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Foreword

Many problems arising in optimization and optimal control may be reduced to the following nonlinear mathematical programming problem:

$$\text{minimize } \{J(u) : u \in \mathcal{U} \ , \ \ G(u) \in K \}$$

where $\mathcal{U}$ is a metric space, K is a subset of a Banach space X and $J\colon \mathcal{U} \to \mathbf{R} \bigcup \{+\infty\}$, $G\colon \mathcal{U} \to X$ are given functions. The author proves a general Kuhn-Tucker type necessary condition for minima. This general multiplier rule allows to prove, in particular, the maximum principle for a semilinear problem with nonconvex end points constraints and necessary conditions for optimality for a nonconvex ill-posed problem.

The results were exposed during the Comcon Workshop (Montpelier, 1988) on the optimization of flexible structures.

Alexander B. Kurzhanski
Chairman
System and Decision Sciences Program

A GENERAL MULTIPLIER RULE FOR INFINITE DIMENSIONAL OPTIMIZATION PROBLEMS WITH CONSTRAINTS

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1. Introduction

Many problems arising in optimization and optimal control may be reduced to the following nonlinear mathematical programming problem:

$$\text{minimize } \{J(u)\colon u \in \mathcal{U},\ G(u) \in K\} \tag{1.1}$$

where $\mathcal{U}$ is a metric space, K is a subset of a Banach space X and $J\colon \mathcal{U} \to R \bigcup \{+\infty\}$, $G\colon \mathcal{U} \to X$ are given functions.

A vast literature exists on the necessary conditions associated with (1.1) in some concrete cases. Usually the methods rely either on subdifferential calculus of convex analysis (see for example ([18] [5]) or on penalization technique ([17], [4]). Both approaches are somewhat restrictive: the first applies only to convex problems (in particular K has to be convex), the second one applies only to these problems which can be penalized in reasonable way (which in practice yields many assumptions on the set K and, often, the convexity of K).

When K is just a closed set, one is led to apply a different technique. In [9] Fattorini studied some optimal control problems using Ekeland's variational principle [7], [8]. Although this approach is well-known in finite dimensional optimization ([6], [8]), its application to infinite dimensional problems is not immediate.

In Fattorini and Frankowska [10] results of [9] were extended to a very general class of constraints K. Namely K has to be a closed subset of a Hilbert space X satisfying some "variational" assumptions.

We observe here that the very same ideas allow to go beyond Hilbert spaces and to prove a much more general multiplier rule making the class of applications broader. The main aim of this paper is to provide such general rule and to give some new applications. The multiplier rule is proved in Section 2. The application to the maximum principle is given in Section 3. Section 4 is devoted to optimal control of an ill posed semi-linear elliptic system with nonconvex constraints.

2. Multiplier rule for a general optimization problem with constraints

We study here the problem with constraints.

$$\text{minimize} \ \{J(u) : u \in \mathcal{U}, \ G(u) \in K\} \tag{2.1}$$

where

$\mathcal{U}$ is a complete metric space with the metric d

G is a continuos function from $\mathcal{U}$ to a Banach space X

K is a closed subset of X

J is a lower semicontinuous function from $\mathcal{U}$ to $\mathbf{R} \cup \{+\infty\}$

Throughout this section we denote by $||\cdot||$ the norm of X and we assume that it is Gâteaux differentiable away from zero, that is for all $x \in X$, $x \neq 0$ there exists $p_x \in X^*$ such that for all $u \in X$

$$\lim_{h \to 0+} \frac{||x+hu|| - ||x||}{h} = \lim_{h \to 0+, u_h \to u} \frac{||x+hu_h|| - ||x||}{h} = \langle p_x, u \rangle$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing on $X^* \times X$.

We recall first the definitions of Kuratowski's *liminf* and *limsup* of a family of subsets $\{A_\tau\}_{\tau \in T}$ of a Banach space X, where T is a metric space.

$$\mathop{liminf}_{\tau \to \tau_0} A_\tau = \{v \in X : \lim_{\tau \to \tau_0} dist\ (v, A_\tau) = 0\}$$

$$\mathop{limsup}_{\tau \to \tau_0} A_\tau = \{v \in X : \mathop{liminf}_{\tau \to \tau_0} dist\ (v, A_\tau) = 0\}$$

For a point $x \in K$ we denote by $x' \to_K x$ the convergence to x in K.

DEFINITION 2.1. LET $x \in K$.

i) CONTINGENT CONE TO K AT x IS DEFINED BY

$$T_K(x) = \limsup_{h \to 0+} \frac{K-x}{h}$$

ii) TANGENT CONE (OF CLARKE) TO K AT x IS DEFINED BY

$$C_K(x) = \liminf_{\substack{x' \to_K x \\ h \to 0+}} \frac{K-x'}{h}$$

In the other words $v \in T_K(x)$ if there exist sequences $h_i \to 0+$, $v_i \to v$ such that $x+h_iv_i \in K$. Similarly $v \in C_K(x)$ if for all sequences $h_i \to 0+$, $x_i \to_K x$ there exists a sequence $v_i \to v$ such that $x_i+h_iv_i \in K$. It is well known that $C_K(x) \subset T_K(x)$ are closed cones, $C_K(x)$ is convex and when $dim X < \infty$

$$C_K(x) = \liminf_{x' \to_K x} T_K(x')$$

(see [1], [6]). When K is convex we have

$$T_K(x) = C_K(x) = \bigcup_{\lambda \geq 0} \lambda\ (K-x) = \liminf_{x' \to_K x} T_K(x')$$

When K is closed we always have

$$\liminf_{x' \to_K x} T_K(x') \subset C_K(x)$$

(see [21]).

Computation of elements of contingent cone is simpler than that of tangent vectors in the sense of Clarke. In many concrete cases computation of $C_K(x)$ may be a very difficult task. This is why we formulate here results using both notions of tangent cones.

For all $u \in \mathcal{U}$, $h>0$, let $B_h(u)$ denote the closed ball in $\mathcal{U}$ of center u and radius h.

