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Institute of Mathematics and Mechanics, Sverdlovsk,
Ural Branch of the Academy of Sciences of the U.S.S.R.

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

FOREWORD

This paper deals with a specific inverse problem of dynamics for a system described by a parabolic inequality. The aim is to reconstruct the input (the control) of the system on the basis of an on-line measurement corrupted by an error.

The techniques applied to the solution are a combination of those developed in positional control theory and the theory of ill-posed problems. This paper was contributed by the author during his visit to the SDS Program.

A. Kurzhanski
Program Leader
System and Decision Sciences Program.

Inverse Problem of Dynamics for Systems Described by Parabolic Inequality

Yu.S. Osipov

Institute of Mathematics and Mechanics
of the Ural Scientific Center
Academy of Sciences of the USSR, Sverdlovsk

The considered problem is concerned with the following questions.

Let t be the time variable. Consider an evolutional system Σ on an interval $T = [t_0, \theta]$. We are interested in some unknown characteristic $\xi_1(t)$, $t \in T$ of the system (e.g., ξ_1 may be a collection of some parameters of the system, or of some disturbances acting on the system or of controls etc.). We are to reconstruct $\xi_1(t)$ on the basis of measurements of some other characteristic $\xi_2(t)$, $t \in T$ of the system Σ. The results of measurements $\varsigma(t)$ are not precise, the error being estimated by h.

The smaller h is, the more precise should be the reconstruction (in the appropriate sense). This is the stability property of the reconstruction algorithm D_h.

We consider two types of reconstruction problems. In the problems of the first type (which we call problems of program reconstruction) the measurements $\varsigma(t)$ are known for all $t \in T$ at once. Hence the input of the reconstruction algorithm is the function $\varsigma(t)$, $t_0 \leq t \leq \theta$. The output of D_h is a function $\xi_1^{(h)}(t)$, $t_0 \leq t \leq \theta$ close (in a suitable sense) to the characteristic $\xi_1(t)$, $t_0 \leq t \leq \theta$ for h small enough.

In problems of the second type (we call them problems of dynamical reconstruction) the characteristic ξ_1 is to be restored simultaneously with the process of system motion. Here in every current moment t the input of the algorithm D_h is the previous history $\varsigma_t = \varsigma_t(\cdot) = \{\varsigma(\tau), t_0 \leq \tau < t\}$ of the measurements ς made prior to the moment t. The output of D_h in the moment t is a function

$$\xi_{1t}^{(h)}(\cdot) = \{\xi_1^{(h)}(\tau), t_0 \leq \tau < t\} ,$$

which approximates (in the proper sense) the characteristic

$$\xi_1(\tau),\ t_0 \leq \tau \leq t\ , \text{ for small } h\ .$$

Here D_h is to satisfy the property of physical realizability [2], [3]: if $\varsigma^{(1)}(\tau)$, $t_0 \le \tau \le t_1$ and $\varsigma^{(2)}(\tau)$, $t_0 \le \tau \le t_2$ are such that

$$\varsigma_{t_*}^{(1)} = \varsigma_{t_*}^{(2)}, \; t_* \le \min \{t_1, t_2\} \; ,$$

then the functions $D_h\varsigma^{(1)}(\cdot)$, $D_h\varsigma^{(2)}$ are equal on $[t_0, t_*)$.

Below we consider a problem of the second type for a system described by a parabolic inequality. We develop further the method for dealing with such kind of problems proposed in [1-3]. The method is based on some ideas of positional control theory [14-17] and ill-posed problems theory [18].

The present paper is connected with [1-13].

Let $\mathbf{V}$ and $\mathbf{H}$ be real Hilbert spaces, $\mathbf{V}^*$ and $\mathbf{H}^*$ be the spaces dual to $\mathbf{V}$ and $\mathbf{H}$ respectively. We identify $\mathbf{H}$ with $\mathbf{H}^*$. It is supposed that $\mathbf{V} \subset \mathbf{H}$ is dense in $\mathbf{H}$ and is embedded into $\mathbf{H}$ continuously. Denote by $(\cdot,\cdot)_{\mathbf{H}}$ and $|\cdot|_{\mathbf{H}}$ ($(\cdot,\cdot)_{\mathbf{V}}$ and $|\cdot|_{\mathbf{V}}$) the scalar product and the corresponding norm in $\mathbf{H}$ (in $\mathbf{V}$).

