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Aspiration Based Decision Analysis and Support Part I: Theoretical and Methodological Backgrounds

Lewandowski, A. and Wierzbicki, A.P.

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Andrzej Lewandowski
IIASA

and

Andrzej Wierzbicki
Institute of Automatic Control
Warsaw University of Technology
Warsaw, Poland

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

Foreword

In the interdisciplinary and intercultural systems analysis that constitutes the main theme of research in IIASA, a basic question is how to analyze and support decisions with help of mathematical models and logical procedures. This question - particularly in its multi-criteria and multi-cultural dimensions - has been investigated in System and Decision Sciences Program (SDS) since the beginning of IIASA. Researches working both at IIASA and in a large international network of cooperating institutions contributed to a deeper understanding of this question.

Around 1980, the concept of reference point multiobjective optimization was developed in SDS. This concept determined an international trend of research pursued in many countries cooperating with IIASA as well as in many research programs at IIASA - such as energy, agricultural, environmental research. SDS organized since this time numerous international workshops, summer schools, seminar days and cooperative research agreements in the field of decision analysis and support. By this international and interdisciplinary cooperation, the concept of reference point multiobjective optimization has matured and was generalized into a framework of ***aspiration based decision analysis and support*** that can be understood as a synthesis of several known, antithetical approaches to this subject - such as utility maximization approach, or satisficing approach, or goal - program - oriented planning approach. Jointly, the name of quasisatisficing approach can be also used, since the concept of aspirations comes from the satisficing approach. Both authors of the Working Paper contributed actively to this research: Andrzej Wierzbicki originated the concept of reference point multiobjective optimization and quasisatisficing approach, while Andrzej Lewandowski, working from the beginning in the numerous applications and extensions of this concept, has had the main contribution to its generalization into the framework of aspiration based decision analysis and support systems.

This paper constitutes a draft of the first part of a book being prepared by these two authors. Part I, devoted to theoretical foundations and methodological background, written mostly by Andrzej Wierzbicki, will be followed by Part II, devoted to computer implementations and applications of decision support systems based on mathematical programming models, written mostly by Andrzej Lewandowski. Part III, devoted to decision support systems for the case of subjective evaluations of discrete decision alternatives, will be written by both authors.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program.

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Aspiration Based Decision Analysis and Support Part I: Theoretical and Methodological Backgrounds

1. INTRODUCTION

1.1 What is multiobjective analysis?

Various methods for multiobjective optimization and decision making that have been developed since the work of Pareto (1896) have been summarized in many books; see, for instance, Luce and Raiffa (1957), Fandel (1972), Bell, Keeney and Raiffa (1977), Hwang and Masud (1979), Peschel (1980), Rietveld (1980), Spronk (1981), Dinkelbach (1982), Zeleny (1982), Sawaragi, Nakayama and Tanino (1985). In general terms, these methods deal with the situation where one or more persons must generate and choose between various alternatives that cannot be evaluated on the basis of a single aspect, an attribute or a scalar performance measure (a "single-objective") alone. Instead, the evaluation must involve a number of aspects, of attributes, or of performance characteristics ("multiple objectives") which are often not commensurable.

Such situations often arise when technological, economic, social or political decisions are made, and are usually resolved either by intuition, or by the collective processes of choice that have grown up throughout human history. Thus there is nothing new in multiobjective decision making - people have been doing it for thousands of years. However, this term has recently taken on a new and much more specific meaning with the applications of mathematical methods to the problem. These methods are generally designed to clarify the decision making situation and to generate useful alternatives; sometimes they involve considerable use of computers and computerized models. However, in none of these methods can a single practical decision be made without the involvement and approval of people - and the authors hope that this will never happen, except in the most routine of situations. To call this group of methods "multiobjective decision making" without further qualification is therefore semantically misleading; we should perhaps rather refer to it as *multiobjective analysis.*

Researchers concentrating on the mathematical part of the multiobjective analysis prefer to speak of multiobjective optimization. However, this would limit the field of study to a particular area of mathematics, while the motivation and importance of multiobjective analysis come not from mathematics but rather from applied problems. Thus, for methodological clarity, we should consider multiobjective analysis as a part of the multidisciplinary applied science called *applied systems analysis.*

Some readers might object to the definition of "applied systems analysis" as a "multidisciplinary applied science". For example, Rietveld (1980) defines systems theory more traditionally as a new science concerned with the functioning of systems in general, and the word system itself has a very old meaning as a description of a set of elements and the relationships between them. However this definition is too broad: on this basis Ptolemy, Copernicus and Bohr were systems analysts, since the first two investigated the solar system, while the third studied the atomic system. The new factor in contemporary systems analysis is the realization that certain methodological principles and mathematical tools can be applied to systems in a multidisciplinary fashion.

Contemporary systems analysis also lays great importance on the applied or empirical aspects of research. Mathematical systems theory is a new and still developing branch of applied mathematics which includes the theory of dynamical systems, optimization theory, some aspects of economic equilibrium theory, game theory and multiobjective decision theory. Though the initial practical motivation (for example, mechanics, electronics, economics) underlying any part of mathematical systems theory is responsible for the basic concepts, the theory still remains a branch of applied mathematics, where the fundamental questions are those of syntactical correctness and completeness of mathematical language; questions of semantic importance are considered valid only in the sense of motivation. This interpretation of mathematics as a language in which empirical statements can be formulated and transformed, but never validated in the empirical sense is quite clear in the modern philosophy of science. Thus, it is the applied nature of systems analysis that holds the real meaning, for all the beauty of the mathematical language that we can use to describe it.

An empirical scientific statement is one that purports to explain some observations made in the real world and admits an empirical falsification test. In pure sciences, such statements may not have any immediate uses, at least none that can be easily perceived. By contrast, applied sciences concentrate on producing empirical statements of perceived direct usefulness, though these might be limited in their precision and validity. Some researchers distinguish between science and technology on the understanding that science is interested in the universal questions of general validity, while technology considers questions of an approximate, "good enough", "mostly", "can do" character see, e.g., Rose (1982). On this basis, systems analysis is a multidisciplinary methodology for technological thought. However, this understanding of technology is peculiar to the English language; more modern usage and most other languages prefer the broader term of applied sciences. When using this phrase, however, we must avoid narrow interpretations in terms of utilitarian science. This can be illustrated by the classical anecdote about three people who, not knowing anything about electricity, observed that amber sometimes attracts pieces of paper. One of them, a utilitarian scientist, concluded that this amusing fact could have no possible uses. Another one, a technologist, started to produce toys based on this observation. Finally, the third individual, a motivated basic scientist, decided to study the phenomenon, with the result that he discovered electricity and all its potential applications.

To summarize these initial remarks, we can state that multiobjective analysis is a part of a multidisciplinary applied science called systems analysis, and is concerned with situations in which complex decisions involving many objectives must be made. Its purpose is to clarify the problem by constructing prototypes of decision situations, using certain fundamental concepts based on empirical observations. After the prototype situations and related concepts have been chosen, they are described in mathematical language, and mathematical tools can then be used to suggest how these situations should be handled. While the development of mathematical methods for multiobjective analysis is an important element of this scientific discipline, it is even more important that any statement in the multiobjective analysis should be validated by repeated empirical falsification tests. Since we consider multiobjective analysis to be an empirical scientific discipline, we must choose mathematical tools and language that, while syntactically correct, yield statements that are both empirically testable and semantically valid. If we develop decision analysis methods, we cannot stop at mathematical idealizations, we must test our assumptions and methods on applications.

1.2 Why Interactive Decision Analysis?

The progress in integrated circuits and computer technology over the last forty years has prepared ground for a new era; parallel processing, fifth generation computer principles, user-friendly software and large scale production of microcomputers open almost unbounded opportunities for applications in information (data and knowledge) processing, research and development, automatization and robotization of technological, administrative, engineering design processes. However, it will take some decades to take the full advantage of this new technology: new generations must assimilate the new computerized culture, learn to use and live with computers, adjust to the requirements of an era of advanced information processing in the professional and private life. The use of computers is no longer just a professional speciality; especially microcomputing is pervading our social structure - see, e.g., Hazan (1984). Researchers, designers, and educators have unprecedented opportunities - and responsibilities - in helping this technology to meet its potential.

A broad class of computer applications is concerned with decision, policy and strategy analysis. Computerized mathematical models of various aspects of human activity have long been used for these purposes. However, the principles behind computerized decision analysis and support are by no means universally agreed upon, and there are many different schools of thought about how computers should be used. Some support the paradigm of predictive models, which give unique answers but with limited accuracy or validity; some weaken this paradigm by scenario analysis. Some believe in normative models that prescribe how things should happen (based on some theory), and reinforce this by exploiting the tools of mathematical optimization and game theory. Others criticize this approach for its lack of realism and put forward instead the idea of descriptive, behavioral modeling; this criticism is often directed without discrimination at both the normative methodological assumptions and the mathematical tools. Some instinctively dislike any models that imply hierarchical organizations; others take hierarchy in organizations for granted and develop methods and tools for handling hierarchical models.

There are even various schools and approaches with regard to mathematical tools: some prefer static models, while others claim that without accounting for dynamic effects any decision analysis is doomed to failure; the different mathematical descriptions of dynamic processes (e.g., difference equations, ordinary or partial differential equations, equations with delay, differential inclusions, integral equations) all have their adherence. The proponents of linear versus nonlinear models, differentiability versus nondifferentiability, and various methods of handling uncertainty also create diversity. Some prefer to handle uncertainty using statistical models, some using deterministic models with scenarios and interval analysis, other broader probabilistic and stochastic approaches, others adaptive and learning procedures, while yet others argue for the use of fuzzy sets.

The authors of this book believe that a diversity of mathematical tools is necessary, and we should develop many of them. However, mathematical models and methods should play the role of decision *support* only; even if we could construct very precise models of reality, they will never incorporate all human concerns. Thus, an essential part of decision analysis are issues of *interaction* between a human decision maker and computerized models and decision support systems; and this interaction must be dynamic, in several senses. One sense is that human decision makers typically *learn* when using a decision support system, and we cannot assume that a decision maker comes to the system with fixed preferences. Another sense is that dynamic models are of particular importance in any long-term or strategic planning; most applications in this book belong to such category. With all these divisions, increasing numbers of mathematical modellers and systems analysts have come to the conclusion that mathematical models for decision or policy analysis must be built and used *interactively*, that is involving the users at all stages of the process. Again, there are various interpretation of what is meant by interaction. Some understand it to mean simply some way of improving communication between a user and a computerized model. Others stress the educational, learning and adaptive aspects of computerized simulation, experimenting with models, computerized simulated gaming, and procedures for organizing interaction between groups of experts, users and decision makers. Others understand interaction as a tool in decision making, and combine multiobjective optimization with normative decision theory to construct an interactive decision support system. Other try to broaden the principles of interaction while preserving some mathematical rigour and exploiting a wide range of existing mathematical tools. Such a heterogeneity of approaches is not only an inevitable, but also a desirable consequence of the turbulent history of computer modeling. However, new directions can often be found by trying to bridge the gaps between existing approaches.

In this book, we try to present various aspects of interactive use of models in decision analysis. However, we do prefer a particular view on the role of interactive decision analysis: we see it mostly as a tool for *learning* about various aspects of a novel decision problem rather as a tool of selecting one *optimal* decision.

Our preference is supported by recent studies on the differences in the style of decision making between novices and experts - Dreyfus (1984). A true expert or master in a given field does not make his decisions analytically: he evaluates entire relevant information by "Gestalt" and arrives intuitively at the decision, in contrast to novices who have to order available information into analytical categories before painfully reaching a decision. First when an expert faces a novel aspect in a decision situation, he starts to deliberate: but even then he does not come back to an analytical decomposition of the problem, but attempts to find a new intuition by examining new angles and approaches. Thus interactive decision analysis - used either in teaching novices to become experts, or helping experts to perceive new aspects of a novel decision situation - should not concentrate solely on providing an analytical framework for ordering available information and reaching an optimal decision; it should much rather help in learning about various aspects of the decision situation, in forming an intuitive understanding that results in the decision. This decision must be made by the human decision maker that is responsible for it, not by the computerized system. In this sense, we tend to believe in computerized, interactive decision analysis and support, much rather than in computerized (even if interactive) decision making.

1.3 About this book

This book presents a methodology of interactive decision support developed mostly by its authors, but with invaluable help of many friends and co-workers during several years of research in the International Institute for Applied Systems Analysis in Laxenburg, Austria. Research on decision analysis problems originated in this Institute under its first Director, Howard Raiffa; subsequent Directors, Roger Levien, Crawford Holling and Thomas Lee gave unceasing encouragement and support to the research on decision analysis, its theory, methodology, software tools, decision support systems and their applications. An important step in the development of methodology of decision support at IIASA was the concept of reference-point or reference-trajectory multiobjective optimization introduced around 1980 - see Wierzbicki (1979, 1980) and Kallio at all. (1980). Later, many other authors both in IIASA and in independent research institutions - see, e.q. Korhonen (1985) - further extended and developed this basic idea.

The concept of reference-point optimization determined an international trend of research pursued in many countries cooperating with IIASA as well as in many research programs at IIASA - such as energy, agricultural, environmental research. By this international and interdisciplinary operation, the concept of reference-point multiobjective optimization has matured and was generalized into a framework of *aspiration-based decision analysis and support* that can be understood as a synthesis of several known, antithetical approaches to this subject - such as utility maximization approach, or satisficing approach or goal- and program-oriented planning approach. Jointly, the name of *quasi-satisficing* approach can be also used, since the concept of aspirations comes from the satisficing approach, but is augmented by reference point optimization.

This lead to the creation of a family of aspiration based decision support systems in IIASA as well as in cooperating institutions in Poland, Bulgaria and other NMO countries. Directly involved in this process were, besides the authors, Markku Kallio and William Orchard-Hays, later Manfred Grauer who contributed significantly to many ideas in this book, at IIASA, then Tomasz Kreglewski, Tadeusz Rogowski, Marek Makowski, Janusz Sosnowski, Janusz Majchrzak, Grzegorz Dobrowolski, Henryk Gorecki, Jerzy Kopytowski, Tomasz Rys, Maciej Zebrowski and many other researchers from various Polish research institutions, Manfred Peschel from the GDR, then Leo Schrattenholzer, Manfred Strubegger, Sabine Messner, Stephan Kaden, Sergei Orlowski, Ernö Zalai and many other co-workers and friends at IIASA. Most of the development of this methodology was hosted by the System and Decision Sciences Program at IIASA, first under the leadership of Andrzej Wierzbicki, then of Alexander Kurzhanski who gave us support and encouragement to finalize this book.

Aspiration based decision support systems, in various stages of experimental development, were transferred to over 50 collaborating research institutions in many countries of the world and tested on a variety of substantive examples in many applied areas of systems analysis. The authors of this book and their friends have written numerous papers and gave many presentations on international conferences on the subject of this development; finally, we felt that it is the highest time to summarize these experiences in a book form.

We believe that the main advantage of this book is a broad and synthetical outlook obtained from an extensive East-West collaboration on comparing various approaches or even cultures of interactive decision analysis. This was possible because IIASA supported, over many years, several conferences and workshops in this field, such as in the recent four years:

- the Task Force Meeting on Multiobjective and Stochastic Optimization in 1981 at IIASA in Laxenburg, Austria,
- the International Workshop on Interactive Decision Analysis in 1983 at IIASA also in Laxenburg, Austria and
- the International Summer Study on Plural Rationality and Interactive Decision Processes in 1984 in Sopron, Hungary,
- the International Workshop on Large-Scale Modelling and Interactive Decisions in 1985, Wartburg, Eisenach, G.D.R.
- the Seventh International Conference on Multiple Criteria Decision Making, held in Kyoto, Japan in 1986,
- the International Workshop on Methodology and Software for Interactive Decision Support in 1987 in Albena, Bulgaria,
- a sequence of task force meetings New Advances in Decision Support Systems held either in IIASA or in cooperating countries during 1986 and 1987.

The monograph is divided into three parts. The first part which constitutes the content of this Working Paper was written mostly by Andrzej Wierzbicki. The second part relating to the implementation issues of DIDAS family and several applications and the third part relating to the issues of group decision support systems are now being written by Andrzej Lewandowski. The second and third part will be published as separate Working Papers.

In the second chapter we classify basic decision situations. The centralized single-actor situation is considered first, with a discussion of its main concepts: the decision maker, the supporting team of analysts and the "substantive knowlegde" or mathematical model of the problem. Next the situation with centralized decisions and multiple actors is analyzed in terms of hierarchy versus consensus. The role of expert advice is also presented in this context. The autonomous multiple-actor situation is then described against a background of game-theory, gaming and conflict resolution. The impact of diverging perceptions on conflict is analyzed. This chapter concludes with a discussion of the the issues of uncertainty and dynamic planning.

The third chapter presents major theoretical frameworks for rational decision making. It starts with the concept of utility maximization, describing its origins, and discussing the main developments and techniques, experience and criticisms associated with this framework. The origins and concepts of aspiration formation and satisficing behavior are then discussed and the main developments in satisficing decision making are described. Further, the origins and concepts of hierarchical rational decision making, so called "goal-and-program-oriented planning" are analyzed. Finally, a synthesis of these three frameworks that assumes a broader type of behavior of the decision maker, called quasisatisficing behavior, is conceptually presented.

The fourth chapter presents mathematical foundations for quasisatisficing behavior and reference point optimization. It starts with the issue of completeness and constructiveness of characterizations of efficient solutions; leading to an almost complete and constructive characterization by maximization of order-consistent scalarizing achievement functions. Various types of such functions are then discussed in detail, together with functionals needed for multiobjective dynamic trajectory optimization. Basic concepts of organization and phases of decision support based on the quasisatisficing framework are then discussed. This section ends with an analysis of convergence issues in quasisatisficing interactive processes, including the aspects of learning and adaptation of preferences by the decision maker.

2. BASIC TYPES OF DECISION SITUATION AND OTHER ISSUES

2.1 Centralized Single-Actor Situations

Most of the work in multiobjective analysis is based on the prototype decision situation illustrated in Figure 2.1a. This involves a "decision maker" (a single person who has the authority and experience to take the actual decision); an "analyst" or team of analysts responsible for the analysis of the decision situation; and a "substantive model of the problem" that is supposed to represent all the pertinent knowledge that the analyst(s) can muster. It should be emphasized that the term "model" is used here in a very broad sense. It is not necessarily a computerized mathematical model; it may just be a collection of relevant knowledge, data and hypotheses. But this is still a model, not reality, and this fact should be stressed very strongly when examining the methodological implications of the basic prototype. The model is based on the analyst's perception of the decision problem, and this perception may be wrong, or inconsistent with that of the decision maker. Thus, the model should be validated before use. However, before this the model must first be built.

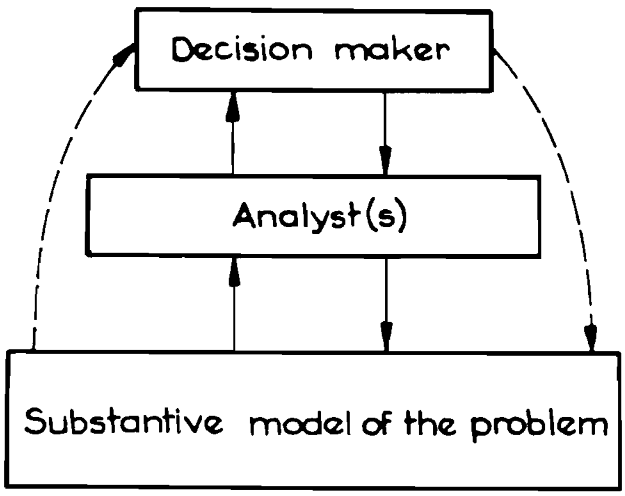

Figure 2.1a. A simple prototype decision situation.

The methodology of model building is itself a separate subject in systems analysis, with its own extensive literature - see, for example, Wierzbicki (1977) and Lewandowski (1982). Here we shall list only a few general principles.

1. The ultimate purpose of the model should be the most important consideration in model building; the model should also be the simplest possible that serves the purpose. One of the most important tasks of model building is to identify the relevant information, hypotheses, etc.

2. Models should be built in an iterative fashion, at each iteration developing and executing falsification tests examining internal consistency, consistency with other information, consistency with available empirical data, and consistency with new data

gathered specifically for falsification purposes.

3. Models should be built interactively, involving not only analysts but also decision makers, so that the decision maker's perceptions of the problem, of the relevant data, and of the model validity can be taken into account.

Unfortunately, these principles are not observed in many system-analytic studies, with multiobjective analysis being one of the worst offenders. A possible reason for this is that multiobjective analysis is often influenced by economic traditions and it is known that the methodological principles of empirical science are sometimes not followed in economic studies (see, for example, a recent critical essay by Leontief, 1982). However, important as the subject is, there is no place here for a detailed discussion of model building. We must assume that the substantive model of the problem has already been built and validated, and concentrate on the second stage: the use of the model to clarify the decision situation.

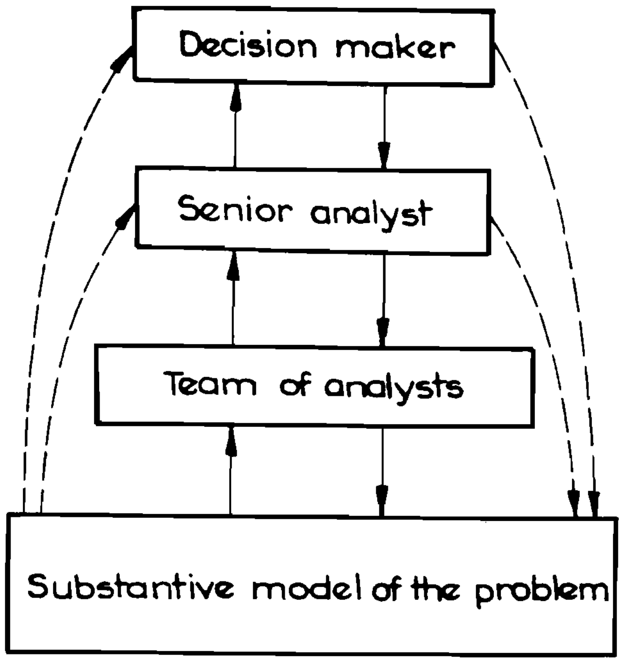

Figure 2.1b. A prototype decision in situation with a hierarchy of analysts.

Before we do this, however, it should be noted that the prototype situation shown in in Figure 2.1a is usually oversimplified. Much more common is the situation shown in Figure 2.1b, where there is an additional link, a senior analyst responsible for explaining the situation to the decision maker. In other cases individual experts may be involved in evaluating the alternatives proposed by the analysts, as in Figure 2.1c. The elements of these nontrivial variants of the first prototype can also be combined in other ways. In addition, the "decision maker" from Figure 2.1a could actually be a "senior analyst" or "expert" or "politician". These distinctions apply mostly to complex, non-repetitive decision situations, often classified as *strategic planning*. In repetitive decision situations of *operational planning*, the prototype decision structure can often be simplified and involves only the problem and the decision maker. However, the main feature of all these prototypes is that decision making is actually centralized, and that one decision-maker (the *single actor*) is responsible for the decision.

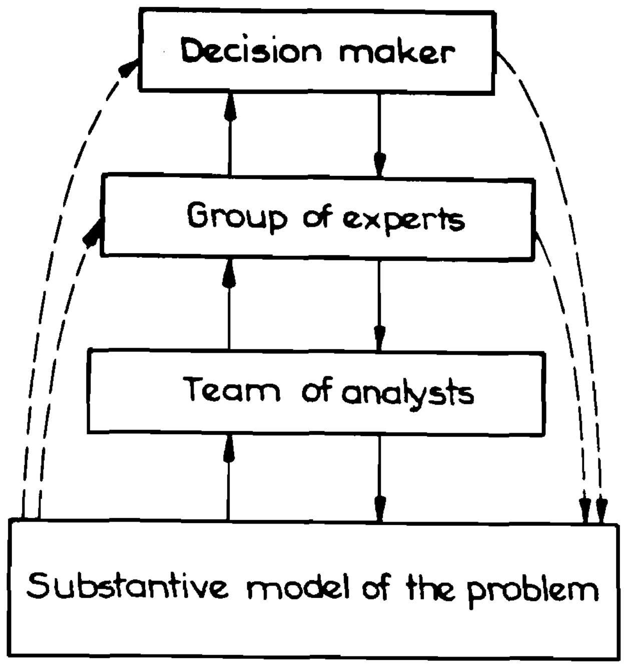

Figure 2.1c A prototype decision situation with a group of experts.

Now, it is the duty of the team of analysts not only to clarify the substantive aspects of the decision situation, but also to formulate proposals taking into account the institutional aspects of this situation i.e., the characteristics of the political process that will lead to the actual decision. This principle is not generally followed in contemporary multi-objective analysis, where attention is concentrated primarily on the prototype situation from Figure 2.1a. However, there are some notable exceptions.

One of the most common aspects of political processes is that neither the decision maker nor even the experts have much time to study the very detailed reports prepared by the analysts. Even if this is not the case, the decision-making process is usually split into two phases. The first phase is usually performed by the team of analysts with some possible interaction from the decision maker, and involves the *generation* of a small number of *alternatives*. The second phase is the responsibility of the decision maker (possibly with the help of experts and senior analysts) and concerns the *choice between alternatives*. Both phases have characteristic features.

Clearly, the stronger the interaction with the decision maker in the first phase, the easier is the second phase. However, in many situations the substantive model is not sufficiently formalized to allow easy interaction. A team of analysts can sometimes have no option but to generate (more or less intuitively) a number of alternatives that seem professionally sound, and submit them to the decision maker.

On the other hand, if the substantive model can be formulated in mathematical terms and computerized, and if the decision maker or experts or even the senior analyst can work interactively with the model to generate alternatives, the chances that the alternatives will be satisfactory are greatly improved. In such a case, it is important to computerize not only the substantive model, but also an interactive decision support system to help the user work with the substantive model (see Figure 2.1d). It is important to have a clear understanding of the role of interactive decision support systems in this situation. Firstly, they stimulate the work of the team of analysts in Figure 2.1b, generating alternatives in response to the requirements of the senior analyst. A model user, although supported by the system, must either have some general analytical knowledge about the problem, or work with an analyst who helps him to interact with the model. Thus, Figure 2.1d represents a situation functionally similar to that illustrated in the lower part of Figure 2.1b. Secondly, the interactive decision support system enables the user to learn about possible alternatives, and assists him in choosing a set of alternatives for the next stage of the decision process. This second phase, choice between alternatives, can be very rarely suppressed by making the decision via interaction with the model. With these qualifications, however, interactive decision support systems are much more effective than analysts trying to prepare alternatives for the decision maker without his participation.

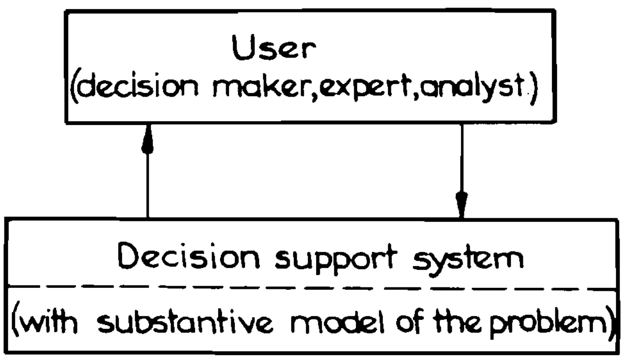

Figure 2.1d. A prototype decision situation with a decision support system.

Thus, the decision maker should be involved in the generation of alternatives; conversely, analysts should be involved in the decision making process. If the actual choice of a decision is made by a single top-level expert intuitively, the analyst should try to understand what aspects of the problem are influencing the intuition of the decision maker and how to help him to understand these aspects more profoundly. If the actual choice is a result of a political process, this does not make it irrational; the analyst should try to understand the rationality of this process and help to choose a decision following this rationality. We should perhaps stress that we do not limit "rationality" to its traditional normative or economic meaning; political processes have their own (mostly procedural) rationality, which arises from experience in making political and social decisions. The best example of procedural rationality is given by the procedures of evidence in courts of law and, more generally, by the rationality of law: this is built on long experience with methods of handling controversial evidence and social disputes. An analyst who understands the rationality of the underlying processes is in a better position to represent the substantive aspects of the problem.

Although there are several methods of multiobjective analysis that can help the analyst to clarify differences of opinion between experts (Keeney and Raiffa, 1976), or even to obtain consensus between decision makers (Rietveld, 1980), most of these methods are based on classical notions of normative or maximizing (Debreu, 1959) economic rationality. A study of procedural rationality and its possible applications in multiobjective decision making would be an important complement to existing methods for multiobjective analysis. However, these comments become truly relevant when we address the second basic type of decision situation, the centralized multiple-actor situation.

2.2 Centralized multiple-actor situation

In many practical situations, a decision is made by some form of a committee. If either the nature of the decision (for example, how to divide a joint budget) or the committee charter (for example, calling for a centralized plan of a transportation system development) prohibit the division and implementation of autonomous decisions by committee members, we still address a centralized decision situation. In an oversimplified way, we could treat the committee as a single body (see Figure 2.2a) and demand just that the decision support system should serve them jointly.

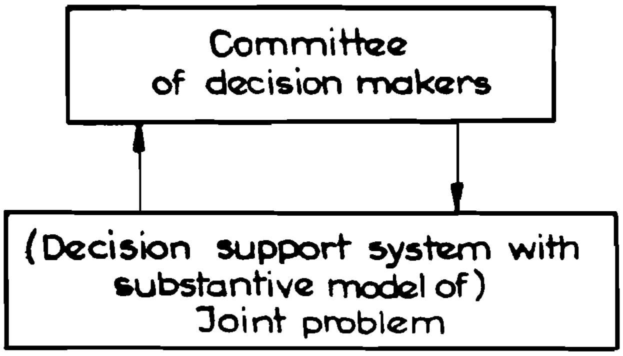

Figure 2.2a A prototype decision situation with a committee of decision makers.

Typically, however, committees are established to represent diverse constituencies: various research groups in a research institute, professors and students at a faculty, various ministries at the national planning level. If the interests represented by committee members are strongly antagonistic, one could imagine a situation where each of them has his own group of advisers - or experts and analysts, his own substantive model of the problem and his own decision support system (see Figure 2.2b).

Thus, each of the principal actors of this situation - each committee member, representing its own constituency - could select a decision which he thinks would be best for his constituency, and propose the decision as his preferred alternative to the committee. If each of the committee members sticks to his alternative, the committee would not get anywhere. In order to get the committee's job done, the chairman of the committee would ask for arguments, learn about the concerns of each constituency, and ask his own advisors to extend their substantive model of the situation to include these diverse concerns. At this moment, it becomes rational for each committee member to ask his advisers for a close cooperation with the chairman advisers - otherwise their concerns might not be correctly represented in the overall model or summary of substantive knowledge.

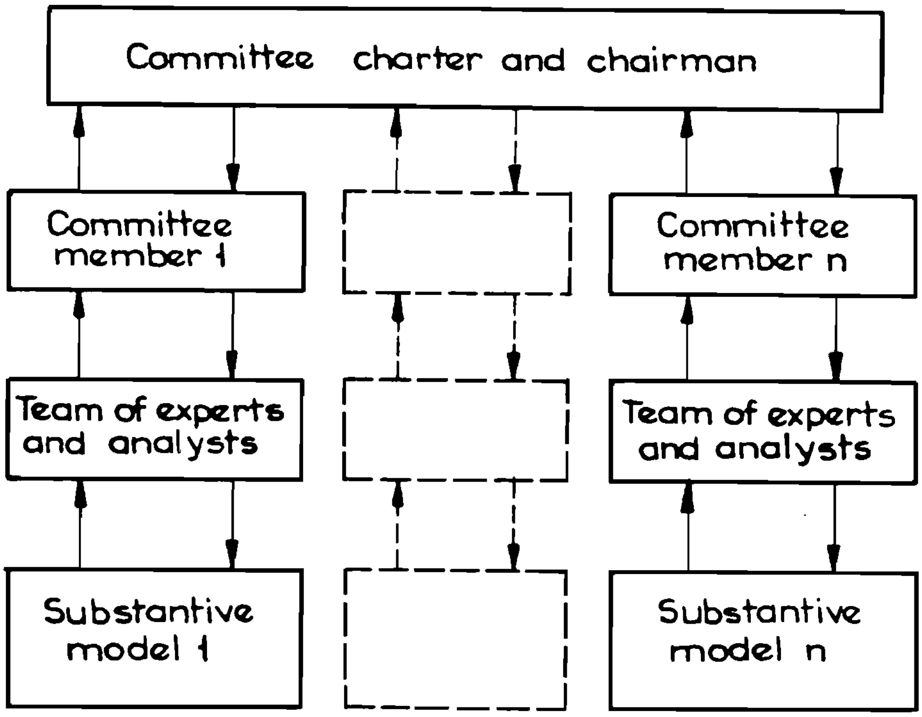

Figure 2.2b A committee decision situation with disjoint decision support.

Arguments would develop not about the interests and concerns, but, much more constructively, about an adequate representation of them in the overall model.

Thus, a decision situation when a centralized decision must be taken with no possibility of division, leads necessarily to information sharing between committee members and between advisory groups. This does not necessarily mean that the information exchanged would not be distorted, biased in favor of any of the constituencies, which leads to the question of incentive compatibility in information exchange: how to organize procedures for information exchange that motivate truthful reporting. However, this question has been until now analyzed only in a very simple, single-objective decision situation; generally, we have to assume that sufficient means of cross-checking the information exchanged are available. Under this assumption, an overall substantive model including all pertinent aspects of the decision situation can be built. This leads to the situation depicted in Figure 2.2c, where all committee members have agreed upon the use of a joint substantive model, but they preserved their independent advisory groups, or an independent use of a joint decision support system.

Using such a support system, each committee member would still select an alternative decision that is preferable for his own constituency; but he knows now the concerns of other members and can see how much they would lose under his preferred decision. This constitutes a starting point for various negotiations, forming coalitions, proposing reasonable compromises. There are, in fact, many possible procedures for organizing such processes of consensus seeking or finding an acceptable compromise; some of them might also be supported by a computerized decision analysis system.

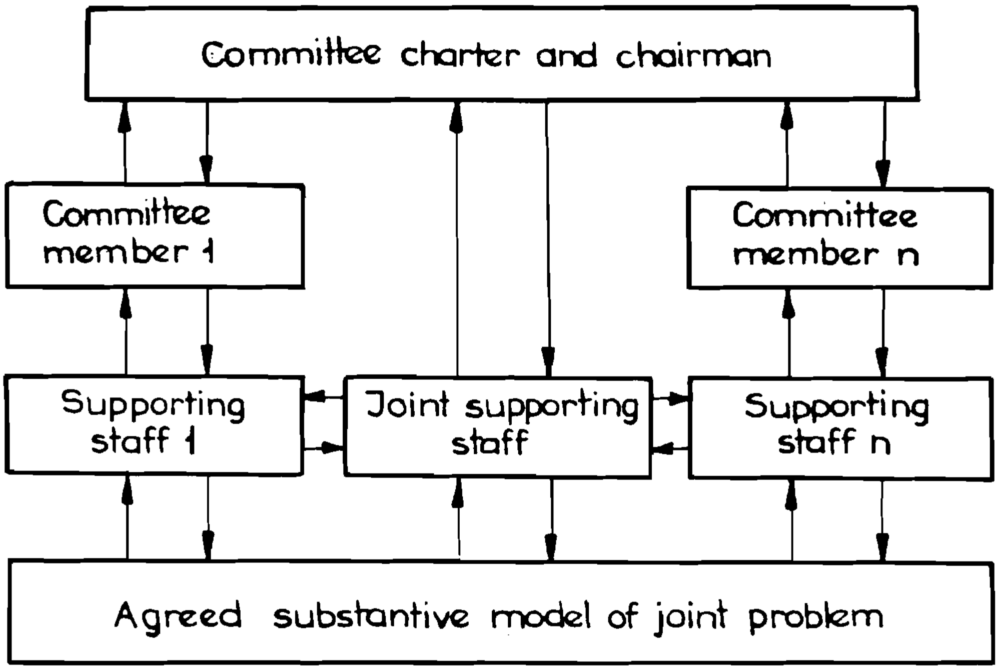

Figure 2.2c. A committee decision situation with joint decision support.

2.3 Autonomous multiple-actor situations

Decisions are often made by independent actors (or "players") who, bearing in mind the fact that the behavior of other actors might influence the final outcome, must choose whether to act independently or to agree on joint action with others. Typical examples are two nations negotiating trade agreements or two regional authorities, one dealing with ecological protection, the other with industrial development.

This situation is typically represented by the prototype in Figure 2.3a. However, although this prototype has been studied in considerable depth by game theory - see for example, Aubin (1979), Germeer (1979), Young et al (1981) - it is not a good representation of a typical decision situation since it assumes that decisions are prepared, evaluated and implemented directly by the principal actors or decision makers. A much more realistic prototype is illustrated in Figure 2.3b. Here the decision analysis is performed by teams of analysts, possibly with senior analysts serving as links between the teams and the principal actors.

The situations in Figures 2.3a and 2.3b may be greatly complicated by antagonism between the actors. Actors and analysts who have common goals or share a cultural background (whether it be political, disciplinary or whatever) can agree relatively quickly on some common model of the problem. They would share their substantive knowledge of the problem, information about the political aspects or even about their real goals. This, however, does not apply to truly antagonistic situations. The strategic aspect of information is the most important difference between the single-actor situation in which all information is assumed to be shared, and multiple-actor situations in which any information (and, in particular, information about the importance of goals) given to other actors

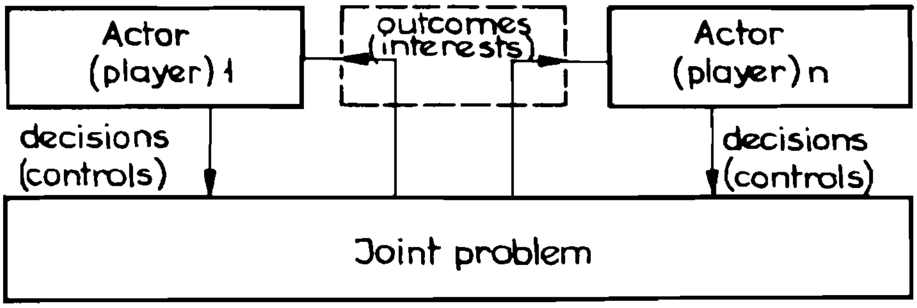

Figure 2.3a. A prototype gamelike decision situation.

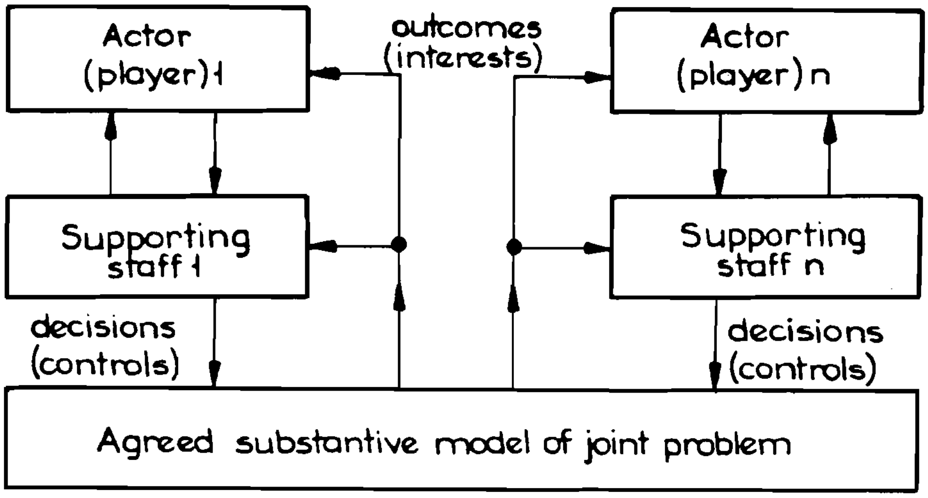

Figure 2.3b. A gamelike decision situation with decision support by a joint model.

might change the outcome of the decision process. This is illustrated by Figure 2.3c, where another aspect of the situation - that of correctness of own problem perception and uncertainty about perceptions of others - becomes also apparent. The autonomous multiple-actor situation is complicated by the fact that there is no compelling reason - except for some common interests - for the actors to act jointly, to arrive at a decision acceptable for all; they can act independently, and their positions in a possible conflict might be affected by information exchange.

In highly antagonistic situations, it is possible that the teams of analysts cannot agree on a joint model of the substantive aspects of the problem, or do not want to exchange substantive information because even this might be too revealing. If a joint decision analysis is necessary in a situation where the actors come from completely different cultural backgrounds (not necessarily from different countries; it can be observed

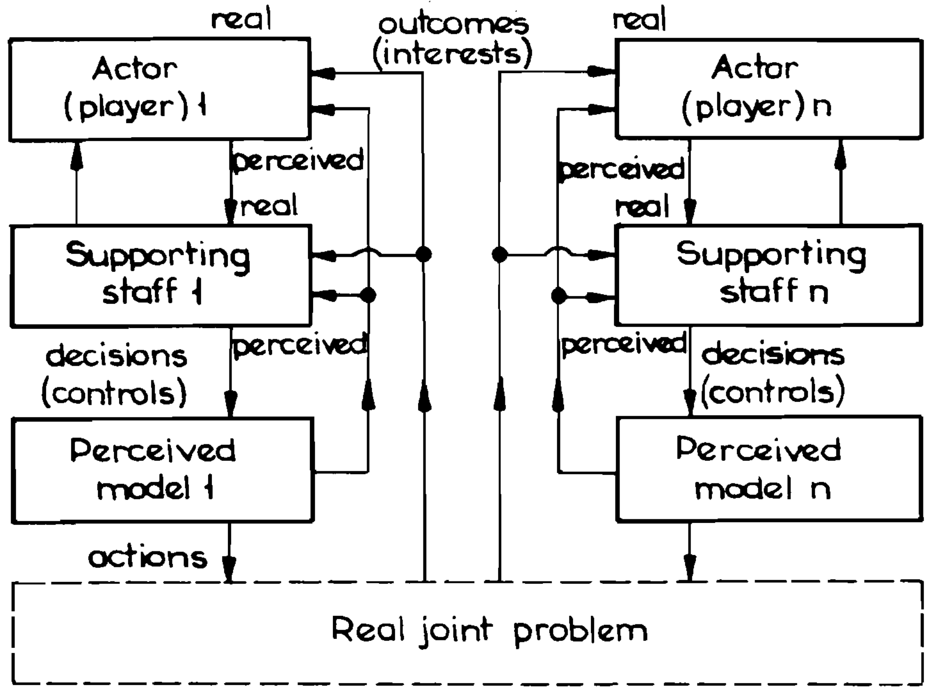

Figure 2.3c. A gamelike decision situation with decision support and disjoint models.

that even economists from different political backgrounds understand each other better than, say, an economist and a lawyer from the same university), then a neutral mediator (see Figure 2.3d) has to be employed, even to assist in joint model building. Such mediations might result in a model that incorporates the models of all interested parties; however, the various parties may or may not agree to the mediator transferring information about their models to the other parties. Clearly, a mediator could theoretically be corrupted by some party; but if his prestige and other benefits depend on the negotiations being successful, he has a strong incentive to remain natural – if his bias were detected then negotiations might be broken off.