DEFINITION 2.2. CONSIDER A FUNCTION F FROM $\mathcal{U}$ TO A BANACH SPACE Y AND A POINT $u \in \mathcal{U}$.

i) THE (FIRST ORDER) CONTINGENT VARIATION OF F AT u IS THE SUBSET OF Y DEFINED BY

$$\mathrm{V}_F(u) = \limsup_{h \to 0+} \frac{F(B_h(u')) - F(u')}{h}$$

ii) THE (FIRST ORDER) VARIATION OF F AT u IS THE SUBSET OF Y DEFINED BY

$$\mathcal{V}_F(u) = \liminf_{\substack{u' \to u \\ h \to 0+}} \frac{F(B_h(u')) - F(u')}{h}$$

IN THE OTHER WORDS $v \in V_F(u)$ IF THERE EXIST SEQUENCES $h_i \to 0+$, $v_i \to v$ SUCH THAT

$$F(u) + h_i v_i \in F(B_{h_i}(u))$$

AND $v \in \mathcal{V}_F(u)$ IF FOR ALL SEQUENCES $h_i \to 0+$, $u_i \to u$ THERE EXISTS A SEQUENCE $v_i \to v$ SUCH THAT

$$F(u_i) + h_i v_i \in F(B_{h_i}(u_i))$$

It is clear that $V_F(u)$ and $\mathcal{V}_F(u)$ are closed starshaped at zero sets and $\mathcal{V}_F(u) \subset V_F(u)$. It was proved in [13] that $\mathcal{V}_F(u)$ is convex.

Recall that the negative polar of a set $P \subset X$ is defined by

$$P^- = \{ \xi \in X^* : \forall p \in P, \ < \xi ,p> \ \le 0\}$$

and the normal cone (of Clarke) to K AT x is defined by

$$N_K(x) = C_K(x)^-$$

We assume that (2.1) is feasible, i.e., for some $u \in U$ satisfying $G(u) \in K$ we have $J(u) \neq +\infty$

THEOREM 2.3. LET u_0 BE A SOLUTION OF PROBLEM (2.1). ASSUME THAT FOR SOME $\rho > 0$, $\gamma > 0$ AND A COMPACT $Q \subset X$ THE FOLLOWING HOLDS TRUE: FOR ALL $x \in K$ NEAR $G(u_0)$, AND $u \in \mathcal{U}$ NEAR u_0

$$\rho B \subset cl\,(\pi_X\,(\overline{co}\,\mathcal{V}_{(J,G)}(u) \cap [-\gamma\,,\gamma] \times X) - \overline{co}(T_K(x) \cap \gamma B) + Q) \tag{2.2}$$

WHERE π_X DENOTES THE PROJECTION OF $\mathbf{R} \times X$ ON X.

THEN THERE EXIST

$$\lambda \ \ge 0, \ \xi \ \in \ (\liminf_{y \to_K G(u_0)} T_K(y))^- \ , \ (\lambda, \xi) \neq 0 \tag{2.3}$$

SUCH THAT

$$\forall \ (j,g) \in \liminf_{u \to u_0} \overline{co}\,\mathcal{V}_{(J,G)}(u), \ \ \lambda\, j + < \xi\,, g > \ \ge 0 \tag{2.4}$$

$$\forall \ M > 0, \quad \xi \in (\liminf_{x \to_K G(u_0)} \overline{co}(T_K(x) \cap MB)) \tag{2.5}$$

MOREOVER, IF THE NORM OF X IS FRECHET DIFFERENTIABLE ON $X \backslash \{0\}$, THEN $\xi \in N_K(G(u_0))$ AND

$$\forall \ (j,g) \in \mathcal{V}_{(J,G)}(u_0), \ \lambda \ j + < \xi \ ,g > \ \geq \ 0 \tag{2.6}$$

Remark.

i) *Observe that when X is a finite dimensional space, then the condition (2.2) is always satisfied with Q equal to the unit ball and $\rho = 1$.*

ii) *When J is Lipschitzian on a neighborhood of u_0, then the assumption (2.2) may be replaced by : for all $u \in \mathcal{U}$ near u_0 and all $x \in K$ near $G(u_0)$.*

$$\rho B \subset cl \ (\overline{co} V_G(u) \ - \ \overline{co}(T_K(x) \cap \gamma \ B) \ + \ Q) \tag{2.2'}$$

iii) *When K is convex, the vector ξ from (2.3) verifies $\xi \in T_K(G(u_0))^-$, i.e. ξ is a normal to K at $G(u_0)$ in the sense of convex analysis.*

THEOREM 2.4. LET u_0 BE A SOLUTION OF PROBLEM (2.1) AND ASSUME THAT J IS LIPSCHITZIAN NEAR u_0. FURTHER ASSUME THAT THERE EXIST SUBSETS $Z(u) \subset \overline{co} \ V_G(u)$ SUCH THAT THE MAP $u \to Z(u)$ IS CONTINUOUS AT u_0. IF FOR SOME COMPACT SET $Q \subset X$, $\rho > 0$, $\gamma > 0$ *AND ALL* $x \in K$ *NEAR* $G(u_0)$

$$\rho \ B \subset cl \ (\overline{co} \ Z(u_0) \ - \ \overline{co} \ (T_K(x) \cap \gamma \ B) \ + \ Q) \tag{2.7}$$

THEN THE SAME ASSERTIONS AS IN THEOREM 2.3 ARE VALID.

COROLLARY 2.5. ASSUME THAT $J = \varphi \circ \Phi$, $G = g \circ \Phi$ WHERE Φ IS A FUNCTION FROM $\mathcal{U}$ TO A BANACH SPACE Y, LIPSCHITZIAN NEAR u_0 AND $\varphi : Y \to \mathbf{R}$, $g : Y \to X$ ARE C^1 at $\Phi \ (u_0)$. IF THERE EXIST $\rho > 0$, $\gamma > 0$ AND A COMPACT SET $Q \subset X$ SUCH THAT FOR ALL $x \in K$ NEAR $G(u_0)$ AND ALL $u \in \mathcal{U}$ NEAR u_0 THE INCLUSION (2.2)' HOLDS TRUE, THEN THERE EXIST λ, ξ SATISFYING (2.3), (2.5) SUCH THAT

$$\forall \ w \in \liminf_{u \to u_0} \ \overline{co} \ V_\Phi \ (u), \ <\lambda \ \varphi \ '(\Phi \ (u_0)) + g'(\Phi \ (u_0))^* \ \xi \ ,w > \ \geq 0$$