Let t be the time variable, $t \in T = [t_0, \theta]$. Consider on T a control system Σ. The state of the system is $y(t) \in \mathbf{V}$. The evolution of the state is given by the following conditions for almost all $t \in T$ the inequality holds ([19,20]):

$$(\dot{y}(t),\, y(t) - \omega)_{\mathbf{H}} + a(y(t),\, y(t)) + \phi(y(t)) - \phi(\omega) \le (Bu(t) + f(t),\, \omega)_{\mathbf{H}} \quad \forall \omega \in \mathbf{V} \tag{1.1}$$

and

$$y(t_0) = y_0 \; . \tag{1.2}$$

Here $a(\omega_1, \omega_2)$ is a continuous on $\mathbf{V}$ bilinear symmetrical form satisfying for some $c_1 > 0$ the condition

$$a(\omega,\omega) \ge c_1 |\omega|_{\mathbf{V}}^2 \; ; \tag{1.3}$$

$\phi: \mathbf{V} \to (-\infty, +\infty]$ is a convex proper lower semicontinuous function (or $\phi: \mathbf{H} \to (-\infty, +\infty]$ is a convex proper lower semicontinuous function satisfying the regularity condition [21,22]; $B : \mathbf{U} \to \mathbf{H}$ is a linear continuous operator, $\mathbf{U}$ is a uniformly convex real Banach space; $f \in L^2(T;\mathbf{H})$; $u(\cdot)$ is a control, i.e. measureable on T function for almost all $t \in T$ having values in bounded closed convex set $P \subset \mathbf{U}$; $y_0 \in \{\omega \in \mathbf{V} : \phi(\omega) < +\infty\}$. Under the above assumptions in $\mathbf{W}^{1,2}(T;\mathbf{H}) \cap L^2(T;\mathbf{V})$ there exists a unique function $y(t) = y(t; t_0, y_0, u(\cdot))$, $t \in T$, satisfying (1.1), (1.2) (see [19-22]). We call it a motion of system Σ from the initial state y_0 corresponding to control $u(\cdot)$.

Consider the following problem of dynamical reconstruction. Let $\mathbf{V} = \mathbf{H}_0^1(\Omega)$ (or $\mathbf{V} = \mathbf{H}^1(\Omega)$), $\mathbf{H} = L^2(\Omega)$,$\mathbf{U} = L^2(\Omega)$,B be the identity operator (see notation in [19,20]). Now in (1.1) we take

$$y(t) = y(t,\cdot) = \{y(t,x),\ x\in\Omega\}\ ,$$

$$\dot{y}(t) = \partial y(t,\cdot)/\partial t,\ u(t) = u(t,\cdot)\ .$$

Let the control u be of the form

$$u(t) = u(t,x) = \chi_{G(t)}(x) \times u^0(t,x) \tag{1.4}$$

Here $G(t) \subset \Omega$ is such that the set $\{(t,x) : t\in T,\ x\in G(t)\}$ is Lebesgue measureable; χ_G is the characteristic function of G; the function u^0 satisfies the inequality

$$0 < \beta_1 \le u^0(t,x) \le \beta_2,\ t\in T,\ x\in\Omega\ , \tag{1.5}$$

where β_1, β_2 are positive numbers.