During joint decision analysis or actual negotiation, the role of a neutral mediator would be even more important. Empirical experience in negotiations (see, for example, Fisher and Ury (1981)) shows that, although the interested parties do not like to disclose their real interests to each other, a mediator often finds that their interests are not as antagonistic as they suspect, and that attractive compromises are possible. This empirical evidence contradicts the usual perceptions of antagonists, who tend to believe the worst of their opponents and view negotiations as a zero-sum game in which they should take hard positions and have a definite, single objective mind.

However, if life were really like this even the simplest negotiations over prices would almost always be unsuccessful. For, if both seller and buyer had the single objectives, say, of charging no less and laying no more than the market value, they could agree only on the current market price, without profit for either of them; there would be no reason for the general observation that both the buyer and the seller conclude the bargaining with a

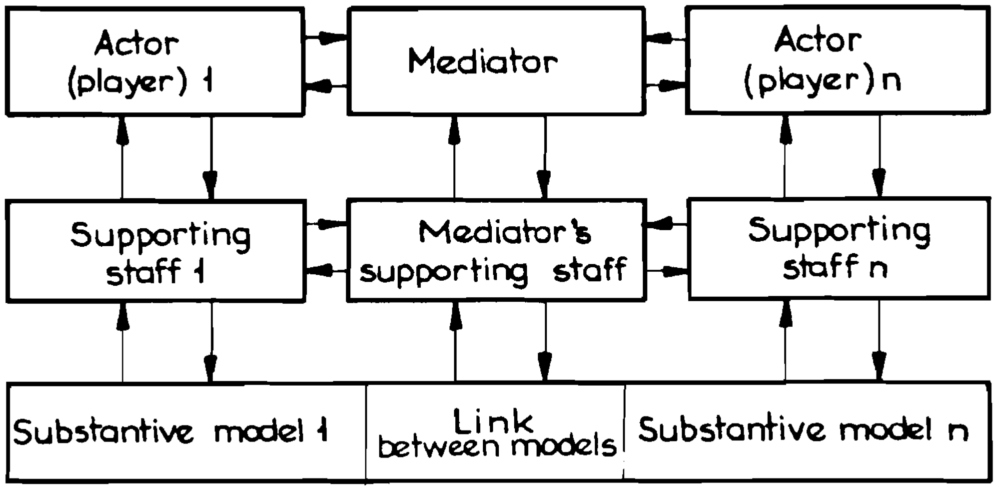

Figure 2.3d. A gamelike decision situation with a mediator.

feeling of satisfaction. To explain this effect, it is necessary to assume that both sides are working with more than one objective. The buyer might want a present for his wife, he might have taken a fancy to the object in question, or he might be a collector who needs the object to complete his collection. The seller might not have had any customers that day, might have liquidity problems, or might want to renew his stock. Thus, there is not a single price, but a range of prices at which both sides would conclude the bargaining, while the ritual of bargaining directs the price to this range by gradually disclosing the strength of interests on either side.

It should be pointed out that earlier analytical work on bargaining was concerned mostly with normative solutions to actually single-objective problems (see Nash (1950), Kalai and Smorodinsky (1973)). Our analytical understanding of the multiobjective, multiparty decision situation is as yet rather poor and has begun to improve only recently - see Raiffa (1982). Much work has yet to be done if we are to describe such situations analytically. An attempt in this direction was given in Wierzbicki (1983a), where the principles of constructing a decision and mediation support system, indicated here in Figure 2.3e, are discussed; see also Grauer et al (1983).

2.4 Hierarchical multi-actor situations and other cases.

Although it has long been recognized that decisions are made within hierarchical structures, the prototype decision situations in which the hierarchy of decisions are investigated have until now been influenced more by the syntactic possibilities of the language of mathematics than by their semantic relevance. Two prototypes have received particular attention. The first assumes fully coordinated interests and single objectives, and is such that an upper-level decision maker can influence and modify the (single) objectives of various lower-level decision makers (see Figure 2.4a), thus maximizing his own objective; another interpretation is a competitive market where the actions of each actor can be

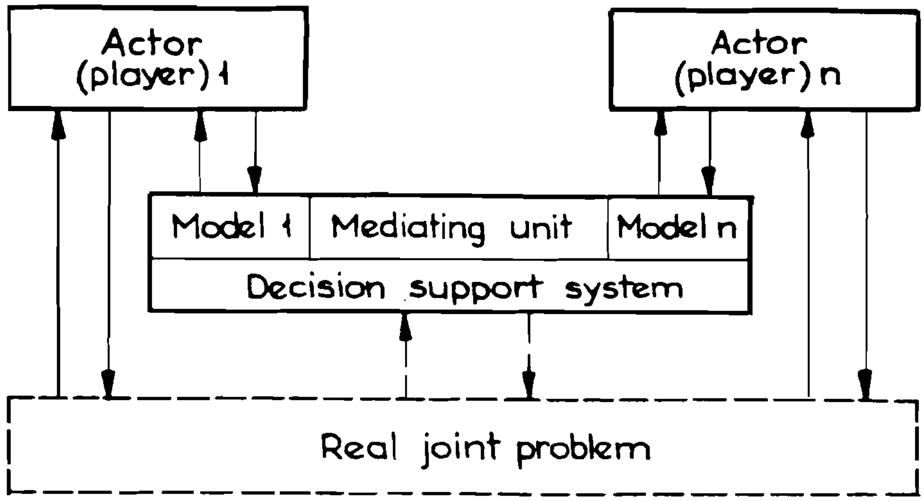

Figure 2.3e. A gamelike decision situation with a mediating role of a decision support system.

made independent once a correct coordinating price is established; however, the price can be established only by a hierarchically dominating market mechanism.
The second prototype assumes shared information, noncoordinated interests and single objectives, and is such that an upper-level decision maker cannot influence the lower-level decision makers but is fully informed of their interests (single objectives); he can plan his moves to maximize his objective assuming that the lower-level decision makers make their maximizing responses (see Figure 2.4b).

The first prototype began with the Dantzig-Wolfe decomposition principle - see Dantzig and Wolfe (1960) - and was developed mostly by Findeisen et al. (1980), the second began with the concept of Stackelberg equilibrium in game theory - see Stackelberg (1938) - and was developed mostly by the Germeer school, see Germeer (1976); both have since been the subject of very considerable extensions and discussions. If we assume full coordination as in the first hierarchical prototype (the hierarchical optimization prototype), we must also describe the means by which the upper-level decision maker influences the choices and preferences of the lower-level decision makers; this is particularly difficult in multiobjective cases, see Seo and Sakawa (1980). It is questionable whether we could often adopt the assumptions of the second hierarchical prototype (the hierarchical game prototype) without modifications, since the assumption that the higher-level decision maker has full information on the preferences of the lower-level, institutionally independent decision makers is not usually justified by empirical evidence. Much more research based on empirical falsification tests must be done before we can formulate prototypes for hierarchical decision situations that are both realistic and mathematically tractable.

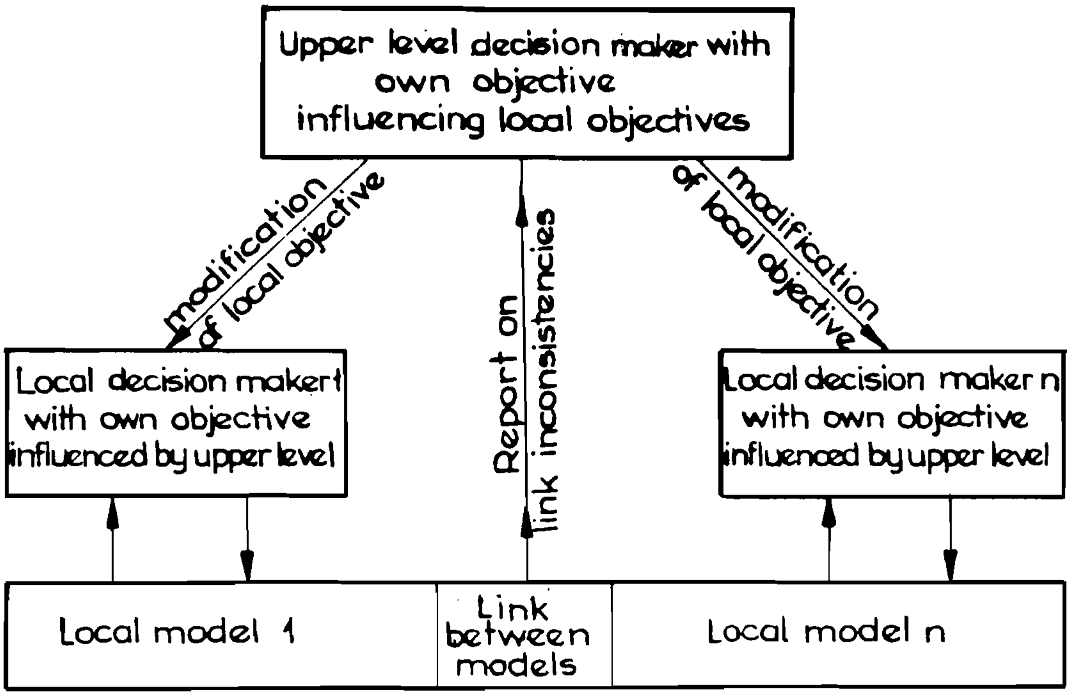

Figure 2.4a. A hierarchically coordinated decision situation with influencing local objectives.

2.5 The issue of uncertainty

The word "uncertainty" has actually many meanings. The outcome of any decision might be uncertain; but there is a world of difference between decisions that might be many times repeated and thus averaged (either over time, or over a set of similar decisions) and a decision that is made only once or only a few times in life. When playing a card game, buying a lottery ticket, or an insurance, we might consider statistical evidence and reasoning, but who would give us insurance against the effects of a nuclear war? In multi-actor decision situations, the uncertainty is compounded by the lack of information about the interests of other actors; in hierarchical situations, by the issues of information aggregation and authority delegation; in learning situations, by the changing preferences of the decision maker.

Yet there is a tendency in the classical decision analysis to reduce all multiple aspects of uncertainty to a probabilistic framework. Possibly, this tendency could be attributed to an indiscriminate belief in mathematics (at least, in statistics) as a semantically valid description of reality. Once we accept, however, that mathematics is only a powerful tool of proving for syntactical correctness in our reasoning, we have to address the question in which situations a probabilistic framework is also semantically adequate.

The probabilistic framework is certainly adequate in all situations where we can give an empirical interpretation to the mechanism of averaging - provided we are sure that our assumptions about this mechanism and probability distributions of events are correctly specified. Thus, an individual might average the outcomes of a game for reasonable stakes or of minor financial decisions ("win some, lose some"); averaging the aggregated information about a large population has much broader applicability, and the information

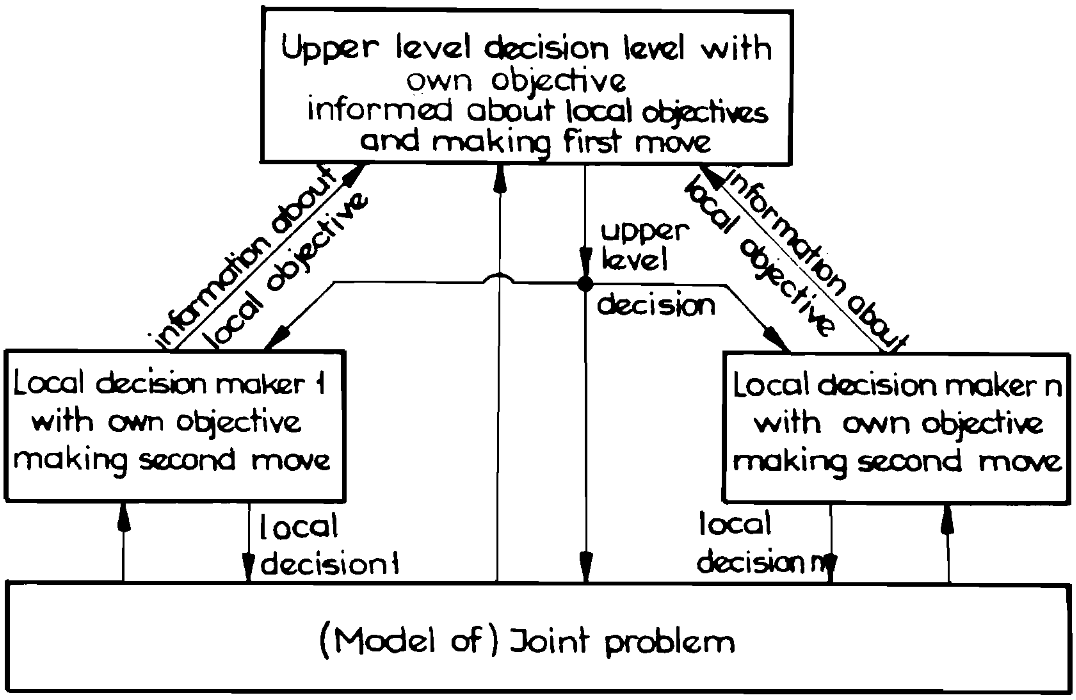

Figure 2.4b. A hierarchical gamelike situation with full information on the upper level.

about probability distributions can be much more reliable in such cases. However, there are several traps even in situations with good empirical interpretation of averaging. The most elementary trap is the confusion of averaging over a set with averaging over time. It is obvious that a nationwide insurance company covering, say, losses due to floods, cannot base its statistics on even arbitrarily long records of data from some arid region of the country. However, in less obvious cases it is easy to fall into such a trap and assume the ergodicity hypothesis - that is, taking the average over time as an estimate of the average over a set - without questioning, while the hypothesis is valid only under rather stringent conditions on the properties of underlying stochastic processes. Another trap is to use probabilistic models without sufficient thought and data concerning probability distributions: we then replace one unknown with another, more complicated unknown. It is easy to fall into this trap when working with subjective probabilities. A subjective estimate of probability is fully substantiated when we confront a problem with well understood probabilistic properties but without full information about probability distributions; say, a player of dice might not know at the beginning of a game whether the dice is loaded, but he can take an enlightened guess after observing the course of the game - and he does not need full statistical data analysis, he can do it intuitively. However, speaking about subjective probability of a nuclear power plant accident is without meaning - if we do not specify very carefully what probabilistic components we estimate and how do we estimate them. This trap is particularly inviting if we erroneously assume that, once we evaluate subjectively the probability, we could also forget about a clear interpretation of what we are averaging.

If we consider the uncertainty of human preferences and learning processes, it is rather difficult to give a clear empirical interpretation of the averaging operation. Therefore, it is questionable whether the most adequate representation of uncertainty in human preferences - as opposed to an uncertainty in a substantive model of reality - is the probabilistic framework. Some other frameworks, such as interval analysis, or fuzzy set theory (which is based on the logic of fuzzy intersection of sets, not on averaging over them) can be more suitable for this purpose.

2.6 The issue of dynamic planning

In many decision situations - say, in policy analysis and planning - the issue of uncertainty is compounded by the fact that all available information usually cannot be processed, must be aggregated, and even if it could, there are always gaps in information and knowledge. When building an aggregated model, the analyst has to consciously neglect various factors that he considers either to be irrelevant or not sufficiently understood to be modelled. These neglected or unpredictable factors might be represented by a probabilistic model or by interval characterization, but such representations are often inadequate.

Before we can examine the effects of uncertainty on planning and policy analysis, we first have to consider what these terms actually mean. Most definitions of planning are in basic agreement (see, for example, Dror (1963)): "planning is the process of preparing a set of decisions for action in the future, directed at achieving goals by preferable means". Therefore, planning is in its essence concerned with process dynamics, with predicting future consequences of current and future actions. Substantive models for planning purposes must have dynamic character, that is, describe the evolution of the state of affairs under the impact of decisions distributed in time, where the concept of state has a very good mathematical idealization in the dynamic systems theory: it corresponds to the set of initial conditions that must be specified when running a dynamic model. Necessarily, the outcomes of planning when using dynamic models have the form of trajectories, that is, graphs over time of selected actions, of evolution of state or of other relevant outcomes. A basic question here is how we evaluate trajectories as outcomes?

A classical approach to this question is conditioned by the mathematical technique of providing for scalar factors - often called performance indices - by computing a weighted average over time of one or several trajectories or of a function of momentary values of these trajectories. When applying this technique, any trajectory evaluation can be converted to an evaluation of a number of scalar factors, such as in the case of static decision analysis. However, the main difficulty with this approach is that experts do not evaluate trajectories this way and feel uncomfortable when presented with a set of averaged indices: they prefer to be presented with full graphs of trajectories and thus, presumably, evaluate them by "Gestalt". Experiments with evaluation of outcomes of dynamic computer simulation indicate also that experts sharpen their intuition by learning and forming *aspiration trajectories,* that is, mental images of what good trajectories look like for a given case. This facilitates the evaluation by "Gestalt" since only the departures from the aspiration trajectories have to be accounted for.

The issue of uncertainty in dynamic models for planning has many aspects. These are not only the questions of an adequate representation of uncertainty - in a probabilistic (or, more precisely in the dynamic case, stochastic) framework, or through interval analysis or fuzzy sets. The most important question here is how to represent the fact that, in the future, we will observe additional data, gain additional information and might correct our planned actions. Should we plan while disregarding these future corrections or should we account for them in planning? Obviously, it would be better to account for them, but this might be a very complicated task - and there are situations in which a separation principle applies, that is, we cannot do any better than planning for an ideal course of affairs without accounting for future corrections and then considering a correction mechanism separately.

On the other hand, such a separation principle does not apply if we can learn in the future some truly new facts - for example, change substantially some nonlinear characteristics in our dynamic model. In such cases, we must plan while accounting for future corrections. There are many possible mechanisms of such corrections, depending on the mathematical framework used to account for uncertainty. In stochastic optimization -see Ermoliev and Wets (1984) - there is the concept of optimization with recourse; more generally, we could utilize the concept of optimal feedback control for stochastic process. However, the stochastic framework requires rather detailed information about probability distributions of stochastic processes - which, together with computational complexity, accounts for formidable difficulties when applying stochastic optimization for multiobjective planning. An alternative, set representation of uncertainty in dynamic processes has been also studied intensively and might provide in the future for more adequate tools - as for example, the concept of guaranteed control, Kurzhanski (1977). However, an effective framework of dealing with uncertainty in dynamic multiobjective planning has not been developed as yet. These difficulties are reflected in the discussions about the concept of policy - clearly related to the issue of dealing with uncertainty through feedbacks or other corrective mechanisms.

There is a rather wide disagreement on the definition of policy. Ranney (1968) states that "policy is a course of action conceived as deliberately adopted, after a review of possible alternatives, and pursued or intended to be pursued", but for many other definitions - see Harrison (1964) - stress either the political process of policy formulation or the implementation aspects. Nonetheless, there is a great similarity in the definitions of planning and policy. As a basis for discussion, therefore, we shall assume that planning is the process of policy formulation or policy specification (in the case when a higher-level policy is accepted as a basis for more detailed planning), while the concept of policy includes both formulation and implementation aspects.

To obtain a comprehensive definition of a policy, we will distinguish between two types of uncertainty: predictable uncertainty and unpredictable uncertainty. The first can be included in a model by probabilistic means, supported by empirical data, while the second should be understood in a pragmatic and semantical (rather than syntactical) sense: due to lack of empirical data, or because of model simplifications, we accept that there are aspects of the problem that can be be predicted in the basic model that we intend to use for policy analysis. Having made this distinction, we can now define the various elements that comprise a policy - see Figure 2.5a.

The first of these elements is the *substantive content of policy* - selected knowledge about real situation (economic, ecological, technological, regional) addressed by the policy; the second is the *political process* - the institutional and sociopolitical aspects of policy formation and implementation. Both of these elements are included in the analysis only to a limited degree: both involve neglected, unpredictable or unknown factors as well as known or predictable factors. For this reason, the concept of policy also contains two other elements:: a *normative core* and a *procedural belt*. The normative core includes everything that is known and predictable about the policy content and political process; the procedural belt describes *implementation procedures* for handling the neglected and unpredictable aspects.

While the concepts of policy content and political process are well-known in policy analysis, the concepts of the normative core and procedural belt are newer - see Wierzbicki, (1984) - and require further discussion. There are many reasons for introducing these ideas: for example, the discussion on the merits of various planning approaches (blueprint versus process planning, etc.) is clearly related to the lack of any distinction between what we call here the normative core and the procedural belt of a policy. Rational comprehensive planning is clearly concerned with the normative core aspects of a

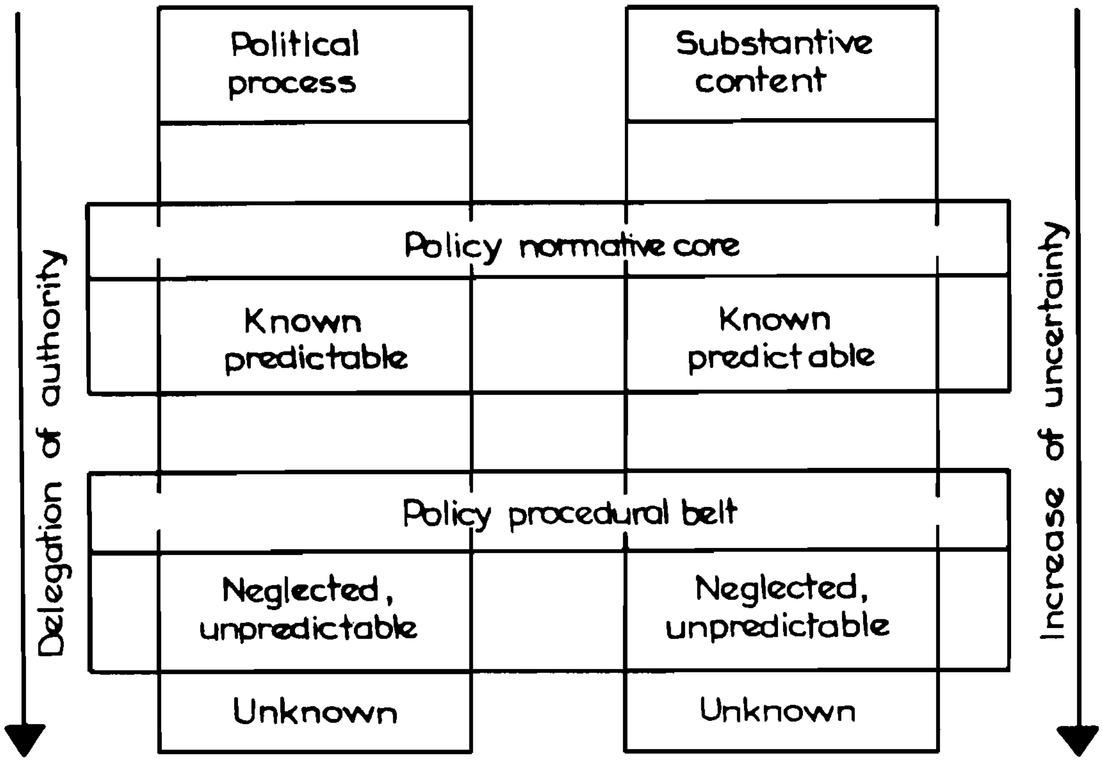

Figure 2.5a. Two aspects of the concept of policy: normative core and procedural belt

policy: set a goal and decide in general how to achieve it, assuming that the world will behave as predicted. However, if anything can go wrong, it will: some aspects are always neglected or unpredictable and must be dealt with by providing specific implementation procedures as well as general normative directions, and by authorizing a 'man on the spot' to deal with developing situations as he finds appropriate.

There are many areas of human activity in which much time is spent considering *what could go wrong* and in devising procedural responses, i.e. in emphasis is on the procedural belt. For example, one of the lessons of the Three Mile Island nuclear reactor accident was that the operating procedures were not rich enough; another was that the personnel were not trained in various emergency actions. Consider the case of the shop owner who says "It is our policy not to accept cheques": it is clear that the common language interpretation of "policy" includes the procedural belt and even concentrates on it. In economics, many widely disputed issues, such as the relative advantages of market and planned economies, are really related more to the robustness of the procedural belt than to the efficiency of the normative core. In control sciences, procedural belt issues correspond to the problems of stabilizing feedback systems, and these have been investigated quite widely. However, in only a few cases (e.g. Wierzbicki (1977)) is a mode of analysis adopted that could encompass both the normative core and the procedural belt.

Now, how can we investigate something that is unpredictable? In the same way that we train pilots: by imagining the most dangerous - if improbable - situations that can develop, and exposing the pilots to them on a flight simulator. In terms of building models for decision analysis, this approach would mean constructing ***two models*** (see Figure 2.5b) - a ***basic model*** and an ***extended model***. The first represents the known and predictable, while the second contains possible answers to the question: which of the aspects of reality neglected in the basic model could have the most negative impact on the implementation of the policy? It should be stressed that the extended model is not a better representation of reality, it is simply a different representation of reality, a falsification hypothesis constructed to check the robustness of the conclusions derived from the basic model. When checking this robustness, we would really like to know which implementation procedures to choose; there are usually many implentation procedures that are consistent with the course of action suggested in the basic model, but these procedures might give quite different results when applied to an extended model.

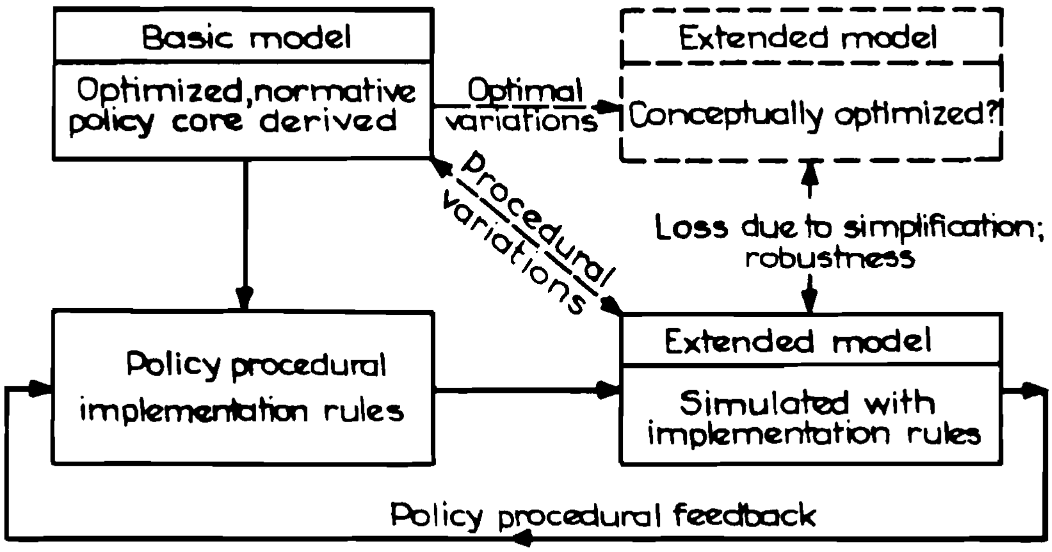

Figure 2.5b. The role of the basic and the extended model in examining robustness of procedural policies.

This framework immediately suggests several research questions. First, how should implementation procedures be generated? Second, how should the consistency of an implementation procedure with respect to the basic model (normative core) be characterized? Third, how should the robustness of an implementation procedure be defined operationally? The most natural definition would be the losses that result from the fact that the policy was devised using the basic model rather than the extended one. However, this might not be feasible, since it would involve deriving the normative policy for each extended model, and then comparing the results of applying two policies to the extended model (one policy should be derived from the basic model, with some implementation procedure, and the other derived from the extended model). If such simulation experiments are to be performed on several extended models, the time necessary for robustness analysis might be excessive. This results in further questions: how could such a definition be made operational? How should we organize robustness analysis? Are there any mathematical methods that would enable us to compare the robustness of various implementation procedures without requiring many solutions for the extended model and the calculation of its normative policy cores?

It turns out that all these questions have an answer, at least for single-objective decision problems - see Wierzbicki (1977), Snower and Wierzbicki (1982) - however, the question whether these results can be extended to the multiobjective case has not been answered yet.

For some models, particularly those of a probabilistic nature, the distinction between the normative and the procedural aspects of a policy can be less sharp. For example, if we have a stochastic process model with some parameters that are not known a priori, we can derive an adaptive optimal feedback policy, i.e., a procedure that both responds to perturbations and can learn by accumulating information - see Walters (1981) for an empirical application of this mathematical idea. Surely this would be equivalent to a joint solution of the normative and procedural aspects of a policy, and, in this case, is the distinction really necessary?

In the above case, we really assume predictability: the world will behave largely as we expect, although there may be some nasty stochastic effects and we cannot predict its behavior fully. There is no place here for really unpredictable events. Thus, although such cases include some procedural features, they really belong to the normative core: for example, the unique optimal feedback policy for a stochastic process might turn out to be wrong if there were some unpredictable parameter change of a type not assumed in the basic model. This is a known paradox in control theory: the optimal stochastic feedback policy suggests proportional controller forms, although a great deal of experimental evidence shows that if we are to achieve robustness we must partly neglect optimality and adopt, typically, proportional-integral controller forms. This implies a multiobjective approach: even if there is a unique implementation procedure that is consistent with the normative core of a policy, we might accept a decrease in the normative efficiency of another procedure if it guarantees a substantial increase in robustness in unpredictable cases. Finally, we should stress that efficiency and robustness might not be the only objectives; another could be adaptability, the ability to learn from experience. Thus, we might try to design policies in a way that takes all three objectives into account.

After this discussion of the procedural belt and normative core, it would perhaps be useful to formulate an extended definition of policy. Policy is a course of action, assumed to include a basic normative direction and procedural implementation rules, which has been deliberately adopted after review of possible alternatives and assessment of predictable and unpredictable aspects of both substantive content and political process. This definition, together with the framework discussed above, still leaves many questions for research; however, it seems to be a constructive point of entry to many important problems. For example, the issue of 'process planning' can clearly be investigated in what we call the procedural belt of policy.

2.7 The scope of this book

After this rather broad discussion of various issues in decision analysis and planning, the reader must be warned that we cannot cover all of those issues in this book. The main emphasis here in on *interactive decision analysis* - from issues that motivate it, through theoretical foundations and analytical tools, to the *design of decision support systems, examples of applications,* details of *computer implementation.* Thus, most of the book will be concentrated on *single-actor* decision situations. The design of decision support systems, their implementation and most of applications examples concern, however, the *dynamic* aspects of planning - since we have found these applications of decision analysis especially useful in practice. Most of the book does not address the issue of uncertainty, under an implicit assumption that a separation principle holds, at least approximately, and our first task is to *plan for a middle course of events*; in this sense, we consider mostly planning for a normative core of a policy. However, some aspects of uncertainty are also discussed briefly, and some other methodological extensions are indicated.

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3. FRAMEWORKS AND TOOLS FOR RATIONAL DECISION MAKING

3.1 The concept of plural rationality

One of the major motivations for concentrating on interactive aspects of decision analysis, or for stressing the role of learning when using decision support systems, is the perception that there are diverse meanings of what is a rational decision - and this diversity reaches much deeper than a simple difference of interests or tastes, it corresponds rather to basic, culturally motivated differences of perceiving what constitutes rational behavior. The individualistic rationality of the industrialized West assumes the superiority of man over nature and differs much from the rationality of Bhuddist culture that stresses the unity of man and nature, both animated and nonanimated, and thus has far-reaching collectivistic consequences (see Zsolnai and Kiss 1984); other cultures might produce intermediate or quite different concepts of rationality. Even in one country, various social groups or various professions can have difference concepts of rational behavior (see Schwarz and Thompson 1984). Small entrepreneurs confronting a broad competitive market will perceive rationality differently than managers in centrally planned economies.

Therefore, when trying to develop interactive decision support systems, we should attempt to accommodate various possible perceptions of rationality. This requirement becomes even more important in multi-actor decision situations, where the parties involved might not share the same rationality; yet even quite culturally diverse actors can achieve agreement if they recognize their diversity, are willing to learn and exchange information, and agree on the legitimacy of some negotiation procedure, or on some principles of fairness for use in mediation.

There exist several major frameworks for rational decision making; although they have often abstractive and mathematical character, they also express some deeply rooted cultural differences. Before proceeding to them, some general comments are still necessary.

The role of value judgement in decision analysis. Following Weber (see, e.g., Weber, 1968), many decision theorists take it for granted that any serious scientific analysis must be value-free – although any actions based on this analysis are typically value-dependent. In order to better understand this apparent paradox we must return to some reservations made by Weber himself. He admits that any concept or assumption used in scientific analysis, and indeed the choice of subject of study itself, might be influenced by value judgements. However, he considers these reservations to be minor. On the other hand, the development of the theory of cognition after Weber has stressed the very basic dependence of concepts on language and thus on deeply rooted cultural values. On this basis, it should not appear surprising that the very concept of rational behavior might be quite different in different cultures. Therefore, Weber's postulate of value-free science must be understood as a methodological ideal worth striving for, but typically achieved to only a limited degree. These reservations, although not really minor, do not imply a totally relativistic attitude; for example, values such as global responsibility, tolerance, the pursuit of truth, understanding, and learning are upheld throughout the community of scientists, independent of their cultural background. Therefore, our purpose should be to ***understand*** plural perceptions of rationality rather than to judge them. In this spirit, we should not call irrational or pathological a behavior that does not conform with a particular perception of rationality but might be termed rational when following other perceptions.

The dialectical triad of cognitive processes. Much attention has recently been paid to the distinction between the ***descriptive, normative*** (a better word might be ***abstractive***, see Raiffa, 1984) and ***prescriptive*** schools of decision analysis. This is a very important distinction, since every process of human cognition involves observation and description, then abstraction (in order to identify the important features of the observed phenomenon), and finally prescription (in order to test or utilize the acquired knowledge). Individual researchers attach different importance to these stages, some concentrating on abstraction, others on observation and critical analysis, and yet others on tests and utilization. However, we could equally well speak about passively empirical, theoretical, and actively empirical stages; most sciences treat these three stages as iterative steps necessary for progress, in the sense of the dialectical triad of thesis, antithesis and synthesis. Examples abound: empirical knowledge of the limited speed of light motivated Einstein to work on relativity theory, and the theoretical understanding acquired in this way prompted not only observations of the deflection of solar radiation by Mercury, but also many other experiments of an even more active nature.

A mature science uses all these three stages of research. Therefore, we cannot use this distinction as evidence of plural rationality - although it is true that a number of perceptions of rationality with different methodological emphases have been developed. As decision analysis matures, however, we might expect these three stages of research to be accepted as equals.

Methods of decision-making and types of rationality. When trying to classify various methods of rational decision making, we must first distinguish between *holistic* and *analytical* ways of making a decision. The holistic approach is based on the decision maker's reaction to the situation as a whole; it is not necessary to identify elements or information before making the decision (Dreyfus, 1984). Purist decision theorists may question whether such a method of decision making is rational at all, since heuristic assumptions and intuition could play a significant role. This leads us to the question of what we mean by "rationality". A broad concept of this type can be restricted by the development of mathematical theory that analyzes only chosen aspects of the concept. However, such a restriction might be detrimental to the development of an applied science; hence, we prefer to use the word "rationality" in its broader, more conventional sense. A rational decision does not have to be based on all the available information, nor does it have to be optimal. It should only take into account the possible consequences of the decision and be intended not to be detrimental to the values and interests of the decision maker. As a reasonable compromise, we can define various types of rationality: *super-rationality* (ability to deal with the paradoxes of rationality), *optimizing rationality, satisficing rationality* (Simon, 1953), *procedural rationality* (see, for example, Dobell, 1984), and so on. Using this broader definition, an adaptively formed decision rule or procedure can lead to quite rational decisions; the effectiveness of various decision rules in multi-actor decision situations is a very interesting subject for study (Rapaport, 1984). Moreover, it can be argued that most day-to-day decisions are made in a holistic way (Dreyfus, 1984) and even that the holistic approach is often superior in the long run, as shown by computer tournaments of repetitive prisoners' dilemma games (Axelrod, 1984). However, decisions based on inadequate experience or involving new issues often require a *calculating* or *analytical* approach to decision making, i.e. a systematic evaluation of possible alternatives and related outcomes before making a decision. Several frameworks for analytical decision making have been developed.

3.2 Utility maximization

Without implying any value judgement, we can say that the *utility maximization framework* originated from the very core of the individualistic culture of the West - from the belief that every individual, when striving for his maximal welfare, contributes positively to the welfare of the society, as exemplified by Adam Smith's "invisible hand" concept and substantiated by the study of ideal markets. The abstractive aspects of this framework has been developed for a long time and it has now the strongest theoretical and mathematical foundations. Partly because of its abstractive elegance and partly because of implied, deeply-rooted cultural convictions, this framework is now widely accepted as a basis of defining rationality - with an unfortunate consequence that other possible definitions of rationality are often looked upon as somehow "less rational".

To introduce mathematical foundations of the concept of utility, consider first the *decision space* E_x and an *admissible decision set* $X_0 \subset E_x$. At this moment, no mathematical structure of this space and set is needed. The set X_0 might consist of a finite number of alternative decisions (the case of *discrete alternatives)* or of a continuum of possible decisions (the case of *continuous alternatives*), the elements x of this set might be alternative, verbal descriptions of proposed plans of action, of bundles of commodities that a consumer can buy under a restricted budget. The last is the classical example that motivated much of the development of utility theory.

We assume that a decision maker (in this example, a consumer) can compare any pair of elements $x', x'' \in X_0$, in terms of "better", "worse", "indifferent". Let $x' \geq x''$ denote that x' is better or indifferent to x''. This constitutes a *preference relation* between the elements of X_0. Typically, three abstract axioms of such a relation are postulated:

(i) Reflexivity:

$x' \geq x'$ for all $x' \in X_0$

(ii) Transitivity:

$x' \geq x''$ and $x'' \geq x'''$ implies $x' \geq x'''$ for any x', x'', $x''' \in X_0$.

(iii) Completeness:

for all x', $x'' \in X_0$, at least one of the relations $x' \geq x''$ or $x'' \geq x'$ holds.

Reflexivity has an obvious meaning: x' should be indifferent to itself. *Transitivity* implies consistency of pairwise comparisons done by a decision maker, no matter how many such comparisons he is making. However, decision makers are notoriously inconsistent - for many reasons, the simplest one being that it is the privilege of humans to change their mind. *Completeness* implies that there is no pair of alternative decisions about which the decision maker is uncertain and would not like to express his opinion. A stronger requirement than that of completeness is that any alternative outside the set X_0 might be termed irrelevant and should not influence the preference relation for $X \in X_0$. This requirement of *independence of context*, though in a slightly different form - is a central requirement in expected utility theory and will be discussed later. Another modification of this requirement in decision theory is called the axiom of independence of irrelevant alternatives. The requirement of independence of context is also related to that of transitivity; in the process of comparing alternatives and when coming across an unattainable but attractive alternative, or an avoidable but dangerous one, a decision maker is apt to change his mind and make the following comparisons inconsistent with the previous ones.

A relation that satisfies *(i), (ii), (iii)* is called *complete preordering* (sometimes, equivalently, a total quasi ordering), while a relation that satisfies only *(i), (ii)* is called *partial preordering*. We prefer to adhere to the convention that any reflexive and transitive relation, admitting nonsingleton equivalence classes (that is, such that $x' \geq x''$ and $x'' \geq x'$ do not necessarily imply $x' = x''$) is called preordering, complete if it satisfies (iii); incomplete preordering is thus called here partial preordering. Such a relation implies also an equivalence (or indifference) relation:

$$x' \approx x'' \text{ iff } x' \geq x'' \text{ and } x'' \geq x' \tag{3.1}$$

and a strict preference relation:

$$x' > x'' \text{ iff } x' \geq x'' \text{ and } \neg\ x'' \approx x' \tag{3.2}$$

Clearly, $x' \geq x''$ can be also defined as $x' > x''$ or $x' \approx x''$. We preserve here the notation $x' \gg x''$ for a relation that is even stronger than $x' > x''$ which means that x' is preferred to x'', but it might be preferred only because one of its aspects is better, while being indifferent in other aspects; $x' \gg x''$ will mean that x' is preferred to x'' in all its aspects.

The basic question in utility theory is: under which conditions a complete preordering can be faithfully represented by a scalar-valued ***utility function*** u: $X_o \rightarrow R^1$, in the sense that "better" corresponds to "a greater value of $u(x)$"; such a function is called also value function, worth function, preference function, or ordinal utility function (see further comments on expected utility and cardinal utility functions).

$$\begin{aligned} &x' \geq x'' \text{ iff } u(x') \geq u(x''); \\ &x' > x'' \text{ iff } u(x') > u(x''); \\ &x' \approx x'' \text{ iff } u(x') = u(x'') \end{aligned} \tag{3.3}$$

This question depends on the mathematical structure of the space E_x. Suppose it is a topological space, suppose X_o is closed, and postulate another requirement on the preference relation ($\geq$):

(iv). Continuity:

$$\text{for all } x' \in X_o,$$
$$\text{the sets } \{x \in X_o\text{: } x \geq x'\},\ \{x \in X_o : x' \geq x\} \text{ are closed.}$$

Some complete preorderings do not satisfy this requirement. For example, let $E_x = X_o = R^2$ and consider the lexicographic ordering (such as applied in a dictionary) defined by:

$$\begin{aligned} &(x_1', x_2') \geq (x_1'', x_2'') \text{ iff either } (x_1' > x_1'') \text{ or } (x_1' = x_1'' \text{ and } x_2' > x_2''); \\ &(x_1', x_2') \approx (x_1'', x_2'') \text{ iff } x_1' = x_1'' \text{ and } x_2' = x_2'' \end{aligned} \tag{3.4}$$

This ordering does not satisfy the requirement *(iv)* and, therefore, does not admit a faithful representation by a continuous utility function, according to the following theorem:

Theorem 3.1 (Debreu, 1959). If $X_o \subset E_x$ is a closed subset of topological space, then the conditions *(i), (ii), (iii), (iv)* are satisfied if and only if there exists a continuous utility function satisfying (3.3).

