}(\mathbb{R}^s)$, X_o be a closed convex set and fix some $x_o\in X_o$. Assume that μ has a density f_μ and that there exist $\Delta>0$, $\varrho>0$ such that

$$f_\mu(z) \geq \Delta \qquad \text{for all } z\in B(Ax_o,\varrho).$$

Furthermore, assume that there exists $\bar{x}\in X_o$ such that $A\bar{x} \geq Ax_o$ and $A\bar{x} \neq Ax_o$.

Then there exist $\varepsilon_o>0$ and $c>0$ such that (2.8) holds for each $x\in X_o\cap B(x_o,\varepsilon_o)$.

Proof:

First one confirms that, without loss of generality, it is possible to suppose $A\bar{x}\in B(Ax_o,\varrho_1)$ and $[A\bar{x}]_1>[Ax_o]_1$ where $\varrho_1:=\varrho/4$ and $[z]_i$ denotes the i-th component of $z\in\mathbb{R}^s$. Now we choose $\varepsilon_o>0$ such that on the one hand there exists $\xi\in\mathbb{R}$ such that $[A\bar{x}]_1>\xi \geq [Ax]_1$ for all $x\in B(x_o,\varepsilon_o)$ and on the other hand

$$\max_{i=1,\dots,s} |[Ax]_i - [Ax_o]_i| \leq \varrho_1 \quad \text{for all } x\in B(x_o,\varepsilon_o).$$

Denote $a:= [A\bar{x}]_1 - \xi > 0$.

Then we have for arbitrary $x\in X_o\cap B(x_o,\varepsilon_o)$ and $t\in[0,1]$

$$F_\mu(Ax+tA(\bar{x}-x))-F_\mu(Ax) \geq$$

$$\geq \int_{[Ax]_1}^{[Ax]_1+ta} \int_{-\infty}^{[Ax]_2} \cdots \int_{-\infty}^{[Ax]_s} f_\mu(z_1,\dots,z_s)dz_s\dots dz_1$$

$$\geq \int_{[Ax]_1}^{[Ax]_1+ta} \int_{[Ax_o]_2-2\varrho_1}^{[Ax_o]_2-\varrho_1} \cdots \int_{[Ax_o]_s-2\varrho_1}^{[Ax_o]_s-\varrho_1} f_\mu(z_1,\dots,z_s)dz_s\dots dz_1$$

$$\geq t\; a\, \varrho_1^{s-1}\, \Delta$$

Hence, the desired result follows with $c := ([A\bar{x}]_1 - \xi)\Delta(\frac{\varrho}{4})^{s-1}$. □

We remark that Corollaries 2.4 and 2.7 also represent qualitative stability results with respect to weak convergence of probability measures. This is mainly due to the smoothness assumptions imposed on the measures which led to μ-uniformity classes (cf. Remark 2.3). On the other hand, also without such smoothness assumptions conclusions from Theorem 2.1 may be drawn, as can be seen by the following remark where we deal with discrete distributions.

<u>Remark 2.11:</u>
Let $\mu \in \mathcal{P}(\mathbb{R}^s)$ be a discrete measure with countable support, consider (2.1) with X given by (2.7). Let $p_o \in (0,1)$ be such that $\inf_{z\in\mathbb{R}^s} |F_\mu(z)-p_o| > 0$.
Then there exists a neighbourhood U of p_o such that $C_{p_o}(\mu) = C_p(\mu)$ for all $p \in U$ and, consequently, the mapping $p \mapsto C_p(\mu)$ is pseudo-Lipschitzian at each (x_o,p_o) with $x_o \in C_{p_o}(\mu)$.
If the objective in (2.1) is locally Lipschitzian and if there exists a bounded open set $V\subset\mathbb{R}^m$ such that $\Psi_V(\mu)$ is a CLM set for (2.1), we now obtain the stability assertions of Theorem 2.1 with respect to the Kolmogorov metric d_K.

In what follows we indicate the potential of our general results for the situation of unknown distribution μ.