Let the measurement of the system state $y_*(t) = y_*(t,\cdot)$ be possible in every current moment t, the measurement result $\varsigma(t) = \varsigma(t,\cdot)$ satisfying the estimation

$$|\varsigma(t,\cdot) - y_*(t,\cdot)|_{L^2(\Omega)} \le h\ . \tag{1.6}$$

Suppose that the motion being observed is generated by the unique control of the type (1.4), (1.5)

$$u_*(t,x) = \chi_{G_*(t)} u_*^0(t,x),\ t\in T,\ x\in\Omega\ .$$

Consider the problem of dynamical reconstruction with

$$\xi_1(t) = \{u_*(t)\ ;\ S_*(t)\}\ ,$$

$$S_*(t) = \{(\tau,x) : \tau\in[t_0,t),\ x\in G_*(\tau)\}\ ;$$

$$\xi_2(t) = y(t,\cdot)\ .$$

Remark 1.1. Let e.g., (1.1), (1.2) describe the process of diffusion of a substance in a domain Ω and $y(t,\cdot)$ be the concentration of substance in Ω in the moment t. Then we deal with the reconstruction of intensity of the substance sources and their location (see [12]).

We proceed the following way (see [12, 13]). To the system Σ we put into correspondence a control system Σ_1 (the model) which is a copy of Σ.

$$(z(t),\, z(t) - \omega)_{L^2(\Omega)} + a(z(t),\, z(t)) \tag{1.7}$$

$$- \omega) + \phi(z(t)) - \phi(\omega) \leq (v(t) + f(t),\omega)_{L^2(\Omega)} \quad \forall w \in V$$

$$z(t_0) = y_0 \; .$$

The control $v(\cdot) \in L^2(T\,;\, L^2(\Omega))$ in the model is chosen for almost all $t \in T$ from convex bounded closed set P which contains all the $L^2(\Omega)$ functions of the form $\chi_B \cdot g(x)$ where $B \subset \Omega$ is a measurable set, $g(\cdot)$ is a measurable function, $g : \Omega \rightarrow [\beta_1, \beta_2]$.

Consider a partition τ_i of interval T,

$$t_0 = \tau_0 < \tau_1 < \cdots < \tau_m = \theta \; ;$$

$$m = m(h),\, \delta(h) = \max_i(\tau_{i+1} - \tau_i),\, \delta(h) \leq ch,\, c = const > 0 \; .$$

Take

$$v(t) = v^{(h)}(t) = v_i,\, \tau_i \leq t < \tau_{i+1} \, , \quad i = 1,\ldots,m$$

where v_i are (the unique) points of minimum of the functional

$$\psi(p) = 2(z(\tau_i\,;\, t_0,\, y_0,\, v(\cdot)) - \varsigma(\tau_i),\, p)_{L^2(\Omega)} + \alpha(h)|p|^2_{L^2(\Omega)} \; .$$

The function $\alpha(h) > 0$; $\alpha(h) \rightarrow 0$, $h/\alpha(h) \rightarrow 0$ as $h \rightarrow 0$. Form the set

$$S_i^{(h)} = [\tau_i,\, \tau_{i+1}) \times \{x \in \Omega : v_i(x) \geq \mu\} \, , \tag{1.8}$$

where μ is some positive number $\beta_1 \leq \mu \leq \beta_2$.

Denote

$$S^{(h)} = \bigcup_{i=0}^{m-1} S_i^{(h)} \, ,$$

where $d(S_*(\theta),\, S^{(h)})$ is the Lebesgue measure of the symmetric difference of sets S_*, $S^{(h)}$.

Theorem. If $h \rightarrow 0$ then the following is valid

$$|v^{(h)} - u_*|_{L^2(T;L^2(\Omega))} \rightarrow 0$$

$$d(S(\theta),\, S^{(h)}) \rightarrow 0 \; .$$

Remark 1.2. Similar to [12] one can obtain an estimate of reconstruction accuracy.

2. Consider an example. Let ϕ be a convex continuous function under the assumption of Section 1. Then the system (1.1) is equivalent to the equation

$$\frac{\partial y}{\partial t} = Ay + u + f(t,x),\ t \in T,\ x \in \Omega,\ y|_\Gamma = 0 \tag{2.1}$$

Here A is an elliptic coercive operator

$$Ay = \frac{\partial}{\partial x_j}\left(a_{ij}(x)\,\frac{\partial y}{\partial x_i}\right) - q(x)y,\ a_{ij} = a_{ji}\,, \tag{2.2}$$

$$a_{ij} \in L^\infty(\Omega),\ q \in L^\infty(\Omega)\ .$$

For (2.1) consider a concrete variant of reconstruction problem [12].