This function, however, is not defined uniquely; if we take any such utility function and any continuous strictly monotonically increasing function v: $R^1 \rightarrow R^1$, then the function $u'(x)$ defined by $u'(x) = v(u(x))$ is another such utility function. In other words the utility function that faithfully (3.3) represents a continuous complete preordering is defined only on an ordinal scale, up to any order-preserving transformation, and is often called an *ordinal utility function.*

For ordinal utility functions, above considerations apply without much modification if we introduce, beside the decision space E_x, a space E_q of decision outcomes (or objectives, or attributes) and a mapping $f\colon E_x \rightarrow E_q$ that transforms each decision into its outcomes. Since it might be argued that a decision maker evaluates rather outcomes of alternative decisions than the details of those decisions, a plausible form of the utility function is:

$$u(x) = U(f(x)) \tag{3.5}$$

where U: $E_q \rightarrow R_1$ is another utility function. Since the space E_q is usually equipped with a partial preordering relation, we can say much more about the properties of the function U - for example, it should be monotone with respect to this relation (called also order-preserving, see next chapters). If the decision outcomes are characterized by scalar performance indices of positive values, $E_q = R^p$ and $X_o \subset R^p_+$, then typically assumed forms of utility functions often exploit the fact that a norm is order-preserving in the positive ortant of the space. The weighted l_1 norm results then in the following form of utility function:

$$U_1(q)=\sum_{i=1}^{p} \lambda_i q_i \tag{3.6}$$

$$\sum_{i=1}^{p} \lambda_i=1;\ \lambda_i>0 \text{ for } i=1,...p$$

Similarly, a function analogous to the weighted l_k norm but with $k < 1$ is also monotone in the positive ortant and can be directly used as an utility function

$$U_k(q)=(\sum_{i=1}^{p} \lambda_i q_i^k)^{1/k} \tag{3.7}$$

$$\sum_{i=1}^{p} \lambda_i=1;\ \lambda_i> 0 \text{ for } i=1,...p;\ \ k < 1$$

since for $k < 1$ it is concave in the positive ortant - and the concavity of utility functions is usually required in order to obtain uniqueness of maximization results over (strictly) convex sets. Because of this requirement, the Euclidean norm l_2, norms l_k for $k>1$, and the Chebyshev norm l_∞ cannot be used directly as utility functions and must be first modified - by shifting the origin of the space and changing signs appropriately. For example, the so called L-shaped utility function analogous to the Chebyshev norm has the form:

$$U_\infty(q)=\min_{1\leq i\leq p} \lambda_i q_i \tag{3.8}$$

$$\sum_{i=1}^{p} \lambda_i=1;\ \lambda_i > 0 \text{ for } i=1,...p$$

There are also various multiplicative forms of utility functions. However, these convenient forms of utility functions are seldom good predictors of individual behavior, even if their parameters are well estimated statistically. On the other hand, they are rather successfully applied in economics to predict mass phenomena, such as consumers' behavior. In the same sense, welfare economics uses so called welfare functions, or utility functionals defined on a space of trajectories of dynamic decision outcomes $q(t)$, in a typical form:

$$W(q)=\int_0^\infty \exp(-\mu t)\, U(q(t))\, dt \tag{3.9}$$

where $\mu>0$ is interpreted as a discount coefficient for the future. Again, an individual decision maker asked to suggest a reasonable value of such a discount coefficient is typically at loss, even if he would know how to evaluate outcome trajectories $q(t)$ in 'Gestalt'; on the other hand, it is possible to make reasonable estimates for the discount coefficient from observed economic mass phenomena.

The concept of utility functions has been substantially modified and refined by considering decisions under risk - that is, under conditions of uncertainty but with a clear interpretation of averaging and well substantiated probabilistic description. The prototype decision situation considered (von Neumann and Morgenstern, 1944) describes a choice between probability distributions of various outcomes and has the following form. Suppose $X_o=\{1,..j,..J\}$. Suppose n possible outcomes q_i, $i=1,...n$, have been specified from a possible outcome space (which, at this moment, can have an arbitrary nature) and the decision maker can strictly order them according to his preferences, hence

$$q_1 \leq q_2 \leq \cdots \leq q_i \leq \cdots \leq q_{n-1} \leq q_n$$

These possible outcomes are not precisely the outcomes of the decisions $x=1,..j,..J$; the actual outcomes of these decisions are probability distributions

$$P_j=\{p_{1j},...p_{ij},...p_{nj}\}$$

where p_{ij} is the probability of the possible outcome q_i if the decision j is chosen. Such probability distributions are supposed to be known a priori for each decision j and are sometimes called strategies (because of game theoretic interpretations). Suppose the preferences of the decision maker over possible outcomes can be represented by some utility function U, with

$$U(q_1)<\cdots<U(q_i)<\cdots<U(q_n)$$

If we would use for this purpose any ordinal utility function (called a value function in this context), we would have quite a wide choice, since these functions are determined up to a strictly monotone transformation.

The fundamental question of the expected utility theory is how to limit the class of utility functions so that they would give consistent results when assuming that the decision maker is averaging his utility under risk, that is, he is trying to maximize the expected utility of the decision j:

$$u(j)=\sum_{i=1}^{n} p_{ij}U(q_i) \tag{3.10}$$

This assumption is well justified in examples when outcomes have direct monetary interpretation and probabilities are well known, such as buying lottery tickets. Now, it can be seen from the form of (3.10) that an arbitrary monotonous transformation of U will affect the results of maximization of u; for example, a monotonous transformation of U can change it from concave to convex and completely change the order of $u(j)$. The only class of transformations of U that does not change the results of maximizing $u(j)$ over j - nor does it change the concavity or convexity of U - are affine strictly increasing transformations of the form $v(U) = aU+b$, with $a > 0$. Utility functions that are defined up to such a transformation are said to be measured on interval scale (since these transformations preserve the ratios of intervals) and are called *cardinal utility functions*.

In order to elicit from the decision maker his utility function, the expected utility theory applies usually the so called *principle of certainty equivalence*. We can fix arbitrarily $U(q_1)=0$ and $U(q_n)=1$, since this corresponds only to a certain choice of a and b in the affine transformation. The decision maker is asked to compare a possibility of getting the outcome q_i, with subsequent $i=2,...,n-1$, for certain, with a possibility of the result of a lottery of getting q_n with probability π_i or getting q_1 with probability $1 - \pi_i$; the probability π_i is varied until he states his indifference between these two options. Thus obtained probabilities π_i are measures of his utility of q_i, $\pi_i = U(q_i)$; observe that $\pi_1 = 0$ and $\pi_n = 1$ consistently with this definition. When this is done for all i, the optimal j can be selected by maximizing

$$u(j)=\sum_{i=1}^{n} p_{ij}\pi_i.$$

This prototype procedure has been extended to various other situations, including estimations of various functional forms of $U(q)$ both for single- and multi-dimensional E_q (in multiattribute utility theory - see Keeney and Raiffa, 1975) as well as the examination of risk-proneness and risk-aversion of decision makers - equivalent to convexity or concavity of the cardinal utility function - and other related subjects and applications.

As in the case of ordinal utility, cardinal utility theory depends on the conditions of existence of utility functions that are defined up to an affine transformation. Since this is a subclass of ordinal utility functions, the required conditions are much stronger.

Theorem 3.2 (see Fishburn, 1970). Let P be the set of all probability measures with finite support in E_q and let $\geq$ with implied $>$ be a complete preordering in P; let $E(u,R)$ denote the expected value of a function u: $E_q \rightarrow R^1$ under a probability measure $R \in P$. Then, for the existence of such a function u that would satisfy:

$$Q > R \text{ iff } E(u,Q) > E(u,R) \text{ for all } Q,\, R \in P \tag{3.11}$$

it is necessary and sufficient that two following conditions hold:

(i) Independence of side-alternatives :

$$\alpha Q+(1-\alpha)S > \alpha R+(1-\alpha)S$$

for all Q, R, $S \in P$ such that $Q > R$ and all $\alpha \in (0;\, 1)$.

(ii) Continuous interpolation of ordering :

if $Q > R > S$ for some Q, R, $S \in P$

then there exists α, $\beta \in (0;\, 1)$ such that

$$\alpha Q+(1-\alpha)S > R \text{ and } R > \beta Q+(1-\beta)S.$$

Moreover, u is unique up to a strictly increasing affine transformation.

The requirement of independence of side alternatives means that the relation $Q > R$ cannot be influenced by mixing (in the probabilistic sense, since α and β should be interpreted as probabilities in the above theorem) Q and R with any side alternative S, no matter how inferior (or even disastrous) this alternative might be. It can be interpreted as a requirement of strict independence of the decision maker's preferences on context; but decision makers are often known to change their minds depending on the context, particularly when faced with some possibilities of disastrous alternatives. The requirement of continuous interpolation of ordering means that an inferior alternative S, no matter how bad, can dominate a reasonable alternative R when S is mixed with the superior alternative Q - and conversely. This requirement can be violated in many decision situations, since decision makers often resist, for reasons of basic cultural values, the idea of mixing a disastrous alternative with very favorable ones. For example, consider a question such as: "given a lottery of one per cent chance of nuclear destruction of Earth and ninety nine per cent chance of winning a billion dollars, how much are you willing to pay for this lottery?" Most people would refuse to answer such a question on the grounds that there are, after all, values that are incomparable, and not all can be bought with money. In other words, our basic cultural values are often hierarchically (for example, lexicographically) ordered and do not admit continuity assumptions.

When not applied to such extreme cases, when limited to applications of clear statistical and monetary interpretations, expected utility theory has had considerable successes, particularly in applications to clarify conflicting motives of complex decisions. However, for all its theoretical power and analytical elegance, expected utility theory is rather poor empirical predictor of individual decision behavior. One of possible explanations is that maximizing behavior is not necessarily the dominant mode of individual decision making. Under presure of economic mechanisms such as market forces, people might tend in average to maximizing behavior; but in other environments or cultural settings, different modes of behavior might dominate. In order to solve some specific problems, or as an excellent tool for clarifying limits of achievement, people might use optimization; but optimization is not necessarily the main goal and explanation of human behavior.

Another possible explanation is that individual decision makers, even when trying to maximize, do not have constant utility functions and change them depending on context. This is best illustrated by the following paradox, due to Allais (1953). Consider a choice between lotteries l_j, $j = 1,...4$; each of the lotteries can have outcomes

$$q_i,\ q_1 = 0,\ q_2 = 1M,\ q_3 = 5M$$

(where M is a million monetary units, which are assumed to be significant in value for the decision maker) with some probabilities p_{ij}. The probabilities and the straight (unweighted) expected values of these lotteries are given in the following table:

Outcomes q_i	0	1M	5M	Straight expected values
Lotteries l_j	Probabilities P_{ij}			
l_1	0.00	1.00	0.00	1.00M
l_2	0.01	0.89	0.10	1.39M
l_3	0.90	0.00	0.10	0.50M
l_4	0.89	0.11	0.00	0.11M

The decision maker has to compare pairwise l_1 with l_2 and l_3 with l_4. Almost all people prefer l_3 to l_4 but a majority of them would also prefer l_1 (1M for sure) to l_2 (expected 1.39M, but a chance one in hundred to get nothing). However, there does not exist a utility function that is consistent with both of these choices.

There are various attempts to explain the above paradox, as well as many variants of this paradox. Here we adopt the simplest explanation that the utility function of the decision maker is not constant, but depends on the expectations or aspirations formed relative to the pair of lotteries assessed. Assume, for example, that the decision maker takes the lower expected value of the pair of lotteries compared as his aspiration $\bar{q}$; thus $\bar{q} = 1.00M$ for the comparison of l_1 and l_2, while $\bar{q} = 0.11M$ for the comparison of l_3 and l_4. Assume that the decision maker has a strong *regret* (similar ideas are presented, for example, by Kahneman and Tversky, 1982) when the actual outcome falls below his aspirations; if the regret coefficient is, say, 100, we can postulate the following utility function for his choice:

$$u(q,\bar{q}) = \begin{cases} q - \bar{q} & \text{if } q > \bar{q} \\ 100(q - \bar{q}) & \text{if } q \leq \bar{q} \end{cases}$$

where

$$\bar{q} = \min_{j'j''} E(q \mid l_j)$$

and

$$\bar{q}^A = 1.00M \text{if} j' = 1,\ j'' = 2,\ \bar{q}^B = 0.11M$$

if $j' = 3$, $j'' = 4$. Take the expected values of this non-classical utility function (which depends, through the aspiration levels $\bar{q}$, on the probabilities p_{ij} and thus is not constant). We obtain then

$$E(u(q,\bar{q}^A) \mid l_1) = 0.00,\ \ E(u(q,\bar{q}^A) \mid l_2) = -0.50,$$

and l_1 is preferred to l_2;

$$E(u(q,\bar{q}^B) \mid l_3) = -9.41,\ \ E(u(q,\bar{q}^B) \mid l_4) = -9.69,$$

and l_3 is preferred to l_4.

Many decision theorists would protest against calling $u(q,\bar{q})$ a utility function, because of its dependence on probabilities p_{ij} through $\bar{q}$. Therefore, we shall adopt here another name and call it an ***achievement function***, see further paragraphs. The form of $u(q,\bar{q})$ used above is not cardinal; but we can make it cardinal, independent of affine transformations of q by simply subdividing it through the range $\Delta_q = q_3 - q_1 = 5M$ of outcomes:

$$u(q,\bar{q}) = \begin{cases} (q - \bar{q})/\Delta_q & \text{if } q > \bar{q} \\ 100(q - \bar{q})/\Delta_q & \text{if } q \leq \bar{q} \end{cases}$$

Thus, the cardinality of a utility function is a different issue than its independence on broadly understood context, while the latter concept includes also expected outcomes and aspirations formed relative to them. We see that there might be at least two reasons for decision makers behaving differently than predicted by classical utility theory: one is that they do not necessarily maximize, and the second one, that their preferences might depend on broadly understood context and aspirations, even if they try to maximize.

3.3. Satisficing behavior and aspiration formation.

The hypothesis that people seldom maximize when preparing individual decisions was analyzed first in considerable detail by Simon (1957, 1958), though many other researchers (such as Boulding, (1955), Galbraith, (1967), Kornai, (1977), March, (1958), Sauerman and Selton, (1962), Tietz, (1983)) contributed to an advance of this thesis and to so called behavioral school of decision analysis. The main arguments of this school are:

A. Bounded rationality. People cannot maximize their utility in individual decisions, because of many reasons. Optimization problems can be very difficult to solve, and people do not necessarily have time and ability to solve them; the cost of solving optimization problems might outweight the gains from solving them. The information about the state of the world and about other people intentions, that is necessary to solve optimization problems, is typically not fully available.

B. Institutional and behavioral approach. When facing various institutional limitations in the complex life of administrative and large industrial organizations, people develop systems of rules and procedures for decision making. These historically formed procedures (legal, legislative, administrative, etc) allow for plausible inference under uncertainty, for information collection and learning, but are often difficult to understand from an abstract "normative" point of view - and are not easy to change. Thus, a decision analysts' task is to observe decision behavior and to construct plausible, if often ad hoc, models of this behavior.

C. Aspirations and satisficing behavior. When following such empirical direction of research, a recurrent observation is that people, while learning about the state of the world and the results of the actions of others, tend to summarize their learnings by forming aspirations on desirable outcomes of their decisions. When predicted outcomes of their decisions fail to satisfy their aspirations, people tend to work hard and seek ways to improve the outcomes; however, when their aspirations are satisfied, people turn their attention to other matters.

Intense discussions and research that resulted from this antithesis to the utility maximization framework had modified somehow the original meaning of these arguments, particularly the argument about bounded rationality. This argument states that people could not maximize, because the problems are too complex; however, this argument is not entirely convincing today. Research on deliberative, holistic decision making (see Dreyfus, 1984) has shown that expert decision makers can intuitively, by "Gestalt", process all available information and make optimal decisions. If they need analytical support in novel decision situations, the modern development of computer technology, optimization techniques and ways of treating uncertainty has enormously extended the class of optimization problems that can be solved computationally - far beyond the classical formulations of the utility maximization school. Moreover, these developments provided for a methodological reflection on the use of optimization tools. We know that most of optimization problems are solved up to a given accuracy; that however complicated an optimization problem might be, it can always be approximated by a simpler one; that there exist heuristic and artificial intelligence techniques for solving approximately optimization problems. Thus, when treating optimization as a tool, not as a goal or main model of behavior, we can support even very complex decisions.

The relevant question is, therefore, *not whether people could, but whether they should optimize*. Observe, first, how strong is the cultural background of the maximizing school that any departure from it must be called "bounded", somewhat less than perfect; but this background is deeply rooted in the individualistic culture of belief in the Adam Smith's "invisible hand". On the other hand, there are at least two reasons why people *should not maximize without reservations.*

D. Collective rationality. The fact that satisficing behavior in big industrial organizations is related to collegial decision making was noted already by Galbraith (1967). We can give now much more reasons for this relation. When facing any multiactor decision situation other than a perfect market - say, in any nonzero-sum n person game with $n<\infty$ - an individual decision maker can fall into a *social trap* (Rapoport, 1985) or even start a *conflict escalation* (Wierzbicki, 1983). Both these phenomena are characterized by the fact that unmodified individual maximizing behavior leads to much worse results than foregoing individual maximization and seeking some measures of cooperation. A social trap involves non-cooperative equilibria that are much worse for all concerned than results of cooperative action; such examples as the "tragedy of commons" or the "prisoners' dilema" have been known for some time, but strategies for repetitive situations of this type have been only recently developed and studied. Conflict escalation occurs in a more complicated situation when the noncooperative equilibria are nonunique - as in so-called "game of chicken", or in many environmental simulation games - and each player tries to select a different equilibrium that is advantageous for him, which results in persistent disequilibria much worse than even the non-cooperative equilibria. We can give abstract and mathematical models of such phenomena; moreover, recent studies show that there exist non-maximizing strategies that give an individual much better chances of evolutionary survival than purely maximizing strategies (see Axelrod, 1984); one of the most elegant and effective strategies of this type, proposed by Rapoport and called "tit for tat", can be described as "non-naive altruism that gives best chances for survival" - see Rapoport in Grauer, Thompson, Wierzbicki, 1985). However, a detailed discussion of these developments is beyond the scope of this book; it is sufficient to note here that the need of foregoing individual maximization and developing some rules of collective rationality has been long recognized in the historical social development of mankind, in our ethical systems, laws and customs.

In many discussions of utility theory, the issue of collective rationality is dealt with by an argument that an individual can introduce such attributes as compassion, cooperation, etc, in his utility. However, such an argument serves only to avoid a deeper analysis of this issue: no matter whether an individual would include cooperation in his utility, there would be still multiactor situations resulting in social traps or conflict escalation, and we better study them explicitly in order to understand them. Collective rationality means placing some values, such as preservation of Earth for the human race, hierarchically higher than others, such as monetary gains; already this violates axioms of utility theory and leads to discontinuous utility functions. Collective rationality means also learning about concerns and interests of others and thus an adaptive dependence of own decisions on the context, which again violates axioms of utility theory. All this leads to the conclusion that, when trying to account for collective rationality, we cannot use utility functions in their classical forms and must at least operationalize the concept of their dependence on the context; moreover, we must assume that an individual will have, in certain social situations, to forego his tendency to maximization.

However, it is only fair to add that collective rationality has its own traps. These traps occur because the *concept of fairness* has no reasonable absolute meaning, can be defined only relatively to a status quo situation and develops historically. Through many centuries, people fought for *equality before law*; later, *equality of chances* has become the major objective of social struggles. Out of human compassion, we tend to think that equality of chances should be accompanied by at least some measure of *equality of results* (for example, in the standards of living); the issue of an appropriate measure of equality of results has been a major element of social struggles of the last century. However, an absolute meaning of fairness - an ultimate equality of results - could be only achieved if all men would think alike, consume the same, make the same inventions and publish the same books and papers; such a society could not develop any further nor adapt to change. Thus, the trap of collective rationality is that one could always demand more fairness - and stop all human development, learning and adaptation.

E. Learning and adaptation. Though supported by other aspects of satisficing behavior, learning and adaptation have their own independent meaning. The phenomenon of human curiosity - the propensity to learn much more than it is needed for direct applications - was perhaps the decisive factor of the evolution of human civilization, but cannot be consistently explained by utility maximization. An explanation that curiosity might be one of the attributes of individual utility is tautological and leads to similar inconsistencies as the attempts to include compassion and cooperation into utility. Thus, learning must be considered as an independent factor in human decisions.

Except in the most simple cases, learning is done at the cost of optimality. This general observation has been formalized mathematically for the case of learning understood as quantitative adaptation. If a structure of an adequate model of a given process is known but the parameters of this model are not, then, parallely to trying to control the process optimally, we must identify the parameters. For the basic case when the model is linear, the unknown parameters relate only to the initial state of the process or to an additive perturbation of known stochastic properties, and the costs are quadratic, Kalman (1960) has established the *principle of separation*: one can go on with the optimal control while parallely estimating the parameters, without any loss of optimality. However, the principle of separation does not apply to any more complicated case, for example, when the model is nonlinear or even if it is necessary to estimate unknown parameters that enter multiplicatively an otherwise linear model. In general, any more complicated situation requires active experimentation and probing in order to identify parameters. If we try to control optimally, we should adhere to optimal decisions and forego probing; but if we do not estimate correctly the parameters, we might end up applying decisions that are optimal but for a different problem. Thus, there is a dynamic trade-off between the quality of learning and optimality: it pays to forego optimality first and learn more at the beginning in order to be better off at the end. These concepts have been formalized mathematically by Feldbaum (1962) who called them the *principle of dual control* (with dual purpose: that of learning and that of optimization).

In cases of qualitative learning, when we face a new situation and try to probe its numerous aspects in order to determine which of them are truly important, the same observation is valid though much more difficult to formalize mathematically. Various existing models of learning processes might be yet not fully satisfactory, but both these models and experimental research indicate some general conclusions:

- learning consists of probing and we learn mostly by making mistakes;

- two basic cases should be distinguished: a customary situation, in which we have an adequate framework or model and need only to fill in details, and a novel situation, in which we have to devise a framework;

- in customary situations, learning can be described as a nonstationary but convergent process of assessing some basic parameters;

- in novel situations, there are two phases of learning: the search for a framework, terminating in an "aha" effect, and then the resulting customary situation of filling in the details;

- one of the greatest difficulties of learning is the recognition that a situation is novel, since adherence to an old framework typically prevents such a recognition. Master experts - such as chess champions of international level - are particularly sensitive to this need of recognizing novel and potentially dangerous developments, feeling an uneasiness that forces them to search for new angles.

The last observation on the difficulties of learning applies also to the recognition of the satisficing framework of rationality: while it was accepted as a "bounded" rationality concept, as a description of possible departures from "true" rationality, the developments of abstract foundations of decision theory continued to be concentrated on the utility maximization framework. Thus, mathematical tools for the satisficing framework have been considerably less developed than for the utility framework - with some important exceptions. Mesarovic et al. (1970) gave first mathematical formalization of satisficing decision making. The dynamics and impacts of aspiration levels on decision processes have been thoroughly studied (see Sauerman and Selton, 1962, Tietz, 1983, 1985). In economic theory, satisficing equilibria of markets have been studied see, e.g., Kortanek and Phouts, (1982). In multiobjective optimization theory, techniques of goal programming (see Charnes and Cooper, 1975) and of displaced ideal point (Salukvadze, 1971, Yu and Leitmann, 1974, and Zeleny, 1973) have been developed. We present here shortly the main ideas of the technique of goal programming.

Let the decision space be $E_x = R^n$ and the admissible decisions belong to the set

$$X_o = \{x \in R^n \colon g_i(x) \leq 0,\ i=1,...m\}$$

Let the decision outcome space be $E_q = R^p$ and the decision outcomes be characterized by $f\colon R^n \rightarrow R^p$, so that the set of attainable outcomes is $Q_o = f(X_o)$. Suppose all the outcomes improve for the decision maker if the values of the corresponding outcome functions $f_i(x)$ increase, where

$$f(x) = (f_1(x), .. f_i(x), .. f_p(x))$$

this case is sufficiently general, since we can transform to it most other cases (when the decision maker prefers to decrease some outcome functions, or to keep their values at some specified level) by suitably modifying the form of the function f.

The *goal programming technique* assumes that the decision maker specifies goals $\bar{q}_i$, $i=1,...p$, jointly denoted by $\bar{q}$; the goals can be equivalently interpreted as aspiration levels for all outcomes. The typical formulation of goal programming technique assumes that a decision support system solves the following mathematical programming problem in response to the aspiration levels $\bar{q}$ stated by the decision maker:

$$\text{minimize}\ \ h(q^+, q^-) = \{\sum_{i=1}^{p} \alpha_i (q_i^+ + q_i^-)^k\}^{(1/k)} \tag{3.12a}$$

subject to constraints

$$f(x) + q^+ - q^- = \bar{q} \tag{3.12b}$$

$$x \in X_o$$

$$q^+, q^- \geq 0$$

where $\alpha_i > 0$ and the function $h(q^+, q^-)$ is, in fact, equivalent to the weighted l_k norm of the difference $f(x) - \bar{q}$. The use of q^+ and q^-, interpreted as overachievement and underachievement of $\bar{q}$, stresses the possibility of transforming the mathematical programming problem (3.12a,b) to a linear programming problem provided that the function f is linear or affine, the set X_0 is described by linear inequalities, and $k=1$, the l_1 norm is used. In this particular case, goal programming technique has been widely applied. The following, elementary theorem characterizes the results of the goal programming technique:

Theorem 3.3. Let the set $Q_0 = f(X_0)$ be compact. Then a solution $\hat{x}$ of the mathematical programming problem (3.12 a, b) exists and $\hat{q} = f(\hat{x})$ has the following properties: if $\bar{q} \in Q_0$ - we say then that $\bar{q}$ is attainable - then $\hat{q} = \bar{q}$; if $\bar{q} \notin Q_0$ - is not attainable - then $\hat{q}$ is an element of Q_0 that is closest to $\bar{q}$ in the l_k norm. If additionally (see Theorem 4.8) Q_0 is convex and $\bar{q}_i > \hat{q}_i$, $i = 1,..p$, then $\hat{q}$ is an efficient outcome: a component $\hat{q}_i$ of it cannot be improved without deteriorating other components $\hat{q}_j$, $j \neq i$.

Clearly, the goal programming technique does not necessarily suggest decisions that are maximizing a monotonous utility function or efficient; it only suggests decisions that have outcomes closest to the goals. Although the goal programming technique apparently represents precisely the rationality of satisficing decision making, it also expresses the inconsistencies of interpretations of this framework, related to the question whether a decision maker could not or should not maximize. If we adhere to the interpretation that a decision maker could not maximize because of complexity of decision problems, the goal programming technique directly contradicts this interpretation, since it uses a - hopefully adequate - mathematical model of the decision situation and a mathematical programming technique that could support the decision maker in maximization. If we say that the decision maker could, but perhaps should not maximize in certain situations, then the goal programming technique is inadequate as a decision support tool - because the decision maker should then choose to forego maximization on the basis of full information available, that is, he should be informed how much he could gain if he chose not to forego maximization. This inadequacy of goal programming can be overcome in the *quasisatisficing rationality framework*, described in detail later in this book, and was, in fact, one of the motivations for developing this framework.

Beside the prototype formulation (3.12a,b) of goal programming, there are many refinements and further developments of this technique (see, e.g., Masud and Hwang, 1981, Ignizio, 1983). For example, the overachievements q^+ and underachievements q^- can be further transformed by so called achievement functions; however, the question of desirable properties of achievement functions that would help to overcome the basic inconsistency of goal programming was addressed only in research on quasisatisficing framework (Wierzbicki, 1982). A hierarchy of goals expressed either by the weighting coefficients α_i or by a lexicographical ordering of objectives can be also introduced in goal programming; however, these technical possibilities do not express a clear-cut hierarchical approach to decision making, such as represented by the next rationality framework - that of goal- and program-oriented planning and management.

3.4 Goal- and program-oriented planning and management

We have already observed that hierarchical, lexicographic ordering of fundamental values is a typical characteristic of human cultures. This observation forms a basis of an alternative framework of rational decision making, developed in the Soviet Union by Glushkov (1972), Pospelov and Irikov (1976) and others, but also perceived independently as a reasonable framework for rational action by researchers from other cultures (see, e.g., Umpleby, 1983). This framework distinguishes between (at least) two groups of objectives: ***primary objectives*** or ***goals***, and ***secondary objectives*** or ***means***; both of them can be treated dynamically, in which case we speak about a ***program*** of goals or of means. A rational plan of action is such that guarantees the attainment of aspired values for primary objectives or goals due to a reasonable choice of secondary objectives or means. In other words, if a goal appears in the first round of analysis to be not attainable, we should not concentrate on devising trade-offs between primary objectives, but much rather on finding such constraints that should be shifted - as means - in order to make this goal attainable. As a culturally determined perception of what constitutes rational decisions, this framework is related to the culture of planning; thus, it is quite different than the utility maximizing framework and also different - although perhaps closer methodologically - than the satisficing framework. Formally, if we consider the primary goals as constraints and address the question of reasonable choice of means via utility maximization, we could reduce the goal- and program-oriented planning to maximizing framework (as we could do also with satisficing decisions, if we introduced some form of disutility of further maximization; such formal reductions do not increase our understanding of different perceptions of rational decision making, but might be useful in mathematical formalizations). Similarly, we could reduce the goal- and program oriented planning to the satisficing behavior - possibly, with a better behavioral reason - by addressing the question of a reasonable choice of means via satisficing. However, a goal- and program oriented decision maker concentrates his attention on the question why his aspired values of primary goals might be not attainable and modifies these aspirations much less readily than those for secondary means.

A mathematical formalization of the goal- and program-oriented decision making that addresses the question of reasonable choice of means in partly satisficing and partly maximizing way is as follows. Let the space of primary objectives or goals (or goal programs) be denoted by E_{q1} and take, as a simple example, $E_{q1} = R^{p1}$; denote the aspired values (or trajectories in case of dynamic goal programs) of these objectives by $\bar{q}_1$. Similarly, denote the space of secondary objectives or means by E_{q2} and take, as a simple example, $E_{q2}=R^{p2}$; suppose that the decision maker forms also aspiration levels $\bar{q}_2$ for secondary objectives. Observe that ***means*** should not be confused with detailed ***decisions*** from the admissible set X_o in the decision space E_x. Take, as a simple example,

$$X_o = \{x \in E_x : g_j(x) \leq 0, j = 1,...m\},$$

where $g_j : E_x \rightarrow R^1$ are constraint functions called shortly constraints; these constraints can be later redefined as means, and only in simple cases, when constraints have the form of simple bounds on some decisions, means and decisions can coincide. Let the outcomes of decisions be described by the mappings $f_1 : X_o \rightarrow E_{q1}$ and $f_2 : X_o \rightarrow E_{q2}$; if $E_{q1} = R^{p1}$ and $E_{q2} = R^{p2}$, then $f_1(x) = (f_{1_1},...,f_{1_{p1}})$ and $f_2 = (f_{2_1},...,f_{2_{p2}})$. Denote the sets of attainable outcomes for primary objectives by $Q_{1_o} = f_1(X_o)$ and for secondary objectives by $Q_{2_o} = f_2(X_o)$.

The essence of goal- and program oriented decision making lies in a test of goals' attainability, whether $\bar{q}_1 \in Q_{1_o}$; if not, first the space E_{q2} together with the constraints of the set X_o and then possibly even the space E_{q1} (or the aspired goals $\bar{q}_1$) must be interactively redefined. For example, suppose that the test of goals' attainability gives first negative results, but we can identify some active constraint g_j of the set X_o that can be considered as a secondary outcome or a component of means. We redefine then the set X_o and the space E_{q2} by taking the function g_j out of the set of constraints and putting it as an additional dimension, $f_{2_{p2+1}}(x)=g_j(x)$, of the space of secondary outcomes, together with an appropriate aspiration level (which can be defined, at first, as $\bar{q}_{p2+1}=0$, since the aspirations for secondary objectives need not be attainable). After this redefinition, we test again the attainability of goals. If this test fails again, we might look for other constraints to be redefined as means or secondary outcomes; first when there are no more constraints that could be considered as means, we can check whether some of the primary goals could not be moved to the category of secondary outcomes or, if this also fails, whether we should modify the aspired values for goals. Under quite general assumptions (since, in the end, we can also modify the aspired values for goals, hence it is sufficient to assume that the set Q_{1_0} is not empty), this process of problem redefinition ends with aspired values of goals that are attainable.

Now comes the phase of selection of actual decisions and means that result in attainable goals. In this phase, goals can be treated as additional constraints and we face a multiobjective decision with outcomes in the space of secondary objectives or means. This secondary phase can be solved by following either the maximizing or satisficing framework; if we assume that the aspiration levels for secondary objectives are not attainable, we can use a norm of the difference between these levels and the actual values of secondary objectives as a disutility function, similarly as in the goal programming technique. Without this assumption, in a more general setting of quasisatisficing decision making, we can use special achievement functions in an interactive technique of selection of means, see next sections and chapters; in fact, the goal- and program- oriented decision making was also one of the motivations for developing the quasisatisficing framework. This framework is also helpful in the first phase of goal- and program-oriented decision making, since it provides for easy tests of attainability together with a measure of under- or overattainment of goals; although goals are typically specified at rather high levels and an overattainment of them is not very probable, there are specific cases when a measure of overattainment is useful. For example, if the decision maker specifies two goal levels: a *reservation level*, which must be reached, and an *aspiration level*, which should be reached if possible, then the goal- and program-oriented decision making loses its strict hierarchical nature and a measure of overattainment of reservation levels for primary objectives must be compared with measures of attainment of aspirations for secondary objectives.

3.5 Quasisatisficing decision making

The quasisatisficing framework of decision making was developed (mainly by the authors of this book, but also in cooperation with many others, whose contributions are presented later) in order to provide decision support for decision makers that adhere either to the maximizing, or satisficing, or goal- and program-oriented perceptions of what constitutes rational decisions; in this sense, it is a generalization of the three preceding frameworks. We say that a decision maker behaves in a quasisatisficing way if, aware of his objectives (together with possible distinctions between primary and secondary objectives, such as in the goal- and program-oriented decision making), aware of the scales of attainability of these objectives, aware of his aspirations (together with possible distinctions between aspiration and reservation levels for his objectives), he tries to reach the aspiration (or reservation) levels by maximizing when the outcomes of admissible decisions fall below these levels, but, when the aspiration (reservation) levels are attainable, he can choose either to further maximize in order to reach efficient outcomes, or to forego maximization for additional good reasons (such as reaching cooperative solutions in multiactor decision situations).

This definition has two essential elements: first is the awareness of objectives, their importance, scales of attainability and aspirations for these objectives, which implies adaptive learning of the decision maker about the decision situation with possible changes of his aspiration levels (also, if he has any utility function, this function might be changing during the learning process); second, the assumption that the decision maker can choose between satisficing and further maximization upon reaching his aspiration levels. This specific assumption is of particular importance in multiactor autonomous decision situations; in centralized decisions, either with a single decision maker or in collegial decision making where objectives of all members of the group are jointly considered, reaching aspiration levels for certain objectives might strongly influence trade-offs and priorities between objectives, but does not as a rule prevent searching for efficient solutions.

Here we return to the question whether a decision maker could not or should not maximize in certain situations and the relation of this question to the concept of efficiency. Recall that a decision is called ***efficient*** with respect to a certain number of outcomes or objectives if there does not exist another admissible decision that is as good as the efficient one in all outcomes and strictly better at least in one outcome; in other words, an efficient decision cannot be improved in one of its outcomes without deteriorating other outcomes. The concept of efficiency is very naturally related to that of rationality and can be expressed as the following ***axiom of efficiency***: if a decision maker is certain that he has listed all relevant outcomes, there exist no rational reasons for him to be satisfied with outcomes there are not efficient. Since a simple way to guarantee efficient decisions is to maximize a function which is strictly increasing with improvements of all outcomes - for example, an utility function - the efficiency axiom has been used as one of the main arguments of the proponents of maximizing against satisficing.

However, the concept of efficiency is relative to the completeness of decision outcomes considered, which fact is often not stressed enough in decision analysis. If the decision outcomes considered are not complete, there might be rational reasons for foregoing complete efficiency. For example, the traditional argument that a decision maker could not maximize is based on the assumption that the costs of optimization or of procuring additional information are implicit additional outcomes that cannot be precisely assessed. In the quasisatisficing framework, this assumption is not valid: a decision maker, aware of his objectives and their scale of attainability, can assess the costs of information and optimization, if these are relevant outcomes. On the other hand, there are other good reasons for incompleteness of outcomes considered, indicating the situations in which a decision maker should not necessarily maximize. One, in autonomous multiactor decision situations, is the unwillingness to consider or uncertainty about the outcomes of interest for other actors. Another, common also for centralized decision situations, is the uncertainty about own intentions and the necessity to learn; hence, in the beginning phase of a quasisatisficing decision process, a decision maker must be prepared to make mistakes in the specification of his objectives and aspirations and to learn by them, before he is sure that he specified objectives completely.

However, a decision support system that is designed to help in such learning should not misinform the decision maker by proposing to him outcomes that are inefficient relative to the current specification of objectives; if the decision maker wishes to change the concept of efficiency or to stop at a seemingly inefficient solution for any reason, he should do it consciously, as an effect of his own learning and not of an inefficient decision support. In this sense, a traditional goal programming technique is not an adequate tool for quasisatisficing decision support, since it does not necessarily propose efficient decisions if the aspiration levels are attainable - because, in order to propose efficient decisions starting from attainable aspiration levels, we would have not to minimize but to maximize the distance of actual outcomes from these aspiration levels.

Another concept in the quasisatisficing framework of rationality, relating it to utility maximization, is the following ***principle of interactive reference point optimization***.

Suppose the decision maker is maximizing his utility, but he does not have full information about the admissibility of alternatives and about their possible outcomes; he has only some mental model of them. Still, suppose he is an expert and can intuitively, holistically maximize his utility function over this mental model of decision situation. He arrives then at some "best" decision and outcome that is not necessarily attainable; let us call this outcome his aspiration or ***reference point***. He communicated his reference point either to his supporting staff of advisers - his team of analysts - or to a decision support system; what should be the function of a good decision support system in such a case?

The staff of advisers or a decision support system should gather all pertinent information about the decision situation-alternatives, their constraints, their outcomes; this leads to the concept of the ***substantive model*** of the decision situation. The decision support system should then take the reference point as a guideline and try to find efficient solutions corresponding to this point, ***optimize in response to a reference point.*** If the reference point is not attainable, the decision maker should be informed and presented with alternatives that give outcome possibly close to the reference point. If the reference point is attainable, the decision maker should also be informed about it; if there are efficient alternatives with better outcomes than the reference point, they should be presented to the decision maker.

In view of the above discussion, quasisatisficing decision support needs a concept of a function that:

a) is similar to an utility function and, when maximized, produces efficient decisions relative to the current list of objectives; moreover, can be used as an approximation to a class of utility functions;

b) is explicitly dependent on aspiration levels stated and modified by the decision maker and thus makes operational the concept of adaptive dependence of utility on learning and context;

c) corresponds to the minimization of a distance of decision outcomes to aspiration levels, if the latter are not attainable, and to the maximization of such a distance, if the aspiration levels are attainable; however, can be modified by the decision maker, if he wishes to forego maximization;

d) can be modified by the decision maker to express his hierarchy of goals, such as in the goal- and program-oriented decision making;

e) can be easily used to test attainability and efficiency of aspirations;

f) can be easily generalized to the case of dynamic outcomes in form of trajectories.

Such a class of functions exists and is, in fact, a result of long research and development (see Wierzbicki, 1975, 1977, 1978, 1980, 1982, 1984). We shall call here these functions ***order-consistent achievement functions*** or, in short, ***achievement functions.*** An axiomatic definition of this class of functions will be given later; here we start with an example of two useful members of this class.

Consider a decision maker who, supported by a team of analysts or a decision support system, faces a problem of choosing a decision out of an admissible set X_0 and has specified p objectives; without loss of generality, suppose that all outcomes of decision improve if the values of objectives increase. Assume that the decision maker, either relying on his own experience or supported by the team of analysts, has established (judgementally or analytically) a mapping $f : X_0 \rightarrow E_q = R^p$ that specifies the outcomes of decisions and, with possible help of the decision support system, has learned about the ranges of attainability $(q_{i,\min}; q_{i,\max})$ of each decision outcome. These ranges need not be very tight bounds of the set of attainable decisions $Q_0 = f(X_0)$; they might give rather broad bounds for this set, or even broad bounds of its reasonable subset - for example, of the set of efficient outcomes; it is only important that they have been accepted as reasonable outcome ranges by the decision maker.

Suppose that, in this situation, the decision maker specifies his ***reservation level*** $\bar{q}_i'$ and his ***aspiration level*** $\bar{q}_i''$, where these levels satisfy

$$q_{i,\min} < \bar{q}_i' < \bar{q}_i'' < q_{i,\max}$$

for each objective or outcome, $i=1,...p$. What can the team of analysts or the decision support system conclude about the preferences of the decision maker on the basis of such information?

One way would be to select an established theoretical tool; to this type of information, the most appropriate tool would be the theory of fuzzy sets - see, e.g., Sakawa, (1983). Membership functions for the assessment of satisfying decision makers requirements on each outcome could be postulated in the form:

$$\mu_i = \begin{cases} 0 & \text{if } q_i \le \bar{q}_i' \\ (q_i - \bar{q}_i')/(\bar{q}_i'' - \bar{q}_i') & \text{if } \bar{q}_i' < q_i < \bar{q}_i'' \\ 1 & \text{if } \bar{q}_i'' \le q_i \end{cases} \tag{3.13a}$$

where $q_i = f_i(x)$ denotes the i-th outcome of the decision; the convolution of these membership functions could be interpreted in the sense of minimum operation:

$$\mu = \min_{1 \le i \le p} \mu_i \tag{3.13b}$$

The level sets of this function in case $p = 2$ are shown in Figure 3.1.

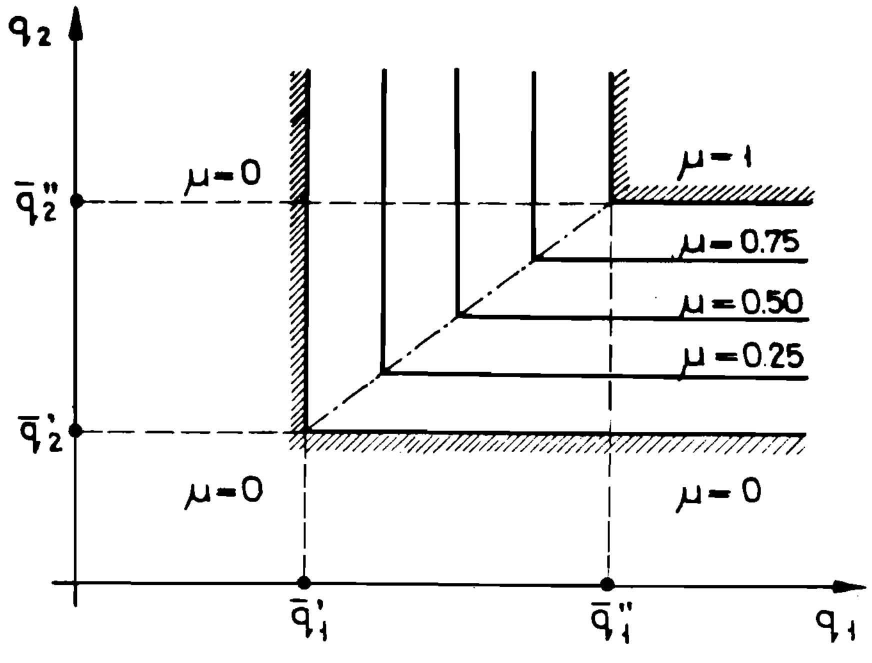

Figure 3.1. Level sets of the membership function (3.13).