MOREOVER IF THE NORM OF X IS FRECHET DIFFERENTIABLE THEN $\xi \in N_K(G(u_0))$ AND

$$\forall \ w \in \mathcal{V}_\Phi \ (u_0), \ < \lambda \ \varphi \ ' \ (\Phi \ (u_0)) + g'(\Phi \ (u_0))^* \ \ \xi \ , \ w > \ \geq 0 \tag{2.8}$$

Proof. *For all $n \geq 1$ define functions*

$$\begin{cases} f_n: \mathcal{U} \to \mathbf{R}, & f_n(u) = \max\{0, J(u) - J(u_0) + 1/n^2\} \\ F_n: \mathcal{U} \times K \to \mathbf{R}, & F_n(u,x) = \sqrt{f_n(u)^2 + \|G(u) - x\|^2} \end{cases}$$

Then F_n is a nonnegative lower semicontinuous function on the complete metric space $\mathcal{U} \times K$ and $F_n(u_0, G(u_0)) = 1/n^2$. Hence we may apply the Ekeland variational principle [8] to F_n and the point $(u_0, G(u_0))$ to prove the existence of $u_n \in \mathcal{U}$, $x_n \in K$ such that

$$d(u_n, u_0) \leq \frac{1}{n}, \quad \|G(u_0) - x_n\| \leq \frac{1}{n}, \quad F_n(u_n, x_n) \leq \frac{1}{n^2} \tag{2.9}$$

and for all $(u,x) \in \mathcal{U} \times K$

$$F_n(u_n, x_n) \leq F_n(u,x) + \frac{1}{n}(d(u,u_n) + \|x - x_n\|) \tag{2.10}$$

Since u_0 is a solution, by definition of F_n, we always have $F_n(u_n,x_n) \neq 0$. The Gâteaux differentiability of the norm of X away from zero implies that for all n such that $G(u_n) \neq x_n$, there exists $p_n^* \in X^*$ satisfying $\|p_n^*\| = 1$ and for all $w \in X$

$$\lim_{h \to 0+, w' \to w} \frac{\|G(u_n) - x_n + hw'\| - \|G(u_n) - x_n\|}{h} = \langle p_n^*, w \rangle$$

Setting $p_n^* = 0$ when $G(u_n) = x_n$ and $p_n = \|G(u_n) - x_n\| p_n^*$, we have $\|p_n\| = \|G(u_n) - x_n\|$. Fix $n \geq 1$. Then for all $h_i \to 0+$, $w_i \to w$, $j_i \to j$ we have

$$\begin{cases} \|G(u_n) - x_n + h_i w_i\|^2 = \|G(u_n) - x_n\|^2 + 2h_i \langle p_n, w \rangle + o(h_i) \\ \max\{0, J(u_n) + h_i j_i - J(u_0) + 1/n^2\}^2 = f_n(u_n)^2 + 2h_i f_n(u_n) j + \bar{o}(h_i) \end{cases} \tag{2.11}$$

where $\lim_{i\to\infty} o(h_i)/h_i = 0 = \lim_{i\to\infty} \bar{o}(h_i)/h_i$. Define $\lambda_n \geq 0$, $\nu_n \geq 0$, $\xi_n \in X^*$ by

$$\nu_n = F_n(u_n, x_n), \quad \lambda_n = \frac{f_n(u_n)}{\nu_n}, \quad \xi_n = \frac{p_n}{\nu_n}$$

and observe that $\sqrt{\lambda_n^2 + \|\xi_n\|^2} = 1$. We shall prove the following inequalities:

$$\begin{cases} i) \ \forall (j,w) \in V_{(J,G)}(u_n, x_n), & \lambda_n j + \langle \xi_n, w \rangle \geq -1/n \\ ii) \ \forall y \in T_K(x_n), & \langle \xi_n, y \rangle \leq \|y\|/n \end{cases} \tag{2.12}$$

Indeed setting $x = x_n$ in (2.10) yields

$$\forall u \in \mathcal{U}, \quad F_n(u_n, x_n) \leq F_n(u, x_n) + \frac{1}{n} d(u, u_n) \tag{2.13}$$

Pick any $(j,w)\in V_{(J,G)}(u_n,x_n)$. *Then for some* $h_i\to 0+$, $(j_i,w_i)\to(j,w)$ *we have* $(J(u_n),G(u_n))+h_i(j_i,w_i)\in(J,G)(B_{h_i}(u_n))$. *From (2.13), (2.11) we obtain*

$$\nu_n\le\sqrt{\nu_n^2+2h_if_n(u_n)j+2h_i<p_n,w>+o(h_i)}+h_i/n =$$

$$\nu_n(1+2h_i(\lambda_nj+<\xi_n,w>)/\nu_n+o(h_i)/\nu_n^2)^{1/2}+h_i/n =$$

$$\nu_n(1+h_i(\lambda_nj+<\xi_n,w>)/\nu_n+\bar{o}(h_i)+h_i/n = \nu_n+h_i(\lambda_nj+<\xi_n,w>)+\bar{o}(h_i)\nu_n+h_i/n$$

where $\lim_{i\to\infty}o(h_i)/h_i=0=\lim_{i\to\infty}\bar{o}(h_i)/h_i$. *This implies that for some* $\epsilon_i\to 0+$

$$\nu_n\le\nu_n+h_i\lambda_nj+h_i<\xi_n,w>+\epsilon_ih_i+\frac{1}{n}h_i$$

Dividing by h_i *and taking the limit when* $i\to\infty$ *we obtain (2.12) i). Set next* $u=u_n$ *in (2.10). Then*

$$\forall x\in K,\ F_n(u_n,x_n)\le F_n(u_n,x)+\frac{1}{n}||x-x_n|| \qquad (2.14)$$