Let Ω be a two-dimensional domain

$$0 < x_1 < \ell_1\ ,\ 0 < x_2 < \ell_2\ ;\ f = 0\ ,\ q = 0$$

and

$$Ay = a^2 \cdot \partial^2 y/\partial x_1^2 + b^2 \cdot \partial^2 y/\partial x_2^2\ .$$

For the sake of simplicity we confine the considerations to the case of reconstruction of location $G(t)$, $t \in T$. Let it be known a priori that the control being restored satisfying the inequality $|u(t,\cdot)|_{L^2(\Omega)} \le R$.

A closed ball in $L^2(\Omega)$ of radius R is taken as P. Then

$$v_i = [\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))]\ /\ \alpha(h)\ \text{ if}$$

$$|\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))|_{L^2(\Omega)} \le R\cdot\alpha(h)\ ,$$

$$v_i = R\cdot[\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))]\ /\ |\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))|_{L^2(\Omega)}\ ,\ \text{ if}$$

$$|\varsigma(\tau_i) - z(\tau_i\ ;\ t_0,y_0,v(\cdot))|_{L^2(\Omega)} > R\cdot\alpha(h)\ .$$

For the considered variant of the problem the calculations were carried out for the following data

$$a^2 = b^2 = 0.1\ ,\ \ell_1 = \ell_2 = 10\ ,\ t_0 = 0\ ,\ \theta = 1\ ,\ R = 100\ ,$$

$$y_0 = 0\ ,\ \beta_1 = \beta_2 = 10\ ,\ \delta(h) = h\ ,\ \alpha(h) = \sqrt{h}\ ,\ h = 0.1\ .$$

The motions of the dynamical system and the auxiliary model were calculated with the help of an explicit difference scheme with constant time step $\tau = \delta(h)$ and constant spatial steps γ_1 and γ_2 in x_1 and x_2 respectively.

The set $G(t_0)$ is depicted in Fig. 1 and Figs. 2 and 3 show the results of reconstruction of the set

$$G(t) = \{(x_1,x_2) : 0.01 \le x_1 \le 9.99, x_1(t,x_1) \le x_2 \le x_2(t,x_1)\} ,$$

where

$$x_1(t,x_1) = 3.5 + \cos(0.5\cdot x_1 - 5\cdot t) + 0.3\cdot\cos(5\cdot x_1 + t/h)\cdot\sin(3.2\cdot x_1 + t/h) ,$$

$$x_2(t,x_1) = 6.5 + \cos(0.5\cdot x_1 - 5\cdot t) + 0.3\cdot\cos(10\cdot x_1 + t/h)\times\sin(3.2\cdot x_1 + t/h) ,$$

at the moments $t = 0.5$, $t = 0.9$ respectively for

$$\gamma_1 = \gamma_2 = 10/16 .$$

The unknown set is reconstructed with the help of rectangles with centres in the mesh nodes and sides γ_1 and γ_2 parallel to axes x_1, x_2 respectively.