However, the logic of fuzzy sets is still too sharp to describe fully the preferences of the decision maker: the membership function does not describe his disutility of not reaching his reservation levels, nor his possible utility of reaching more than his aspiration levels; this would require an extension of the membership function (3.13a,b). Such specific utility function, explicitly dependent on the aspiration and reservation levels of the decision maker, can be constructed when accepting following assumptions:

(i) The decision maker prefers outcomes that satisfy all his reservation levels to any outcome that does not satisfy at least one of his reservation levels; similarly for aspiration levels;

(ii) The satisfaction of the decision maker at reaching (all, or the last of) his reservation levels can be measured by 0, while his satisfaction at reaching (all, or the last of) his aspiration levels can be measured by 1;

(iii) The satisfaction of the decision maker at reaching the maximum of the range of all outcomes can be measured by $1+\beta$, where $\beta \geq 0$ is a parameter (if $\beta=0$, then the decision maker behaves in a strict satisficing way); the (dis)satisfaction of the decision maker at reaching the minimum of the range of at least one of the outcomes can be measured by $-\gamma$, where $\gamma>0$ is another parameter;

(iv) Since all available information for the construction of this special utility function has been already used, the simplest form of this function that would satisfy (i), (ii), (iii), obtained through linear interpolation, is postulated.

Such a function has the following form:

$$\mu_i = \begin{cases} \gamma((q_i - q_{i,min}) \,/\, (\bar{q}_i' - q_{i,min}) - 1) & \text{if } q_i \leq \bar{q}_i' \\ (q_i - \bar{q}_i') \,/\, (\bar{q}_i'' - \bar{q}_i') & \text{if } \bar{q}_i' < q_i < \bar{q}_i'' \\ \beta(q_i - \bar{q}_i'')/(q_{i,max} - \bar{q}_i'')+1 & \text{if } \bar{q}_i'' \leq q_i \end{cases} \tag{3.14a}$$

$$s(q,\bar{q}',\bar{q}'') = \min_{1\leq i\leq p} \mu_i \tag{3.14b}$$

The function s above is called an *achievement function* (also an *achievement scalarizing function* or a *scalarizing function*, see Wierzbicki, 1977, 1982) since it belongs to the class of *order representing achievement functions*, defined axiomatically in the next chapter; its level sets - see Figure 3.2 for the case of $p=2$ - coincide with the shifted positive cone R_+^p that defines the partial preordering of the outcome space in case of maximization of all outcomes.

This function has several interpretations. One of them is a L-shaped utility function, consistently summarizing the information contained in the points $q_{\min}$, $\bar{q}'$, $\bar{q}'$, $q_{\max}$ and thus serving as an approximation to the preferences of the decision maker. Observe that this particular function is not an ordinal, but a cardinal utility function (it is defined by ratios of intervals and thus independent of any positive monotone affine transformations of outcomes); therefore, it can be even used for statistical averaging or in interpersonal comparisons of utility in collegial decision making.

Another interpretation of this function is a transformed and weighted (with changing, but piece-wise constant weighting coefficients) Chebyshev or l_∞ norm of the difference between the point $q_{\max}$ and the actual outcome $q=f(x)$. To illustrate this interpretation more clearly, consider a case when the points $q_{\min}$, $\bar{q}'$ are not specified and denote $\bar{q}''=\bar{q}$; the function (3.14a,b), after an affine transformation, simplifies then to the form:

$$s(q,\bar{q})= \min_{1\leq i\leq p} (q_i-\bar{q}_i)/(q_{i,max}-\bar{q}_i)-1 \tag{3.15a}$$

which can also be written (after subtracting a constant term) in the form:

$$s(q,\bar{q})=- \max_{1\leq i\leq p} \alpha_i(q_{i,max}-q_i) \tag{3.15b}$$

$$\alpha_i=1/(q_{i,max}-\bar{q}_i)$$

Since we assume that $q_i < q_{i,max}$ for all $1 \leq i \leq p$, the achievement function corresponds in this case to the weighted Chebyshev norm with changed sign; however, the weights α_i are not specified explicitly by the decision maker, but defined implicitly through his statement of aspiration levels $\bar{q}$ as compared to the upper bound point $q_{\max}$. Interpreted as an achievement function, this function was used, for example, by Wierzbicki, (1984) and

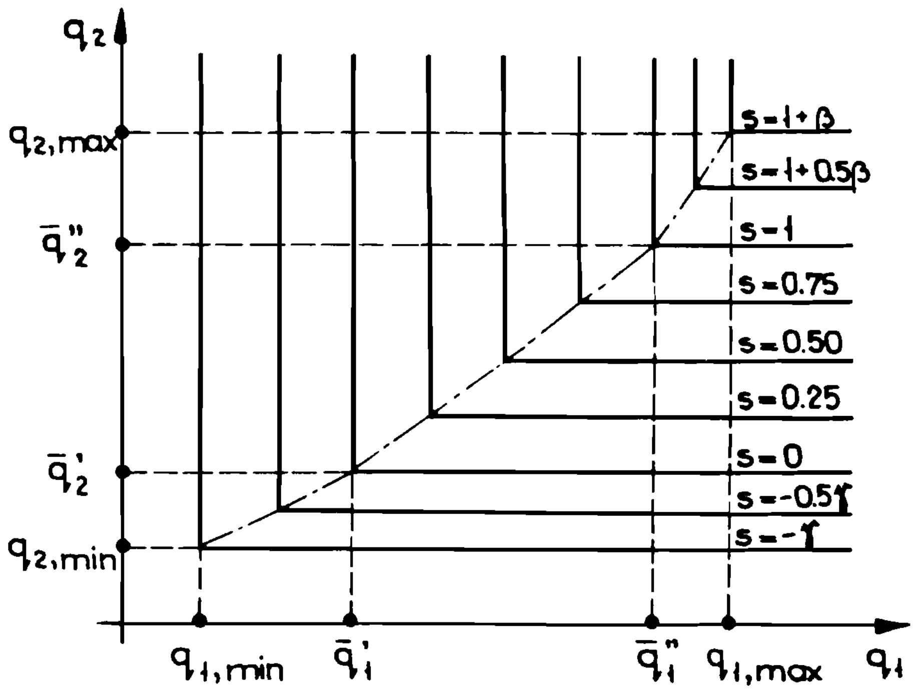

Figure 3.2. Level sets of the achievement function (3.14)

Lewandowski, Toth, Wierzbicki, (1985); as a Chebyshev norm, by Nakayama, (1984) and Steuer and Choo, (1983). The case when a decision maker specifies a reservation and an aspiration point was also investigated, though not in the form of the achievement function (3.14a,b), by Gorecki et al., (1984), and by Weistroffer, (1984).

The simplification (3.15a,b) of the achievement function (3.14a,b) is not the only one possible. If the decision maker knows both upper bound point $q_{\max}$ and the lower bound point $q_{\min}$ but specifies only one reference point $\bar{q}$, this point should be interpreted as an aspiration level point rather than reservation level point; but in mathematical modifications of (3.14a) we must treat it as reservation level, $\bar{q}=\bar{q}'$, and let the aspiration level coincide with the upper bound, $\bar{q}'' = q_{\max}$ (otherwise the function (3.14a) would become discontinuous, if we would let $\bar{q}''=\bar{q}'$). If both aspiration and reservation levels are specified but the lower bounds are not available, the definition of the function (3.14a) for $\bar{q}_i' < q_i < \bar{q}_i''$ must be used also for $q_i \leq \bar{q}_i'$.

An achievement function can be interpreted also in various other ways, as a penalty function (Wierzbicki, 1975, 1977, 1978, Weistroffer, 1984), as a tool of characterizing efficient solutions (Wierzbicki, 1977), as an utility function of an ideal team of staff in response to aspirations set by the boss (Wierzbicki, 1982), as a tool for organizing interaction with the decision maker in decision support systems (Wierzbicki, 1980, Kallio, Lewandowski and Orchard-Hays, 1980, Wierzbicki, Grauer and Lewandowski, 1982); some of these interpretations will be discussed in detail in further chapters.

The achievement function (3.14a,b) or (3.15a,b) has, however, one disadvantage. It is not strictly monotonous with respect to the decision outcomes $q_i=f_i(x)$; in fact, it is constant when one of the outcomes increases much above other outcomes that are kept constant. This implies, theoretically, that the maxima of this function might be not efficient but only weakly efficient (see next chapter) and, behaviorally, that the decision maker does not pay any attention to overachievements, however large, in some outcomes, as long as other outcomes show underachievements. Since most decision makers would allow at least some degree of compensation of underachievements by large overachievements in other outcomes, it is necessary to modify the function (3.14a,b). We can do it when adding the following postulate to the list (i), (ii), (iii), (iv):

(v) If an outcome shows underachievement when compared to its reservation (or aspiration) level, and other outcomes show overachievements, the decision maker is willing to accept a compensation of the underachievement by the average overachievement in other outcomes (all measured relative to the scales implied by points $q_{\min}$, $\bar{q}''$, $\bar{q}'$, $q_{\max}$) with a weighting coefficient ρ, where $0 < \rho \leq p$.

This postulate leads to the following form of the achievement function (which is an example of *order approximating achievement functions*, see next chapter):

$$s(q,\bar{q}', \bar{q}'') = \{\min_{1\leq i\leq p} \mu_i + (\rho/p)\sum_{i=1}^{p} \mu_i\}/(1+\rho) \tag{3.16}$$

with μ_i defined as in (3.14a). This achievement function is also a cardinal utility function. Its maxima are not only efficient, but also properly efficient (with a priori bounded tradeoff coefficients, see next chapter). Its level sets - see Figure 3.3 for the case p=2 - approximate from outside the shifted positive cone R^p_+. Beside these properties, it has all the interpretations of the function (3.14a,b).

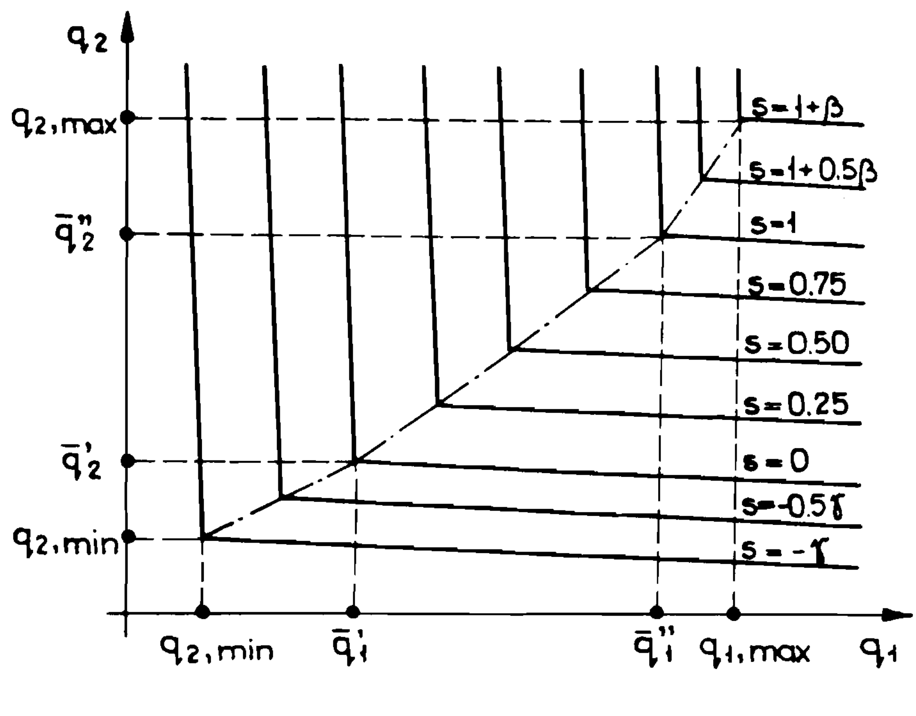

Figure 3.3. Level sets of the achievement function (3.16).

Functions of this and similar types have been extensively used in so called DIDAS decision support systems, based on the quasisatisficing framework of rationality and often called *interactive reference point methods*. The use of achievement functions in decision support is motivated not only by the fact that they constitute a reasonable approximation of preferences of the decision maker, explicitly dependent on his aspiration levels which he can modify when learning with the support of the system. We can also construct the following mental model of a decision support process. Suppose the decision maker knows well what he wants and is an expert in his field, however, does not have full information about present and future alternatives of decision and all their consequences. Since he has some information and is an expert, he can holistically optimize his utility function on an imagined set of alternatives; their way, he arrives at some hypothetical decision outcomes that are expressed as aspiration levels. The decision support system contains a much more adequate model of present and future decision alternatives and their outcomes constructed by a staff of analysts and called here the substantive model. Now, the construction and maximization of an achievement function is a tool of organizing interaction between the substantive model and the decision maker, who learns from the substantive model about decision alternatives and their consequences but preserves his full sovereignty of preferences and final decisions. Thus, the maximization of achievement function is only a tool in organizing good interaction.

Each maximum of an achievement function is (weakly or properly) efficient; but achievement functions can be also used to test attainability and efficiency of any given aspiration point (hence they are very useful in goal- and program-oriented decision making). If, say, a reservation point is (weakly or properly) efficient, then an (order representing or order approximating) achievement function achieves its maximum, equal zero, at this point. If a reservation point is not attainable, then the maximum of an achievement function over attainable outcomes is negative, and conversely; if this maximum is positive, then the reservation point is attainable and not efficient, that is, dominated by attainable points. These properties are related to the question of completeness and constructiveness of characterizations (necessary and sufficient conditions) of efficient decisions, discussed in the next chapter.

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4. QUASISATISFICING AND REFERENCE POINT OPTIMIZATION - MATHEMATICAL AND PROCEDURAL FOUNDATIONS

4.1 Completeness and constructiveness of characterizations of efficient solutions.

Given a mathematical model of the decision problem and a list of objectives specified by the decision maker and represented in the model as outcome variables, we assume that a decision support system should respond to any further information supplied by the user (that is, the decision maker or expert or analyst) by proposing some decisions and outcomes that are efficient with respect to the list of objectives. If the user would prefer inefficient outcomes, it means that he has other, yet not specified objectives in mind. He can then either supplement the list or, in the case when his additional objectives are not represented in the model, change the ordering of the already specified objectives in such a way that some of them should not be maximized nor minimized, but kept close to some given levels. This modifies the meaning of efficiency of solutions and enables the user to choose solutions according to his preferences; however, this does not change the basic requirement that the decision support system should "try to do its best" and respond only with efficient solutions once the list of objectives and the character of their ordering is given by the user. Thus, the basic theoretical questions when constructing multiobjective decision support systems are how to characterize mathematically efficient solutions and which characterizations can be constructively used for computations in a decision support system.

4.1.1. Basic concepts.

We consider here the basic case when the decision space is $E_x = R^n$ and the set of admissible decisions $X_0 \subset R^n$ is compact. Let there be p objectives or outcomes of interest to the decision maker, $E_q = R^p$, and let the outcome mapping (vector-valued objective function) $f: X_0 \to R^p$ be continuous, hence the set of attainable outcomes $Q_0 = f(X_0)$ be also compact. If the decision maker wants to maximize all outcomes (when he wants to minimize some of them, we can simply change the signs of corresponding components of objective function), then we say that the partial ordering of the outcome space is implied by the positive cone $D = R^p_+$ - which, in other terms, means that the inequality

$$q' \geq q'' \iff q' - q'' \in D \tag{4.1a}$$

is understood in the sense of simple inequalities $q_i' \geq q_i''$, $i = 1,...k$ for each component of vectors q', q'' while the *strict inequality*

$$q' > q'' \iff q' - q'' \in \tilde{D} = D\backslash\{0\} \tag{4.1b}$$

means that $q_i' \geq q_i''$ for all $i = 1,...k$ but there exists $j = 1,...k$ such that $q_j' > q_i''$, and the *strong inequality*

$$q' \gg q'' \iff q' - q'' \in \text{int } D \tag{4.1c}$$

means that $q_i' > q_i''$ for all $i = 1,..k$.

However, we admit here also the possibility that the decision maker would like to maximize (or minimize) only first p_0 of outcomes, while the last $p - p_0$ outcomes, numbered here by $p_0+1,....,p$, are to be kept at some given levels. In this case, we redefine the positive cone to the form

$$D = \{q \in R^p \colon q_i \geq 0,\ i = 1,...p_o,\ q_i = 0,\ i = p_o + 1,...p\} \tag{4.2}$$

Observe that the cone D does not have an interior in this case; however, it is still a closed convex cone, independently whether $p_o < p$ or $p_o = p$ (in which case, clearly, the definition (4.2) reduces to $D = R^p_+$). If the cone D is closed and the set Q_o is compact (in fact, much weaker assumptions are sufficient here, see Wierzbicki, 1977, and Benson, 1978), then there exist ***D−efficient*** or ***D−optimal*** elements of Q_o. These are such elements $\hat{q} \in Q_o$ that

$$Q_o \cap (\hat{q}+\tilde{D}) = \phi \tag{4.3}$$

where $\tilde{D} = D \backslash \{0\}$ is the cone D except its origin ($\tilde{D}$ is called strictly positive cone) and $\hat{q} + \tilde{D}$ denotes the set $\tilde{D}$ shifted by $\hat{q}$. If $D = R^p_+$, then D-efficient elements are called also Pareto-optimal; in other words, they can be defined as such outcomes that none component of them can be improved without deteriorating other components. The corresponding decisions $\hat{x} \in X_o$ such that $\hat{q} = f(\hat{x})$ are called D-efficient or Pareto-optimal as well. The set of all D-efficient outcomes

$$\hat{Q}_o = \{\hat{q} \in Q_o \colon Q_o \cap (\hat{q} + \tilde{D}) = \phi\ \} \tag{4.4}$$

is called the D-efficient set (D-optimal set, Pareto set) in objective or outcome space. Several other concepts of efficiency are essential for the discussion of characterizations. The ***weakly D-efficient*** elements belong to the set

$$\hat{Q}_{ow} = \{\hat{q} \in Q_o \colon Q_o \cap (\hat{q} + \text{int } D) = \phi\} \tag{4.5}$$

If $D = R^p_+$, then the weakly D-efficient elements are such that they cannot be improved in all outcomes together, in the sense of the strong inequality

$$q' \gg q'' \Longleftrightarrow q' - q'' \in \text{int } D$$

while the D-efficient are such that they cannot be improved in any outcome, in the sense of the strict inequality

$$q' > q'' \Longleftrightarrow q' - q'' \in \tilde{D}$$

Although important for theoretical considerations, weakly D-efficient elements are not useful in practical decision support, since there might be too many of them: observe, for example, that if $p_o < p$ and the interior of D is empty, then all elements of Q_o are weakly D-efficient.

Another concept is that of ***properly D-efficient*** elements; there are many almost equivalent definitions of such elements (see Sawaragi et al, 1985). We adopt here the definition of Henig (1982) that characterizes properly ***D-efficient elements*** as D'-efficient elements for any cone D' that contains D in its interior

$$\hat{Q}_{op} = \bigcup_{D' \in OD} \{\hat{q} \in Q_o \colon Q_o \cap (\hat{q} + \tilde{D}') = \phi\ \};\ OD = \{D' \colon D' \supset \text{int } D\} \tag{4.6}$$

If $D = R^p_+$, then properly Pareto-optimal elements have bounded trade-off coefficients that indicate how much one of the objectives must be deteriorated in order to improve another one by a unit. In applications, it is more useful to restrict further the concept of properly Pareto-optimal elements and consider only such that have trade-off coefficients bounded by some a priori number. This corresponds to the concept of ***properly D-efficient elements with a priori bound*** ϵ or ***D_ϵ-efficient elements*** that belong to the set

$$\hat{Q}_{o\epsilon} = \{\hat{q} \in Q_o \colon Q_o \cap (\hat{q} + \tilde{D}_\epsilon) = \phi\ )\} \tag{4.7}$$

where

$$D_\epsilon = \{\ q \in R^p \colon \text{dist}(q,\ D) \leq \epsilon ||q||\ \};\ \tilde{D}_\epsilon = D_\epsilon \backslash \{0\} \tag{4.8}$$

while $\epsilon > 0$ is some small given number - see Wierzbicki (1977). The D_ϵ-efficient elements are properly efficient with trade-off coefficients bounded (approximately) to the interval $(\epsilon;\ 1/\epsilon)$. Beside ***weakly D-efficient, properly D-efficient*** and ***D_ϵ -efficient elements,*** there is also the concept of ***strongly D-efficient elements*** - such that the set $\hat{Q}_o$ consists of a single element. However, problems with strongly D-efficient outcomes are rare and, in a sense, trivial, since the essential difficulty of multiobjective decisions relates precisely to the selection among many elements of the efficient set. In order to better distinguish D-efficient elements defined by (4.2), (4.4) as opposed to properly or weakly D-efficient elements, we shall call them here (strictly) D-efficient. The sets $\hat{Q}_{o\epsilon}$, $\hat{Q}_{op}$, $\hat{Q}_o$, $\hat{Q}_{ow}$ are contained in each other, $\hat{Q}_{o\epsilon} \subset \hat{Q}_{op} \subset \hat{Q}_o \subset \hat{Q}_{ow}$, and the relation between them is illustrated in Figure 4.1 for the case when $D = R^2_+$.

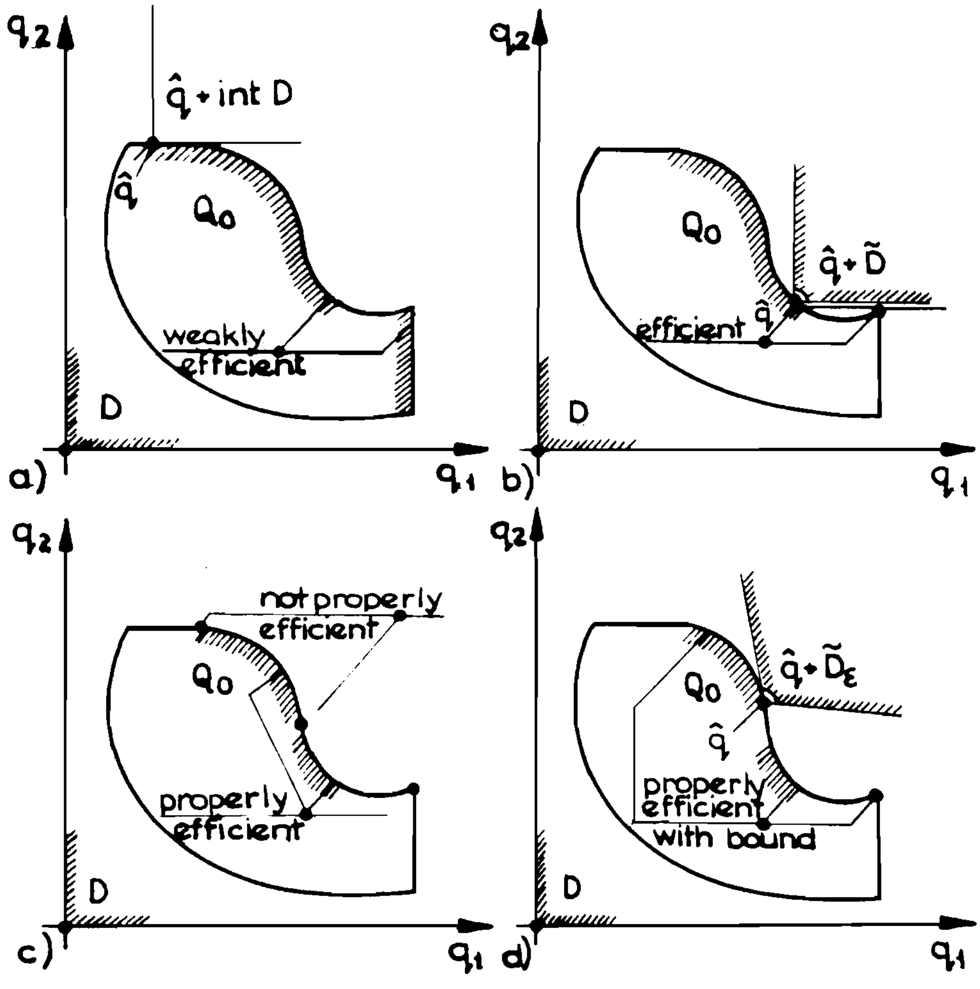

Figure 4.1. The concepts of weak efficiency, efficiency, proper efficiency and proper efficiency with a priori bound.

We see from this example that a practical decision maker would be mostly interested in D_ϵ-efficient elements which do not admit very large trade-off coefficients between outcomes.

4.1.2. Parametric characterizations and representations, their completeness, computational robustness and controllability.

In mathematical language, a characterization means any conditions that are both necessary and sufficient. In multiobjective optimization, most characterizations are related to the use of some substitute scalarizing function that typically depends not only on the objective values but also on some additional parameters. There are two classes of such parameters that are important for applications in decision support systems: weighting coefficients and objective function levels (which can be interpreted as reference, aspiration or reservation levels). Generally, we consider a set A of such parameters in R^p (although R^{p-1} would suffice and R^{p+1} is sometimes used). Let a substitute scalarizing function be denoted by s: $Q_o \times A \rightarrow R^1$; important examples are (bi)linear functions, norms, achievement functions. Such a function should desireably have two basic properties, denoted here by (S) and (N).

(S) The sufficiency property: for each $\alpha \in A_s$

$$\operatorname*{Argmax}_{q \in Q_o \cap Q_s(\alpha)} s(q, \alpha) \subset \hat{Q}_o \tag{4.9}$$

where Argmax denotes the set of maximal points of a scalar-valued function (obviously, by changing the sign of this function, minimal points could be used as well), while argmax will denote the same set if it contains only one point; A_s is a subset of A for which the condition (4.9) holds and $Q_s(\alpha)$ represents possible additional constraint set. An analogous property could characterize weakly efficient points (with A_s replaced by A_{sw} and $\hat{Q}_o$ by $\hat{Q}_{ow}$) or properly efficient points (with A_s replaced by A_{sp} and $\hat{Q}_o$ by $\hat{Q}_{op}$). If (S) holds, then a point-to-set mapping of A_s (A_{sw}, A_{sp}) into $\hat{Q}_o$ ($\hat{Q}_{ow}$, $\hat{Q}_{op}$) can be defined:

$$\Psi(\alpha) = \operatorname*{Argmax}_{q \in Q_o \cap Q_s(\alpha)} s(q,\alpha) \tag{4.10}$$

Such a mapping is typically used as a basis of interaction between a decision maker and a decision support system. In such applications, however, we need a single point in the set $\Psi(\alpha)$, that is, a selection $\psi(\alpha) \in \Psi(\alpha)$. The decision maker, called also the user, specifies some $\alpha \in A_s$ and the system responds with an efficient outcome $\hat{q} = \psi(\alpha) \in \hat{Q}_o$; hence, parameters α will be called controlling parameters. Clearly, the system should respond also with the corresponding efficient decisions $\hat{x} \in \hat{X}_o$; although we limit here most of discussion to the outcome space only, we shall keep in mind that one of the main difficulties of multiobjective optimization consists in the fact that the set Q_o is defined implicitly and we can only compute its elements corresponding to given $x \in X_o$. The mapping $\Psi(\alpha)$, or its selection $\psi(\alpha)$, will be called here a ***parametric representation*** of $\hat{Q}_o$.

However, the scalarizing function $s(q,\alpha)$ that implies a parametric representation should also desireably have the following property:

(N) The necessity property: for each $\hat{q} \in \hat{Q}_o$, there exists $\hat{\alpha} \in A_n$ such that:

$$\hat{q} \in \operatorname*{Argmax}_{q \in Q_o \cap Q_s(\hat{\alpha})} s(q,\hat{\alpha}) \tag{4.11}$$

This property can be also modified for weakly efficient or properly efficient points (with A_n substituted by A_{nw} or A_{np} and $\hat{Q}_o$ by $\hat{Q}_{ow}$ or $\hat{Q}_{op}$). The necessity property is typically used for checking the efficiency of a given $\hat{q} \in \hat{Q}_o$. Observe that, if $Q_n(\alpha) = Q_s(\alpha)$ and $A_n = A_s$, then we check both necessary and sufficient conditions when (N) and (S) both hold. The pair of conditions (S) and (N) will be called here a ***parametric characterization*** of solutions to multiobjective optimization problems. Such parametric characterizations of vector optimality have certain peculiarities when compared to optimality conditions for other problems, where necessary conditions are typically used to generate some a priori unknown candidate for optimal solution and sufficient conditions are used to check optimality of a given solution. In multiobjective optimization, sufficient conditions can be typically used directly to generate a priori unknown efficient solutions, but it is desireable that necessary and sufficient conditions coincide in the above sense when testing efficiency of given solutions. Therefore, we shall say that (S) and (N) completely characterize parametrically the efficient set $\hat{Q}_o(\hat{Q}_{ow}, \hat{Q}_{op})$ if the same function s is used in both (S) and (N) and if $A_s = A_n$, $Q_s(\alpha) = Q_n(\alpha)$ for all $\alpha \in A_s$.

The sets $\hat{Q}_{ow}$ of weakly efficient solutions and $\hat{Q}_{op}$ of properly efficient solutions have several characterizations that are complete. Characterizations of the (strictly) efficient set $\hat{Q}_o$ are either almost complete (in the sense that A_n is the closure of A_s or that different, but convergent to each other functions s_s and s_n are used in (S) and (N)) or they have other drawbacks. The sets A_s and A_n might depend on the set Q_o and thus on computational accuracy; the intersections of Q_o and $Q_s(\alpha)$ or $Q_n(\alpha)$ might become empty by computational inaccuracies at some α; the mathematical operations required in characterizations might be unreasonable from a computational point of view. Thus, we shall say that a characterization of the type (S), (N) is ***robustly computable*** if it satisfies the following conditions:

(i) The conditions (S), (N) do not contain additional requirements of p-time repetition of maximization (if the computational effort required increases too strongly with the dimensionality, then it prohibits applied extensions to large-dimensional cases of multiobjective trajectory optimization), nor requirements of uniqueness of minima (because we do not have dependable computational tests of uniqueness). If the sets A_s or A_n depend on Q_o, then the characterization should be valid when using internal points of A_s, A_n only, that is, it should not depend on precise information about the set Q_o.

(ii) The intersection of Q_o with $Q_s(\alpha)$ or $Q_n(\alpha)$ should not become empty when the set Q_o is slightly perturbed. In other words, for any $\hat{q} \in \hat{Q}_o$ there should be a neighborhood $U(\hat{q})$ such that the intersection of of Q_o, $U(\hat{q})$ and $Q_s(\hat{\alpha})$ or $Q_n(\hat{\alpha})$ contains more points than $\hat{q}$ alone, that is, for example,

$$Q_o \cap Q_n(\hat{\alpha}) \cap U(\hat{q}) \backslash \{\hat{q}\} \neq \phi$$

If Q_o is of arbitrary, a priori unknown shape, this however means that for each $\hat{q} \in \hat{Q}_o$ and the corresponding $\hat{\alpha}$ in (N), there is a neighborhood $U(\hat{q})$ such that:

$$U(\hat{q}) \subset Q_n(\hat{\alpha}) \tag{4.12}$$

Thus, if a characterization is robustly computable, (S), (N) cannot contain additional constraints that might be active at any $\hat{q} \in \hat{Q}_o$: all such constraints should be included in the form of the function $s(q,\alpha)$, say, by penalty techniques. Unfortunately, completeness and robust computability of characterizations of (strict) efficiency do not coincide, which will be shown later in an impossibility theorem.

Beside robust computability, there is also a special issue of ***constructive computability*** of the necessary conditions (N). Some of them specify, in their proof, the value of parameters $\hat{\alpha} \in A_n$ for which these conditions should be checked; such necessary conditions will be called direct. Other asssure us only of the existence of such $\hat{\alpha}$ while searching for this $\hat{\alpha}$ might be computationally cumbersome; such necessary conditions will be called indirect.

Another important aspect of parametric characterizations is their ***controllability***. If a characterization is complete, then the related parametric representation has a specific "onto" property:

$$\bigcup_{\alpha \in A_s} \Psi(\alpha) = \hat{Q}_o \tag{4.13}$$

which, in fact, can be taken as a precise definition of completeness of characterizations. For incomplete characterizations, the equality sign in (4.13) must be substituted by an inclusion; for almost complete characterisations, a limit or a closure must be added on the left-hand side of (4.13). This can be interpreted that complete or almost complete characterizations provide for a kind of ***global controllability*** of the parametric representation by a user: he can reach (almost) all $\hat{q} \in \hat{Q}_o$ by suitably changing α.

However, a user of a decision support system needs also ***local controllability*** of a parametric representation in the sense of being able to easily and continuously influence his selection of $\hat{q} \in \hat{Q}_o$; otherwise, he might become frustrated by his attempts to obtain a desireable outcome that the system can produce theoretically but does not produce in actual interaction. This means that the computable selection $\psi(\alpha) \in \Psi(\alpha)$, ψ: $A_s \rightarrow \hat{Q}_o$, should be Lipschitz-continuous:

$$\| \psi(\alpha') - \psi(\alpha'') \| \leq \beta' \| \alpha' - \alpha'' \| \text{ for all } \alpha', \alpha'' \in A_s \tag{4.14}$$

which, in turn, necessitates a Lipschitz-continuity of the mapping Ψ, for example, in the sense of Hausdorff distance:

$$\text{dist}_H(\Psi(\alpha'), \Psi(\alpha'')) \leq \beta'' \| \alpha' - \alpha'' \| \text{ for all } \alpha', \alpha'' \in A_s \tag{4.15}$$

with reasonably small values of Lipschitz coefficients β', β''. Unfortunately, there are until now very few results on Lipschitz-continuity of parametric characterizations. We give later an example of such result for a simple case; in other cases, intuitive or negative statements can be still made, based on logical evaluation or simple counterexamples.

4.1.3. Other aspects of constructiveness of characterizations.

Beside robust computability and controllability, there are several other aspects of constructiveness of parametric characterizations and representations of efficient solutions to multiobjective optimization problems. Some of these aspects can be expressed mathematically, some have purely subjective form.

One of such aspects of characterizations is their independence on a priori information. Many characterizations use information about so-called *ideal* or *utopia point*. Abstractly, this point is defined as the strict upper bound to the efficient set or as the unique (strong) D-maximal point of the set $\{ q \in R^p : Q_o \subset q - D\}$, see Figure 4.2(a). However, such points do not exist if the set Q_o has an interior and the cone D does not. Therefore, it is more useful to define the utopia point as the composition of results of scalar maximization of each objective function separately:

$$\hat{q}_{max} = (\hat{q}_{1,max}, \ldots \hat{q}_{i,max}, \ldots \hat{q}_{p,max}) \tag{4.16}$$

$$\hat{q}_{i,max} = \max_{q \in Q_o} q_i, \; i = 1, \ldots p$$

and remember that this point should be interpreted with care if some objective functions are not maximized but kept close to given levels. A characterization should not depend on the precise knowledge of the utopia point, because it would not then be robustly computable. As long as only approximate information about the utopia point is required in a characterization, it does not constitute an excessive dependence on a priori information, because an approximate utopia point can be computed once for entire Q_o. The issue of using approximate upper bounds instead of precise utopia points becomes rather important in multiobjective trajectory optimization, when the number of objectives, if not infinite in any computational approximation, is nevertheless rather large; in such a case, computing upper bounds for each computed trajectory point would require rather extensive effort and approximate upper bounds for entire trajectories should be computed instead.

Some interactive decision processes use also (not for characterizations, but for other purposes) lower bounds of the efficient set $\hat{Q}_o$. The strict lower bound, called ***nadir point***, is defined as the D-maximal point of the set $\{q \in R_p : \hat{Q}_o \subset q + D\}$, but this strict nadir point is not constructively computable (except in cases of discrete optimization). However, even an approximate lower bound to the efficient set might be useful in an interactive decision process; such an approximate nadir point can be expressed (in the case of maximization of all objectives) as:

$$\hat{q}_{min} = (\hat{q}_{i,min}, \dots \hat{q}_{i,min}, \dots \hat{q}_{p,min})$$
$$\hat{q}_{i,min} = \min_{1 \le j \le p} \hat{q}_i(j)$$
$$\hat{q}(j) \in \operatorname*{Argmax}_{q \in Q_o} q_j \tag{4.17}$$

where $\hat{q}(j)$ is an arbitrary selection from the set of elements maximizing q_j, see Figure 4.2 (b).

While the use of approximate bounds to the set $\hat{Q}_o$ is quite constructive, the requirements of further a priori knowledge of $\hat{Q}_o$ are not. For example, if a priori knowledge of entire $\hat{Q}_o$ is used in (N), it makes the necessary condition rather useless, since we cannot then apply (N) to check whether $\hat{q}$ belongs to a priori unknown $\hat{Q}_o$ (if we knew $\hat{Q}_o$, it would be simpler to check $\hat{q} \in \hat{Q}_o$ by more direct means). (N) shall be called *tautological* in such a case.

Experience in applications of parametric representations in multiobjective optimization and interactive decision support has led now most authors to agree more or less explicitly on several subjective attributes of constructiveness of such characterizations and representations. These attributes are:

Simplicity. A parametric representation should be conceptually simple and easy to grasp mentally.

Generality. A parametric representation should be, if possible, applicable not only to linear and convex problems, but also to nonconvex, discrete and dynamic problems of multiobjective trajectory optimization.

Interpretability of parameters. The parameters in the sets A_s should have an easy and reasonable interpretation for the user (who needs such an interpretation when changing these parameters in order to control the parametric representation), not for theorists only.

Computability. Beside the requirements of robust computability and directness of necessary conditions, parametric representations should be computable by means of algorithms that do not require excessive computer time and are can be relied upon to produce results without the need of adjustment by the user.

4.1.4. Alternative characterizations and parametric representations of efficient solutions.

There are many characterizations that imply various parametric representations. We shall subdivide them into three classes: (A) those based on weighting coefficients used in (bi)linear functions and various norms; (B) those based on aspiration or reservation levels used in various norms and achievement functions; (C) other possible characterizations. We shall discuss here only the classes (A) and (B).

(A) *Characterizations by weighting coefficients.* These characterizations are obtained if α is a vector composed of weighting coefficients α_i used in (bi)linear functions or

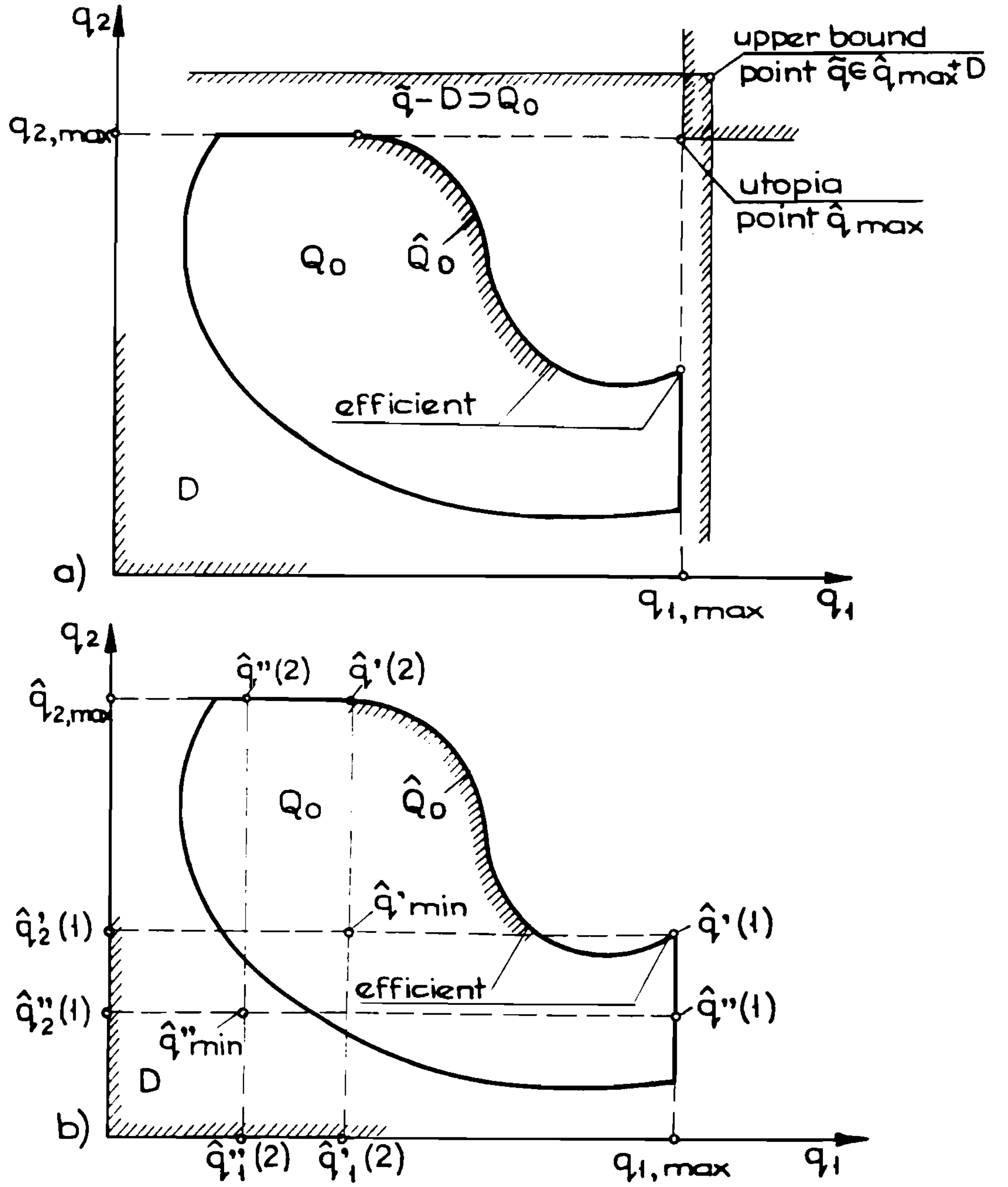

Figure 4.2. Concepts of the utopia point (a) and the nadir point (b).

various norms that scalarize the components q_i of the outcome vector. All characterizations in this class have one fundamental disadvantage in common: weighting coefficients are not easy to be understood well and interpreted by an average user, since they actually belong to a dual space to the space of outcomes and the relations between the dual and the primal spaces are not necessarily easy to interpret. In particular, users find it difficult to interpret weighting coefficients for objective trajectories. The typical interpretation of weighting coefficients through their relation to the trade-off coefficients does not help much since the concept of trade-off itself is, in a sense, dual to the concept of preference. On the other hand, weighting coefficients are well understood by mathematicians, hence the theory of characterizations based on weighting coefficients is best developed; for a review, see Sawaragi et al., (1985) or Jahn, (1985). For all characterizations of this class we assume that $D = R^p_+$; modifications to other forms of D are possible but not necessarily straightforward, since they require a consistent use of dual spaces and cones.