Consider $y\in T_K(x_n)$ *and let* $h_i\to 0+$, $y_i\to y$ *be such that* $x_n+h_iy_i\in K$. *Then from (2.14), applying (2.11) with* $w_i=-y_i$, *we obtain*

$$\nu_n\le\sqrt{\nu_n^2+2h_i<p_n,-y>+o(h_i)}+\frac{1}{n}h_i||y_i||$$

and as in the proof of (2.12) i) this implies that

$$\nu_n\le\nu_n+<\xi_n,-h_iy>+o(h_i)+\frac{1}{n}h_i||y_i||$$

Dividing by h_i *and taking the limit when* $i\to\infty$ *we obtain (2.12) ii). Since* $||(\lambda_n,\xi_n)||=1$, *taking a subsequence and keeping the same notations we may assume that for some* $\lambda\ge 0$, $\xi\in X^*$

$$\lambda_n\to\lambda;\ \xi_n\to\xi \text{ weakly }-*$$

Then, from (2.12) i) we deduce that ξ *verifies (2.4). We prove next that* $(\lambda,\xi)\ne 0$. *Indeed if* $\lambda=0$ *then* $||\xi_n||\to 1-$. *By (2.12) for all* $(j,w)\in\overline{co}V_{(J,G)}(u_n,x_n)$ *and for all* $y\in\overline{co}(T_K(x_n)\cap\gamma B)$,

$$\lambda_nj+<\xi_n\,,\,w-y>\ge-\frac{\gamma+1}{n}$$

Let $\varsigma_n\in X$ *be such that* $||\varsigma_n||\le 1$ *and* $\xi_n,\varsigma_n>\to 1-$. *By the assumption (2.2) there exist* $\epsilon_n\to 0$, $(j_n,v_n)\in\overline{co}V_{(J,G)}(u_n)$, $|j_n|\le\gamma$, $y_n\in\overline{co}(T_K(x_n)\cap\gamma B)$, $q_n\in Q$ *such that*

$-\rho\varsigma_n = \epsilon_n + v_n - y_n + q_n$. *Let* $\{q_{n_i}\}$ *be a subsequence converging to some* $q \in Q$. *Then, from the last inequality, we deduce that* $<\xi_{n_i}, -\rho\varsigma_{n_i} - q_{n_i}> \geq -\frac{\gamma+1}{n} - \lambda_n\gamma - ||\epsilon_n||$. *Taking the limit we obtain*

$$-\rho \lim_{i\to\infty} < \xi_{n_i}, \varsigma_{n_i} > - <\xi,q> = -\rho - <\xi,q> \geq 0$$

This implies that $\xi \neq 0$. *From (2.12) ii) we derive (2.3).*

Fix $M>0$. *Inequality (2.12) ii) implies that for all* $y \in T_K(x_n) \cap MB$ *we have* $<\xi_n, y> \leq M/n$. *Obviously it holds true also for all* $y \in \overline{co}(T_K(x_n) \cap MB)$ *and (2.5) follows by the limit procedure.*

Assume next that the norm of X is Fréchet differentiable away from zero. Then for every n satisfying $G(u_n) \neq x_n$, *there exists a function* $o_n : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $\lim_{h\to 0+} o_n(h)/h = 0$ *and for all* $b \in B$, $||G(u_n) - x_n + hb|| \leq ||G(u_n) - x_n|| + h<p_n^*, b> + o_n(h)$. *Hence for all* $n \geq 1$ *and* $b \in B$

$$||G(u_n) - x_n + hb||^2 \leq ||G(u_n) - x_n||^2 + 2h<p_n, b> \tag{2.15}$$

$$+h^2 + 2||G(u_n) - x_n|| o_n(h) + o_n(h)^2 + o_n(h^2)$$

To prove (2.6) fix $(j,w) \in \mathcal{V}_{(J,G)}(u_0)$ *and let* $h_n \to 0+$ *be such that*

$$\begin{cases} \lim_{n\to\infty} h_n/\nu_n = 0; \ sup_{||x|| \leq ||w||+1} o_n(h_n||x||)/h_n \leq 1/n; \\ sup_{||x|| \leq ||w||+1} o_n(h_n^2||x||^2)/h_n^2 \leq 1 \end{cases} \tag{2.16}$$

Let $(j_n, w_n) \to (j,w)$ *be such that for all n,* $(J(u_n), G(u_n)) + h_n(j_n, w_n) \in (J,G)(B_{h_n}(u_n))$. *Then from (2.13), (2.15) we obtain*

$$\nu_n \leq \sqrt{\nu_n^2 + 2h_n(f_n(u_n)j + <p_n, w_n>) + \nu_n^2 o(h_n/\nu_n)} + h_n/n =$$

$$\nu_n(1 + 2h_n(\lambda_n j + <\xi_n, w_n>)/\nu_n + o(h_n/\nu_n))^{1/2} + h_n/n =$$

$$\nu_n(1 + h_n(\lambda_n j + <\xi_n, w>)/\nu_n + \tilde{o}(h_n/\nu_n)) + h_n/n = \nu_n + h_n(\lambda_n j + <\xi_n, w>) + \tilde{o}(h_n)$$

where $\lim_{n\to\infty} \frac{\tilde{o}(h_n)}{h_n} = 0 = \lim_{n\to\infty} \frac{o(h_n)}{h_n} = \lim_{n\to\infty} \frac{\nu_n}{h_n} \tilde{o}(\frac{h_n}{\nu_n})$. *This implies that*

$$0 \leq h_n <(\lambda_n, \xi_n), (j,w)> + \tilde{o}(h_n)$$

We already know that (λ_n, ξ_n) *has a subsequence converging weakly - * to* $(\lambda, \xi) \neq 0$. *Dividing by* h_n *the last inequality and taking the limit we obtain that* $\lambda j + < \xi, w> \geq 0$. *Since*

$(j,w)\in\mathcal{V}_{(J,G)}(u_0)$ is arbitrary, this proves (2.6). To prove that $\xi\in N_K(G(u_0))$ fix $w\in C_K(G(u_0))$ and let $h_n\to 0+$ be such that (2.16) holds true. Pick a sequence $w_n\to w$ such that for all n, $x_n+h_nw_n\in K$. Then from (2.14), (2.15) we obtain

$$\nu_n\leq\sqrt{\nu_n^2+2h_n<p_n,-w_n>+\nu_n^2 o_n(h_n/\nu_n)}+h_n||w_n||/n=$$

$$\nu_n\sqrt{1+2h_n<\xi_n,-w_n>/\nu_n+o_n(h_n/\nu_n)}+h_n||w_n||/n=$$

$$\nu_n(1+h_n<\xi_n,-w>)/\nu_n+\bar{o}(h_n/\nu_n))+h_n||w_n||/n=\nu_n+h_n<\xi_n,-w>+o(h_n)$$

Hence we proved that

$$0\leq -h_n<\xi_n,w>+o(h_n)$$

Dividing by h_n and taking the limit yields $<\xi,w>\leq 0$. Since $w\in C_K(G(u_0))$ is arbitrary, this implies that $\xi\in N_K(G(u_0))$ and ends the proof.