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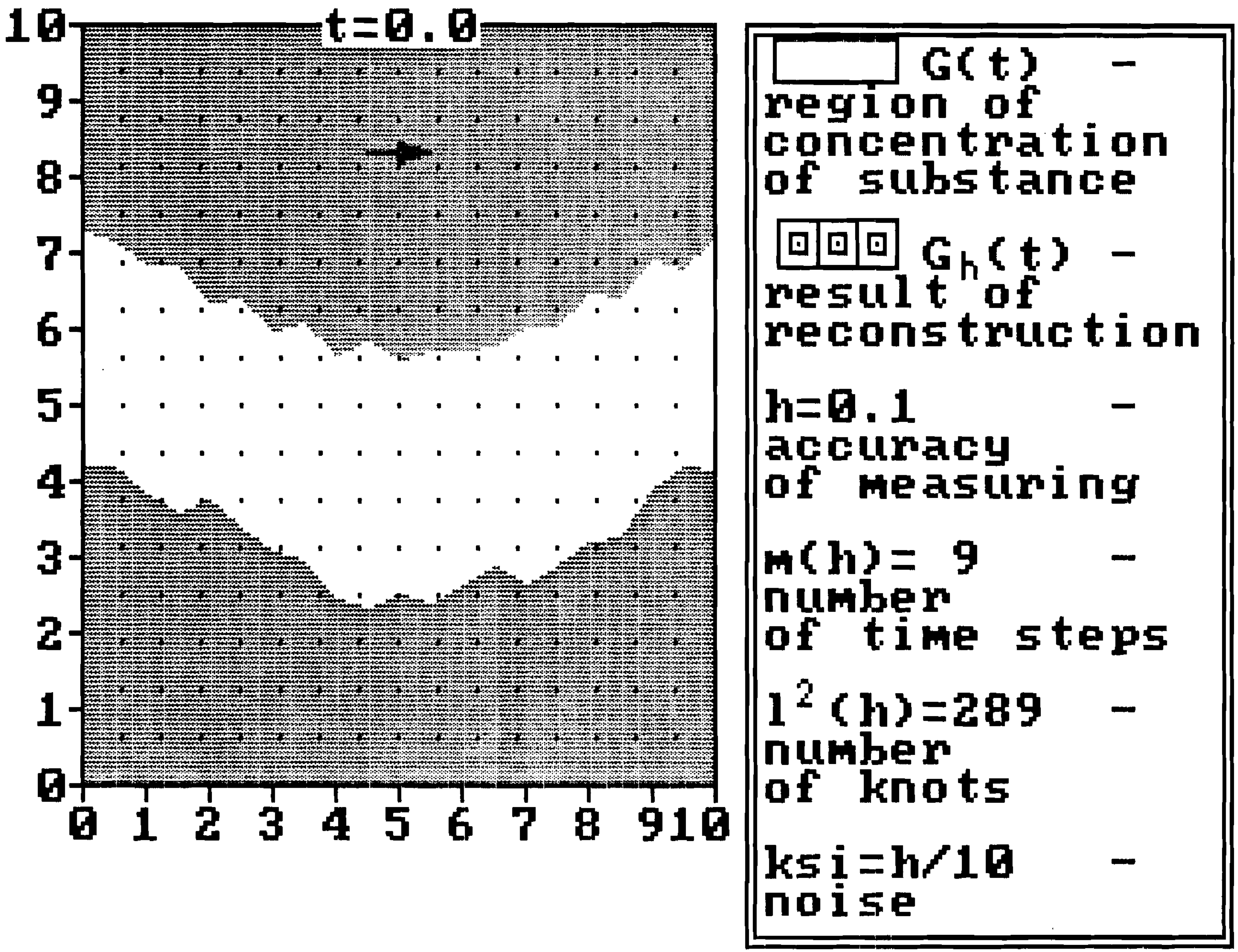
t=0.0
0 1 2 3 4 5 6 7 8 9 10
G(t) - region of concentration of substance
G_h(t) - result of reconstruction
h=0.1 - accuracy of measuring
m(h)= 9 - number of time steps
l^2(h)=289 - number of knots
ksi=h/10 - noise