(A1) *(Bi)linear functions* used as substitute scalarizing functions have the following form:

$$s(q,\alpha) = \sum_{i=1}^{p} \alpha_i q_i \tag{4.18}$$

with $\alpha = (\alpha_1,..\alpha_i,..\alpha_p)$; the sets A_s, A_n are defined by:

$$A_s = \left\{\alpha \in \text{int } R^p_+ : \sum_{i=1}^{p} \alpha_i = 1\right\}; \quad A_n = \left\{\alpha \in R^p_+ : \sum_{i=1}^{p} \alpha_i = 1\right\} \tag{4.19}$$

Theorem 4.1. Let s, A_s, A_n be defined as above. If $\alpha \in A_s$, then each $\hat{q}$ that maximizes $s(q,\alpha)$ over $q \in Q_o$ is efficient. If $\hat{q}$ is efficient and Q_o is convex, then there exists $\hat{\alpha} \in A_n$ such that $\hat{q}$ maximizes $s(q,\hat{\alpha})$ over $q \in Q_o$. Moreover, $\hat{\alpha}$ is in such a case the normal vector to a supporting ("from above") hyperplane of Q_o at $\hat{q}$; if such hyperplane is unique (the boundary of Q_o being smooth at $\hat{q}$), then the trade-off coefficients at $\hat{q}$ are defined by:

$$\lim_{\Delta_{\hat{q}} \to 0} \frac{\Delta \hat{q}_j}{\Delta \hat{q}_i} = \frac{\alpha_i}{\alpha_j}; \quad \hat{q} + \Delta \hat{q} \in \hat{Q}_o \tag{4.20}$$

If $\alpha \in A_n$, then each $\hat{q}$ that maximizes $s(q,\alpha)$ over $q \in Q_o$ is weakly efficient; if $\hat{q}$ is weakly efficient and Q_o is convex, then there exists $\hat{\alpha} \in A_n$ such that $\hat{q}$ maximizes $s(q,\hat{\alpha})$ over $q \in Q_o$. If $\alpha \in A_s$, then each $\hat{q}$ that maximizes $s(q,\alpha)$ over $q \in Q_o$ is properly efficient; if $\hat{q}$ is properly efficient and Q_o is convex, then there exists $\hat{\alpha} \in A_s$ such that $\hat{q}$ maximizes $s(q,\hat{\alpha})$ over $q \in Q_o$.

For the proofs of various parts of this well-known theorem see, for example, Sawaragi et al. (1985) or Jahn (1985); originally, this characterization dates back to Koopmans (1951), Kuhn and Tucker (1951) and Geoffrion (1968). See also Wierzbicki (1977) for extensions to arbitrary convex closed cones D in linear topological or Banach spaces.

We see that, for convex cases, this characterization is complete for weak and proper efficiency and almost complete (since A_n is the closure of A_s) for (strict) efficiency. Moreover, it is easy to see that these characterizations are robustly computable but indirect for necessary conditions (since determining a supporting hyperplane is not necessarily straightforward). They are also independent of a priori information, conceptually simple, rather general (with the restriction of necessary conditions to the convex case) and easily computable for sufficient conditions. The main drawback of them, beside bad interpretability of weighting coefficients, is the fact that the related parametric representations are not Lipschitz-continuous for such basic cases as when Q_o is a convex polyhedral set, which can be easily seen on simplest examples, see Figure 4.3. Thus, these representations are not locally controllable by the user.

Similar properties to the above characterizations have those based on a l_1 norm:

$$s(q,\alpha) = \sum_{i=1}^{p} \alpha_i |q_i - \bar{q}_i|; \quad \alpha \in A_s \tag{4.21}$$

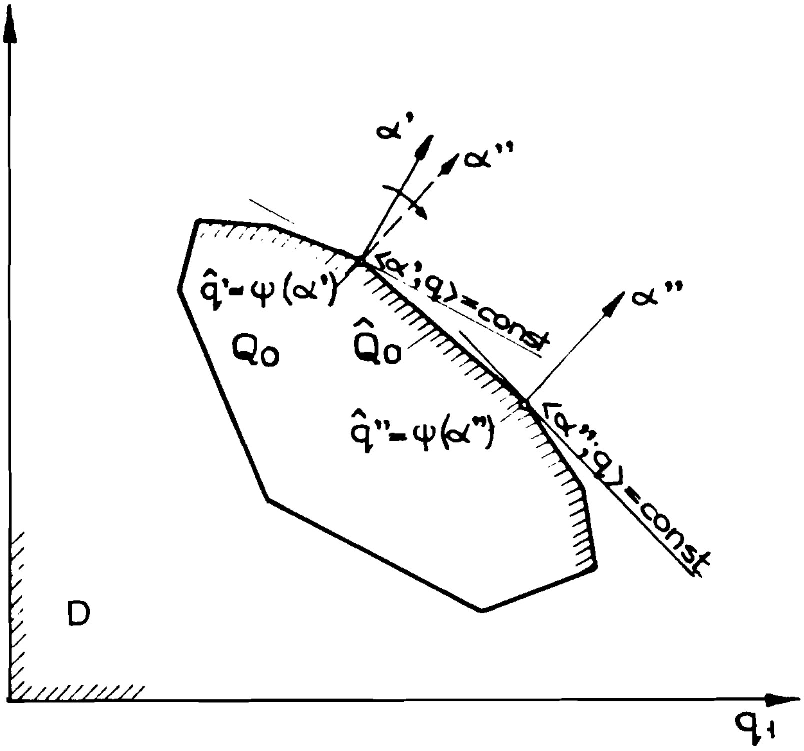

Figure 4.3. An example of the discontinuity of the parametric representation $\psi(\bar{q})$ when using linear substitute scalarizing functions.

with A_s defined as in (4.19) and a upper bound point $\bar{q}$ restricted by:

$$\bar{q} \in \hat{q}_{\max} + \text{int } R^p_+ \tag{4.22}$$

Actually, $\bar{q} \geq \hat{q}_{\max}$ would suffice, but the strong inequality in (4.22) is assumed to obtain computational robustness. These characterizations will not be discussed separately.

(A2) *A weighted l_k-norm* is also often used as a substitute scalarizing function:

$$s(q,\alpha) = \left\{\sum_{i=1}^{p} \alpha_i | q_i - \bar{q}_i |^k\right\}^{\frac{1}{k}}; \quad \alpha \in A_s \tag{4.23}$$

with A_s defined as in (4.19) and $\bar{q}$ restricted as in (4.22); the parameter $k \in (1; +\infty)$ can be also treated as $(p+1)$-th component of the parameter vector.

Theorem 4.2. Let $s(q,\alpha)$, $\alpha,k,\bar{q}$, be selected as above. Then each $\hat{q}$ that minimizes $s(q,\alpha)$ over $q \in Q_0$ is properly efficient. If $\hat{q}$ is efficient, then for each $\epsilon > 0$ there exist such $\alpha \in A_s$, such $k \in (1; +\infty)$ and such $\hat{q}'$ with $\| \hat{q} - \hat{q}' \| < \epsilon$ that $\hat{q}'$ minimizes $s(q,\alpha)$ over $q \in Q_0$.

This form of this theorem is due to Gearhart (1983); earlier, similar results were given by Salukvadze (see, for example, 1979), Zeleny (1973), Yu and Leitmann (1974), while a most general and early form of the sufficiency part of this theorem for Banach spaces with suitable assumptions on the cone D was given by Rolewicz (1975), see also Wierzbicki (1977). We see that this characterization is almost complete for proper and (strict) efficiency also in non-convex cases; in this sense, it is stronger than this by (bi)linear functions.

This characterization is robustly computable, but the necessary condition is indirect. The Lipschitz-continuity of the related parametric representation has not been studied, but we might suspect local controllability. The characterization depends on a priori information, but not excessively and is not tautological. It is not quite simple conceptually, but rather general. The interpretability of the parameter pair (α,k) for an average user is bad; moreover, this representation might be not easily computable if k is very large, since it leads to badly conditioned nonlinear programming problems.

(A3) *A weighted* l_∞ *(Chebyshev)* norm is a very useful substitute scalarizing function:

$$s(q,\alpha) = \max_{1\le i\le p} \alpha_i |q_i - \tilde{q}_i|; \quad \alpha \in A_s \tag{4.24}$$

where A_s is defined as in (4.19) and $\tilde{q}$ restricted as in (4.22).

Theorem 4.3. Let $s(q,\alpha)$, $\tilde{q}$, A_s be defined as above. Then each $\hat{q}$ that minimizes $s(q,\alpha)$ over $q \in Q_0$ is weakly efficient. If the minimum is unique, then such $\hat{q}$ that minimizes $s(q,\alpha)$ over $q \in Q_0$ is efficient. If $\hat{q}$ is weakly efficient, then there exists such $\hat{\alpha} \in A_s$ that $\hat{q}$ minimizes $s(q,\hat{\alpha})$ over $q \in Q_0$. If $\hat{q}$ is efficient, then there exists such $\hat{\alpha} \in A_s$ that $\hat{q}$ uniquely minimizes $s(q,\alpha)$ over $q \in Q_0$.

This theorem is due to Dinkelbach (1971) and Bowman (1976). We see that this characterization is complete for weak efficiency and also for (strict) efficiency even in a nonconvex case, but at the cost of the requirement of uniqueness and thus lost of robust computability for (strict) efficiency. Beside this basic drawback, this characterization depends on a priori information but not excessively and is not tautological, is rather simple conceptually, general, and easily computable (since we can substitute the operation max in (4.24) by p inequalities; recently, nondifferentiable optimization algorithms are becoming more dependable and could possibly in future be also applied for this case).

The basic drawback of all weighting coefficient methods - their bad interpretability - can be overcome in this case by making these coefficients dependent on aspiration or reference levels $\bar{q}_i$ for objective function, that is, by introducing a transformation $\alpha(\bar{q})$. Under the restriction that $\bar{q}_i < \tilde{q}_i$, we can take:

$$\alpha_i(\bar{q}) = (1/(\tilde{q}_i - \bar{q}_i)) / \sum_{j-1}^{p} 1/(\tilde{q}_j - \bar{q}_j) \tag{4.25}$$

which has the interpretation that the closer is an aspiration or reference level $\bar{q}_i$ to the upper bound level $\tilde{q}_i$, the more important is the objective, see Figure 4.4(a). When checking necessary conditions in Theorem 4.3, the application of (4.25) with $\bar{q}_i = \hat{q}_i$ makes these conditions directly computable, see Figure 4.4(b). This modification has been used by Steuer (1983), Nakayama (1985) and others; however, if aspiration or reference levels are used as the controlling parameters, then the method belongs to another class since the norm (4.24) changes its form of dependence on controlling parameters and should be interpreted as an achievement function. In this sense, we shall show later that the corresponding parametric representation is Lipschitz-continuous and thus locally controllable.

(A4) *A composite norm*, in particular - a combination of weighted l_1 and l_∞ norms is one of the strongest scalarizing functions:

$$s(q,\alpha) = \max_{i\le i\le p} \alpha_i |q_i - \tilde{q}_i| + \alpha_{p+1} \sum_{i=1}^{p} \alpha_i |q_i - \tilde{q}_i|$$
$$\alpha \in A_s, \quad \alpha_{p+1} \in (0;\, 1] \tag{4.26}$$

where A_s is defined as in (4.19) and $\tilde{q}$ restricted as in (4.22).

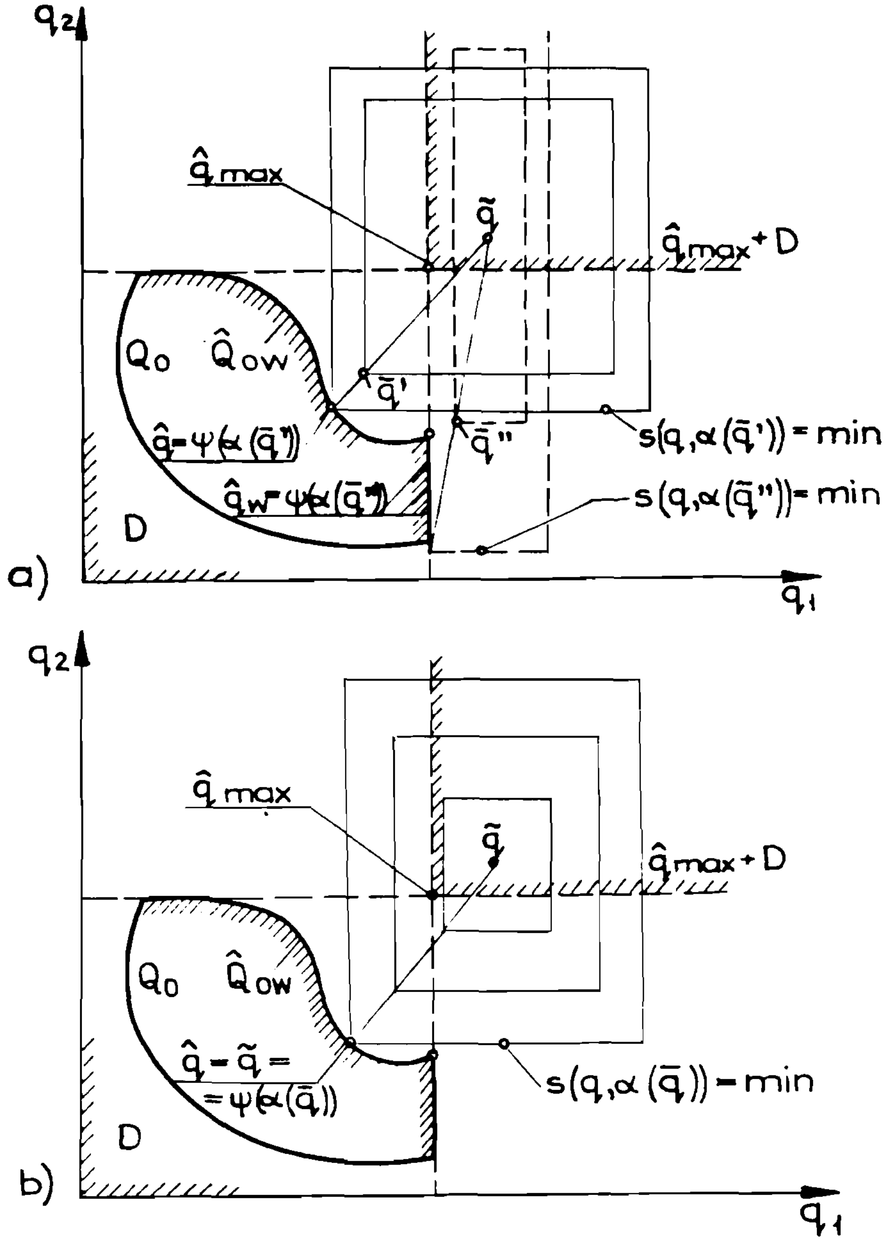

Figure 4.4. Defining the weighting coefficients for a Chebyshev norm with help of aspiration levels: a) sufficiency; b) necessity.

Theorem 4.4. Let $s(q,\alpha)$, $\bar{q}$, α_{p+1} and $\alpha \in A_s$ be defined and restricted as above. Then each $\hat{q}$ that minimizes $s(q,\alpha)$ over $q \in Q_0$ is properly efficient; if $\hat{q}$ is properly efficient, then there exists a (sufficiently small) α_{p+1} and $\alpha \in A_s$ such that $\hat{q}$ minimizes $s(q,\alpha)$ over $q \in Q_0$.

This theorem is due to Iserman and Dinkelbach (1973). We see that this completely characterizes proper efficiency without convexity assumptions; since (4.26) converges to (4.24) with $\alpha_{p+1} \to 0$, we obtain also an almost complete characterization of (strict) efficiency. This characterization is robustly computable and its necessary condition becomes direct if we apply (4.25) with $\bar{q}_i = \hat{q}_i$ and choose α_{p+1} smaller than an a priori bound on (the inverse of) trade-off coefficients.

This characterization depends on a priori information but not excessively and is not tautological. It is not quite simple conceptually but rather general and easily computable (again, we can transform the problem of minimizing (4.24) to a simpler one by using $p+1$ inequalities; if Q_o is a convex polyhedral set, this leads to a linear programming problem). Thus, it might be one of the best characterizations - provided, however, that we use the transformation (4.25) of weighting coefficients in order to assure easy interpretability and local controllability. This has been used by Wierzbicki (1985), Lewandowski et al. (1985) and applied in the Dynamic Interactive Decision Analysis and Support (DIDAS) system, although not as a norm but as an achievement function.

(B) *Characterizations by objective function levels.* These characterizations assume that α is a vector composed of objective function levels that are interpreted either as reservations (values that must be achieved), aspirations (values that should be desirably achieved) or reference values (which can be, in fact, interpreted as aspirations). While much better interpretable for an average user than the characterizations by weighting coefficients, most of the characterizations by objective function levels have several disadvantages that can be overcome first by introducing the concept of order-consistent achievement functions - a class that includes functions such as (4.24) under the transformation (4.25) but is much more general.

(B1) *Directional search.* If a direction $w \in R^p_+$ and the utopia point $\hat{q}_{\max}$ are given, we can construct a substitute scalarizing function for the directional search:

$$s(q,w,t) = \| \hat{q}_{\max} - q - tw \|; \quad w \in A_n = R^p_+ \tag{4.27}$$

with an arbitrary norm in R^p and with t selected as the smallest value of $t \in [0;+\infty)$ for which the minimum of s over $q \in Q_o$ is equal zero. This is actually an additional minimization requirement; moreover, $\hat{q}_{\max}$ should be known exactly in the corresponding sufficient condition, hence the following incomplete characterization is certainly not robustly computable:

Theorem 4.5. (N) Let $\hat{q} \in \hat{Q}_o$ be (strictly) efficient and let an upper bound point $\tilde{q} \geq \hat{q}_{\max}$ be given. If $\hat{w} = \tilde{q} - \hat{q}$, then $t = 1$ is the lowest value of t such that $q - t\,\hat{w} \in Q_o$; the minimum of the function s above with $\tilde{q} = \hat{q}_{\max}$ over $q \in Q_o$ is then equal zero. (S) If $p = 2$ and Q_o is convex and compact, then, for each $w \in R_+{}^p$, the smallest value of $t > 0$ such that $\hat{q}_{\max} - t\,w \in Q_o$ results in a (strictly) efficient $\hat{q} \in \hat{Q}_o$. If $p > 2$, counterexamples show that an analogous sufficient condition cannot be proven even under convexity assumptions.

The above theorem is well known, but we give the proof of it in Appendix to illustrate why it is impossible to obtain a complete characterization by directional search if $p > 2$. Thus, the above characterization cannot be used to generate a priori unknown efficient solutions in response to user requirements; it is only a very good tool for checking efficiency of given $\hat{q}$.

(B2) *Reservation levels or constraints on objective functions.* Here several simple substitute functions and constraints are used:

$$s_k(q, \bar{q}) = q_k; \quad q \in Q_o \cap A_k(\bar{q}) \tag{4.28}$$

where:

$$A_k(\bar{q}) = \{q \in R^p : q_i \geq \bar{q}_i,\ i = 1,..p,\ i \neq k\} \tag{4.29}$$
$$\bar{q} = (\bar{q}_1,..\bar{q}_i,..\bar{q}_p) \in \hat{q}_{\max} - R^p_+$$

Theorem 4.6. Let $D = R^p_+$ and s_k, $A_k(\bar{q})$ be defined as above. If, for some $k = 1,...p$, $\hat{q}$ maximizes s_k over $q \in Q_o \bigcap A_k(q)$ with some $\bar{q} \in \hat{q}_{\max} - R^p_+$, then $\hat{q}$ is weakly efficient; if $\hat{q}$ is weakly efficient, then there exists $k = 1,...p$ such that $\hat{q}$ maximizes s_k over $q \in A_k(\bar{q})$ with $\hat{q} = \bar{q}$. If, for all $k = 1,...p$, $\hat{q}$ maximizes s_k over $q \in Q_o \cap A_k(\bar{q})$ with some $\hat{q} \in \hat{q}_{\max} - R^p_+$, then $\hat{q}$ is (strictly) efficient; if $\hat{q}$ is (strictly) efficient, then $\hat{q}$ maximizes s_k over $q \in A_k(\hat{q})$ with $\bar{q} = \hat{q}$ for all $k = 1,...p$. Let Q_o be convex. Then $\hat{q}$ is properly efficient if and only if the problems of maximizing s_k over $Q_o \bigcap A_k(\hat{q})$ are stable, that is, the perturbation functions:

$$s_k(\hat{q}) = \min_{q \in Q_o \bigcap A_k(\hat{q})} s_k(q,\hat{q}) \tag{4.30}$$

are Lipschitz-continuous for all $k = 1,...p$.

The proof of this theorem, due to earlier results of Changkong and Haimes (1978) and Benson and Morin (1977) can be found in Sawaragi et al. (1985). This characterization is complete and rather general (valid without any convexity assumptions but not easy to generalize for the case of trajectory optimization). However, it is not robustly computable for (strict) efficiency, because of the requirement of p-times repeated maximization and because $Q_0 \cap A_k(\hat{q})$ becomes a singleton set $\{\hat{q}\}$ in necessity statements. Thus, only the weak efficiency part of this characterization has found broader applications. Moreover, the proper efficiency part of this characterization is not direct. On the other hand, we can suspect local controllability, although Lipschitz continuity of this representation has not been investigated.

The characterization depends on a priori information but not excessively and is not tautological; it is conceptually simple and the parameters are easily interpretable as reservation levels for objective values. For the weak efficiency part of this characterization, it is also easy to compute.

Another, early variant of characterization by using reservation levels is related to one of two possible interpretations of goal programming: this of trying to improve given attainable lower bounds or reservations for objective values. Originally suggested by Charnes and Cooper (1961), further developed by Fandel (1972) and Ecker and Kuada (1975), it has been studied extensively in various modifications - see Gal (1982) for a survey. Its prototypical formulation is:

$$s(q,\bar{q}) = \sum_{i-1}^{p} \alpha_i(q_i - \bar{q}_i);\ \bar{q} \in Q_0;\ q \in Q_0 \cap Q(\bar{q}) \tag{4.31}$$

$$Q(\bar{q}) = \{q \in R^p : q_i \geq \bar{q}_i, 1 = 1,...p\}$$

with some fixed $\alpha \in A_s$ defined as in (4.19). This gives a complete characterization of (strictly) efficient solutions:

Theorem 4.7. Let $D = R^p_+$ and $s(q,\bar{q})$, $Q(\bar{q})$ be defined as above with $\bar{q} \in Q_o$. If $\hat{q}$ maximizes $s(q,\bar{q})$ over $q \in Q_o \cap Q(\bar{q})$, then $\hat{q}$ is (strictly) efficient; if $\hat{q}$ is (strictly) efficient and we set $\bar{q} = \hat{q}$, then $\hat{q}$ maximizes $s(q,\bar{q})$ over $q \in Q_0 \cap Q(\hat{q})$.

This theorem is well known, but we give its proof in Appendix in order to illustrate the basic drawback of this characterization: it is not robustly computable. In fact, we use here $Q_o \cap Q(\hat{q}) = \{\hat{q}\}$ in necessary conditions and this singleton set might become empty by any, however slight, perturbation of Q_o, see Figure 4.5. Lipschitz-continuity of the related parametric representation has not been investigated (this representation is obviously Lipschitz-continuous with coefficient 1, if we consider only $\bar{q} \in \hat{Q}_0$, in which case, however, it is not computationally robust; for $\bar{q} \in Q_0 \backslash \hat{Q}_0$ the problem of Lipschitz-continuity is more complicated). Except for these essential drawbacks, this characterization does not depend on a priori information, is simple conceptually, very general (no convexity assumptions are needed and a generalization to multiobjective trajectory optimization is easy), well interpretable and easily computable if we do not come with $\bar{q}$ too close to $\hat{Q}_o$.

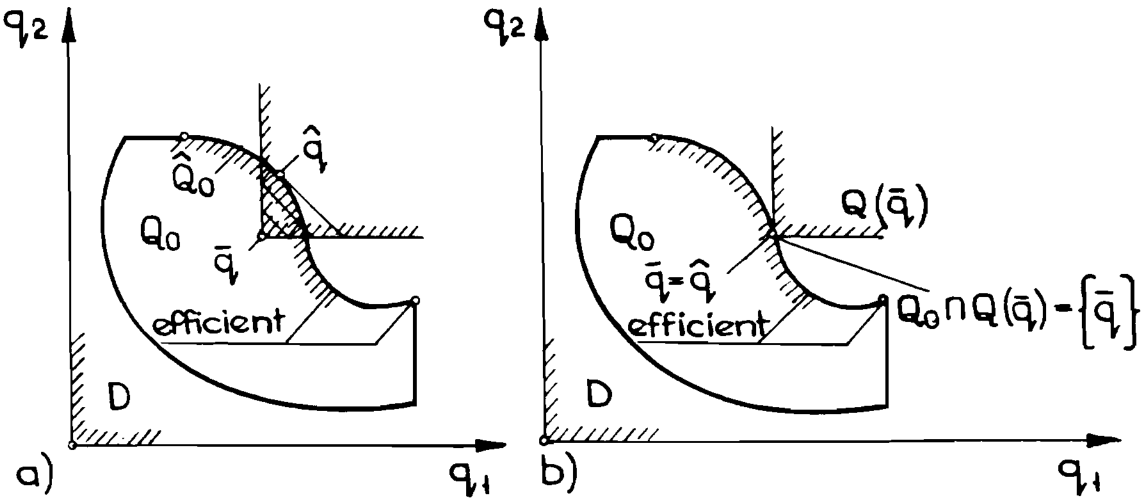

Figure 4.5. Characterization of efficient points by reservation levels $\bar{q}$ treated as constraints: a) sufficiency; b) necessity.

The drawbacks of this otherwise excellent class of characterizations could be overcome when substituting constraints by penalty functions - but this leads to the concept of an achievement function. Before adressing this concept, yet another class of characterizations must be considered.

(B3) *Aspiration levels with various norms.* This class consists of two subclasses. The first subclass, called compromise programming, corresponds to the case where aspiration levels for objective function values are above utopia point and thus far from being attainable, $\bar{q} \gg \hat{q}_{\max}$. This is actually the case of classes A2, A3, A4 with the upper bound point $\tilde{q}$ - called in this case the displaced ideal - treated as the controlling parameter and interpreted as aspiration level point; this case will not be considered here any further (see, for example, Zeleny 1982 or 1984 for more detailed discussion). The second subclass corresponds to the second, widely used interpretation of goal programming: this of trying to come close to given aspiration levels or goals which are typically not far from being attainable. In fact, consider formula (4.23) with another interpretation:

$$s(q,\tilde{q}) = (\sum_{i-1}^{p} \alpha_i \mid \tilde{q}_i - q_i \mid^k)^{1/k} \tag{4.32}$$

where $\alpha \in A_s$ is treated not as the controlling parameter but as a constant and $\tilde{q}$ is the controlling parameter instead. The limit case when $k \to \infty$ is a form similar to (4.24). If $k = 1$ and $\tilde{q} = \bar{q}$, we obtain a form similar to (4.31); however, there is a basic difference: the function above should be minimized and not maximized as it was the case with (4.31). Theorem 4.7 implies that *one must maximize a norm or a measure of improvement from attainable reservation levels in order to get to the efficient set; from unattainable aspiration levels, however, one must minimize the distance to the attainable and efficient set.* Thus, there are two precisely opposite interpretations of goal programming techniques used as a tool for reaching efficient solutions. To distinguish between them, we must have additional means of checking that their boundary - the efficient set - has been crossed.

Theorem 4.8. Let $D = R^p_+$, $s(q,\tilde{q})$ be defined as above with any $k \in (1; \infty)$, and Q_o be convex. If $\hat{q}$ minimizes $s(q,\tilde{q})$ over $q \in Q_o$ and $\hat{q}_i < \tilde{q}_i$ for all $i=1,..p$, then $\hat{q}$ is properly efficient. If $\hat{q}$ is properly efficient, then there exists such $\tilde{q}$ with $\tilde{q}_i > \hat{q}_i$ for all $i = 1,..p$ that $\hat{q}$ minimizes $s(q,\tilde{q})$ over $q \in Q_o$.

The proof of this complete characterization of proper efficiency is given in Appendix; similar results are given by Jahn (1985). By admitting limit cases $k = 1$, $k = \infty$ and weak inequalities between (some, but not all) $\hat{q}_i$ and $\tilde{q}_i$, a similar complete characterization of weak efficiency and an almost complete characterization of (strict) efficiency can be proven. For convex compact Q_o, we can use this characterization constructively but not neccesarily directly. Suppose $\tilde{q}$ is arbitrary and the minimum of the distance (4.40), equal zero, is attained at $\hat{q} = \tilde{q}$; thus we know that $\hat{q}$ is attainable but do not know whether it is efficient. By moving $\tilde{q}$ in a direction from int R^p_+ sufficiently far (or several times) we can be sure that finally $\tilde{q}-\hat{q} \in$ int R^p_+ is obtained which, in the convex case, implies that the efficient set has been crossed - see Figure 4.6. The necessary condition is even less direct since the trade-off coefficients at $\hat{q}$ must be first computed in order to determine the direction $\tilde{q} - \hat{q}$ that is needed when checking the efficiency of $\hat{q}$.

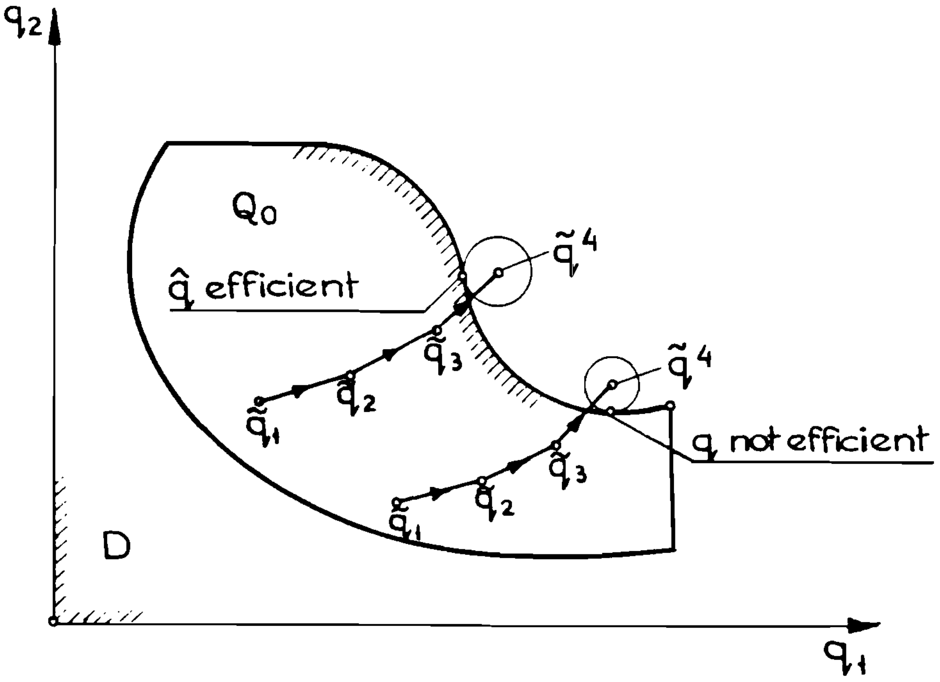

Figure 4.6. The impact of convexity on the efficiency of goal programming solutions obtained by increasing goals.

On the other hand, goal programming is simple conceptually, easily interpretable and relatively easily computable; therefore, it has been widely used, see Charnes and Cooper (1975), Dyer (1972), Ignizio (1983). In the terminology of goal programming, the components $|q_i - \tilde{q}_i|$ of the distance function are often called achievement functions (or under-achievement and over-achievement functions, if the sign of $q_i - \tilde{q}_i$ is taken into account). The drawbacks of goal programming suggest, however, that a strenghtening of this concept would be useful: we need such achievement functions that would preserve monotonicity when $\bar{q}$ crosses the efficient boundary, since we would not then have to change from maximization to minimization. Such functions are called here order-consistent achievement functions and are discussed in detail in the next section.

4.1.5. Concepts and properties of order-consistent achievement functions.

When trying to specify a class of characterizations based on objective function levels that would have good properties in applications for decision support, it is essential to choose first appropriate concepts that correspond to the nature of the vector optimization problem. We address here two such concepts: this of monotonicity, essential for sufficiency parts of characterizations, and that of separation of sets, essential for the necessity parts of characterizations.

The role of monotonicity in vector optimization is explained by the following basic theorem:

Theorem 4.9. Let a function $r : Q_0 \longrightarrow R^1$ be strongly monotone, that is, let $q' > q''$ (equivalent to $q' \in q'' + \tilde{D}$) imply $r(q') > r(q'')$. Then each maximal point of this function is efficient. Let this function be strictly monotone, that is, let $q' \gg q''$ (equivalent to $q' \in q'' + \text{int } D$) imply $r(q') > r(q'')$. Then each maximal point of this function is weakly efficient. Let this function be ϵ-strongly monotone, that is, let $q' \in q'' + \tilde{D}_\epsilon$ imply $r(q') > r(q'')$, where D_ϵ, $\tilde{D}_\epsilon$ are defined as in (4.8). Then each maximal point of this function is properly efficient with bound ϵ.

Various parts of this theorem are well-known (see Yu and Leitman, 1974, Wierzbicki, 1977, Jahn, 1984, Sawaragi et al., 1985); to illustrate its basic simplicity, the proof of the proper efficiency part is given in Appendix. Observe that a function constructed with the help of a norm, $r(q) = \| q - \bar{q} \|$, is strictly monotone for all $q \leq \bar{q}$ if the Chebyshev norm is used and strongly monotone for all $q \leq \bar{q}$ if any other norm is used; a composite norm of the form (4.28) where $r(q) = s(q,\alpha)$ with some $\alpha \in A_s$ is ϵ-strongly monotone for all $q \leq \bar{q}$ if ϵ is sufficiently small when compared to α_{p+1}.

The second concept, that of separation of sets, is actually used implicitly or explicitly whenever necessary conditions of scalar or vector optimality are derived. We say that a function r: $R^p \longrightarrow R^1$ *strongly separates two disjoint sets* Q_1 and Q_2 in R^p, if there is such $\beta \in R^1$ that $r(q) \geq \beta$ for all $q \in Q_1$ and $r(q) < \beta$ for all $q \in Q_2$. Since the definition of efficiency (4.4) requires that the sets Q_0 and $\bar{q} + \tilde{D}$ are disjoint (or Q_0 and $\hat{q}$ + int D for weak efficiency, or Q_0 and $\hat{q} + \tilde{D}_\epsilon$ for proper efficiency with bound), they could be separated by a function. If Q_0 is convex, these sets can be separated by a linear function of the form (4.18); this separation of sets is precisely the primal concept beyond the dual concept of weighting coefficients. If Q_0 is not convex, the sets Q_0 and $\hat{q} + \tilde{D}$ could still be separated at an efficient point $\hat{q}$, but we need for this a nonlinear function with level sets $\{q \in R^p\text{: } r(q) > \beta\}$ which would closely approximate the cone $\hat{q} + D$. There might be many such functions; we shall define first their desirable properties and then give several exampels of them

(B4) *Order-representing achievement functions* are defined generally as such continuous functions $s{:}Q_0 \times A \longrightarrow R^1$ that $s(q,\bar{q})$ is strictly monotone (see Theorem 4.9) as a function of $q \in Q_0$ for any $\bar{q} \in A$ and, moreover, possesses the following property of order representation:

$$\{q \in R^p\text{: } s(q,\bar{q}) > 0\} = \bar{q} + \text{int } D, \text{ for all } \bar{q} \in A \tag{4.33}$$

which implies, together with the continuity of $s(q,\bar{q})$, that:

$$s(q,\bar{q}) = 0 \text{ for all } q = \bar{q} \in Q_o \tag{4.34}$$

Here we assume $A = R^p$ or any reasonably large subset of R^p containing Q_o or, at least, $\hat{Q}_{ow}$; the controlling parameter $\bar{q}$ is interpreted as aspiration level point that might be attainable or not. A simple example of such a function is:

$$s(q,\bar{q}) = \min_{1 \le i \le p} \alpha_i(q_i - \bar{q}_i) \tag{4.35}$$

with $A = R^p$ and some fixed $\alpha \in A_s$ defined as in (4.19). Other examples are functions (3.14a,b) or (3.15a,b); still other examples of order-representing functions will be given later. At any weakly efficient point $\hat{q}$, an order representing function with $\bar{q} = \hat{q}$ strictly separates the sets $\hat{q} + \text{int } D$ and Q_o. However, an order-representing function cannot be strongly monotone, since it could not be continuous in such a case.

(B5) ***Order-approximating achievement functions*** are defined generally as such continuous functions s: $Q_o \times A \longrightarrow R^1$ that $s(q,\bar{q})$ is strongly monotone (see Theorem 9) as a function of $q \in Q_o$ for any $\bar{q} \in A$ and, moreover, possesses the following property of order approximation:

$$\bar{q} + D_{\bar{\epsilon}} \subset \{q \in R^p \colon s(q,\bar{q}) \ge 0\} \subset \bar{q} + D_\epsilon, \text{ for all } \bar{q} \in A \tag{4.36}$$

with some small $\epsilon > \bar{\epsilon} \ge 0$, for some reasonably large set A containing Q_o or, at least, $\hat{Q}_o$; the requirement (4.36) implies also (4.34). A simple example of order-approximating function is:

$$s(q,\bar{q}) = \min_{1 \le i \le p} \alpha_i(q_i - \bar{q}_i) + \alpha_{p+1} \sum_{i=1}^{p} \alpha_i(q_i - \bar{q}_i) \tag{4.37}$$

with $A = R^p$, $\alpha_i > 0$, $i = 1,...p$ and some $\alpha_{p+1} > 0$ that is sufficiently small as compared to ϵ and large as compared to $\bar{\epsilon}$; this function is not only strongly monotone, but also $\bar{\epsilon}$-strongly monotone. Another example is function (3.16); other examples of order-approximating functions will be given later. At any point $\hat{q}$ that is properly efficient with bound ϵ, an order-approximating function with $\bar{q} = \hat{q}$ strictly separates the sets $\hat{q} + \tilde{D}_{\bar{\epsilon}}$ and Q_o.

Order-representing and order-approximating functions are jointly called ***order-consistent achievement functions***. When the concepts of monotonicity and separation of sets are used, the following theorem that characterizes efficient solutions by maxima of order-consistent functions might appear simple to the point of triviality; but this is precisely the power of arguments based on separation of sets that they simplify complex problems.

Theorem 4.10. Let $s(q,\bar{q})$ be an order-representing function. Then, for any $\bar{q} \in A$, each point that maximizes $s(q,\bar{q})$ over $q \in Q_o$ is weakly efficient; if $\hat{q}$ is weakly efficient (or efficient), then the maximum of $s(q,\bar{q})$ with $\bar{q} = \hat{q}$ over $q \in Q_o$ is attained at $\hat{q}$ and is equal zero. Let $s(q,\hat{q})$ be an order-approximating function with some $\bar{\epsilon}$, ϵ as in (4.36). Then, for any $\bar{q} \in A$, each point that maximizes $s(q,\bar{q})$ over $q \in Q_o$ is efficient; if $\hat{q}$ is properly efficient with bound ϵ (D_ϵ-optimal), then the minimum of $s(q,\bar{q})$ with $\bar{q} = \hat{q}$ over $q \in Q_o$ is attained at $\hat{q}$ and is equal zero. Let, in addition, $s(q,\bar{q})$ be $\bar{\epsilon}$-strongly monotone in q; then each point that minimizes $s(q,\bar{q})$ over $q \in Q_o$ is properly efficient with bound $\bar{\epsilon}$.

Parts of this theorem were given earlier (Wierzbicki 1977, 1980, 1982), also for infinite-dimensional normed spaces. In Appendix, we give only the proof of necessary conditions for proper efficiency with bound.

The essential difference between the use of displaced ideal or goal programming techniques, based on norms, and the use of order-consistent achievement functions - even if simple forms (4.35), (4.37) of achievement functions strongly resemble norms (4.24), (4.26) - is that the aspiration point $\bar{q}$ needs not to be above the utopia or ideal point, as in the case of displaced ideal, nor to be unattainable in order to achieve efficiency, as in the case of goal programming. No matter whether the aspiration point $\bar{q}$ is attainable or not, the results of maximizing an order-consistent achievement functions are efficient (weakly or strictly or properly with bound ϵ), because such a function possesses an appropriate monotonicity property. Somewhat simplifying, we can say that an order-consistent achievement function switches automatically from norm minimization to maximization when the aspiration point $\bar{q}$ crosses the efficient boundary and becomes attainable. On the other hand, the characterization by Theorem 4.10 is obtained without convexity assumptions, because the order-representing or order-approximating properties of achievement functions result in a constructive though nonlinear separation of sets Q_0 and $\hat{q} + \tilde{D}$ (or $\hat{q}$ + int D, or $\hat{q} + \tilde{D}_\epsilon$) even in nonconvex cases - see Figure 4.7. Therefore, this characterization can be also used when Q_0 is a discrete set.

Therefore, classes (B4, B5), without any convexity assumptions nor restrictions on controlling parameters $\bar{q}$, completely characterize weakly efficient elements and almost completely characterize properly efficient elements (if we take the closure of sets of maximal points of an order-approximating achievement function as $\epsilon \longrightarrow 0$). By adding the requirement of uniqueness of minima in Theorem 10, we could make this characterization complete also for efficient solutions, but we forego this generalization because it would mean the loss of robust computability. The requirement that $\bar{q} = \hat{q}$ in necessary conditions is not tautological, if we want to use these conditions to check the efficiency of a given element: it is direct and robustly computable, since we do not assume any a priori knowledge of Q_0, nor do we limit the maximization to a single point.

These characterizations are not quite simple conceptually, but the controlling parameters $\bar{q}$ and the values of the achievement function $s(q,\bar{q})$ are very well interpretable: while $\bar{q}$ is interpreted as aspiration levels, the sign of the maximum of achievement function indicates whether these aspirations are attainable or not, and the value zero of this maximum indicates that aspirations are attainable and efficient. These characterizations are also very general, valid not only for nonconvex and discrete cases, but also easy to extend for problems of multiobjective trajectory minimization - see Wierzbicki (1977, 1980) for appropriate extensions of Theorem 4.10. Computationally, their applications are either simple - if Q_0 is a convex polyhedral set, then the problem of maximizing (4.35) or (4.37) can be rewritten as a linear programming problem - or more complicated for nonlinear or nonconvex problems. In such cases, we must either represent (4.35) or (4.37), by additional constraints, or apply nondifferentiable optimization techniques, since the definitions of order-consistent achievement functions imply their nondifferentiability at $q = \bar{q}$.

These characterizations are also, most probably, locally controllable; before establishing Lipschitz-continuity of a parametric representation corresponding to the simple achievement function (4.35) we must, however, indicate the use of order-consistent functions for checking the uniqueness of maxima. The concept of separation of sets used in Theorem 4.10 implies the following corollary:

Corollary. If $\hat{q}$ is a maximal point of an ϵ-strongly monotone order-approximating function $s_1(q,\bar{q})$ over $q \in Q_0$ with any $\bar{q} \in A$, then $\hat{q}$ is also the unique maximal point of an order-representing function $s_2(q,\bar{q})$ with $\bar{q} = \hat{q}$ over $q \in Q_0$.

This corollary is an immediate consequence of the separation of the sets $\hat{q} + \tilde{D}$ and Q_0 by the cone $\hat{q} + D_{\bar{\epsilon}}$. On one hand, this confirms only an easy theoretical conclusion that an order-representing function has unique maxima at all properly efficient points. On the other hand, however, the corollary gives a constructive computational way of checking the uniqueness of maxima of an order-representing function.