To prove Theorem 2.4 it is enough to replace Q by $\overline{co}Q$ and to observe that (2.7) continuity of Z at u_0 and the separation theorem imply that for all $u\in\mathcal{U}$ near u_0 and all $x\in K$ near $(G(U_0)$ (2.2)$'$ is satisfied with ρ replaced by $\rho/2$. Hence the result follows from Theorem 2.3.

3. Maximum principle in optimal control of infinite dimensional semilinear systems

We consider below the problem

$$\text{minimize } \varphi(y(0),y(T)) \tag{3.1}$$

over the solutions of semilinear initial value problem

$$\begin{cases} y'(t)=Ay(t)+f(t,y(t),u(t)), & 0\leq t\leq T\\ y(0)=y_0\end{cases} \tag{3.2}$$

$$[0,T]\ni t\rightarrow u(t)\in U \text{ is measurable} \tag{3.3}$$

satisfying the end point constraints

$$(y(0),y(T))\in K \tag{3.4}$$

where

U is a topological space.

A is the infinitesimal generator of a C_0 -semigroup $\{S(t)\}_{t\geq 0}$ in a separable Banach space E with the norm Fréchet differentiable away from zero.

$f:[0,T]\times E\times U\rightarrow E$, $\varphi:E\times E\rightarrow \mathbf{R}$ are continuous functions with $f(t,.,u)$ differentiable for all $t\in[0,T]$, $u\in U$

K is a closed subset of $E\times E$.

We assume that for some $a>0$ and all $t\in[0,T]$, $u\in U$

$$||f(t,y,u)||\leq a(||y||+1) \tag{3.5}$$

and that for every bounded set $C\subset E$ there exists a constant $L>0$ such that

$$\forall x,\ y\in C\ ,\ u\in U\ ,\ t\in[0,T],\ ||f(t,x,u)-f(t,y,u)||\ \leq\ L||x-y||$$

i.e. $f(t,\cdot,u)$ is Lipschitz continuous on C uniformly in (t,u).

A continuous function $y:[0,T]\rightarrow E$ is called a mild solution of (3.2) if for all $t\in[0,T]$

$$y(t)=S(t)y_0+\int_0^t S(t-s)f(s,y(s),u(s))\,ds$$

Our assumptions imply that for every $u(\cdot)$ as in (3.3) the system (3.2) has a unique, mild solution.

Remark. *Recall that the problem*

$$\text{minimize } g(z(0),z(T)) + \int_0^T L(t,y(t),u(t))\,dt$$

over the solutions of (3.2) - (3.4) can easily be reduced to the problem (3.1) - (3.4) by a simple changing of variables.

Let z be a solution of (3.2), (3.4) corresponding to a control $\bar{u}$ and consider the linearized control system

$$\begin{cases} w'(t) = Aw(t)+\dfrac{\partial f}{\partial y}(t,z(t),\bar{u}(t))w(t)\ +\ v(t) \\ w(0)=0\ ,\ \ v(t)\in\overline{co}f(t,z(t),U)\ -\ f(t,z(t),\bar{u}(t)) \end{cases} \tag{3.6}$$

Let $R^L_{z,\bar{u}}(T)$ denote its reachable set at time T, i.e.

$$R^L_{z,\bar{u}}(T)\ =\ \{w(T):w \text{ is a mild solution of (3.6)}\}.$$

Then

$$R^L_{z,\bar{u}}(T) = \{\int_0^T S_{z,\bar{u}}(T,t)v(t)dt:\ v(t)\in\overline{co}f(t,z(t),U)-f(t,z(t),\bar{u}(t))\ \textit{is measurable}\}$$

where $S_{y,u}(t,s)$ is the solution operator of the linear equation

$$z'(t)=(A+\frac{\partial f}{\partial y}(t,y(t),u(t)))z(t)$$

That is, the only strongly continuous solution of the operator equation

$$S_{y,u}(t,s)z = S(t-s)z+\int_s^t S(t-\sigma)B_{y,u}(\sigma)S_{y,u}(\sigma,s)zd\sigma$$

in $0\leq s\leq t\leq T$ with $B_{y,u}(\sigma) = \frac{\partial f}{\partial y}(\sigma,y(\sigma),u(\sigma))$

Denote by $S_{y,u}(T,0)_B$ the restriction of the linear operator $S_{y,u}(T,0)$ to the closed unit ball B.

THEOREM 3.1 LET z BE A SOLUTION OF (3.1) - (3.4) AND $\bar{u}$ BE THE CORRESPONDING CONTROL. ASSUME THAT φ IS CONTINUOUSLY DIFFERENTIABLE ON A NEIGHBORHOOD OF $(z(0),z(T))$ AND FOR ALMOST ALL $t\in[0,T]$, $\frac{\partial f}{\partial y}(t,\cdot,\bar{u}(t))$ IS CONTINUOUS AT $z(t)$. FURTHER ASSUME THAT FOR SOME $\bar{\rho}>0$, $\bar{\gamma}>0$ AND A COMPACT SET $\bar{Q}\subset E\times E$ AND FOR ALL $x\in K$ NEAR $(z(0),z(T))$

$$\bar{\rho}B \subset cl(graph\ S_{z,\bar{u}}(T,0)_B + \{0\}\times R^L_{z,\bar{u}}(T)-\overline{co}(T_K(x)\cap\bar{\gamma}B)+\bar{Q}) \tag{3.7}$$

THEN THERE EXIST $\lambda\geq 0$ AND $\xi=(\xi_1,\xi_2)\in N_K(z(0),z(T))$ NOT BOTH EQUAL TO ZERO SUCH THAT THE FUNCTION

$$p(t) = S_{z,\bar{u}}(1,t)^*(\lambda\frac{\partial\varphi}{\partial x_2}(z(0),z(T))+\xi_2) \tag{3.8}$$

SATISFIES THE MINIMUM PRINCIPLE

$$<p(t),f(t,z(t),\bar{u}(t))> = \min_{u\in U} <p(t),f(t,z(t),u)> \tag{3.9}$$

AND THE TRANSVERSALITY CONDITION

$$(-p(0),\ p(T))=\lambda\varphi'(z(0),z(T))+(\xi_1,\xi_2)\in(\lambda\varphi'(z(0),z(T))+N_K(z(0),z(T)) \tag{3.10}$$

COROLLARY 3.2. LET z, $\bar{u}$, φ, f BE AS IN THEOREM 3.1. AND ASSUME THAT

$K=K_1 \times K_2 \subset E \times E$. FURTHER ASSUME THAT THERE EXIST $\bar{\rho}>0, \bar{\gamma}>0$ AND A COMPACT $\bar{Q} \subset E$ SUCH THAT FOR ALL $x \in K_2$ near $z(T)$

$$\bar{\rho}B \subset cl(R^L_{z,\bar{u}}(T) - \overline{co}(T_{K_2}(x) \cap \bar{\gamma}B) + \bar{Q} \tag{3.11}$$

THEN THE CONCLUSION OF THEOREM 3.1 IS VALID.