Let $\xi_1, \xi_2, \ldots$ be independent random variables on a probability space $(\Omega, \mathcal{A}, P)$ with values in $\mathbb{R}^s$ and common distribution μ. Consider the empirical measure μ_n which is given by

$$\mu_n(\omega) := n^{-1} \sum_{i=1}^{n} \delta_{\xi_i(\omega)} \quad (\omega \in \Omega, n \in \mathbb{N}),$$

where $\delta_z \in \mathcal{P}(\mathbb{R}^s)$ denotes the measure with unit mass at $z \in \mathbb{R}^s$. Then it is known that (see e.g. [8] and the references therein)

$$d_K(\mu_n(\omega), \mu) = O\left(\left(\frac{\log \log n}{n}\right)^{1/2}\right) \quad \text{P-almost surely} \tag{2.9}$$

and

$$P(\{\omega : d_K(\mu_n(\omega), \mu) > \varepsilon\}) \leq C_1 \exp(-C_2 \varepsilon^2 n) \tag{2.10}$$

where $C_1 > 0$ and $0 < C_2 < 2$ are some constants.
Inequality (2.10) often is referred to as Dvoretzky-Kiefer-Wolfowitz inequality.
Our quantitative stability results together with relation (2.9) now give rise to rates for the almost sure convergence of optimal values and optimal solutions if the unknown distribution μ is estimated by empirical distributions.

Let us finally illustrate how to combine our Lipschitz (or Hölder) stability results with inequality (2.10). Suppose for instance you have a result of the type

$$|\varphi_V(\mu) - \varphi_V(\nu)| \leq L d_K(\mu, \nu) \quad \text{whenever } d_K(\mu, \nu) < \delta$$

(Corollary 2.7, Remark 2.11). Then we obtain

$$P(\{\omega : |\varphi_V(\mu_n(\omega)) - \varphi_V(\mu)| > \varepsilon\})$$
$$\leq P(\{\omega : \varepsilon < L d_K(\mu_n(\omega), \mu)\}) + P(\{\omega : d_K(\mu_n(\omega), \mu) > \delta\})$$

and in view of (2.10) we can continue

$$\leq 2C_1 \exp(-C_2(\min\{\tfrac{\varepsilon}{L}, \delta\})^2 n).$$

Following the above way, in principle, it is possible to derive corresponding estimates for optimal solutions or feasible sets. In the latter case one then arrives at results which are in the spirit of Theorem 3 and Proposition 1 in [33].

3. Conclusions

To ensure a certain level of reliability for solutions to optimization problems containing random data it has become an accepted approach to introduce probabilistic (or chance) constraints into the model. In applications, however, one is often faced with incomplete information on the underlying probability distributions. Therefore, applicable models should at least enjoy some kind of stability with respect to variations of the distributions involved. This gives rise to investigating distribution sensitivity of the models. Compared to earlier work ([29],[27],[28]) the present paper deals with more practicable models, and it gives sufficient conditions for (also quantitative) stability of optimal values and optimal solutions which are easier to verify.
For a quite large class of distributions (Corollaries 2.4 and 2.7, Lemma 2.10, Remark 2.11) we obtain upper semicontinuity of the optimal-set-mapping and Lipschitz continuity of the optimal value function. Under more restrictive assumptions it is possible to quantify the upper semicontinuity of the optimal-set-mapping (Remark 2.5).
The material developed in Section 2 applies to a number of practical models which are known from the literature (the STABIL model [19], a flood control model [20], a model for water resources system planning [6]).

For the load dispatch model presented in Section 1 we may derive the following conclusions:
If we assume that we have approached the true distribution of the demand with sufficient accuracy then the optimal production policies behave upper semicontinuous and the optimal costs are Lipschitz continuous if either:

- we know that the true distribution has a certain convexity property (cf. (2.4)) and there exists a Slater point (Corollary 2.4), or
- the true distribution is a discrete one (Remark 2.11), or
- the true distribution has a density which is uniformly bounded below by a positive number on some neighbourhood related

to the set of optimal solutions and among the optimal policies (with respect to the true distribution) there is no one which exhausts the full generation capacity (see constraint (1.3)) (Corollary 2.7, Lemma 2.10). (In practice, the latter requirement on the optimal generation policy is always fulfilled, since, due to lower demand during the night, there is usually at least one power station which, during at least one hour, does not work with maximum capacity.)

An examination of the objective in the optimal-load-dispatch model shows that is possible to fulfil condition (2.6) with $q = 2$ and $|x|_* := \|y\|_2$ (here $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^{NK}$). Hence, in presence of the assumptions made in Corollary 2.4, Remark 2.5 applies, and we have Hölder continuity (with exponent 1/2) of the optimal generation policies in the thermal plants.

When the original distribution is estimated by empirical ones then the presented stability results together with the considerations at the end of Section 2 yield rates of convergence for optimal values and optimal solutions.

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