If $\hat{q}$ is, for example, a maximal point of function (4.35), we can take function (4.37) with some small α_{p+1} and $\bar{q} = \hat{q}$ and maximize the latter function; if we obtain the same

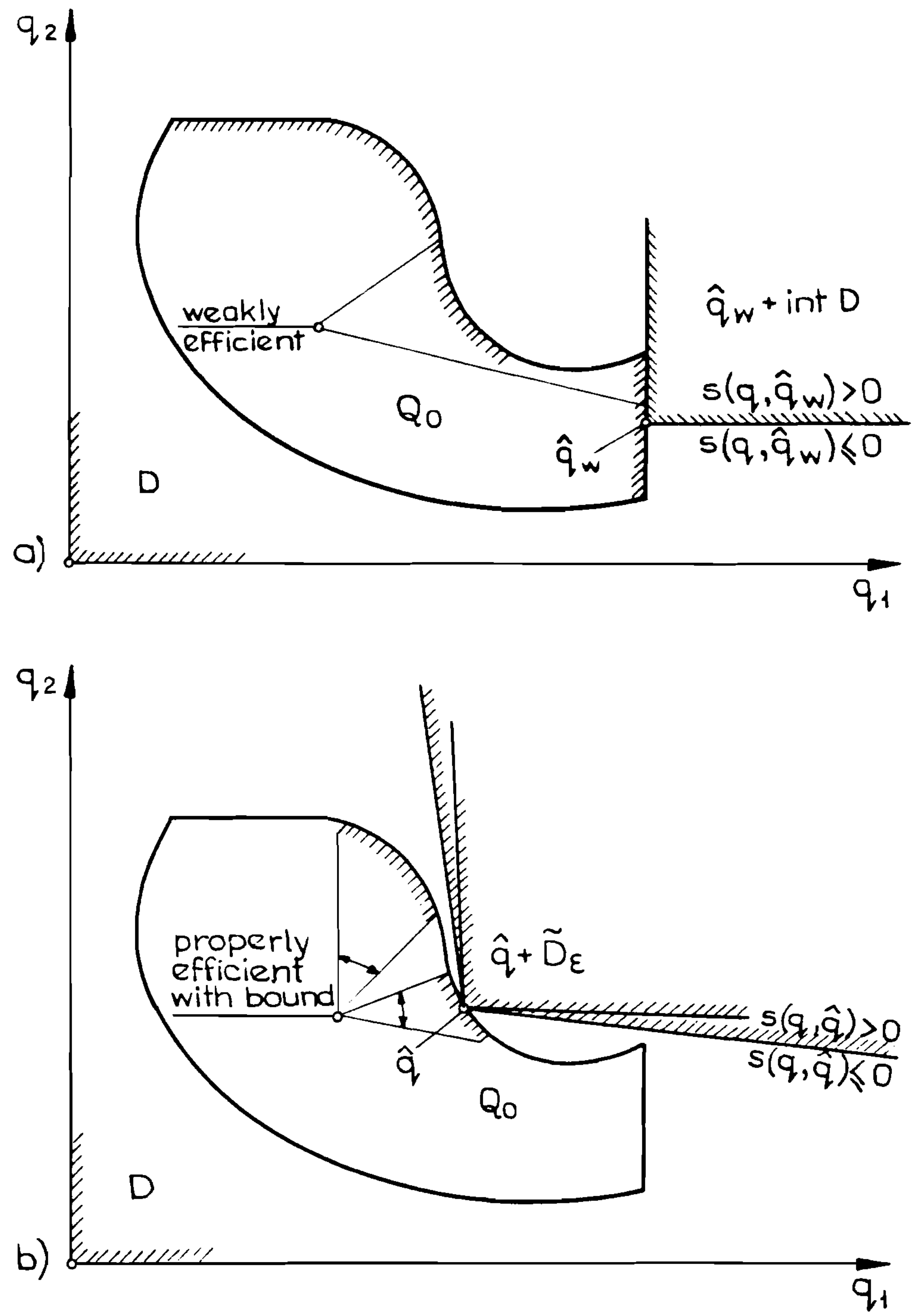

Figure 4.7. The principle of separation of sets by an order-consistent achievement function: (a) at weakly efficient elements; (b) at properly efficient elements with bound ϵ.

result of this second maximization, we are sure that the maximum of the former function is unique. This applies, however, only to order-consistent functions in multiobjective minimization, and is by no means a general way of checking the uniqueness of maxima of other functions, for which task we do not have constructive computational methods.

The above corollary explains also why we can use rather strong assumptions in the following theorem.

Theorem 4.11. Let the order-representing function $s(q,\bar{q})$ be defined as in (4.35) and consider the set A of such $\bar{q} \in R^p$ that the maxima of this function are properly efficient elements of Q_0, that is, are unique. Then the parametric representation:

$$\hat{q} = \psi(\bar{q}) = \arg\max_{q \in Q_o} s(q,\bar{q}) \tag{4.38}$$

is Lipschitz-continuous with the Lipschitz constant 4, that is,

$$\| \psi(\bar{q}') - \psi(\bar{q}'') \| \leq 4 \| \bar{q}' - \bar{q}'' \|$$

where the Chebyshev norm is used, which implies also Lipschitz-continuity in any other norm in R^p.

The proof of this theorem is given in Appendix. Finally, next theorem explains the impossibility of complete and robustly computable characterization of efficient elements $\hat{q} \in \hat{Q}_o$.

Theorem 4.12. Let $s: Q_o \times A \longrightarrow R^1$ be a continuous substitute scalarizing function for vector minimization problems over an arbitrary set $Q_o \subset R^p$.

(a) Suppose that for each efficient $\hat{q} \in \hat{Q}_o$ there exists an $\hat{\alpha} \in A_n \subset A$ such that $\hat{q}$ is a maximal point of $s(q,\hat{\alpha})$ over $q \in Q_o \cap Q(\hat{\alpha})$, where $Q(\hat{\alpha})$ is an additional constraint set, and that each maximal point of $s(q,\alpha)$ over $q \in Q_o \cap Q(\alpha)$ is weakly efficient for any $\alpha \in A_s \subset A$; let $A_s \cap A_n \neq \emptyset$. If, for each $\hat{q} \in \hat{Q}_o$ and the corresponding $\hat{\alpha} \in A_n$, the set $Q(\hat{\alpha})$ contains a neighborhood $U(\hat{q})$ of $\hat{q}$, then the function $s(q,\alpha)$ has the following property of local order-representation:

$$\{q \in U(\hat{q}): s(q,\alpha) > s(\hat{q},\alpha)\} = (\hat{q} + \text{int}\ D) \cap U(\hat{q}) \quad \text{for all} \quad \alpha \in A_n \cap A_s \tag{4.39}$$

(b) If a continuous function $s(q,\alpha)$ has the property (4.39) then, for sets Q_o of arbitrary form, there exist maximal points $\hat{q}'$ of this function over $q \in Q_o \cap U(\hat{q})$ that are weakly efficient but not efficient.

Hence, a complete characterization of efficiency by maximal points of such a function is impossible, if we do not apply additional conditions of uniqueness or repetitive maximization. This theorem indicates that the class of characterizations by order-consistent achievement gives, in a sense, strongest possible characterizations of efficiency for sets Q_o of arbitrary form: we cannot then obtain a complete characterization of (strictly, as opposed to weakly or properly) efficient solutions without foregoing the computational robustness of this characterization.

4.2. Examples and properties of order-consistent achievement functions

As it was shown in the previous chapter (point 3.5), an achievement function can be interpreted as a non-stationary approximation to a utility function, explicitly dependent on a changing context, while the influence of this context is summarized by a special parameter of this function - the aspiration or reference point $\bar{q}$. In order to obtain good properties of parametric representation and characterization of efficient solutions through maximal points of such a function, we have added abstract requirements of order-consistency to the definition of an achievement function. However, the definitions of order-consistent functions do not require that all level sets of the function $s(q,\bar{q})$ should represent or approximate order; only the zero level set should have this property. Hence there are many examples of order-consistent functions.

4.2.1. Order-representing achievement functions.

We consider here achievement functions first for the case when $D = R^p_+$; for the more complicated positive cone (4.2), order-representing functions are anyway not interesting, since their maxima would occur at any admissible decision and attainable outcome, all attainable outcomes being weakly efficient for positive cones without interior. If $D = R^p_+$, a general form of an order-representing function can be written as follows:

$$s(q,q) = \begin{cases} v(q - \bar{q}) & \text{if } q - \bar{q} \in D \\ -\text{dist}\,(q - \bar{q},\, D) & \text{if } q - \bar{q} \notin D \end{cases} \tag{4.40}$$

where $v: R^p \longrightarrow R^1$ is a strongly monotone value (or utility) function with the property that $v(q-\bar{q}) = 0$ for all $q - \bar{q} \in D\backslash \text{int } D$, and any norm in R^p can be taken to define the distance. If we take a multiplicative form of v - for example, the Nash (1950) compromise function - and use the norm l_k with $k \geq 2$, then the function $s(q,\bar{q})$ is differentiable except for $q - \bar{q} \in D\backslash \text{int } D$:

$$s(q,\bar{q}) = \begin{cases} \prod_{i=1}^{p} (q_i - \bar{q}_i) & \text{if } q - \bar{q} \in D \\ -[\sum_{i=1}^{p} (\bar{q}_i - q_i)_+^k]^{1/k} & \text{if } q - \bar{q} \notin D \end{cases} \tag{4.41}$$

where $(\bar{q}_i - q_i)_+ = \max(0, \bar{q}_i - q_i)$.

Another form of order-representing function is piece-wise linear and can be interpreted as an exact penalty function for the characterization (4.31) of efficient solutions:

$$s(q,\bar{q}) = \min(\rho \min_{1 \leq i \leq p} \alpha_i(q_i - \bar{q}_i),\, (1/p)\sum_{i=1}^{p} \alpha_i(q_i - \bar{q}_i)) \tag{4.42}$$

with some $\alpha_i > 0$. This function is determined by the sum only for such $q - \bar{q} \in D$ that

$$(1/p)\sum_{i=1}^{p} \alpha_i(q_i - \bar{q}_i) < \rho \min_{1 \leq i \leq p} \alpha_i(q_i - \bar{q}_i)$$

which is possible only when $\rho > 1$ - see Figure 4.8.

The above function is useful when applied to linear vector optimization problems, where Q_0 is a convex polyhedral set. In such cases, we rewrite the problem of maximizing (4.42) by using additional variables $z_i = \alpha_i(q_i - \bar{q}_i)$, $i = 1,..p$, $z_{p+1} = s(q,\bar{q})$, to the following form:

$$s(q,\bar{q}) = z_{p+1},\ z_{p+1} \leq (1/p)\sum_{i=1}^{p} z_i,\ z_{p+1} \leq \rho z_i,\ i = 1,..p \tag{4.43}$$

Observe that the additional variables z_i, z_{p+1} are not restricted in sign; hence, when using standard linear programming codes, additional modifications might be needed. This function has been used in the DIDAS system of decision support - see Lewandowski et al. (1982), Grauer et.al.(1984) and further chapters. Similar transformations are possible for all concave or concave-like - see Jahn (1984) - piece-wise linear functions $s(q,\bar{q})$, such as (4.35), (4.37) or their further modifications given below.

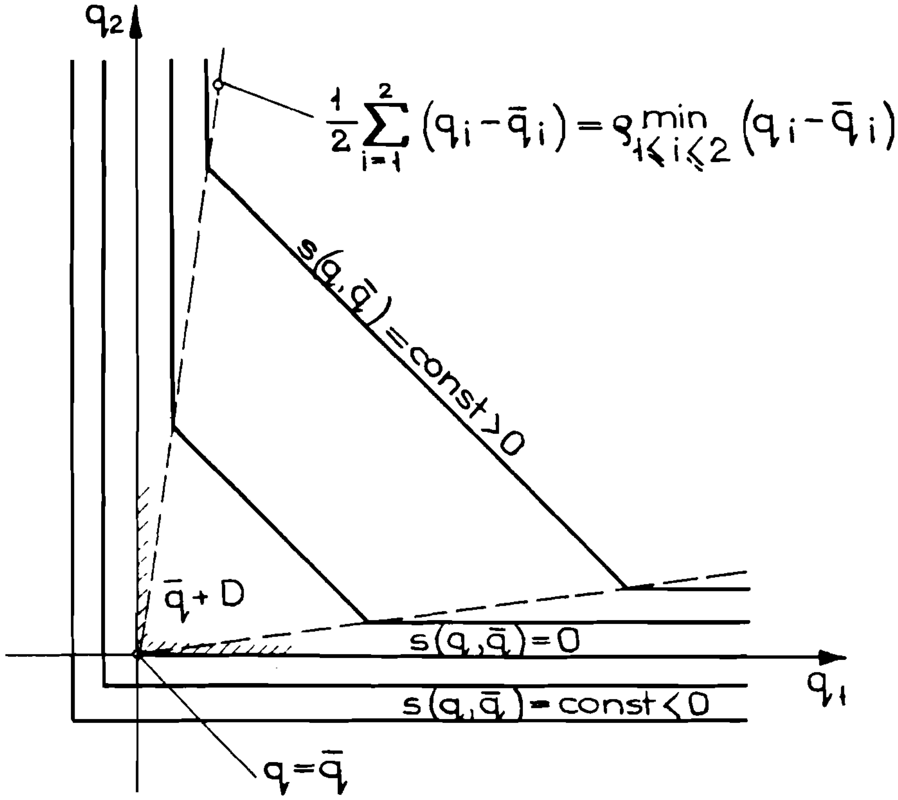

Figure 4.8. Level sets of the order-representing function (4.42).

The prototype order-representing function (4.35) has also several modifications in case when additional information about Q_0 is available. The function (3.15a,b), or the Chebyshev norm (4.24) under transformation (4.25), after slight modifications can be written in the form:

$$s(q,\bar{q}) = \min_{1\le i\le p}(q_i-\bar{q}_i)/(\tilde{q}_{i,max} - \bar{q}_i) \tag{4.44}$$

where $\tilde{q}_{i,max} > \hat{q}_{i,max}$, $i = 1,...p$, is an upper bound higher than utopia point component for the i-th outcome or objective function. In applications to interactive decision support, when the user can change the controlling parameter $\bar{q}$ arbitrarily, an important consideration is that $\bar{q}_i$ should be always smaller than, and - for computational reasons - not too close to $\tilde{q}_{i,max}$. This can be practically secured by selecting an additional scaling point $\tilde{q}$ such that $\hat{q}_{i,max} < \tilde{q}_i < \tilde{q}_{i,max}$ - a reasonable choice might be, for example,

$$\tilde{q}_{i,max} = \hat{q}_{i,max} + 0.2(\hat{q}_{i,max} - \hat{q}_{i,\min})$$

$$\tilde{q}_i = \hat{q}_{i,max}, + 0.1(\hat{q}_{i,\max} - \hat{q}_{i,\min})$$

and by using this scaling point as an upper bound for aspiration levels. The user should be informed that his aspirations q_i must not exceed $\tilde{q}_i$, but the decision support system should also automatically take $\bar{q}_i = \tilde{q}_i$ if the user specifies $\bar{q}_i > \tilde{q}_i$. This restriction might be considered as a drawback of function (4.44); however, the function has several other advantages. Firstly, the function has a cardinal form (it is independent of positively monotone affine transformations of the outcome space) and can be thus used as an approximation to a cardinal utility function of the user. Secondly, the weighting coefficients $\alpha_i = 1/(\tilde{q}_{i,max} - \bar{q}_i)$, implied by the aspirations $\bar{q}_i$ specified by the user, represent the relative importance of various outcomes or criteria to him: the more close $\bar{q}_i$ to $\tilde{q}_{i,max}$ or $\hat{q}_i$, the more important is the i-th outcome. When using achievement function (4.44), he can much more easily influence the selection of an efficient outcome $\hat{q} = \psi(\bar{q})$ by changing the controlling parameter $\bar{q}$, than when using functions (4.35) or (4.37), see Figure 4.9.

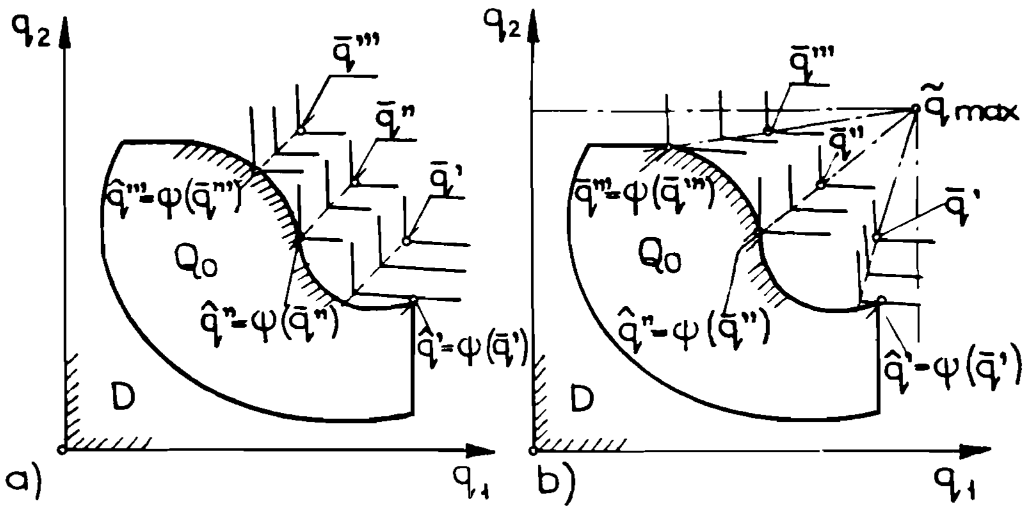

Figure 4.9. Controllability properties of order-consistent achievement functions: (a) functions (4.35), (4.37); (b) functions (4.44), (4.49); observe the difference in $\hat{q}''' = \psi(\bar{q}''')$ in these cases.

Another modification of the prototype order-representing function (4.35) is function (3.14a,b), based on two general information points - an upper bound point $\tilde{q}_{\max} > \hat{q}_{\max}$ and a lower bound point $\tilde{q}_{\min} < \hat{q}_{\min}$ - as well as on two reference points specified by the user - a reservation point $\bar{q}'$ and an aspiration point $\bar{q}''$, whereas

$$\tilde{q}_{i,min} < \bar{q}_i' < \bar{q}_i'' < \tilde{q}_{i,max}, \quad i = 1,...p$$

For the convenience of the reader, we repeat here the definition of this function:

$$\mu_i(q_i,\bar{q}_i',\bar{q}_i'') = \begin{cases} \gamma((q_i - \tilde{q}_{i,min})/(\bar{q}_i' - \tilde{q}_{i,min}) - 1) & \text{if } \tilde{q}_{i,min} \le q_i < \bar{q}_i' \\ (q_i - \bar{q}_i')/(\bar{q}_i'' - \bar{q}_i') & \text{if } \bar{q}_i' \le q_i \le \bar{q}_i'' \\ \beta(q_i - \bar{q}_i'')/(\tilde{q}_{i,max} - \bar{q}_i'') + 1 & \text{if } \bar{q}_i'' < q_i \le \tilde{q}_{i,max} \end{cases}$$

$$s(q,\bar{q}',\bar{q}'') = \min_{1 \le i \le p} \mu_i(q_i,\bar{q}_i',\bar{q}_i'') \tag{4.45}$$

The main controlling parameter, corresponding to the role of $\bar{q}$ in the definition of order-representing achievement functions, is here the reservation level point $\bar{q}'$. However, the use of both aspiration and reservation levels as controlling parameters by the user of a decision support system expresses an important aspect of user's uncertainty in aspirations. Because of the form of this function, the user obtains important information about attainability of his aspirations and reservations, contained in the maximal values of (4.45) over $q \in Q_o$. Since

$$s(\tilde{q}_{\min},\bar{q}',q'') = -\gamma$$
$$s(\bar{q}',\bar{q}',\bar{q}'') = 0$$
$$s(\bar{q}'',\bar{q}',\bar{q}'') = 1$$
$$s(\tilde{q}_{\max},\bar{q}',\bar{q}'') = 1 + \beta$$

the user knows that his reservation levels are attainable and the aspiration levels are not, if the maximal values of this function are contained between 0 and 1.

The above function can be used in decision support systems with subjective evaluation of merits of discrete decision alternatives by a committee of experts - for example, the SCDAS system, see Lewandowski et al. (1985); in such applications, the parameters $\beta,\gamma > 0$ can be chosen arbitrarily to obtain a scale of achievement that is easily interpretable by the user (for example, $\beta = \gamma = 1$, which results in achievement values -1 for the lower bound, 0 for reservation levels, 1 for aspiration levels and 2 for the upper bound). In applications to decision support systems with a substantive model of the outcome mapping $q=f(x)$, however, an important consideration is that all functions μ_i should be concave in q_i which simplifies computational aspects of maximizing $s(q,\bar{q}',\bar{q}'')$ over $q \in Q_0$. This can be achieved by selecting appropriate values of the parameters β,γ and additionally restricting the selection of aspiration and reservation levels. Such additional restrictions (which should obviously be communicated to the user but also imposed automatically by the system) might take, for example, the form:

$$\bar{q}_i'' \leq \tilde{q}_i''$$

where

$$\tilde{q}_i'' = \hat{q}_{i,max} + 0.01(\hat{q}_{i,max}-\hat{q}_{i,min}) < \tilde{q}_{i,max} = \hat{q}_{i,max}+0.02(\hat{q}_{i,max} - \hat{q}_{i,\min})$$

and

$$\bar{q}_i' > \tilde{q}_i'$$

where

$$\tilde{q}_i' = \hat{q}_{i,\min} > \tilde{q}_{i,min} = \hat{q}_{i,min} - 0.01(\hat{q}_{i,max} - q_{i,min})$$

and, finally,

$$\bar{q}_i'' - \bar{q}_i' \geq 0.01(\hat{q}_{i,max} - \hat{q}_{i,min})$$

For the concavity of μ_i, it is sufficient then to take $\beta = 0.01$ or less and $\gamma = 100$ or more.

If the functions μ_i are concave and Q_0 is a convex polyhedral set, defined by a number of linear inequalities or equations, then the problem of maximizing (4.45) over $q \in Q_0$ can be rewritten equivalently as a linear programming problem of maximizing an additional variable z (not restricted in sign), where:

$$s(q,\bar{q}',\bar{q}'') = z \tag{4.46}$$
$$z \leq \gamma((q_i - \tilde{q}_{i,min})/(\bar{q}_i' - \tilde{q}_{i,min}) - 1) \qquad i = 1,\ldots p$$
$$z \leq (q_i - \bar{q}_i')/(\bar{q}_i'' - \bar{q}_i') \qquad i = 1,\ldots p$$
$$z \leq \beta(q_i - \bar{q}_i'')/(\tilde{q}_{i,max} - \bar{q}_i'') + 1 \qquad i = 1,\ldots p$$

4.2.2. Order-approximating achievement functions.

Order-approximating achievement functions can be obtained from order-representing functions by adding linear terms. For example, function (4.45) can be made order-approximating by modifying its form to (3.16), which we repeat here for the convenience of the reader:

$$s(q,\bar{q}',\bar{q}'') = \{\min_{1\le i\le p} \mu_i(q_i, \bar{q}_i', \bar{q}_i'') + (\rho/p)\sum_{i-1}^{p} \mu_i(q, \bar{q}_i', \bar{q}_i'')\}/(1+\rho) \qquad (4.47)$$

where $\rho \in (0;p)$; it is easy to check that this function is ϵ-order-approximating and $\bar{\epsilon}$-strongly monotone with $\bar{\epsilon}$ sufficiently small and ϵ sufficiently large as compared to ρ. Thus, this function can be used to generate properly efficient solutions with bound ϵ in response to reservation levels $\bar{q}'$ and aspiration levels $\bar{q}''$ specified by the user. If the user wishes to specify these two reference points, this function might be in fact most appropriate to be applied in decision support systems either with discrete alternatives or based on a substantive model of linear programming type, because in most practical situations we can restrict the selection of efficient solutions to properly efficient solutions with bound. If μ_i are concave functions of q_i, as discussed above, and Q_0 is a convex polyhedral set, we can rewrite the problem of maximizing (4.47) to the linear programming form by the following transformation:

$$\begin{aligned}
s(q,\bar{q}',\bar{q}'') &= (z_{p+1} + (\rho/p)\sum_{i-1}^{p} z_i)/(1+\rho); \quad z_{p+1} \le z_i, \quad i = 1,\dots p \\
z_i &\le \rho((q_i - \tilde{q}_{i,min})/(\bar{q}_i' - \tilde{q}_{i,min}) - 1) \quad i = 1,\dots p \\
z_i &\le (q_i - \bar{q}_i')/(\bar{q}_i'' - \bar{q}_i') \qquad i = 1,\dots p \qquad (4.48) \\
z_i &\le \beta(q_i - \bar{q}_i'')/(\tilde{q}_{i,max} - \bar{q}_i'') + 1 \qquad 1 = i,\dots p
\end{aligned}$$

where the variables z_i, z_{p+1} are not restricted in sign.

If the user wishes to specify only one reference or aspiration point as the controlling parameter, a similar modification of (4.44) leads to an order-approximating function which can be interpreted as a transformation of the composite norm (4.26):

$$s(q,\bar{q}) = \qquad (4.49)$$

$$= \{\min_{1\le i\le p}(q_i - \bar{q}_i)/(q_{i,max} - \bar{q}_i) + (\rho/p)\sum_{i=1}^{p}(q_i - \bar{q}_i)/(q_{i,max} - \bar{q}_i)\}/(1+\rho)$$

This form has been used in Lewandowski et al.(1985) for evaluating discrete alternatives. Similarly as (4.44), this function results in a better controllability, than the prototype order-representing function (4.37), of the selection $\hat{q} = \psi(\bar{q})$ of an efficient outcome by the user, cf. Figure 4.9. Through a transformation similar to (4.48), if Q_0 is a convex polyhedral set, the problem of maximizing (4.49) can be rewritten as a linear programming problem. The same applies to the prototype order-approximating function (4.37) or to an order-approximating function that can be obtained by modifying (4.42):

$$s(q,\bar{q}) = \qquad (4.50)$$

$$= \{\min(\rho' \min_{1\le i\le p} \alpha_i(q_i - \bar{q}_i), (1/p)\sum_{i=1}^{p}\alpha_i(q_i - \bar{q}_i)) + (\rho''/p)\sum_{i=1}^{p_i}(q_i - q_i)\}/(1+\rho'')$$

This function has been used in DIDAS systems, see further chapters.

Until now, we have considered the case when $D = R_+^p$. In the case when the positive cone D has the more complicated form (4.2), that is, when only first p_0 outcomes should be maximized and the remaining outcomes $p_0+1,\dots p$, should be kept close to their respective reference levels or ***stabilized***, all theoretical concepts and definitions of order-approximating functions together with the appropriate parts of Theorem 4.10 remain valid; however, we must modify then the form of order-approximating achievement functions. The modification of the prototype order-approximating function (4.37) is rather easy:

$$s(q,\bar{q}) = \min_{1 \le i \le p} z_i + \alpha_{p+1} \sum_{i=1}^{p} z_i \tag{4.51a}$$

$$z_i = \alpha_i(q_i - \bar{q}_i) \ , \ i = 1,\ldots p_o$$

$$z_i = -\alpha_i | q_i - \bar{q}_i | \ , \ i = p_o + 1,\ldots p$$

The problem of maximizing this function over a convex polyhedral set can be equivalently rewritten as the following linear programming problem:

$$s(q,\bar{q}) = z_{p+1} + \alpha_{p+1} \sum_{i=1}^{p} z_i' \tag{4.51b}$$

$$z_{p+1} \le z_i' \ , \ i = 1,\ldots p$$

$$z_i' = \alpha_i(q_i - \bar{q}_i) \ , \ i = 1,\ldots p_o$$

$$z_i' \le \alpha_i(\bar{q}_i - q_i) \ ; \ z_i \le \alpha_i(q_i - \bar{q}_i) \ ; \ i = p_o+1,\ldots p$$

where the additional variables z'^i are, in fact, less or equal 0 for $i = p_o+1,\ldots p$.

The order-approximating achievement function (4.49) can be modified for the case when D has the form (4.2) in the following way:

$$s(q,\bar{q}) = \{ \min_{1 \le i \le p} z_i + (\rho/p) \sum_{i=1}^{p} z_i \}/(1 + \rho) \tag{4.52}$$

$$z_i = (q_i - \bar{q}_i)/(\tilde{q}_{i,max} - \bar{q}_i), \ i = 1,\ldots p_o$$

$$z_i = \min(z_i', z_i''), \ i = p_o+1,\ldots p$$

$$z_i' = (q_i - \bar{q}_i)/(\tilde{q}_{i,max} - \bar{q}_i) \ ; \ z_i'' = (\bar{q}_i - q_i)/(q_i - \tilde{q}_{i,min})$$

and the problem of maximizing this function over a convex polyhedral set can be rewritten similarly as (4.51b) to a linear programming form.

Before modifying the order-approximating function (4.47) for the case of positive cones of the form (4.2), it is necessary to define more precisely what does it mean to stabilize an outcome variable between the reference levels $\bar{q}_i'$, $\bar{q}_i''$. Both these levels play then the role of reservation levels and both $\tilde{q}_{i,max}$, $\tilde{q}_{i,min}$ should be then interpreted as lower bounds; but what are aspiration levels and upper bound in this case? We can reasonably assume that the upper bound of achievement in this outcome variable is the arithmetic mean $0.5(\bar{q}_i' + \bar{q}_i'')$, while the aspiration levels correspond to coming close to this mean, for example, as close as between $0.6\bar{q}_i' + 0.4\bar{q}_i''$ and $0.4\bar{q}_i' + 0.6\bar{q}_i''$. Thus, the modification of (4.47) can be defined as follows:

$$s(q,\bar{q}',\bar{q}'') = \{ \min_{1 \le i \le p} z_i + (\rho/p) \sum_{i=1}^{p} z_i \}/(1 + \rho) \tag{4.53a}$$

$$z_i = \mu_i(q_i,\bar{q}_i',\bar{q}_i''), \ i = 1,\ldots p_o \tag{4.53b}$$

$$\mu_i(q_i,\bar{q}_i',\bar{q}_i'') = \begin{cases} \gamma((q_i - \tilde{q}_{i,min})/(\bar{q}_i' - \tilde{q}_{i,min}) - 1) & \text{if } \tilde{q}_{i,min} \le q_i < q_i' \\ (q_i - \bar{q}_i')/(\bar{q}_i'' - \bar{q}_i') & \text{if } \bar{q}_i' \le q_i \le q_i'' \\ \beta(q_i - \bar{q}_i'')/(\tilde{q}_{i,max} - \bar{q}_i'') + 1 & \text{if } \bar{q}_i'' < q_i \le \tilde{q}_{i,max} \end{cases} \tag{4.53c}$$

$$z_i = \min(z_i{}', z_i{}''),\ i = p_o + 1, \ldots p \tag{4.53d}$$

$$z_i{}' = \begin{cases} \gamma((q_i - \tilde{q}_{i,min})/(\bar{q}_i{}' - \tilde{q}_{1,min}) - 1) & \text{if } \tilde{q}_{i,min} \leq q_i < \bar{q}_i{}' \\ (q_i - \bar{q}_i{}')/(\bar{q}''_{i,min} - \bar{q}_i{}')) & \text{if } \bar{q}_i{}' \leq q_i \leq \bar{q}''_{i,mid'} \\ \beta(q_i - \bar{q}''_{i,mid'})/(\tilde{q}_{i,mid} - \bar{q}''_{i,mid'}) + 1 & \bar{q}''_{i,mid'} < q_i \leq \tilde{q}_{i,mid} \end{cases} \tag{4.53e}$$

$$\bar{q}''_{i,mid'} = 0.6\bar{q}_i{}' + 0.4\bar{q}_i{}'' \ ; \ \tilde{q}_{i,mid} = 0.5(\bar{q}_i{}' + \bar{q}_i{}'')$$

$$z_i{}'' = \begin{cases} \gamma((\tilde{q}_{i,max} - q_i)/(\tilde{q}_{i,max} - \bar{q}_i{}'') - 1) & \text{if } \bar{q}_i{}'' < q_i \leq \tilde{q}_{i,max} \\ (\bar{q}_i{}'' - q_i)/(\bar{q}_i{}'' - \bar{q}''_{i,mid''}) & \text{if } \bar{q}''_{i,mid''} \leq q_i < \bar{q}_i{}'' \\ \beta(\bar{q}''_{i,mid''} - q_i)/(\bar{q}''_{i,mid''} - \tilde{q}_{i,mid}) + 1 & \text{if } \tilde{q}_{i,mid} \leq q_i < \bar{q}''_{i,mid''} \end{cases} \tag{4.53f}$$

$$\bar{q}''_{i,mid''} = 0.4\bar{q}_i{}' + 0.6\bar{q}_i{}'' \ ; \ \tilde{q}_{i,mid} = 0.5(\bar{q}_i{}' + \bar{q}_i{}'')$$

If $\bar{q}_i{}'' \leq \tilde{q}_i{}''$, where

$$\tilde{q}_i{}'' = \hat{q}_{i,max} + 0.01(\hat{q}_{i,max} - \hat{q}_{i,min}) < \tilde{q}_{i,max} = $$
$$= \hat{q}_{i,max} + 0.02(\hat{q}_{i,max} - \hat{q}_{i,min})$$

and $\bar{q}_i{}' \geq \tilde{q}_i{}'$, where

$$\tilde{q}_i{}' = \hat{q}_{i,min} > \tilde{q}_{i,min} = \hat{q}_{i,min} - 0.01(\hat{q}_{i,max} - \hat{q}_{i,min})$$

and finally,

$$\bar{q}_i{}'' - \bar{q}_i{}' \geq 0.01(\hat{q}_{i,max} - \hat{q}_{i,min})$$

then it is sufficient to take $\beta = 0.01$, $\gamma = 250$ to guarantee the concavity of the above function. In this case, the problem of maximizing (4.53) over a convex polyhedral set can be equivalently rewritten in a linear programming form similarly as (4.48), (4.51b).

4.2.3. Smooth order-approximating functions.

Order-consistent functions are nondifferentiable at $q = \bar{q}$; this might cause difficulties in applications to decision support systems with nonlinear substantive models of the outcome mapping $q = f(x)$ and with nonlinear constraints defining the set X_o and thus Q_o. These difficulties occur because nondifferentiable optimization algorithms, though already quite advanced, are not yet as robust and reliable as smooth nonlinear optimization algorithms. There are two ways to overcome these difficulties. One is to transform the problem of maximizing a nondifferentiable achievement function through introducing additional inequalities, as it is done in the linear programming case, and then to apply nonlinear programming algorithms. Another way is to modify the concept of order-approximation in order to admit smooth functions (see Wierzbicki, 1980). For this purpose, the concept of approximating the cone D by the cone D_ϵ must be weakened; we need here an approximation of the form:

$$D_{\epsilon\, s} = \{q \in R^p \colon dist(q,D) \leq \epsilon r(\| q \|)\} \tag{4.54}$$

where $r : R^1_+ \longrightarrow R^1_+$ is a positive monotone function such that $\lim_{t \to 0} r(t) = 0$. This class of approximations is much broader than that implied by cones D_ϵ.

A smooth order-approximating function is a continuous and differentiable function $s: Q_0 \times A \longrightarrow R^1$ such that $s(q,\bar{q})$ is strongly monotone in q for all $\bar{q} \in A$ and satisfies, moreover, the following smooth order approximation property:

$$\bar{q} + D \subset \{q \in R^p : s(q,\bar{q}) \geq 0\} \subset \bar{q} + D_{\epsilon s}, \text{ for all } \bar{q} \in A \tag{4.55}$$

for some $\epsilon > 0$.

Each maximal element $\hat{q}$ of a smooth order-approximating function is properly efficiently; however, when using smooth order-approximating functions, necessary conditions of proper efficiency cannot be stated directly as in Theorem 10. On the other hand, we can expect that smooth order-approximating functions might have properties similar to those stated in Theorem 4.2, that is, if $\hat{q}'$ is properly efficient and ϵ is sufficiently small, then a maximal element $\hat{q}$ of $s(q,\bar{q})$ over $q \in Q_0$ with $\bar{q} = \hat{q}'$ should not be too far from $\hat{q}'$, see Figure 4.10.

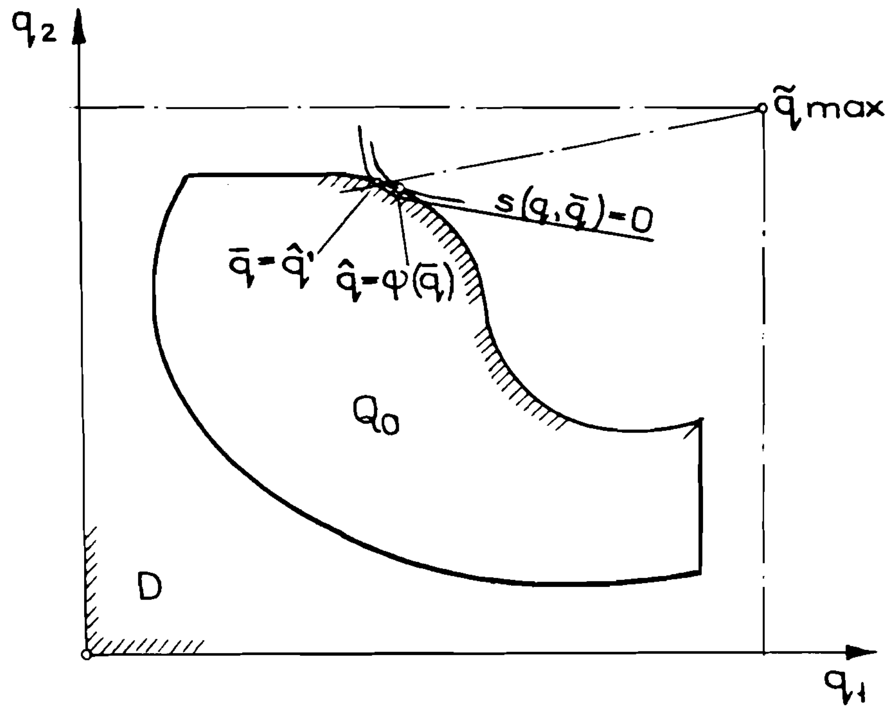

Figure 4.10. The approximation property of efficient solutions by maximal elements of smooth order-approximating functions.

The following smooth order-approximating function:

$$s(q,\bar{q}) = 1 - \{(1/p)\sum_{i=1}^{p} |(\tilde{q}_{i,max} - q_i)/(\tilde{q}_{i,max} - \bar{q}_i)|^k\}^{1/k}; \ \tilde{q}_{\max} \geq \hat{q}_{\max} \tag{4.56}$$

is, in fact, a modification of the norm used in Theorem 4.2; however, the controlling parameter $\bar{q}$ used in this function is different than the weighting coefficient α used in that norm. Observe also that the function (4.56) is not a weighted norm of the distance $\bar{q}_i - q_i$, because it depends on the controlling parameter $\bar{q}$ in a more complicated way. Because of these differences, both attainable and unattainable aspiration points $\bar{q}$ can be used as controlling parameters, as long as $\bar{q} \leq \tilde{q}_{\max}$. This function is a smooth approximation of the order-representing function (4.44) or its order-approximating variant (4.49). To obtain a close approximation, a sufficiently large k should be used (it is easy to check that (4.56) converges to (4.44) for $k \longrightarrow \infty$). However, this would result in badly conditioned nonlinear optimization problems when maximizing this function over $q \in Q_o$, hence $k = 4...8$ is used for applications in decision support systems - for example, in nonlinear optimization extensions of DIDAS methodology, see Kaden and Kreglewski (1986).

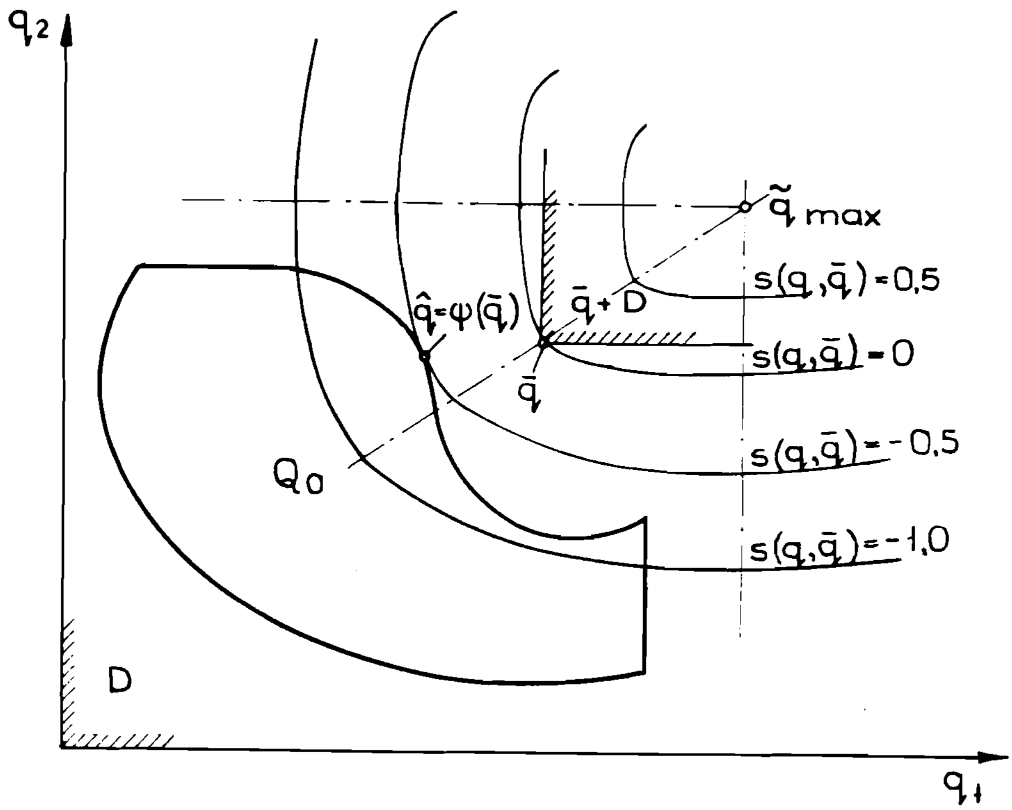

Figure 4.11. Level sets of the smooth order-approximating function (4.56).