Remark 3.3. *Observe that, in particular, (3.11) is satisfied for all $x \in K_2$ near $z(T)$ if one of the following assumptions holds true*

i) $Int\ R^L_{z,\bar{u}}(T) \neq \emptyset$

ii) *K is a convex subset of a closed subspace $H \subset E$ of finite codimension and $Int_H K \neq \emptyset$*

iii) *E is a Hilbert space and for some $\gamma>0, \epsilon>0$ and a closed subspace H of finite codimension*

$$Int_H \bigcap_{\substack{||x-z(T)|| \leq \epsilon \\ x \in K_2}} \pi_H \overline{co}(T_K(x) \cap \gamma B) \neq \emptyset$$

where π_H denotes the orthogonal projection on H

iv) *E is a finite dimensional space.*

Loosely speaking (3.11) means that $cl(R^L_{z,\bar{u}}(T) - \overline{co}(T_{K_2}(z(T)) \cap \gamma B)$ is an open set modulo a compact set Q. Corollary 3.2 and iii) allow to compare results of this paper with those from [10].

To prove the above results set

$$\mathcal{U} = \{u:[0,T] \rightarrow U:\ u \text{ is measurable}\}$$

$$\forall u,v \in \mathcal{U},\ d(u,v) = \mu(\{t \in [0,T]: u(t) \neq v(t)\})$$

where μ stands for the Lebesgue measure. Then $(\mathcal{U},d)$ is a complete metric space (see Ekeland [8]). (Since $d(u,v)=0 \Rightarrow y_u = y_v$, we identify controls equal almost everywhere, here y_u denotes the (mild) solution of (3.2)).

Define continuous maps $J: K \times \mathcal{U} \rightarrow \mathbf{R}$, $G: K \times \mathcal{U} \rightarrow E \times E$ by

$$J(y_0,u) = \varphi(y_0, y_u(T)) \ ;\ G(y_0,u) = (y_0, y_u(T))$$

Then the problem (3.1) - (3.4) may be rewritten as the problem (2.1) considered in the previous section. Hence in order to write necessary conditions for optimality we have to study variations of the map (J,G).

For this aim fix $u \in \mathcal{U}$, $y_0 \in E$ and let y be the solution of (3.2). Consider needle perturbations of u at a point $s \in [0,T]$: Let $v \in U$, $h>0$, and set

$$u_h(t) = \begin{cases} v & s-h \le t \le s \\ u(t) & \text{otherwise} \end{cases}$$

Denote by y_h the solution of (3.2) with u replaced by u_h.

LEMMA 3.4. LET s BE THE LEFT LEBESGUE POINT OF THE FUNCTION $t \to f(t,y(t),u(t))$. THEN

$$\lim_{h\to 0+} \frac{y_h(T)-y(T)}{h} = S_{y,u}(T,s)(f(s,y(s),v)-f(s,y(s),u(s)))$$

For the proof see [9].

Differentiating with respect to the initial condition we obtain easily

LEMMA 3.5. LET $w_0 \in E$ and y_h DENOTE THE SOLUTION OF (3.2) WITH y_0 REPLACED BY y_0+hw_0. THEN

$$\lim_{h\to 0+} \frac{y_h(T)-y(T)}{h} = S_{y,u}(T,0)w_0$$

COROLLARY 3.6. FOR EVERY $u \in \mathcal{U}$, $y_0 \in E$ AND THE CORRESPONDING SOLUTION y OF (3.2) WE HAVE

$$\frac{1}{2}\text{graph } S_{y,u}(T,0)_B + \{0\} \times \frac{1}{2T} R^L_{y,u}(T) \subset \overline{co}\, V_G(y_0,u)$$

Proof. *By Lemma 3.4, for every Lebesgue point s of the function $t \to f(t,y(t),u(t))$ we have*

$$\{0\} \times S_{y,u}(T,s)(\overline{co} f(s,y(s),U) - f(s,y(s),u(s))) \subset \overline{co}\, V_G(y_0,u)$$

Since the set of Lebesgue points has a full measure, integrating the above inclusion we obtain

$$\{0\} \times R^L_{y,u}(T) \subset T \cdot \overline{co}\, V_G(y_0,u)$$

This and Lemma 3.5 yield the result.

Proof of Theorem 3.1. *It is not restrictive to assume that $T=1$. We apply Theorem 2.4 with $J=\varphi \circ G$ and G defined by*

$$\forall\ (y_0,u) \in E \times \mathcal{U},\ \ G(y_0,u) = (y_0, y_u(1))$$

where y_u is the solution of (3.2). By our assumptions G is Lipschitz continuous. From Corollary 3.6 follows that for all $(y_0,u)\in E\times\mathcal{U}$

$$A(y_0,u) := \frac{1}{2}(\text{ graph } S_{y,u}(1,0)_B+\{0\}\times R^L_{y,u}(1)) \subset \overline{co}\, V_G(y_0,u)$$

On the other hand, the map $(y_0,u)\to S_{y,u}(1,0)$ is continuous and $(y_0,u)\to R^L_{y,u}(1)$ is continuous in the Hausdorff metric (here y denotes the solution of (3.2)).