Figure 1

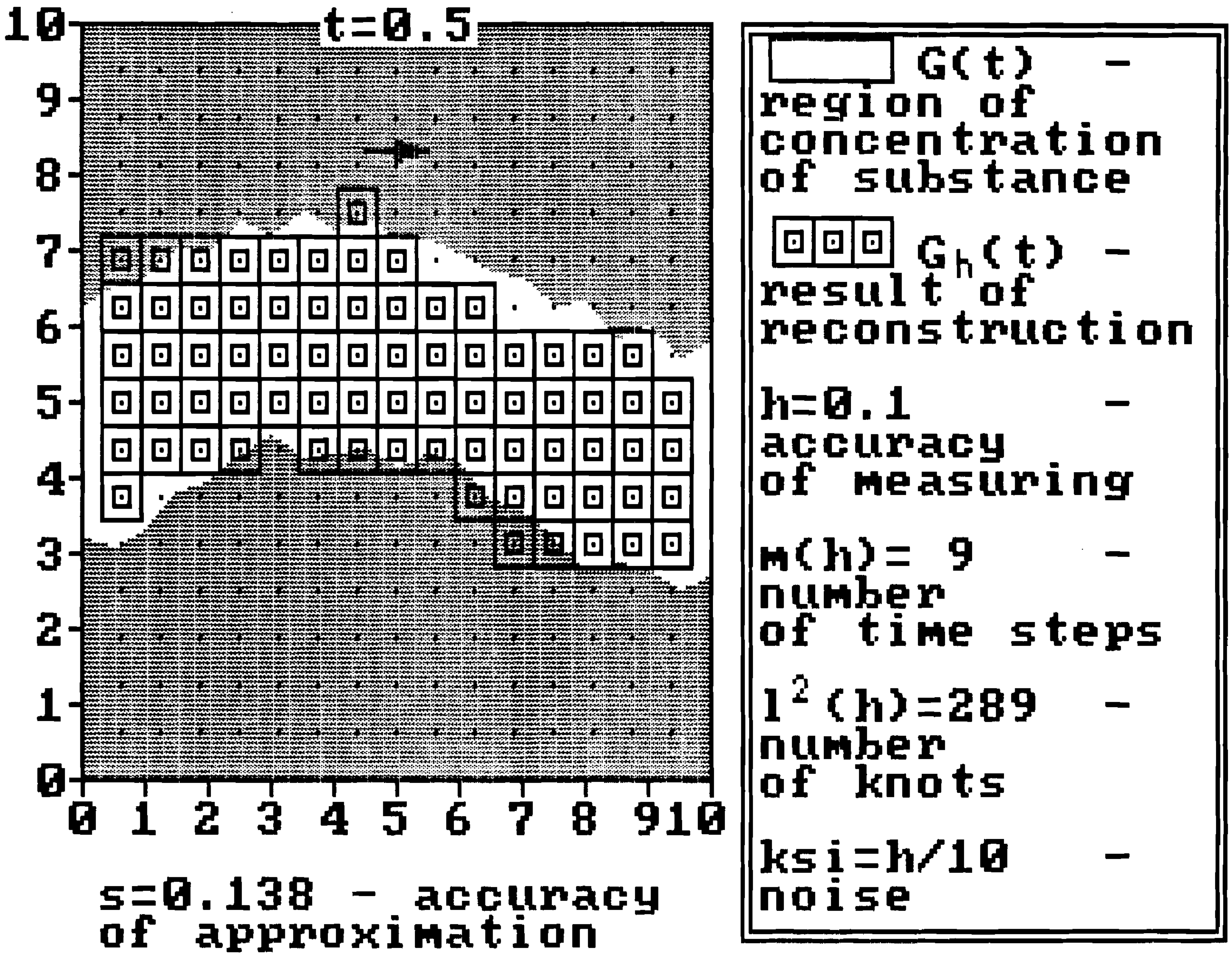
t=0.5
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10
s=0.138 - accuracy
of approximation
G(t) -
region of
concentration
of substance
G_h(t) -
result of
reconstruction
h=0.1 -
accuracy
of measuring
m(h)= 9 -
number
of time steps
l^2(h)=289 -
number
of knots
ksi=h/10 -
noise