A more complicated problem is a smooth approximation of the order-representing achievement function (4.45) or its order-approximating variant (4.50), where two reference points (a reservation point $\bar{q}'$ and an aspiration point $\bar{q}''$) are used as controlling parameters. As before, we assume that $\tilde{q}_{\min} \leq \hat{q}_{\min}$ and $\tilde{q}_{\max} \geq \hat{q}_{\max}$ are given and that $\tilde{q}_{i,min} < \bar{q}_i' < \bar{q}_i'' < \tilde{q}_{i,max}$ for all $i = 1,...p$. Such a smooth order-approximating function has the following form:

$$s(q,\, \bar{q}',\, \bar{q}'') = (\rho'' - \theta)\xi(\tau)\varphi(t) + (1 + \beta - \beta\delta\tau)(1 - \xi(\tau)) + \gamma(-1 + \eta(\rho' - t))(1 - \varphi(t)) \tag{4.57a}$$

where τ, θ, t are shortened denotations of the following norms:

$$\tau = \tau(q,\bar{q}',\bar{q}'') = \{(1/p)\sum_{i=1}^{p}((\tilde{q}_{i,max} - q_i)/(\tilde{q}_{i,max} - \bar{q}_i''))^k\}^{1/k} \tag{4.57b}$$

$$\theta = \theta(q,\bar{q}',\bar{q}'') = \{(1/p)\sum_{i=1}^{p}((\tilde{q}_{i,max}'' - q_i)/(\bar{q}_i'' - \bar{q}_i'))^k\} \tag{4.57c}$$

$$t = t(q,\bar{q}',\bar{q}'') = \{(1/p)\sum_{i=1}^{p}((\tilde{q}'_{i,max} - q_i)/(\bar{q}_i' - \tilde{q}_{i,min}))^k\}^{1/k} \quad (4.57d)$$

whereas $\xi(\tau)$ and $\varphi(t)$ are differentiable spline functions:

$$\xi(\tau) = \begin{cases} 2\tau^2 & \text{if } 0 \leq \tau < 0.5 \\ -1 + 4\tau - 2\tau^2 & \text{if } 0.5 \leq \tau \leq 1 \\ 1 & \text{if } 1 < \tau \end{cases} \quad (4.57e)$$

$$\varphi(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq \rho'-1 \\ 1 - 1(\rho' - 1)^2 + 4(\rho' - 1)t - 2t^2 & \text{if } \rho' - 1 < t < \rho' - 0.5 \\ 2(\rho')^2 - 4\rho' + 2t^2 & \text{if } \rho' - 0.5 \leq t < \rho' \\ 0 & \text{if } \rho' \leq t \end{cases} \quad (4.57f)$$

the additional parameters ρ' and ρ'' are defined by:

$$\rho' = \max_{1 \leq i \leq p} (\tilde{q}_{i,max} - \tilde{q}_{i,min})/(\bar{q}_i' - \tilde{q}_{i,min}) \geq 1 \quad (4.57g)$$

$$\rho'' = \max_{1 \leq i \leq p} (\tilde{q}_{i,max} - \bar{q}_i')/(\bar{q}_i'' - \bar{q}_i')) \geq 1 \quad (4.57h)$$

and the additional scaling points $\tilde{q}'_{\max}$ and $\tilde{q}''_{\max}$ have the form:

$$\tilde{q}'_{\max} = \tilde{q}_{\min} + \rho'(\bar{q}' - \tilde{q}_{\min}) \geq \tilde{q}_{\max} \quad (4.57i)$$

$$\tilde{q}''_{\max} = \bar{q}' + \rho''(\bar{q}'' - \bar{q}') \geq \tilde{q}_{\max} \quad (4.57j)$$

whereas

$$t(\tilde{q}_{\min},\bar{q}',\bar{q}'') = \rho', \quad t(\bar{q}',\bar{q}',\bar{q}'') = \rho' - 1$$

$$\theta(\bar{q}',\bar{q}',\bar{q}'') = \rho'', \quad \theta(\bar{q}'',\bar{q}',\bar{q}'') = \rho'' - 1$$

Moreover, β, $\gamma > 0$ are scaling parameters such that

$$s(\tilde{q}_{\min},\bar{q}',\bar{q}'') = -\gamma \,, \; s(\bar{q}',\bar{q}',\bar{q}'') = 0$$

$$s(\bar{q}'',\bar{q}',\bar{q}'') = 1 \,, \; s(\tilde{q}_{\max},\bar{q}',\bar{q}'') = 1 + \beta$$

The additional parameters δ, $\eta \in [0; 1)$ influence the monotonicity of this achievement function.

The spline functions $\xi(\tau)$ and $\varphi(t)$ are shown in Figure 4.12a; they mix together the norms τ, t, used in the definition of this function in various regions. The sets

$$Q_\tau = \{q \in R^p\colon \tau(q,\bar{q}',\bar{q}'') < 1\}, \quad Q_t = \{q \in R^p\colon t(q,\bar{q}',\bar{q}'') > \rho' - 1\}$$

and

$$\bar{Q} = \{q \in R^p\colon \bar{q}_i' \leq q_i \leq \bar{q}_i'' \,, \, i = 1,\ldots p\}$$

are disjoint. If $q \in \bar{Q}$ or, more generally, if $q \notin Q_t \cup Q_\tau$, then $\varphi(t) = 1$, $\xi(\tau) = 1$ and function (4.57a) is defined by its first term only, which depends on the norm θ scaled by the difference $\bar{q}'' - \bar{q}'$:

$$s(q,\bar{q}',\bar{q}'') = \rho'' - \theta(q,\bar{q}',\bar{q}'')$$

If $q \in Q_\tau$, then function (4.57a) is a differentiable mix of the norms θ and τ; if $q \in Q_t$, then it is a differentiable mix of the norms θ and t. The role of the additional scaling points $\tilde{q}'_{\max}$, $\tilde{q}''_{\max}$ and the construction of this achievement function are illustrated in Figure 4.12b,c.

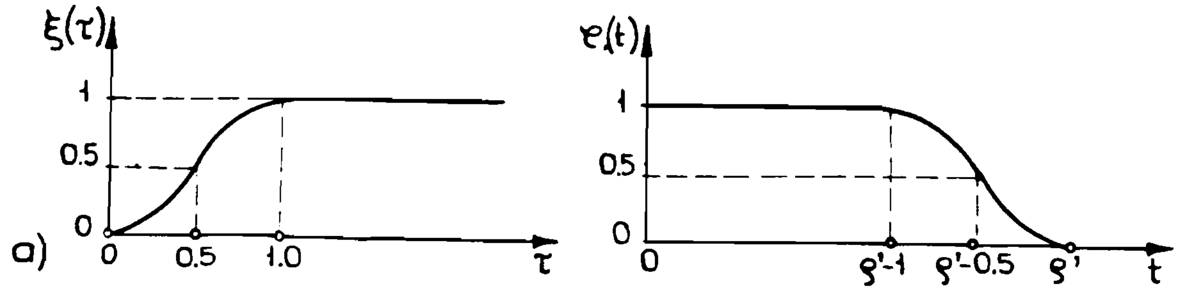

Figure 4.12a. Spline functions $\xi(\tau)$ and $\varphi(t)$.

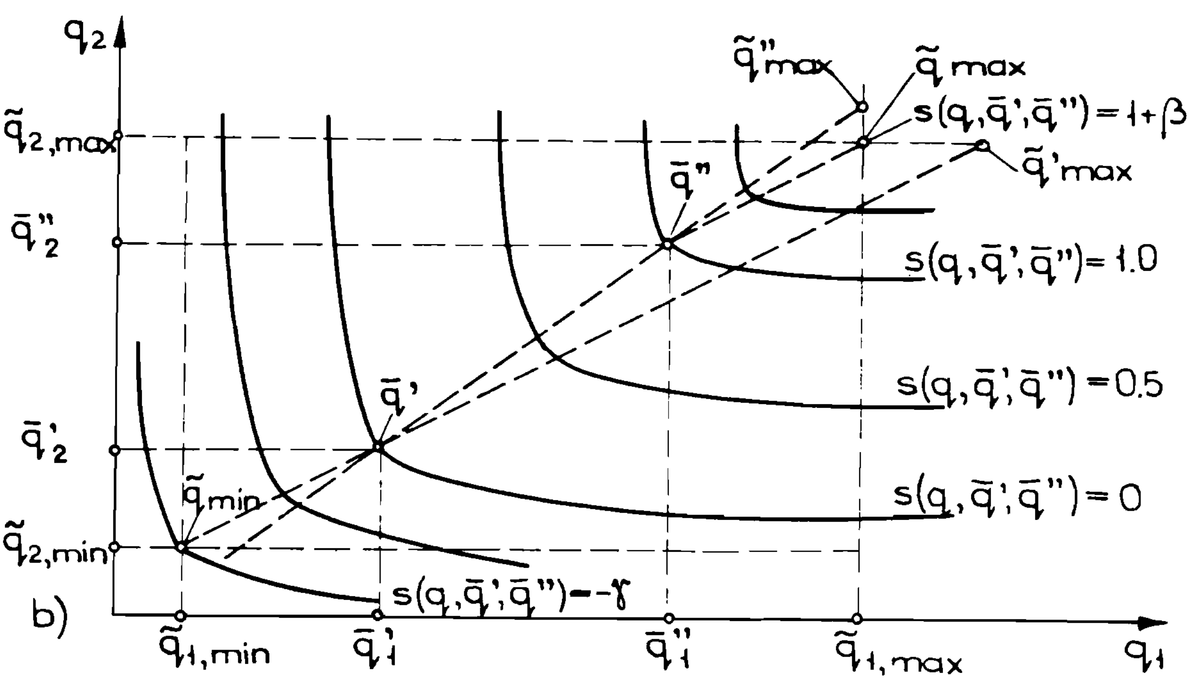

Figure 4.12b. Level sets of the smooth achievement function (4.57a).

As it can be seen from Figure 4.12c, function (4.57a) approximates well, for sufficiently large k, function (4.45), but only for $q \in \bar{Q}$ - that is, for outcomes that are between reservation and aspiration levels; however, this range of decision outcomes is typically most interesting for the user of a decision support system. If parameters β, γ are chosen sufficiently large and δ, η - sufficiently small, then function (4.57a) is strongly monotone and all its maximal points are indeed efficient.

Theorem 4.13. If $q \leq \tilde{q}_{\max}$, $k < \infty$ and

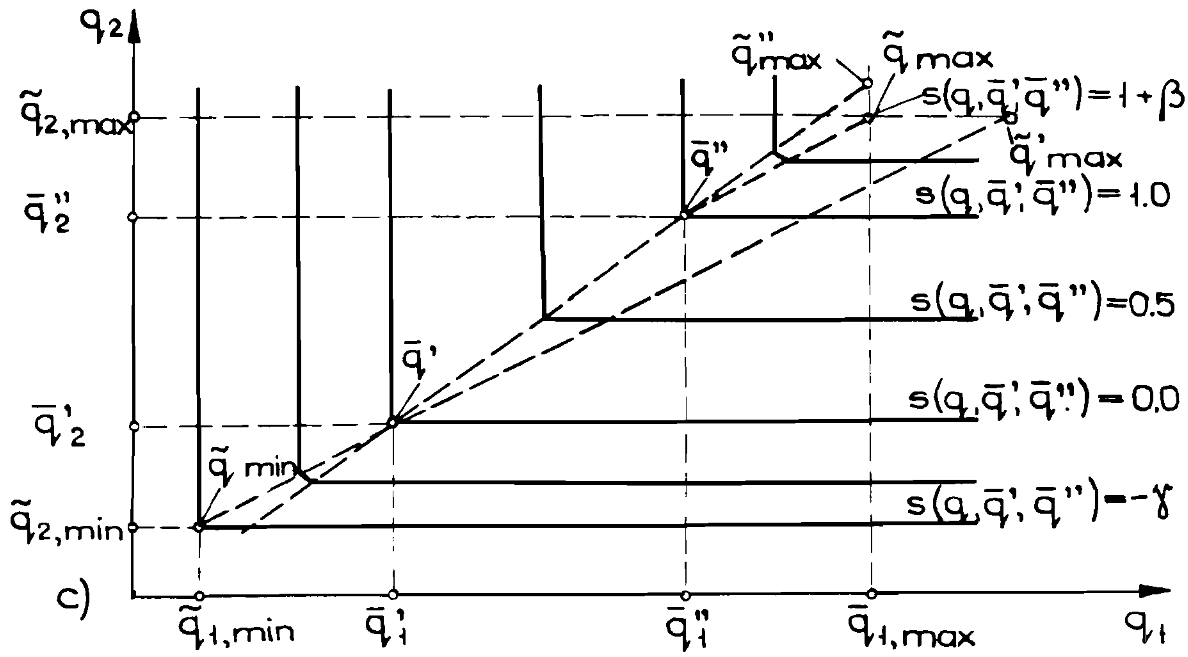

Figure 4.12c. Limits of these level sets as $k \longrightarrow \infty$.

$$\beta \geq \{(1/p)\sum_{i=1}^{p}((\tilde{q}_{i,max} - \bar{q}_i'')/(\bar{q}_i'' - \bar{q}_i'))^k\}^{1/k}/(1-\eta) \tag{4.58a}$$

$$\gamma \geq \{(1/p)\sum_{i=1}^{p}((\bar{q}_i' - \tilde{q}_{i,min})/(\bar{q}_i'' - \bar{q}_i'))^k\}^{1/k}/(1-\delta) \tag{4.58b}$$

then (4.57a) is a strongly monotone function of q.

The proof of this theorem is given in Appendix. This theorem suggests for applications in decision support systems that the minimal difference between aspiration and reservation levels $\bar{q}_i'' - \bar{q}_i'$ should be limited as compared to the range $\tilde{q}_{i,max} - \tilde{q}_{i,min}$, say, to be above $0.01(\tilde{q}_{i,max} - \tilde{q}_{i,min})$. When choosing $\delta = \eta = 0.5$, the values of the parameters $\beta = \gamma = 200$ suffice then for the monotonocity of the achievement function (4.57a). Although the bounds (4.58a,b) are not very tight, counterexamples show that (4.57a) can indeed loose its monotonicity if β and γ are much smaller than indicated by this bounds.

This smooth order-approximating achievement function has quite complicated analytical form; however, this does not matter much, since the user of a decision support system does not need to understand the details of this function definition - it is sufficient for him to know that the maximal values of this function, if contained between 0 and 1, indicate that his reservation levels are attainable and his aspiration levels are not. Therefore, this achievement function might be very useful in decision support systems with substantive models of nonlinear programming type, if the user of this system wishes to control his selection of efficient outcomes by specifying or changing two controlling parameters - the reservation point $\bar{q}'$ and the aspiration point $\bar{q}''$.

4.2.4. Achievement functions for trajectory optimization.

When considering continuous-time dynamic models and their trajectories as decision outcomes, infinite-dimensional outcome spaces and positive cones in them are needed. However, most computational applications in decision support systems are based on discrete-time approximations of such models which reduces the outcome spaces to finite, though large dimensions; the discussion here is limited to such cases. Still, it is necessary to know that no basic theoretical difficulties arise when the number of dimensions grows very large. It is one of the advantages of the approach based on order-consistent achievement functions with reference points used as controlling parameters that the theoretical foundations of this approach, such as Theorem 4.10, are valid without major modifications also in infinite-dimensional spaces - see Wierzbicki (1980, 1982).

Another, more practical advantage is related to this approach. When controlling the selection of efficient outcomes, the user of a decision support system should be able to well interpret his controlling parameters. If he used weighting coefficients as controlling parameters, he might be baffled in their interpretations when their number grows large. When using aspiration or reservation points, he can interpret them as *reference trajectories* by aggregating a number of reference values in a meaningful trajectory and evaluating this trajectory by "Gestalt", see Figure 4.13. If the outcomes of a decision are represented by a solution of dynamical model, there is a natural way of aggregating them into trajectories: we combine the values of the same outcome for consecutive instants of time, and the number of these instants can grow rather large, but we still deal with the same kind of trajectory. Even for models of static type with a large number of outcomes, it is useful to combine these outcomes into meaningful trajectories - for example, distributions of income or patterns of trade in an economic model. Once a meaning of a trajectory of outcomes is well understood, the specification or interpretation of a related reference, aspiration or reservation trajectory becomes easy.

It is a known psychological fact that a human cannot compare or evaluate in his mind more than five to nine objects, depending on their complexity; however, this does not mean that these objects should be characterized each by only a scalar-valued attribute. For example, we can compare five maps of some geographical region, or five photographs of various or the same person, each containing a large amount of information; we have even the concept of a reference photograph - for example, the synthesized picture of a suspect. Thus, the conclusion that no more than five to nine scalar attributes should be compared in a decision system is valid only when defining attributes for subjective evaluation - and even in this case, the attributes might be the results of hierarchical aggregation of a number of lower-level attributes; in decision support systems based on substantive models, we can as well compare five to nine trajectories each containing a large amount of information.

Dynamic models can have various mathematical character, see Kalman et al. (1969). Here, we shall consider only a relatively simple but widely applied class of such models - with concentrated state and discrete time. The prototype form of such a model is as follows:

$$w[t+1] = h(w[t], u[t], t),\ t = 0,1,\dots T-1, (T);\ w[0] - \text{given} \tag{4.59a}$$

$$g(w[t], u[t], t) \le 0,\ t = 0,1,\dots T-1 \tag{4.59b}$$

$$q[t] = f(w[t], u[t], t),\ t = 0,1,\dots T-1, (T) \tag{4.59c}$$

where t is the discrete time variable (counted in days, years, or any other time units). The decision variable $u[t] \in R^n$ is often called *control*, the theory of such dynamic models being closely related to the control theory; but the actual decision variable is here the entire *control trajectory* or *decision trajectory* $u = \{u[0], u[1], \dots u[T-1]\} \in R^{nT}$. The additional variable $w[t] \in R^{m'}$ is called the *state* of the dynamic model, being defined as

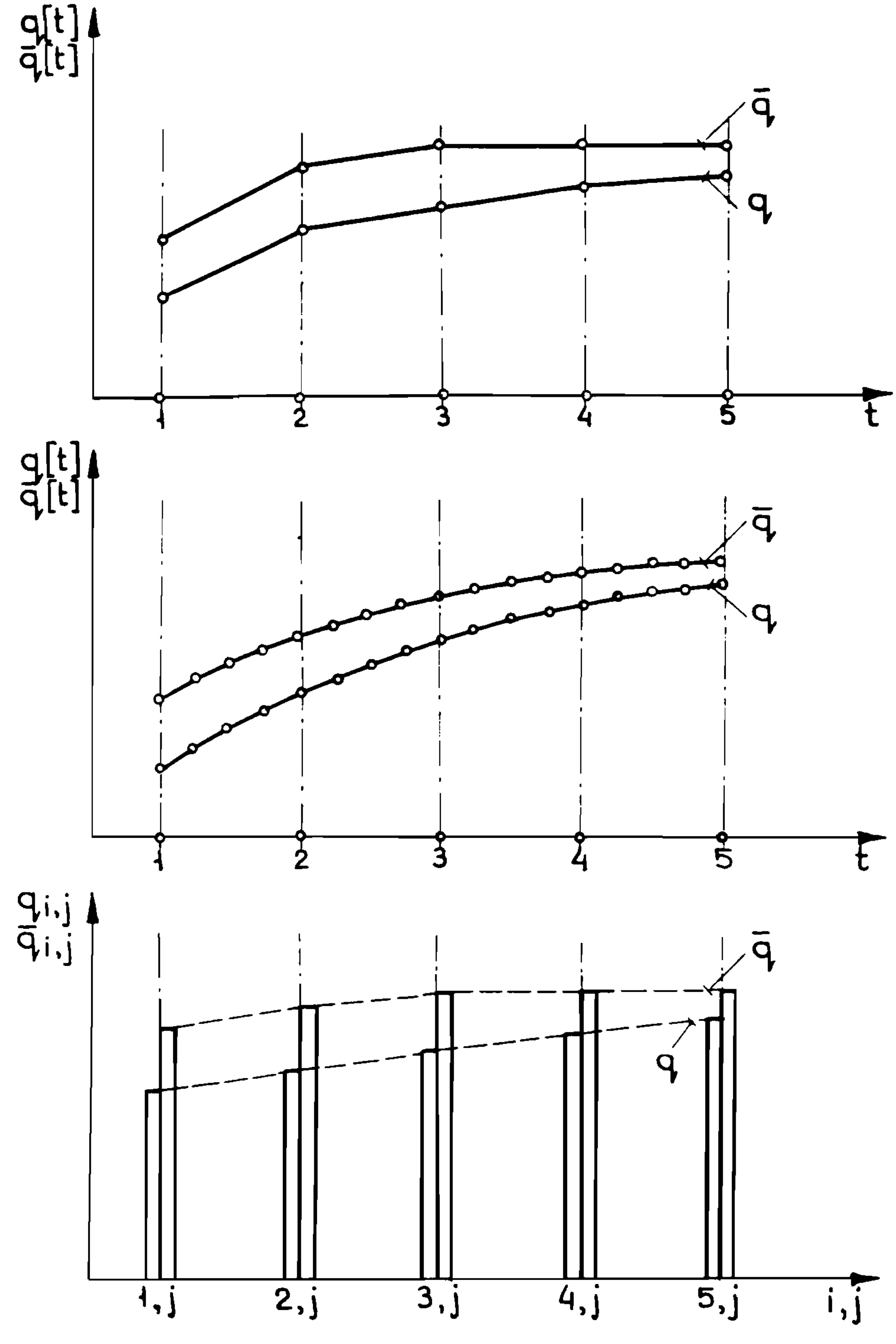

Figure 4.13. Examples of combining outcomes and reference points in trajectories (a) for dynamic models with sparse discrete time; (b) for dynamic models with dense discrete time; (c) for static models.

the set of initial conditions that must be specified in order to solve the model (in this case, $w[0]$ is assumed to be given, but we could start solving this model at any other instant of time); the entire ***state trajectory***

$$w = \{w[0], w[1], ... w[T-1], w[T]\} \in R^{m'(T+1)}$$

is actually one period longer than the decision trajectory, because we must account for dynamic consequences of the decision in the last period, $u[T-1]$. The same applies to the ***outcome trajectory*** (called also output, performance or objective trajectory) $q = \{q[0], q[1], ... q[T-1], q[T]\} \in R^{p(T+1)}$ while $q[t] \in R^p$. Thus, if the number of periods or ***time horizon*** grows, the dimensionality of the outcome space could increase substantially - but, as commented above, this does not really matter as long as the number of outcome trajectories, p, is not too high. Equation (4.59a) is often called the state equation of the dynamic model, while inequalities (4.59b) are called state-dependent constraints (sometimes additional control or decision constraints of the form $u[t] \in U_o[t]$ are taken into consideration) and equation (4.59c) is called the outcome or output equation of this model. On the last period T, the outcome equation should not depend on the variable $u[T]$ which is not included in the definition of the model.

The theory of analysis and optimization of dynamic models is rather extensive - see, e.g., Wierzbicki (1977b) - and will not be presented here in detail. For the purposes of this book it is sufficient to note that an important class of such models are ***linear dynamic models***, where f,g,h are linear or affine functions; a piece-wise linear concave achievement function, when maximized over outcomes of such a model, results in a linear programming problem.

We shall consider here only order-approximating achievement functions for trajectory optimization. The prototype order-approximating function (4.37) can be written for this purpose in the following way:

$$s(q,\bar{q}) = \tag{4.60}$$

$$= \min_{0 \le t \le T} \min_{1 \le i \le p} \alpha_i[t](q_i[t] - \bar{q}_i[t]) + \alpha_{p+1} \sum_{t=0}^{T} \sum_{i=1}^{p} \alpha_i[t](q_i[t] - \bar{q}_i[t])$$

To construct other forms of order-approximating achievement functions, we need often an upper and possibly a lower bound for efficient outcomes. This creates a major difficulty in case of outcome trajectories: a computation of an utopia trajectory and an approximation of a nadir trajectory as in (4.16), (4.17) would require in this case $p(T+1)$ scalar optimization computations, which in most cases is an excessive computational load. However, precise upper bound and lower bound trajectories are not needed in most cases of decision support and their approximate values often suffice, if we use the concept of order-consistent achievement functions. A convenient way of computing such approximate upper and lower bounds, used in DIDAS systems, is to maximize function (4.60) p times with some neutral values of weighting coefficients α_i (they could be set all equal to 1 or $1/(p(T+1))$, if all components of outcomes are expressed in reasonable units of scale) and with p different reference trajectories

$$\bar{q}^{(j)} = \{\bar{q}^{(j)}[0], \bar{q}^{(j)}[1], ... \bar{q}^{(j)}[T-1], \bar{q}^{(j)}[T]\}, j = 1, ... p$$

where the components $\bar{q}_i^{(j)}[t]$ are chosen to be very high if $i = j$ and very low if $i \neq j$. Denote the results of this maximizations - which are properly efficient outcomes, since (4.60) is strongly monotone - by $\hat{q}^{(j)} = \psi(\bar{q}^{(j)})$. Approximate upper bound and lower bound for trajectories can be constructed then as follows:

$$\hat{q}_{i,max}[t] = \hat{q}_i^{(i)}[t],\ t = 0, t, ... T-1,\ i = 1, ... p \tag{4.61}$$

$$\hat{q}_{i,min}[t] = \min_{1 \le j \le p} \hat{q}_i^{(j)}[t],\ t = 0, 1, ... T-1,\ T,\ i = 1, ... p$$

$$\tilde{q}_{\max}[t] = \hat{q}_{\max}[t] + \delta(\hat{q}_{\max}[t] - q_{\min}[t]),\ t = 0, 1, \ldots T - 1, T$$

$$\tilde{q}_{\min}[t] = \hat{q}_{\min}[t] - \delta(\hat{q}_{\max}[t] - \hat{q}_{\min}[t]),\ t = 0, q, \ldots T - 1, T$$

where reasonable values for the additional parameter δ are $\delta = 0.1 \ldots 0.5$.

This way of computing bounds on trajectories has, however, one drawback: while the trajectory $\hat{q}_{i,max}[t]$ comes from an actual trajectory of the dynamic system, the constructed trajectory $\hat{q}_{i,min}[t]$ does not. Another, preferable way of computing $\hat{q}_{i,min}[t]$ is to perform additional p maximizations with reference trajectories chosen in such a way that components $\bar{q}_i^{(j)}[t]$ are very low if $i = j$ and very high if $i \neq j$; $\tilde{q}_{\max}(t)$ and $\tilde{q}_{\min}(t)$ are then defined similarly as above. Once the results of such approximation are combined in an upper bound $\tilde{q}_{\max}$ and lower bound $\tilde{q}_{\min}$ for trajectories (it should be stressed that, precisely speaking, $\tilde{q}_{\max}$ and $\tilde{q}_{\min}$ are not trajectories of solutions of the dynamic model, only trajectories of approximate bounds for such efficient solutions), other forms than (4.60) of achievement functions for mutiobjective trajectory optimization can be specified that result in a better controllability of efficient outcome trajectories by changing reference trajectories. The order-approximating achievement function (4.49) can then be rewritten in the form:

$$s(q,\bar{q}) = \{ \min_{0 \leq t \leq T} \min_{1 \leq i \leq p} z_i[t] + (\rho/p) \sum_{t=0}^{T} \sum_{i=1}^{p} z_i[t] \}/(1 + \rho) \tag{4.62}$$

$$z_i[t] = (q_i[t] - \bar{q}_i[t])/(\tilde{q}_{i,max}[t] - \bar{q}_i[t])$$

In a similar way, other order-approximating achievement functions, such as (4.47), (4.49), or even smooth order-approximating functions, such as (4.56), can be rewritten for the case of multiobjective trajectory optimization. Some other examples of achievement functions for this case are discussed in Wierzbicki (1980).

4.3 Phases and procedures of decision support.

There are many diverse meanings of the concept "decision support". When understanding this concept very broadly, we can include into it many functions or activities, such as just filing and organizing information, aggregating and processing information, interpolating the aggregate results in order to build mathematical models, establishing decision rules for some customary decision-making situations, identifying these decision rules in form of mathematical models, etc. Therefore, the term "decision support system" is understood today in a very broad sense, including computer data base or spreadsheet systems for organizing, aggregating and processing information, knowledge base systems for proposing decisions based on established inference rules, and various other types of systems that support the actual processes of choice or selection among alternatives. This book addresses only a subclass of this last group of decision support systems - designed for cases where a choice among many alternatives is helped by a substantive model of a decision situation, a model that summarizes in a computerized form the knowledge of a group of experts and analysts for a substantive class of decision problems. In this specific case, the decision support process has several important phases that will be discussed here in some detail. The particular arrangement of these phases constitutes the procedural foundation for the quasisatisficing decision approach, at least in the situations with a single decision maker responsible for the final selection of the decision.

4.3.1 Substantive model definition and edition.

A substantive model represents pertinent knowledge about a class of decision problems, such as forecasting and planning forestry economic factors in a country, planning the development of a branch of chemical industry, planning energy sector development strategies, controlling ground water quality and quantity in a region, etc. For each case, a substantive model has to be developed, computerized, tested and validated on relevant data. To be useful for decision support, a substantive model must include relevant decision or control variables and their constraints as well as all outcome variables that might be of interest to the decision maker; thus, it is crucial to involve final users in the phase of model building.

Even if such a model is ready, it must be prepared for the utilization in a decision support system, that is, defined and edited in a format that allows for an easy communication with other parts of the system. Since we assume interactive multicriteria optimization of the outcomes of the model, its format must be adapted to the optimization algorithm used in the system; this algorithm depends on the mathematical type of the model. In contemporary optimization techniques, there are several classes of problems that require distinct optimization tools. Among these are: linear programming or linear optimization problems; nonlinear differentiable optimization problems; nondifferentiable optimization problems; stochastic optimization problems and other approaches to optimization under uncertainty; discrete optimization problems. Here we are mostly concerned with the first two classes of problems, although the mathematical foundations of quasisatisficing can be also applied to other classes and thus decision support systems of DIDAS family described in this book can be also constructed for those classes. In fact, experiments with nondifferentiable and stochastic optimization algorithms for decision support systems of DIDAS family has been made and pilot versions of discrete optimization algorithms for such systems developed. However, reliable and robust optimization algorithms that allow for an easy interaction of the user with the decision support system have been best developed for linear and discrete time dynamic linear optimization problems as well as for nonlinear and discrete time dynamic nonlinear differentiable optimization problems; thus we restrict our attention to these classes.

Defining and editing a linear or dynamic linear model for optimization purposes is typically done in the standard MPS format of linear programming. For nonlinear or dynamic nonlinear models, no such standards exist, and a format of a computer language subroutine (with or without using automatic calculation of derivatives of all functions in the model) is then applied. Since such formats are highly specialized, this phase of decision support requires the participation of an analyst that would prepare the final format of the substantive model for the decision support system. Since the substantive model is usually a result of work of a team of analysts, preferably one of them should edit the model in the desired format. The use of professional microcomputer with more modern standards of user-friendliness of software systems substantiate, however, the possibility that a user himself might define and edit the substantive model in an easy format such as a computerized spreadsheet. Moreover, a given substantive model is typically related to a specific type of graphical representation of results of its analysis; thus, appropriate computer graphic tools must be chosen. These aspects are described in more detail in a further chapter on DIDAS systems implementations.

4.3.2 Specification and initial analysis of a multiobjective problem.

A substantive model can be used to support analysis of various multiobjective problems defined according to the needs of the decision maker. Thus, when the substantive model is defined and edited, the user can proceed to the specification of his multiobjective analysis problem; obviously, the participation of the final decision maker in this phase is crucial. Since the decision maker is free to change his mind about the objectives during the analysis, the system must allow for an interactive, easy specification of the multiobjective analysis problem.

The specification of this problem consists in:

- selecting outcome variables of the substantive model that represent the objectives of the decision maker;
- specifying the ordering in the space of objectives, that is, defining for each objective variable whether it should be maximized, or minimized, or kept close (from both sides) to a desired reference level.

Once a multiobjective problem is defined, the initial analysis of this problem can be performed. It consists in:

- evaluation of the outcomes of some decisions proposed by the user, in order either to test the adequacy of the substantive model or to obtain some reasonable outcomes for comparison with results of further analysis;
- assessment of the best attainable values for each objective (or approximately best attainable trajectory for each objective trajectory in dynamic cases), that is, the assessment of the precise or approximate utopia point in the objective space;
- assessment of the worst reasonable (efficient) values for each objective or objective trajectory, that is, of an approximate nadir point in the objective space.

This initial analysis is a crucial prerequisite of the further multiobjective analysis of the problem, because it gives to the decision maker the possibility to learn about reasonable ranges of each objective or objective trajectory. It can be preformed automatically by the system once a multiobjective problem is specified, but some special features of this analysis must be understood by the user.

Firstly, an evaluation of the outcomes of one or several decisions proposed by the user is optional; the user might omit this, if he wishes to proceed to further stages.

Secondly, a precise assessment of the best attainable values for each objective requires a number of optimization runs equal to that of objectives and might take too long a time if the number of objectives is too large. In dynamic cases, there is no sense to assess precisely the best attainable values for each point of a trajectory, since it would take too much time. Moreover, a collection of such points would not have the interpretation of an attainable trajectory and might give misleading information to the decision maker. Therefore, an approximate assessment of the best attainable trajectory for an objective can be preferably obtained by maximizing an achievement function with reference points corresponding to a 'high' aspiration trajectory for the objective in question and to 'low' aspiration trajectories for all other objectives. The same procedure can be also applied for static objectives; naturally, it must be repeated for each objective or objective trajectory.

Thirdly, a precise assessment of the worst reasonable values for each objective is, in general, not possible, because we restrict our attention to efficient outcomes understood as reasonable. It should be stressed that Eq. (4.17) gives a precise nadir point only in case of two objectives; if there are more than two objectives, it is easy to construct examples when Eq. (4.17) gives results "above" the nadir point and a precise computation of the nadir point would require very large effort and time. Fortunately, the decision maker is usually not interested in the precise nadir point, he needs only to know reasonable ranges of objective values. A reasonable approximation to the nadir point can be computed by maximizing an achievement function based on a very "low" aspiration level (or aspiration trajectory) for the objective in question and "high" aspirations (or trajectories of aspirations) for all other objectives, repeatedly for all objectives or their trajectories. The user must be, however, informed that the results of such computations give only an approximate nadir point. For this reason, it is better to use somewhat broader upper and lower bounds than the utopia and nadir points; this was discussed in more detail in previous sections.

Naturally, one could establish more precise "lower" bounds to all objectives by "minimizing" each of them separately (we use here the terms "lower" and "minimizing" in a relative sense, assuming that the objective is originally one to be maximized; if the objective is to be minimized - as, for example, variables representing costs - "lower" should be replaced by "upper" and "minimized" by "maximized"). However, this would include also inefficient outcomes and might result in unreasonable values. Moreover, some variables in a substantive model might be unrestricted when we move in an unreasonable direction, if the model is not very carefully constructed.

This is related to the third special feature of the initial analysis: to the understanding of bounds of objectives that should be neither maximized nor minimized but kept close to an aspiration level. As shown in previous sections, the meaning of the corresponding components of the utopia or the nadir point in such a case looses its sense and these components must be interpreted differently in a construction of an achievement function. However, the user should still know what values for this objective are reasonable. This can be assessed by minimizing and maximizing values of such objectives separately; thus, the user should know that he can specify such objectives when they are otherwise bounded (if they are not, he should modify the substantive model by adding constraints on such variables, or treat such variables not as objectives but as additional constraints - which, however, has some disadvantages). When assessing utopia or nadir point components for other objectives, variables that should be kept close to a given reference level (and are not treated as constraints) must be considered as free, that is, should not be included into an achievement function used for the purpose of such an assessment.

Once the bounds on the ranges of reasonable objective values have been assessed, they can be used as scaling units for these objectives in further computations. Even before the preliminary analysis, it is useful to ask the user (preferably, the analyst that prepared the substantive model) to specify a priori reasonable scaling units for all outcome variables in the model; any optimization performed on a model with unreasonably scaled variables (say, cosmic distances computed in micrometers) can fail for numerical reasons. After the preliminary analysis, however, the user knows much more about reasonable ranges of outcomes and is well prepared to specify aspiration levels for each objective and thus for further interactive analysis.

4.3.3 Exploration of efficient alternatives and outcomes.

Observe that the decision support system and the analysts constructing the substantive model in the previous stages of the process performed the role of staff in an ideal organization while preparing for the boss or the decision maker not only a summary of the pertinent knowledge in the form of the substantive model but also specifying for his information reasonable bounds on attainable decision outcomes. This analogy can be pursued further - see Wierzbicki, 1982: an ideal staff should be able to respond to repeated instructions of the boss by specifying, upon his request and following his general instructions, what detailed decisions and course of action should be taken in order to attain efficient decisions that are best attuned to these general instructions. A human staff might become too tired if asked to perform such a job repeatedly with changing general instructions, but the boss would be certainly better informed if many such efficient alternatives were elaborated. A computerized decision support system does not tire, it can prepare new plans as many times as the decision maker wishes to change his general instructions and to see a new alternative. Thus, if the user specifies general instructions by setting aspiration levels for all previously defined objectives, the decision support system should test, whether these aspirations are attainable, and:

a) if the aspiration levels are not attainable, inform the user about this fact, but at the same time propose a detailed efficient decision that leads to outcomes in some sense uniformly close to the aspiration levels;

b) if the aspiration levels are attainable and it is possible to surpass them (they do not correspond to efficient outcomes), inform the user about this fact and at the same time propose a detailed efficient decision that leads to outcomes in some sense uniformly better than the aspiration levels;

c) if the aspiration levels, either by chance or as a result of interactive learning by the user, are just attainable and correspond to efficient outcomes, inform the decision maker about this fact and elaborate a detailed efficient decision that leads precisely to these efficient outcomes.

All these goals can be achieved by maximizing an order-consistent achievement function with the aspiration levels taken as the controlling parameters. The test of attainability is then the maximal value of this function, which is negative in case a), positive in case b), and equal zero in case c) - cf. Theorem 4.10.

The maximization of an achievement function is performed in the system by a special optimization algorithm, called here solver, that must be chosen depending on the nature of the substantive model as discussed above; thus, in DIDAS-type systems, robust and reliable solvers for linear and dynamic linear models as well as for nonlinear and dynamic nonlinear differentiable models have been partly developed but mostly especially adapted for the use in decision support.

The achievement function used in such decision support is generally an order-approximating one; but it can have one of several forms discussed in section 4.2. Depending on the form of the achievement function, an important issue might be this of relative scaling of all objectives that defines the sense of uniform approximation of aspirations by efficient outcomes in cases a), b). Some of the forms of achievement functions, such as (4.47), assume automatic scaling of objectives by the differences between aspiration and reservation levels; other forms might admit various scaling factors.

If only one aspiration point is used as the controlling parameter, which is the case in most current implementations of DIDAS-type systems, and an achievement function of the form (4.37) or (4.50) is used, which is typical for linear models, then three types of objective scaling by defining the coefficients are possible:

(i) a user-supplied scaling with $\alpha_i = 1/\Delta q_i$, where Δq_i are reasonable scaling units for objectives supplied by the user or the analyst constructing the substantive model;

(ii) a bound-implied scaling with $\alpha_i = 1/\Delta \tilde{q}_i$, where $\Delta \tilde{q}_i$ are the differences between the upper and lower bounds defined by (or slightly broader than) approximate utopia and nadir point components;

(iii) an aspiration-implied scaling with $\alpha_i = 1/(q_{i,max} - \bar{q}_i)$, where $q_{i,max} - \bar{q}_i$ is the difference between the corresponding component of the upper bound (equal to or somewhat higher than the approximate utopia point) and the currently specified aspiration level. Naturally, $q_{i,max} - \bar{q}_i$ must be positive in such a case and a special automatic modification of aspiration levels that might have been specified too high by the user is necessary in the system.

In the initial analysis phase, only the scaling of type i) is possible. In the phase of interactive exploration of efficient alternatives, the user can also choose between cases (ii) and (iii). Both the theoretical considerations on the controllability of parametric selection of efficient outcomes and practical experience suggest, however, that the use of scaling of the type (iii) is advisable. This scaling results in an achievement function of the form

(4.49) and is also a typical scaling applied for nonlinear differentiable models with an achievement function of the form (4.56).

Until now, we assumed that the decision support system responds with only one efficient solution to each specification of aspiration levels by the user. However, the user might wish to see several alternative efficient solutions as a response to his aspirations. A natural way of preparing such a response is to perturb the aspiration point along each objective (or trajectory of objectives) in the positive or negative direction and then to perform the maximization of the achievement function with such perturbed reference points - see Figure 4.14.

This requires naturally that the user gives a special command to obtain a scan of alternative responses, specifies the sign of perturbations and a coefficient that defines the magnitude of perturbations relative to the current scaling units of objectives, preferably using the bound-implied scaling or the aspiration-implied scaling.

After a number of experiments with changing and/or perturbing aspiration levels and observing the efficient solutions proposed by the system in response to these changes, the user learns typically enough about the shape of the efficient frontier. Sometimes, however, he might wish to learn more by looking at various cuts through the efficient frontier that can be computed by the system and displayed graphically.

If the number of objectives is small, the user might wish to look at cuts through the efficient frontier obtained by changing two specified objectives and keeping other objectives constant. Various procedures can be applied to generate such cuts. One of them is as follows: suppose the user specifies an efficient point $\hat{q}^0$ in the outcome space (obtained, say, as a system response in previous experiments with changing aspiration levels) and two components q_j, q_k of the objective vector that should be varied. Together with these data, the user should specify a coefficient σ that indicates which part of the reasonable range of variation $\Delta\tilde{q}_j$ and $\Delta\tilde{q}_k$ should be covered by the cut as well as a number N of additional points that should be used when constructing the cut. The system can construct now N reference points, say, by the formula

$$\bar{q}_i^{(n)} = \hat{q}_i^0 + \sigma(\hat{q}_i^0 - q_{i,min}) \pm \sigma(q_{i,max} - q_{i,min})(n-1)/(N-1) \qquad n = 1,\ldots N,\ i = j,k \tag{4.63}$$

(where the opposite signs are used for $i = j$ and $i = k$) and use these points consecutively as aspiration levels for q_j, q_k in a special achievement function that will be maximized in order to obtain the efficient points needed for constructing the cut. All other components of objective vector in this achievement function are treated either as constraints or, preferably, as stabilized objectives that should be kept closely to their set values $\hat{q}_i^0$, $i \neq j,k$, with sufficiently large weighting coefficients α_i. After obtaining N efficient points $\hat{q}^{(n)}$ through the maximization of such special achievement function, their components $\hat{q}_j^{(n)}$, $\hat{q}_k^{(n)}$, together with the point $\hat{q}_j^0$, $\hat{q}_k^0$ can be used to illustrate graphically a piecewise-linear approximation to the cut through the efficient frontier. If the user wishes to compare several cuts for varying anchor points $\hat{q}^0$, they can be accumulated in the system and displayed jointly, see Figure 4.15.