Hence we deduce from (3.7) that the assumptions of Theorem 2.4 are satisfied with $\rho=\frac{\overline{\rho}}{2}$, $\gamma=\frac{\overline{\gamma}}{2}$, $Q=\frac{1}{2}\overline{Q}$. Let $\lambda\geq 0$, $\xi=(\xi_1,\xi_2)\in N_K(z(0),z(1))$ be as in the claim of Theorem 2.4. Then

$$\forall\; w \in \frac{1}{2}\text{graph } S_{z,\bar{u}}(1,0)_B+\{0\}\times\frac{1}{2}R^L_{z,\bar{u}}(1) \subset \liminf_{\substack{y_0\to z(0)\\ u\to\bar{u}}} \overline{co}\, V_G(y_0,u)$$

we have

$$<\lambda\varphi'(z(0),\, z(1))+\xi,\, w> \;\geq 0 \tag{3.12}$$

Hence for every measurable selection $v(t)\in\overline{co}f(t,z(t),U)-f(t,z(t),\bar{u}(t))$

$$<\lambda\frac{\partial\varphi}{\partial x_2}(z(0),z(1))+\xi_2,\; \int_0^1 S_{z,\bar{u}}(1,t)v(t)dt >= \tag{3.13}$$

$$\int_0^1 <S_{z,\bar{u}}(1,t)^*(\lambda\frac{\partial\varphi}{\partial x_2}(z(0)\,,\,z(1))+\xi_2)\,,\,v(t)>dt \geq 0$$

Set

$$p(t) = S_{z,\bar{u}}(1,t)^*(\lambda\frac{\partial\varphi}{\partial x_2}(z(0),z(1))+\xi_2).$$

Then (3.13) yields the minimum principle (3.9). On the other hand (3.12) implies that for every $w\in E$

$$<\lambda\varphi'(z(0),z(1))+\xi,\,(w,\, S_{z,\bar{u}}(1,0)w)> \;=\; <\lambda\frac{\partial\varphi}{\partial x_1}(z(0),z(1))+\xi_1+S_{z,\bar{u}}(1,0)^*(\lambda\frac{\partial\varphi}{\partial x_2}(z(0),z(1))+\xi_2),\, w>$$

$$=<\lambda\frac{\partial\varphi}{\partial x_1}(z(0),z(1))+\xi_1+p(0),w>\geq 0$$

Hence $-p(0)=\lambda\frac{\partial\varphi}{\partial x_1}(z(0),z(1))+\xi_1$. Moreover by the definition of $p(\cdot)$, $p(1)=\lambda\frac{\partial\varphi}{\partial x_2}(z(0),z(1))+\xi_2$. This ends the proof.

4. Optimal control of a semilinear system with state constraints

Let Ω be an open bounded subset of $\mathbf{R}^n$ $(n\leq 3)$ with C^2 boundary Γ, X be a Banach space with Frechet differentiable norm and

$$T:\ C_0(\Omega)\rightarrow\mathbf{R}^m\ ,\ L:\ C_0(\Omega)\rightarrow X$$

be C^1 (nonlinear) mappings. Set $Y=H^2(\Omega)\cap H_0^1(\Omega)$ and consider closed sets $K\subset L^2(\Omega)$, $C\subset\mathbf{R}^m$, $D\subset X$ and a continuously differentiable function $J:C_0(\Omega)\times L^2(\Omega)\rightarrow\mathbf{R}$. We study here the problem

$$\text{minimize}\ \ J(y,u) \tag{4.1}$$

over the pairs $(y,u)\in Y\times K$ satisfying the constraints

$$T(y)\in C\quad ,\quad L(y)\in D \tag{4.2}$$

and

$$\begin{cases} Ay+\varphi(y)=u \text{ in } \Omega, \\ y=0 \text{ on } \Gamma \end{cases} \tag{4.3}$$

where

$$Ay=-\sum_{i,j=1}^{n}\partial_{x_j}(a_{ij}(x)\,\partial_{x_i}y)+a_0(x)y,$$

and

$$a_0\in L^\infty(\Omega)\ ,\ a_0(x)\geq 0\ \text{ for a.e. } x\in\Omega,$$

$$a_{ij} \text{ is Lipschitz on } \bar{\Omega}\ (1\leq i,j,\leq n),$$

$$\sum_{i,j=1}^{n}a_{ij}(x)\xi_i\xi_j\geq a_0\|\xi\|^2,\ a_0>0\ ,\ \forall\xi\in\mathbf{R}^n,\ \forall x\in\Omega,$$

$$\varphi:\mathbf{R}\rightarrow\mathbf{R}\ \ \textit{is } C^1 \textit{ nondecreasing function}\ .$$

Remark. *It may happen that to a control $u\in K$ correspond several solutions of (4.3), i.e. we have to deal with an ill posed problem.*

From now on we denote by B_X the closed unit ball in the space X.

THEOREM 4.1. LET $(\bar{y},\bar{u})$ BE AN OPTIMAL SOLUTION OF (4.1) - (4.3) AND ASSUME THAT FOR SOME $\rho>0$, $\gamma>0$ AND A COMPACT $Q\subset X$ WE HAVE

$$\forall d\in D \ \textit{near}\ L(\bar{y})\ ,\ \rho B_X\subset\overline{co}(T_D(d)\cap\gamma B_X)+Q \tag{4.4}$$

THEN THERE EXIST $\lambda \geq 0$, $p \in W_0^{1,s}(\Omega)$, $s < \frac{n}{n-1}$, $l \in \mathbf{R}^m$, $\mu \in X^*$ NOT ALL EQUAL TO ZERO SUCH THAT

$$A^*p + \varphi'(\bar{y})^*p = \lambda \frac{\partial J}{\partial y}(\bar{y},\bar{u})^* + T'(\bar{y})^*l + L'(\bar{y})^*\mu \tag{4.5}$$

$$-\lambda \frac{\partial J}{\partial u}(\bar{y},\bar{u})^* - p \in N_K(\bar{u});\ l \in N_C(T(\bar{y}));\ \mu \in N_D(L(\bar{y})) \tag{4.6}$$

MOREOVER, IF

$$\overline{\mathrm{Im}\ L'(\bar{y})} = X,\ \ \mathrm{Im}\ T'(\bar{y}) = \mathbf{R}^m,\ \ L'(\bar{y})^* N_D(L(\bar{y})) \cap \mathrm{Im}\ T'(\bar{y})^* = \{0\} \tag{4.7}$$

THEN $\lambda + ||p|| > 0$ AND IF IN ADDITION

$$\mathrm{Im}(L'(\bar{y})^* + T'(\bar{y})^*) \cap (A^* + \varphi'(\bar{y})) N_K(\bar{u}) = \{0\} \tag{4.8}$$

THEN $\lambda > 0$.