Figure 2

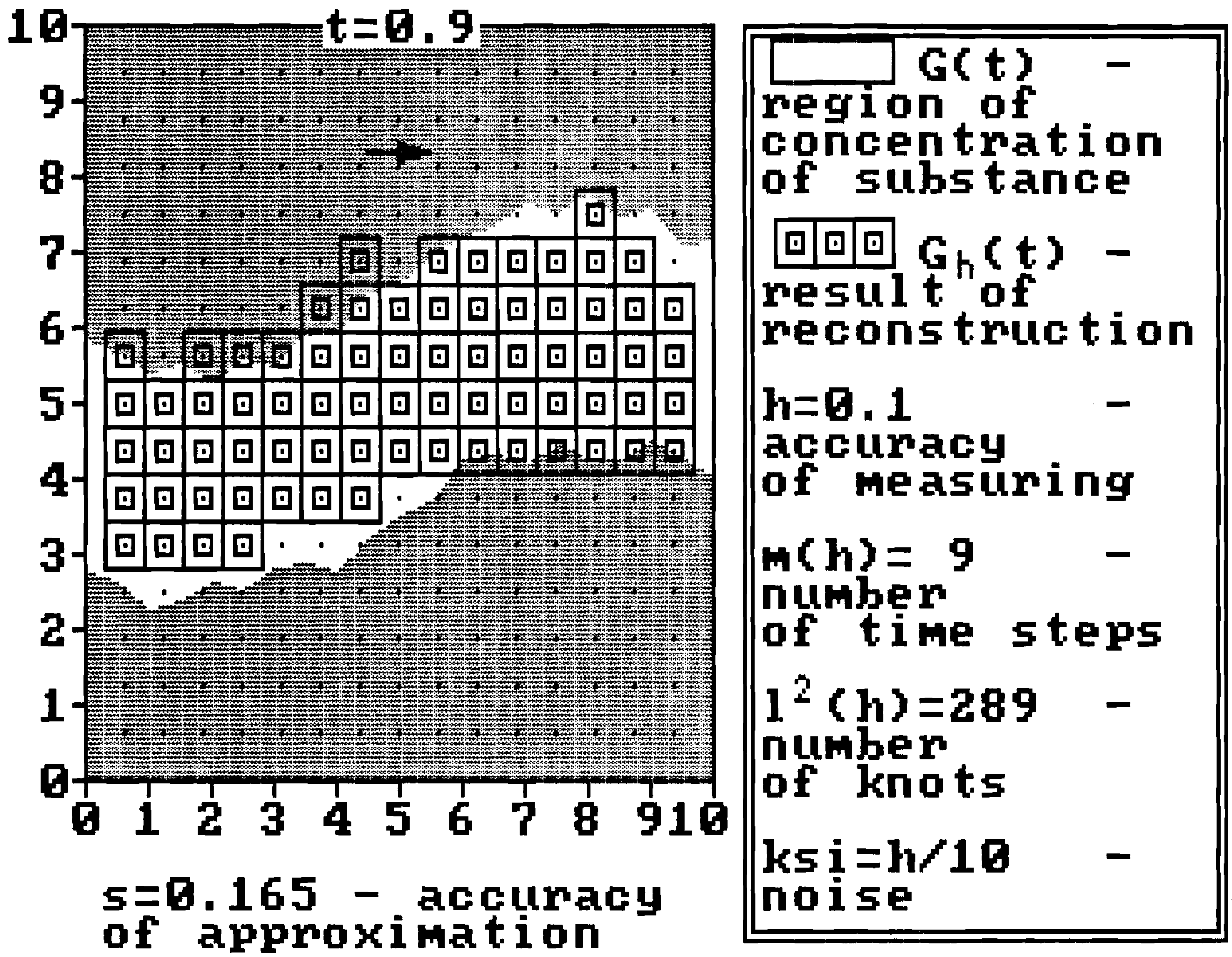
t=0.9
10
9
8
7
6
5
4
3
2
1
0
0 1 2 3 4 5 6 7 8 9 10
s=0.165 - accuracy of approximation
G(t) - region of concentration of substance
$G_h(t)$ - result of reconstruction
h=0.1 - accuracy of measuring
m(h)= 9 - number of time steps
$l^2(h)$=289 - number of knots
ksi=h/10 - noise

Figure 3

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