If the number of objectives is large, the same procedure can be generalized to produce a cut through the efficient frontier approximately along a ray starting from an efficient outcome point $\hat{q}^0$ in a direction $\delta\bar{q}$ specified by the user - see Korhonen (1985). The user should specify, additionally, whether he would like to see a cut only in the positive direction of the ray, or in both directions, a coefficient σ indicating the part of the range of change that the cut should cover and a number N of additional efficient points computed for the construction of this cut. In the case of constructing the cut in the positive direction of the ray, the additional reference points are constructed as:

$$\bar{q}^{(n)} = \hat{q}^0 + \sigma\tau\delta\bar{q}\, n/N \qquad n = 1,\ldots N \tag{4.64a}$$

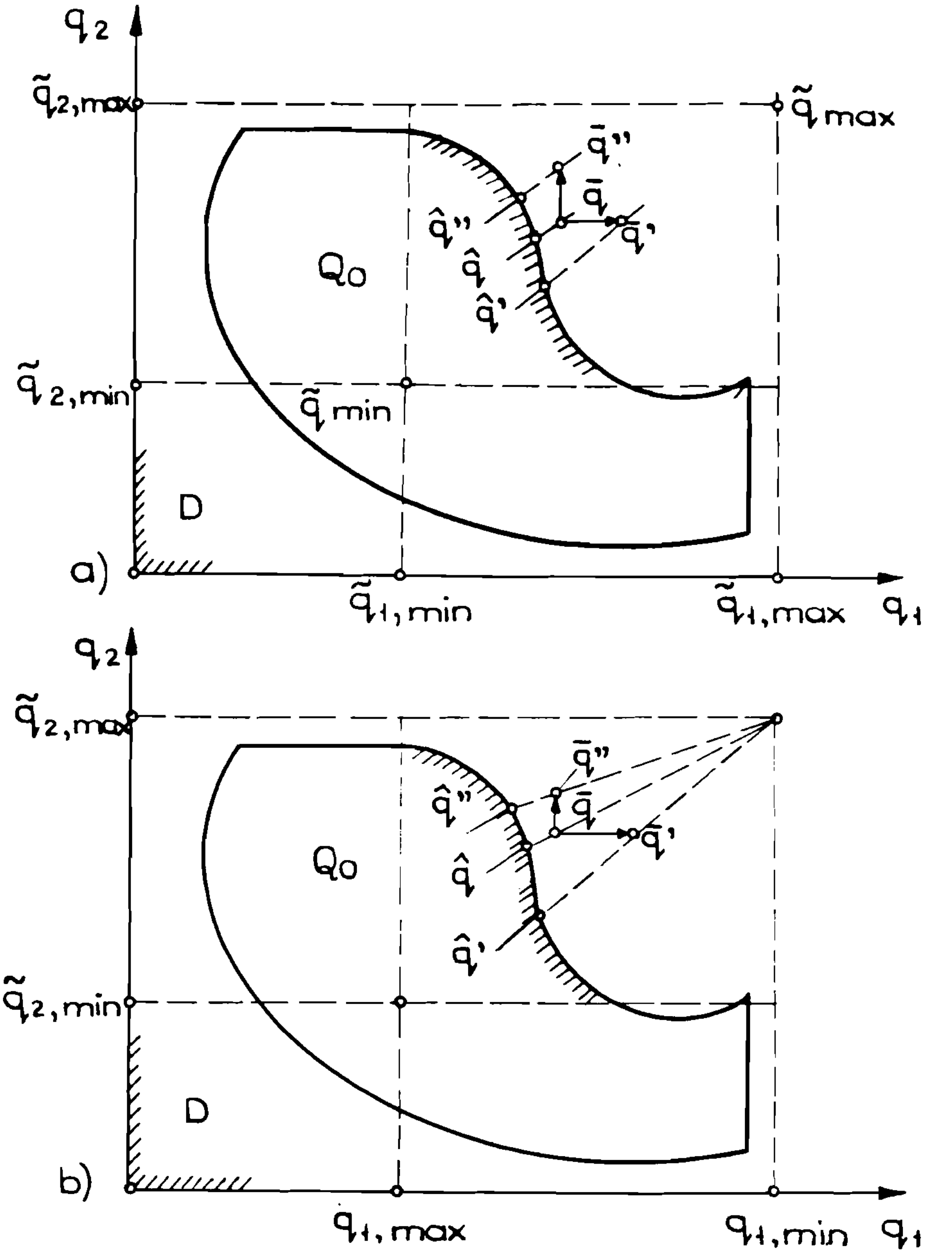

Figure 4.14. Scanning a region on the efficient frontier by perturbing the aspiration point: a) perturbation relative to bound-implied scaling; b) perturbation relative to aspiration-implied scaling.

where

$$r = \min_{1 \le i \le p} r_i; \quad r_i = \begin{cases} (\hat{q}_i^0 - q_{i,min})/\delta\bar{q}_i & \text{if } \delta\bar{q}_i < 0 \\ +\infty & \text{if } \delta\bar{q}_i = 0 \\ (q_{i,max} - \hat{q}_i^0)/\delta\bar{q}_i & \text{if } \delta\bar{q}_i > 0 \end{cases} \tag{4.64b}$$

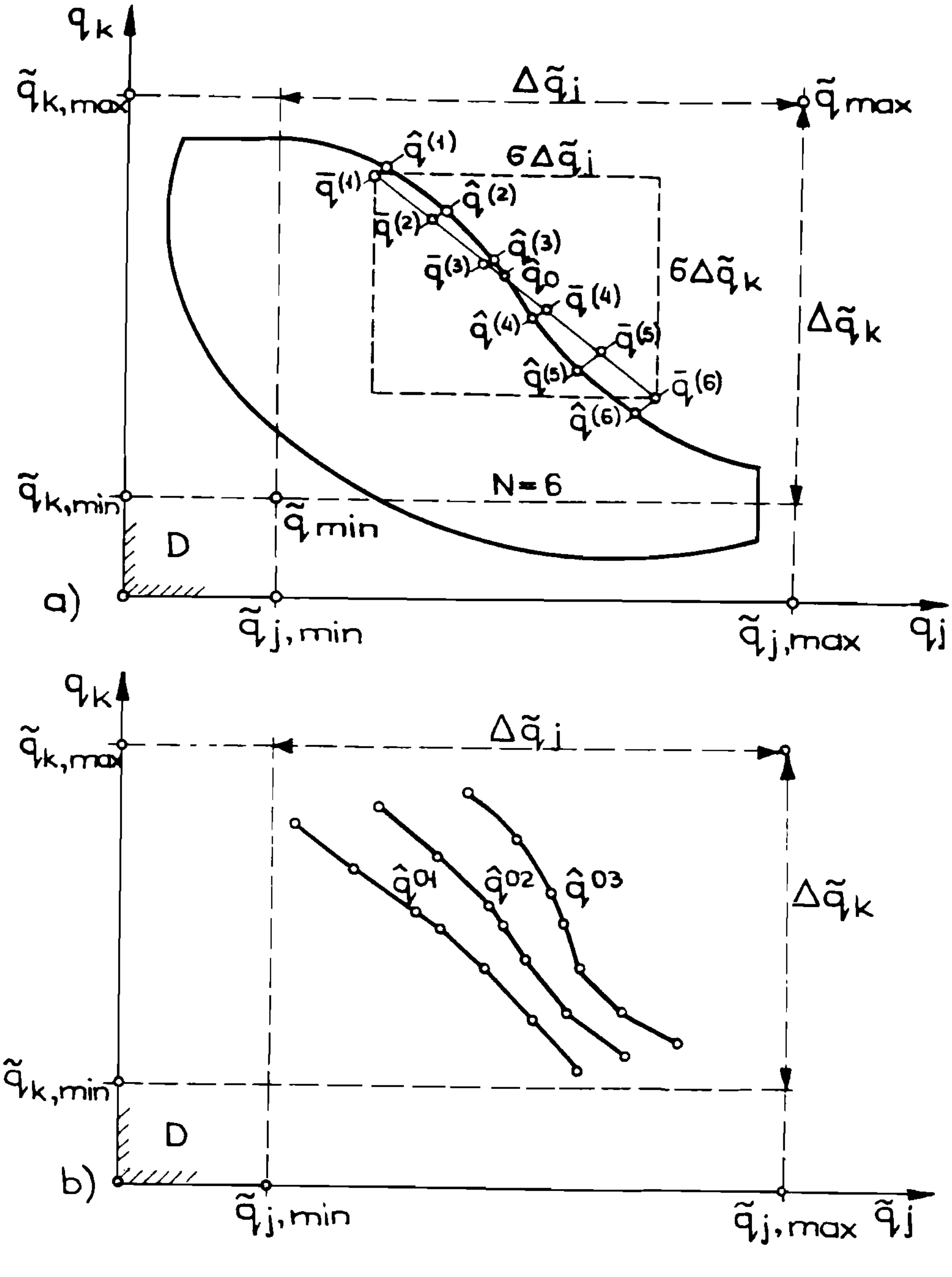

Figure 4.15. Two-variable cuts through the efficient frontier: a) the principle of constructing a cut; b) joint representation of several cuts.

while in the case of constructing the cut in both directions, they modify to:

$$\bar q^{(n)} = \hat q^0 - \sigma r' \delta \bar q + \sigma (r' + r'') \delta \bar q n / N \, , \quad n = 1, \ldots N \tag{4.65a}$$

where

$$r' = \min_{q \le i \le p} r_i' ; \quad r_i' = \begin{cases} (q_{i,max} - \hat q_i^0)/\delta \bar q_i & \text{if } \delta \bar q_i < 0 \\ +\infty & \text{if } \delta \bar q_i = 0 \\ (\hat q_i^0 - q_{i,min})/\delta \bar q_i & \text{if } \delta \bar q_i > 0 \end{cases} \tag{4.65b}$$

$$\tau'' = \min_{1 \le i \le p} \tau_i'' \; ; \; \tau_i'' = \begin{cases} (\hat{q}_i^0 - q_{i,min})/\delta\bar{q}_i & \text{if } \delta\bar{q}_i < 0 \\ +\infty & \text{if } \delta\bar{q}_i = 0 \\ (q_{i,max} - \hat{q}_i^0)/\delta\bar{q}_i & \text{if } \delta\bar{q}_i > 0 \end{cases} \tag{4.65c}$$

Since all objectives are changed when constructing the cut, there is no need to use a special achievement function; the system maximizes then the achievement function used originally in it for the interactive exploration of efficient alternatives, while changing the aspiration point to the consecutive reference points specified above. This results in N additional efficient points that are not precisely but only approximately located along the ray; nevertheless, the graphs of change of values of each objective in these efficient points might give valuable information to the user, see Figure 4.16.

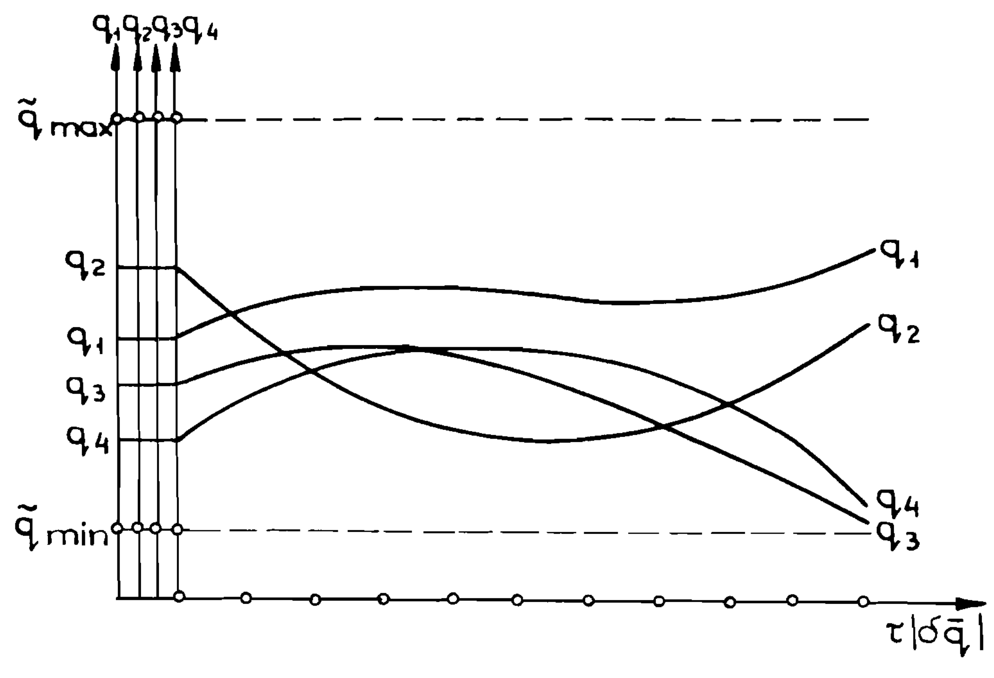

Figure 4.16. Cuts through the efficient frontier along a reference ray $\delta\bar{q}$ (cut in the positive direction of the ray).

Quite often, when the user has explored in detail alternative efficient decisions for a particular definition of multiobjective analysis problem, he comes to the conclusion that he should change the definition. A typical reason for such conclusion is the feeling that he would prefer decisions that are not efficient in the sense of current problem definition but would take into account also other aspects of the decision situation he has in mind. He must then return to the previous phase and try to change or add objectives selected among model output variables that were not used until now for this purpose. Sometimes, the substantive model is not rich enough and must be enhanced for this purpose, either through simple aggregation of variables that already are represented in the model, or by adding quite new variables and functional blocks to the model. In other cases, the user might forego the difficult task of changing the substantive model by correlating in his mind unformalized objectives with a variable represented in the model. In such cases, it is often useful to change the ordering in objective space: such specially interpreted objectives are better not maximized nor minimized, but stabilized, that is, kept close to aspiration levels specified by the user (as indicated before, the introduction of such an objective changes the sense of efficiency by including many decisions that would be considered inefficient if the objective would be maximized or minimized). Naturally, each time either the substantive model or the multiobjective problem formulation is changed, the phases of preliminary analysis and of exploration of efficient alternatives should be repeated, but the latter phase might take much less time if the user is already experienced in the analysis of the substantive model.

4.3.4 Learning and convergence.

The next question is: what should a user or a decision maker do, when he explored sufficiently many efficient alternatives and multiobjective problem formulations to feel that he has learned enough about the substantive problem and the particular decision situation he has in mind?

Many multiple criteria decision analysis systems actually start at this point by assuming that the problem is well defined and the purpose of the system is to support a convergent selection of one 'best' alternative decision, consistent with the preferences of the decision maker which should be somehow identified, typically through a sequence of pairwise comparison questions. This is a standard focus of analytical decision support and a broad array of approaches has been developed to address this issue, some exploiting the tools of utility theory, some taking into account various drawbacks of this theory and proposing new ways of stating pairwise comparison questions that would be psychologically more acceptable to the decision makers and avoid many traps of more classical approaches - see, for example, Larichev (1979), and Saaty (1982).

On the other hand, the investigations of the role of learning in decision processes - cf. Dreyfus (1985) - indicate that the decision maker might not need any further decision support once he has learned enough about the decision situation. An expert decision maker needs the decision support only to learn about some novel aspects of the decision situation; he knows that the models employed in a decision support system are not ideally representing the reality, that not all objectives are formalized; he has enough experience to select the actual decision once he understands sufficiently its possible implications. A novice decision maker also uses a decision support system more for the purpose of learning than actually selecting a decision. A decision maker who comes with a predetermined decision and wants to use the decision support only in order to find rational arguments for his particular choice, would like to learn how to rationalize his opinion not how to change it. Whatever is the particular case, users of decision support systems tend to stop using them when they have learned enough. This is also confirmed by experience in applying DIDAS-type systems to analyze some substantive problems: although some implementations of these systems were equipped with an option of supporting convergence to a "best" solution, this option was seldomly used and most current implementations are not equipped with such an option. Thus, the DIDAS-type systems are mainly aimed at supporting learning by the decision makers.

This does not mean, however, that options of converging to a "best" solution cannot be included in DIDAS-type systems; on the contrary, the framework of quasisatisficing decision making is especially designed to be relatively universal, to be applicable for supporting decisions in other frameworks. If the user wishes to obtain only satisficing decision, he can easily do so in the quasisatisficing framework by simply changing most or all of his objectives to the stabilized type that should not be maximized nor minimized but kept close to aspiration levels. If the user wishes to pursue the framework of goal- and program-oriented planning and management, he would use such type of objectives for the upper-level objective goals and maximize or minimize his lower-level objective means. If the user wishes to let his utility function be identified and modelled, such a model of user's preferences can be incorporated in a DIDAS-type system: an additional optimization solver would then change aspirations automatically until the corresponding efficient outcomes would maximize the utility function.

There might be also cases when a user is not satisfied with generating and learning about efficient alternatives with the support of a DIDAS-like system and wishes that the system would guide him in some easy manner to a "best" alternative. Two aspects of the user-friendliness of such a convergent process are of basic importance: the psychological easiness of questions and robustness of answers required from the user in the convergence process, and the freedom of the user to learn further during this process, to change his mind and be inconsistent but still arrive at some final solution. These aspects are, unfortunately, in conflict.

If we take the position that most important is the psychological easiness of questions put to the user in order to provide for most consistent answers, then the conclusion is - see Larichev (1979) - that the user should compare outcomes that differ in only one objective component at a time. In quasisatisficing framework, this would mean that aspiration levels for subsequent computations of efficient outcomes should differ in only one objective component at a time. Such a process of finding a "best" alternative can be added to DIDAS-type systems and experiments with such a process have been performed. However, such a process takes many iterations and easily tires the user; moreover, it is convergent only under the assumption that the user has already learned enough and does not change his preference structure during the process.

If we admit inconsistencies of the user and concentrate on questions that would guarantee most robust results, then a preferred form of such questions would be - see Saaty (1982) - pairwise comparisons of the importance of improvements of all objectives. When applying this methodology for DIDAS-like systems, one can obtain truly robust and reliable indications, how to change aspiration levels in order to move the efficient alternatives in a preferred direction. However, making pairwise comparisons of the importance of improvements of each objective can take much time and should be repeated at each iteration of the interactive process. On the other hand, such a process could be convergent even if we allow for learning and inconsistencies of the decision maker.

This possibility results from several theorems on the convergence of stochastic optimization algorithms for single- and multicriteria optimization, due mostly to the results of Ermolev and Gaivoronski (1982) and of Michalevich (1986); these theorems will not be quoted here, but only shortly summarized and interpreted.

Suppose the decision maker has a changing utility function which, however, converges to some final function. Consider the utility function in the space of controlling parameters or aspirations $\bar{q}$ as determined by the transformation $u'(\bar{q}) = u(\psi(\bar{q}))$ where $\psi(\bar{q})$ is the parametric representation of efficient solutions in dependence on aspirations $\bar{q}$; suppose this transformed utility function is subdifferentiable and quasiconvex, both at each particular time and in the limit. Suppose a direction of improvement of a contemporary utility function is elicited from the decision maker; this might be done either by generating random directions of decreasing length, showing to him outcomes related to aspirations perturbed along these directions and asking for pairwise comparisons of these outcomes, or by pairwise comparisons of the importance of improvements along each objective, or by any other means; it is important only that these directions should approximate stochastically a direction of a subgradient of the transformed utility function. He might also make random mistakes (with probability less than 0.5) at a pairwise comparison of any two outcomes or at the determination of a direction of improvement of utility. However, if a stochastic optimization algorithm is applied, with step-size coefficients that converge to zero sufficiently slowly (such that the sum of them converges to infinity and suitably defined differences between the contemporary and the final utility functions converge to zero faster than the step-size coefficients; on the other hand, the sum of squares of the step-size coefficients must remain finite), then the outcome of such process converges to an outcome that maximizes the final utility function.

This interesting result substantiates the use of rather simple algorithms for the convergence to some final 'best' alternative even if we allow the decision maker to make mistakes, be inconsistent, change his preferences - as long as he learns sufficiently fast and thus has a convergent utility. We do not need to identify his utility function, we can use the achievement functions that approximate roughly and not necessarily differentiably his utility, as long as we can elicit from him the directions of changing aspiration levels that approximate stochastically a subgradient of his changing utility. The crucial point, however, is to select a procedure for eliciting such directions from him that, on one hand, would give robust results and, on the other hand, would not tire the decision maker too much, since the convergence of stochastic optimization algorithms is known to be rather slow and to require many iterations.

Other approaches to providing for a good convergence procedure for DIDAS-type systems have been also investigated - see Kallio et al. (1980). However, the question of a selection of such a procedure is by no means settled, not because of theoretical difficulties but because of the unwillingness of decision makers to be involved in lengthy iterative procedures with tiring questions; this difficulty is, in a sense, common to all decision support procedures that aim at convergence to a "best" solution.

Appendix to Chapter 4.

Proof of Theorem 4.5. (N) Denote $B = (\tilde{q} - D) \cap (\hat{q} + \tilde{D})$. B is nonempty because $\tilde{q} \geq \hat{q}_{\max}$ and $\hat{q} \in Q_o \subset \hat{q}_{\max} - D$. Since B is convex and $w = \tilde{q} - \hat{q}$, hence $\tilde{q} - tw \in B$ for all $t \in [0;1)$. But $Q_o \cap B = \phi$, according to the definition of efficiency. Hence $t = 1$ is the lowest value of such t that $y - tw \in Q_o$.

(S) If $p = 2$ and Q_o is convex and compact, each ray:

$$T_w = \{q \in \hat{q}_{\max} - R^2_+ \colon q = \hat{q}_{\max} - tw,\ t \geq 0,\ w \in R^2_+\}$$

intersects Q_o. For, suppose otherwise. Then $dist(Q_o, T_w) > 0$ and T_w separates strongly Q_o either from the half-axis T_1 or T_2, where:

$$T_1 = \{q \in \hat{q}_{\max} - R^2_+ \colon q = \hat{q}_{\max} - te_1,\ t \geq 0,\ e_1 = (1,0)\}$$

$$T_2 = \{q \in \hat{q}_{\max} - R^2_+ \colon q = \hat{q}_{\max} - te_2,\ t \geq 0,\ e_2 = (0,1)\}$$

Then there is a positive distance from Q_o to either T_1 or T_2 which contradicts the assumption that $\hat{q}_{\max}$ is the utopia point.

If T_w intersects Q_o for all $w \in R^2_+$, then there exists the lowest value $\bar{t}$ of such $t \geq 0$ that

$$\hat{q}_{\max} - tw \in Q_o$$

Take $\hat{q} = \hat{q}_{\max} - \bar{t}w$ and determine the supporting hyperplane (here - a line) L to Q_o at $\hat{q}$. If L would intersect only one of T_1, T_2, then it would separate strongly the other one from Q_o, which would again contradict the assumption that $q_{\max}$ is the utopia point. Thus, L intersects both T_1, T_2 at some $q' = \hat{q}_{\max} - t'e_1$, $q'' = \hat{q}_{\max} - t''e_2$. If L does not contain either of T_1, T_2, then $t' > 0$, $t'' > 0$ and L has an orthogonal unit vector α with strictly positive components. Therefore $\hat{q}$ maximizes $\alpha_1 q_1 + \alpha_2 q_2$ over $q \in Q_o$, hence $\hat{q}$ is efficient according to Theorem 4.1. If $T_1 \subset L$ or $T_2 \subset L$, then $\hat{q} \in T_1 \cap Q_o$ or $\hat{q} \in T_2 \cap Q_o$; since

$\hat{q}$ is the closest point in these sets to $\hat{q}_{\max}$, it is efficient (other possible points in these sets being weakly efficient).

If $p \geq 3$, even if Q_o is convex and compact, there might be rays T_w that do not intersect Q_o, since for a separation of Q_o from a half-axis we need then a hyperplane, not a ray. To illustrate this, consider $p = 3$ and a ball Q_o in a corner of a room; there are such rays starting from this corner that do not intersect the ball and even such rays that touch the ball at nonefficient points.

Proof of Theorem 4.7. (S) If $\alpha \in A_s$, then function (4.33) is strongly monotone and - see Theorem 4.9. - its maximal arguments are efficient elements of the set $Q_o \cap Q(\bar{q})$, hence - as it is easy to check - also efficient elements of the set Q_o.

(N) Since $Q(\bar{q}) = \bar{q} + D$, hence $Q_o \cap Q(\bar{q}) = \{\bar{q}\}$ according to the definition of (strict) efficiency. On this singleton set, maximum of (4.33) is trivially attained - however, each perturbation of Q_o might make the intersection $Q_o \cap Q(\bar{q})$ empty.

Proof of Theorem 4.8. If $\alpha \in A_s$ and $q_i < \tilde{q}_i$ for all $i = 1,..p$, then

$$s(q,\tilde{q}) = \{\sum_{i=1}^{p} \alpha_i(\tilde{q}_i - q_i)^k\}^{1/k} \tag{4.A.1}$$

and this function is strongly (negatively) monotone; each minimum $\hat{q}$ of such function over $q \in Q_o$ is efficient (Theorem 4.9). However, a normal vector to a supporting hyperplane of Q_o at $\hat{q}$ is is the minus gradient of $s(q,\tilde{q})$ with respect to q at $\hat{q}$. The components of this gradient are:

$$\nabla_{q_i} s(\hat{q},\tilde{q}) = \alpha_i\{\sum_{j=1}^{p} \alpha_j(\tilde{q}_j - \hat{q}_j)^k\}^{1/k}(\tilde{q}_i - \hat{q}_i)^{k-1} \tag{4.A.2}$$

Since these components are all strictly positive and bounded, their ratios that determine marginal substitution rates at $\hat{q}$ are bounded. Therefore, $\hat{q}$ is properly efficient.

Suppose a properly efficient $\hat{q}$ is given. Determine a normal vector to a supporting hyperplane to Q_o at $\hat{q}$; its components are all nonzero and positive, denoted here by ξ_i. Normalize this vector in the dual weighted norm $l_{k''}$ where $k'' = k/(k - 1)$ by taking:

$$\varsigma_i = \xi_i/\{\sum_{j=1}^{p} \xi_j(\xi_j/\alpha_j)^{k''}\}^{1/k''} \tag{4.A.3}$$

Assume any value β of the weighted l_k norm of $\tilde{q} - \hat{q}$ and determine $\tilde{q}$ by:

$$\tilde{q}_i = \hat{q}_i + \beta(\varsigma_i/\alpha_i)^{1/(k-1)} \tag{4.A.4}$$

It is easy to check that for such $\tilde{q}$, with any assumed β, the components of the gradient (4.A.2) are equal ς_i. Since Q_o is convex, this means that $\hat{q}$ minimizes $s(q,\tilde{q})$ over $q \in Q_o$.

Proof of Theorem 4.9, the case of proper efficiency with bound. Suppose the thesis does not hold: let $r(q)$ be ϵ-strongly monotone and $\hat{q}$ maximize $r(q)$ over $q \in Q_o$ but $\hat{q}$ be not D_ϵ-optimal. Than there exists such $q' \in Q_o$ that $q' \in \hat{q} + \tilde{D}_\epsilon$; but, at the same time, $r(\hat{q}) \geq r(q')$ which contradicts the assumption that $r(q)$ is ϵ-strongly monotone.

Proof of Theorem 4.10, necessary condition of proper efficiency with bound. Suppose $\hat{q}$ is properly efficient with bound, $Q_o \cap (\hat{q} + \tilde{D}_\epsilon) = \emptyset$, but the thesis does not hold, $\hat{q}$ does not maximize $s(q,\bar{q})$ over $q \in Q_o$ with $\bar{q} = \hat{q}$. Then there exists such $q' \in Q_o$ that $s(q',\hat{q}) > s(\hat{q},\bar{q}) = s(\hat{q},\hat{q}) = 0$ and $q' \neq \hat{q}$. Thus $q' \in \hat{q} + D_\epsilon$ according to the property of order approximation (4.31); since $q' \neq \hat{q}$, $q' \in \hat{q} + \tilde{D}_\epsilon$. Therefore, $q' \in Q_o \cap (\hat{q} + \tilde{D}_\epsilon) \neq \emptyset$ which contradicts the assumption that $\hat{q}$ is properly efficient with bound ϵ.

Proof of Theorem 4.11. By rescaling the coordinates for both q and $\bar{q}$ equally (which does not change Lipschitz constants for any dependence between them) we can assume $\alpha_i = 1/p$ or even $\alpha_i = 1$ (which in turn does not change the maxima of $s(q,\bar{q})$). Observe that with $s(q,\bar{q}) = \min_{1 \le i \le p} (q_i - \bar{q}_i)$ and $e = (1,..1,..1)$, all points

$$\bar{q} \in A_{\hat{q}} = \{\hat{q} \in R^p : q = \hat{q} + te, t \in (-\infty,\infty)\}$$

lead to the same

$$\hat{q} = \operatorname*{argmax}_{q \in Q_o} s(q,\bar{q})$$

Let $\|.\|$ denote the Chebyshev norm, take any

$$\hat{q}', \hat{q}'' \in \hat{Q}_{op}, \; \hat{q}' \neq \hat{q}''$$

and select $\bar{q}' = \hat{q}' \in A_{\hat{q}'}$ along with such $\bar{q}'' \in A_{\hat{q}''}$ that has the minimal distance from $\bar{q}'$; thus, the distance of any two points in $A_{\hat{q}'}$, $A_{\hat{q}''}$ is not smaller than $\|\bar{q}'' - \bar{q}'\|$. Suppose, without loss of generality, that $\hat{q}_i'' \ge \hat{q}_1'$, and determine $\tilde{q}'' \in A_{\hat{q}''}$ such that $\tilde{q}_1'' = \bar{q}_1' = \hat{q}_1'$. Since

$$\tilde{q}_i'' - \bar{q}_i' = \bar{q}_i'' - \bar{q}_i' - \bar{q}_1'' + \bar{q}_1'$$

we have

$$\|\tilde{q}'' - \bar{q}'\| \le 2\,\|\bar{q}'' - \bar{q}'\|$$

for $\bar{q}'', \bar{q}'$ as defined above as well as for any $\bar{q}'' \in A_{\hat{q}''}$ and any $\bar{q}' \in A_{\hat{q}'}$.

Because $\hat{q}_1'' \ge \hat{q}_1'$, $\tilde{q}'' = \hat{q}'' + t_1 e$ with $t_1 \le 0$. Because $\hat{q}', \hat{q}''$ are efficient, we have at least one $i = 1$ such that $\hat{q}_i'' \le \hat{q}_i'$. Take j such that

$$\hat{q}_j'' - \hat{q}_j' = \min_{1 \le i \le p} (\hat{q}_i'' - \hat{q}_i')$$

Select the point $\tilde{\tilde{q}}'' \in A_{\hat{q}''}$ for which $\tilde{\tilde{q}}_j'' = \tilde{q}_j'$. This point has coordinates

$$\tilde{\tilde{q}}_i'' - \bar{q}_i' = \tilde{q}_i'' - \bar{q}_i' - \tilde{q}_j'' + \bar{q}_j'$$

hence,

$$\|\tilde{\tilde{q}}'' - \bar{q}'\| \le 2\,\|\tilde{q}'' - \bar{q}'\|$$

However, we can represent $\tilde{\tilde{q}}''$ also as $\tilde{\tilde{q}}'' = \hat{q}'' + t_2 e$, where $t_2 \le 0$ since $\hat{q}''^j \ge \hat{q}'^j$. Hence, there exists $\beta \in [0;1]$ such that $\hat{q}'' = \beta\tilde{q}'' + (1-\beta)\tilde{\tilde{q}}''$; since $\hat{q}' = \bar{q}'$, this implies that:

$$\|\hat{q}'' - \hat{q}'\| \le \max(\|\tilde{q}'' - \bar{q}'\|, \; \|\tilde{\tilde{q}}' - \bar{q}'\|) \le 2\|\tilde{q}'' - \bar{q}'\| \le 4\|\bar{q}'' - \bar{q}'\| \tag{4.A.5}$$

for any $\bar{q}'' \in A_{\hat{q}''}$ and any $\bar{q}' \in A_{\hat{q}'}$.

This proves that

$$\|\psi(\bar{q}'') - \psi(\bar{q}')\| \le 4\|\bar{q}'' - \bar{q}'\|$$

for $\bar{q}'', \bar{q}'$ belonging to some lines $A_{\hat{q}''}$, $A_{\hat{q}'}$ passing through points $\hat{q}'', \hat{q}' \in \hat{Q}_{op}$. If a point $\bar{q}$ does not belong to any such line, we translate it to $A_{\hat{q}}$ passing through

$$\hat{q} = \operatorname*{argmax}_{q \in Q_o} s(q,\bar{q})$$

by this translation the Lipschitz inequality will be only strengthened. Observe that if $p = 2$, the Lipschitz constant can be tightened from 4 to 2. The inequality (4.A.5) depends on the use of Chebyshev norm; since all norms are topologically equivalent in R^p, there exist also Lipschitz constants for any other norm.

Proof of Theorem 4.12. (a) Let $\hat{q}$ be efficient; since Q_o is of arbitrary form, any point $q \notin \hat{q} + \tilde{D}$ might belong to Q_o. Since it is assumed that $\hat{q}$ maximizes $s(q,\hat{\alpha})$ over $q \in Q_0 \cap Q(\hat{\alpha})$ for some $\hat{\alpha} \in A_n$, hence $s(q,\hat{\alpha})$ can further increase in q only for points in $\hat{q} + \tilde{D}$; being continuous, it can further increase only for points in the open set $\hat{q} + \text{int } D$. Take this property for all $\hat{q} \in \hat{Q}_0$ and all corresponding $\hat{\alpha} \in A_n$:

$$\{q \in Q(\hat{\alpha}): s(q,\hat{\alpha}) > s(\hat{q},\hat{\alpha})\} \subset (\hat{q} + \text{int } D) \cap Q(\hat{\alpha}), \text{ for all } \hat{\alpha} \in A_n \qquad (4.A.6)$$

It is also assumed that each maximal point of the function $s(q,\alpha)$ over $q \in Q_o \cap Q(\alpha)$, for any $\alpha \in A_s$, is weakly efficient; if a point $q \in Q_o$ is not weakly efficient, $(q + \text{int } D) \cap Q_0 \neq \emptyset$, then it cannot be a maximal point and the function $s(q,\alpha)$ must have the property that it further increases at any point in $q + \text{int } D$. Since Q_0 is of arbitrary form and its weakly efficient points cannot be distinguished from other points before maximizing $s(q,\alpha)$, this property must apply also for $\hat{q}$ that are weakly efficient:

$$\{q \in Q(\alpha): s(q,\alpha) > s(\hat{q},\alpha)\} \supset (\hat{q} + \text{int } D) \cap Q(\alpha), \text{ for all } \alpha \in A_s \qquad (4.A.7)$$

Jointly:

$$\{q \in Q(\alpha): s(q,\alpha) > s(\hat{q},\alpha)\} = (\hat{q} + \text{int } D) \cap Q(\alpha), \text{ for all } \alpha \in A_s \cap A_n \qquad (4.A.8)$$

If $U(\hat{q}) \subset Q(\alpha)$, then the property of local order-representation (4.41) follows from (4.A.8).

(b) If the function $s(q,\alpha)$ has the property (4.41) and is continuous, then:

$$\{q \in U(\hat{q}): s(q,\alpha) \geq s(\hat{q},\alpha)\} \supset (\hat{q} + D) \cap U(\hat{q}), \text{ for all } \alpha \in A_s \cap A_n \qquad (4.A.9)$$

Together with (4.41) and for Q_o of arbitrary form this implies, however, that if a point $\hat{q}$ maximizes $s(q,\alpha)$ over $q \in Q_o$, then this maximum is not necessarily unique: any point $\hat{q}' \in ((\hat{q} + D) \setminus (\hat{q} + \text{int } D)) \cap U(\hat{q})$ might also maximize $s(q,\alpha)$. Suppose Q_0 is such that, beside $\hat{q}$, there is only one such additional maximal point $\hat{q}' \neq \hat{q}$. Even if $s(q,\alpha)$ is strictly monotone as a function of q, which implies that both $\hat{q}',\hat{q}$ are weakly efficient, they cannot both be efficient since $\hat{q}' \in (\hat{q} + \tilde{D}) \cap Q_0$. Hence, the function $s(q,\alpha)$ cannot completely characterize efficient solutions; besides, (4.A.9) implies that such a function cannot be strongly monotone.

Proof of Theorem 4.13. For this proof, we need first a lemma:

Lemma. If $h_1:[t_1; t_2] \longrightarrow R^1$ and $h_2:[t_1; t_2] \longrightarrow R^1$ are strictly monotonically increasing functions of t, such that $h_2(t) \geq h_1(t)$ for all $t \in [t_1; t_2]$, and if $\lambda:[t_1; t_2] \longrightarrow R^1$ is also a strictly monotonically increasing function of t such that $\lambda(t) \in [0;1]$ for all $t \in [t_1;t_2]$, then $h(t) = \lambda(t)h_2(t) + (1 - \lambda(t))h_1(t)$ is also a strictly monotonically increasing function of $t \in [t_1;t_2]$.

Proof of the lemma. Let $t' < t'' \in (t_1;t_2)$; since $\lambda(t)$ is strictly increasing $0 < \lambda(t') < \lambda(t'') < 1$. The difference $h(t'') - h(t')$ can be written as:

$$\begin{aligned} h(t'') - h(t') = {} & \lambda(t')(h_2(t'') - h_2(t')) \\ & + (1 - \lambda(t'))(h_1(t'') - h_1(t')) \\ & + (\lambda(t'') - \lambda(t'))(h_2(t'') - h_1(t'')) \end{aligned} \qquad (4.A.10)$$

where the two first terms are strictly positive and the third is nonnegative. Hence $h(t'') > h(t')$.

For the proof of the theorem, denote:

$$\| q'' - q \|_{(\bar{q}',\bar{q}'')} = \{(1/p)\sum_{i=1}^{p}((q_i'' - q_i')/(\bar{q}_i'' - \bar{q}_i'))^k\}^{1/k} \tag{4.A.11}$$

Hence:

$$\begin{aligned} \tau(q,q',q'') &= \| \tilde{q}_{\max} - q \|_{(\bar{q}'',\tilde{q}_{\max})} \\ \theta(q,q',q'') &= \| \tilde{q}''_{\max} - q \|_{(\bar{q}',\bar{q}'')} \\ t(q,q',q'') &= \| \tilde{q}'_{\max} - q \|_{(\tilde{q}_{\min},\bar{q}')} \end{aligned} \tag{4.A.12}$$

Since $Q_\tau = \{q \in R^p\colon \tau(q,\bar{q}',\bar{q}'') < 1\}$ and $Q_t = \{q \in R^p\colon t(q,\bar{q}',\bar{q}'') > \rho' - 1\}$ are disjoint, we can consider three cases: A, when $q \notin Q_t \cap Q_\tau$, B, when $q \in Q_\tau$, and C, when $q \in Q_t$.

In the case A we have:

$$s(q,\bar{q}',\bar{q}'') = \rho'' - \| \tilde{q}''_{\max} - q \|_{(\bar{q}',\bar{q}'')} \tag{4.A.13}$$

Since $q \leq \tilde{q}_{\max} \leq \tilde{q}''_{\max}$ and $k < \infty$, the norm in (4.A.13) is a stongly decreasing function of q and (4.A.13) is a strongly increasing function.

In the cases B, C, we consider the function s on a ray $q + \omega(q' - q)$ where $q' - q \in R^p_+$ and $\omega > 0$, for ω such that $q + \omega(q' - q) \leq q_{\max}$. In case B, the norm

$$\tau = \| \tilde{q}_{\max} - q - \omega(q' - q) \|_{(\bar{q}'',\tilde{q}_{\max})}$$

is a stricly decreasing function of ω, hence we can introduce a one-to-one map $\omega(\tau)$ and consider s as a function of τ; for the proof of the theorem in this case, it is sufficient to show that this function strictly decreases with τ. This function has the form:

$$\begin{aligned} s(q + \omega(\tau)(q' - q), \bar{q}',\bar{q}'') &= (1 + \beta - \beta\delta\tau)(1 - \xi(\tau)) + \\ &+ (\rho'' - \| \tilde{q}''_{\max} - q - \omega(\tau)(q' - q) \|_{(\bar{q}'',\tilde{q}_{\max})})\xi(\tau) \end{aligned} \tag{4.A.14}$$

Since the norm in (4.A.14) strictly decreases with $\omega(\tau)$ and thus strictly increases with τ, hence we can apply the above lemma to the function $h(\tau) = - s$ (which should be strictly increasing with τ, if s should be strictly decreasing). Take:

$$\begin{aligned} h_2(\tau) &= \| q''_{\max} - q - \omega(\tau)(q' - q) \|_{(\bar{q}'',\tilde{q}_{\max})} - \rho'' \\ h_1(\tau) &= \beta\delta\tau - \beta - 1 \ ; \ \lambda(\tau) = \xi(\tau) \ ; \ \tau \in [0;1] \end{aligned} \tag{4.A.15}$$

It remains to show that $h_2(\tau) \geq h_1(\tau)$ for $\tau \in [0;1]$. We have:

$$h_1(\tau) \leq \beta\delta - \beta - 1 \leq - 1 - \| \tilde{q}_{\max} - \bar{q}'' \|_{(\bar{q}',\bar{q}'')} \tag{4.A.16}$$

where the last inequality follows from the assumption (4.60a) in the theorem. On the other hand, since

$$\rho'' = \| \tilde{q}''_{\max} - \bar{q}'' \|_{(\bar{q}',\bar{q}'')} + 1$$

and

$$q + \omega(\tau)(\bar{q}' - q) \leq \tilde{q}_{\max} \leq \tilde{q}''_{\max}$$

we have

$$h_2(\tau) \geq \| \tilde{q}''_{\max} - \tilde{q}_{\max} \|_{(\bar{q}',\bar{q}'')} - \| \tilde{q}''_{\max} - \bar{q}'' \|_{(\bar{q}',\bar{q}'')} - 1 \tag{4.A.17}$$

which, from the triangle inequality, implies $h_2(\tau) \geq h_1(\tau)$. Thus, the lemma can be applied which finishes the proof in case B.

In case C, we consider the norm

$$t = \| \tilde{q}'_{\max} - q - \omega(q' - q'') \|_{(\tilde{q}_{\min}, \bar{q}')}$$

which is a strictly decreasing function of ω. Hence, we introduce a one-to-one map $\omega(t)$ and consider s as a function of t; for the proof of the theorem in this case, we must show that this function is strictly decreasing with t. This function has the form:

$$s(q + \omega(t)(q' - q), \bar{q}', \bar{q}'') = \gamma(-1 + \eta(\rho' - t))(1 - \varphi(t)) \qquad (4.A.18)$$
$$+ \{\rho'' - \| q_{\max}'' - q - \omega(t)(q' - q)\|_{(\bar{q}'', \tilde{q}_{\max})}\}\varphi(t)$$

Again, we shall apply the above lemma for the function $h(t) = -s$, whereas:

$$h_1(t) = \| \tilde{q}''^{\max} - q - \omega(t)(\bar{q}' - q)\|_{(\bar{q}'', \tilde{q}_{\max})} - \rho'' \qquad (4.A.19)$$
$$h_2(t) = \gamma(-1 + \eta(\rho' - t)); \; \lambda(t) = 1 - \varphi(t); \; t \in [\rho' - 1; \rho']$$

We shall show that $h_2(t) \geq h_1(t)$ for $t \in [\rho' - 1; \rho']$. Since $q + \omega(t)(\bar{q}' - q) \geq \tilde{q}_{\min}$ and $\rho'' = \| \tilde{q}_{\max} - \bar{q}' \|_{(\bar{q}', \bar{q}'')}$, we have:

$$h_1(t) \leq \| \tilde{q}''_{\max} - \tilde{q}_{\min} \|_{(\bar{q}', \bar{q}'')} - \| \tilde{q}''_{\max} - \bar{q}' \|_{(\bar{q}', \bar{q}'')} \leq$$
$$\leq \| \bar{q}' - \tilde{q}_{\min} \|_{(\bar{q}', \bar{q}'')} \qquad (4.A.20)$$

On the other hand:

$$h_2(t) = \gamma + \eta(t - \rho') \geq \gamma(1 - \eta) \geq \| \bar{q}' - \tilde{q}_{\min} \|_{(\bar{q}', \bar{q}'')} \qquad (4.A.21)$$

where the last inequality follows from the assumption (4.60b) of the theorem. Hence, $h_2(t) \geq h_1(t)$ and the lemma can be applied, which finishes the proof for the case C.

References to Chapter 4

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