Remark. *a) Observe that the assumption (4.4) holds true in particular when D is a convex subset of a closed subspace $H \subset X$ of finite codimension and $Int_H D \neq \emptyset$*

b) The above result can be related to [4].

Proof. *Define* $A_1: Y \to L^2(\Omega)$, $J_1: Y \to \mathbf{R}$, $G: Y \to \mathbf{R}^m \times X \times L^2(\Omega)$ by

$$A_1(y) = Ay + \varphi(y),\ J_1(y) = J(y, A_1(y)),\ G(y) = (T(y), L(y), A_1(y))$$

and set

$$\mathcal{K} = C \times D \times K$$

Then our problem may be reduced to the following one.

$$\min\ \{J_1(y) : y \in Y\ ,\ \ G(y) \in \mathcal{K}\}$$

We easily verify that for all $y \in Y$

$$\{(J_1'(y)(w, Aw + \varphi'(y)w),\ T'(y)w,\ L'(y)w,\ Aw + \varphi'(y)w : ||w||_Y \leq 1\} \subset V_{(J,G)}(y)$$

$$Z(y) := \{(T'(y)w,\ L'(y)w,\ Aw + \varphi'(y)w) : ||w||_Y \leq 1\} \subset V_G(y)$$

and for all $c \in C$, $d \in D$, $k \in K$

$$T_{\mathcal{K}}(c,d,k) = T_C(c) \times T_D(d) \times T_K(k)$$

$$C_{\mathcal{K}}(c,d,k) = C_C(c) \times C_D(d) \times C_K(k)$$

Moreover the map Z is continuous in the Hausdorff metric. We apply Theorem 2.4. Since φ *is nondecreasing, for every* $x\in\Omega$ *we have* $\varphi'(\bar{y}(x))\geq 0$. *This and [19] yield that for some* $\varepsilon>0$.

$$\varepsilon B_{L_2(\Omega)} \subset (A+\varphi'(\bar{y}))B_Y \tag{4.9}$$

Set $q=||T'(\bar{y})||+||L'(\bar{y})||+1$ *and observe that from (4.4) follows that*

$$\forall d\in D \text{ near } L(\bar{y}),\ 2qB_X \subset \overline{co}(T_D(d)\cap\frac{2q\gamma}{\rho}B_X)+\frac{2q}{\rho}Q \tag{4.10}$$

Hence from (4.9) we deduce that for all $k\in K$, $c\in C$ *and every* $d\in D$ *near* $L(\bar{y})$

$$qB_{\mathbf{R}^m}\times qB_X\times\varepsilon B_{L^2(\Omega)}\subset Z(\bar{y})+\{0\}\times 2qB_X\times\{0\}\ +2qB_{\mathbf{R}^m}\times\{0\}\times\{0\}\subset$$

$$Z(\bar{y})-\{0\}\times\overline{co}(T_D(d)\cap\frac{2q\gamma}{\rho}B_X)\times\ \{0\}+2qB_{\mathbf{R}^m}\times\frac{2q}{\rho}Q\times\{0\}\subset$$

$$Z(\bar{y})-\overline{co}(T_K(c,d,k)\cap\frac{2q\gamma}{\rho}B)\ +\ 2qB_{\mathbf{R}^m}\times\frac{2q}{\rho}Q\times\{0\}$$

Setting $\bar{\rho}=\min(q,\varepsilon)$, $\bar{\gamma}=\ \frac{2q\gamma}{\rho}$, $\overline{Q}=2qB_{\mathbf{R}^m}\times\frac{2q}{\rho}Q\times\{0\}$ *We obtain that for all* $k\in K$ *near* $(T(\bar{y}),L(\bar{y}),A_1(\bar{y}))$

$$\bar{\rho}B\subset Z(\bar{y})-\overline{co}(T_K(k)\cap\bar{\gamma}B)+\overline{Q}$$

By Theorem 2.4 there exist $\lambda\geq 0$, $l\in N_C(T(\bar{y}))$, $\mu\in N_D(L(\bar{y}))$, $\bar{p}\in N_K\ (A_1(\bar{y}))$ *not all equal to zero such that for every* $w\in B_Y$

$$\lambda J_1'(\bar{y})(w,Aw+\varphi'(\bar{y})w)+<T'(\bar{y})^*l,w>+<L'(\bar{y})^*\mu,w>+<A^*\bar{p}+\varphi'(\bar{y})^*\bar{p},w>\geq 0$$

This yields that

$$A^*(\lambda\frac{\partial J}{\partial u}(\bar{y},\bar{u})+\bar{p})+\varphi'(\bar{y})^*(\lambda\frac{\partial J}{\partial u}(\bar{y},\bar{u})+\bar{p})+\lambda\frac{\partial J}{\partial y}(\bar{y},\bar{u})+T'(\bar{y})^*l+L'(\bar{y})^*\mu=0$$

Setting $p=-\lambda\frac{\partial J}{\partial u}(\bar{y},\bar{u})-\bar{p}$ *we obtain (4.5), (4.6). But from (4.5) we also deduce that* $A^*p\in C_0(\Omega)^*$ *and, consequently, for all* $s<\frac{n}{n-1}$, $p\in W_0^{1,s}(\Omega)$. *Assume for a moment that* $\lambda=0$, $p=0$ *and (4.7) holds true. Then, by (4.5),*

$$L'(\bar{y})^*\mu \in ImT'(\bar{y})^*$$

and, therefore, $L'(\bar{y})^*\mu=0$. *From the injectivity of* $L'(\bar{y})^*$ *follows that* $\mu=0$. *This, (4.5) and injectivity of* $T'(\bar{y})^*$ *yields* $l=0$, *which is not possible. Hence* $\lambda+||p||>0$. *Assume* *next that (4.7), (4.8) hold true. If* $\lambda=0$ *then, by (4.5) , (4.6),* $(A^*+\varphi'(\bar{y}))p \in \mathrm{Im}\,(L'(\bar{y})^* + T'(\bar{y})^*)$ *This implies that* $p=0$ *and, consequently,* $\lambda+||p||=0$. *The obtained contradiction ends the proof.*

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