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FOREWORD

The purpose of this paper is to characterize classical and lower semicontinuous solutions to the Hamilton-Jacobi-Isaacs partial differential equations associated with a differential game and, in particular, characterize closed subsets the indicators of which are solutions to these equations. For doing so, the classical concept of derivative is replaced by contingent epiderivative, which can apply to any function.

The use of indicator of subsets which are solutions of either one of the contingent Isaacs equation allows to characterize areas of the playability set in which some behavior (playability, winability, etc.) of the players can be achieved.

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Jean-Pierre Aubin

1 Introduction

Let us consider a differential game described by

$$(1) \qquad \begin{cases} i) & z'(t) = h(z(t), u(t), v(t)) \\ ii) & u(t) \in U(z(t)) \\ iii) & v(t) \in V(z(t)) \end{cases}$$

where $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ describes the dynamics of the game and where the set-valued $U : \mathbf{R}^n \rightsquigarrow \mathbf{R}^p$ and $V : \mathbf{R}^n \rightsquigarrow \mathbf{R}^q$ are the a priori feedback maps of our players Xavier and Yves which represent the **state-dependent constraints** bearing on the controls of each player.

We denote by K, the playability set, the intersection of the domains of U and V.

The purpose of this paper is to characterize the solutions Φ to the following four Hamilton-Jacobi-Isaacs partial differential equations associated with this differential game.

$$\begin{cases} i) & \inf_{u\in U(z)} \inf_{v\in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z,u,v) = 0 \\ ii) & \sup_{u\in U(z)} \sup_{v\in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z,u,v) = 0 \\ iii) & \sup_{v\in V(z)} \inf_{u\in U(z)} \frac{d\Phi(z)}{dz} \cdot h(z,u,v) = 0 \\ iv) & \inf_{u\in U(z)} \sup_{v\in V(z)} \frac{d\Phi(z)}{dz} \cdot h(z,u,v) = 0 \end{cases}$$

We shall study the properties of the solutions (classical of lower semicontinuous) to these partial differential equations, and in particular, characterize the solutions which are indicators of closed subsets L, defined by

$$\Psi_L(x) := \begin{cases} 0 & \text{if } z \in L \\ +\infty & \text{if } z \notin L \end{cases}$$

which are only lower semicontinuous. For that purpose, we shall weaken the concept of usual derivative and replace it by the concept of contingent epiderivative, which can apply to any function[1], and generalize these Isaacs partial differential equation to corresponding contingent Isaacs equations[2].

The use of indicator of subsets which are solutions of either one of the contingent Isaac's equation allows to characterize areas of the playability set in which some behavior (playability, winability, etc.) of the players can be achieved.

Example Let us consider our two players, Xavier and Yves. Xavier acts on a state space X and Yves on a state space Y. For doing so, they have access to some knowledge about the global state (x, y) of the system and are allowed to choose controls u in a global state dependent set $U(x, y)$ and v in a global state dependent set $V(x, y)$ respectively.

Their actions on the state of the system are governed by the system of differential inclusions:

$$(2) \qquad \begin{cases} a) & \begin{cases} i) & x'(t) = f(x(t), y(t), u(t)) \\ ii) & u(t) \in U(x(t), y(t)) \end{cases} \\ b) & \begin{cases} i) & y'(t) = g(x(t), y(t), v(t)) \\ ii) & v(t) \in V(x(t), y(t)) \end{cases} \end{cases}$$

We now describe the influences (power relations) that Xavier exerts on Yves and vice-versa through **rules of the game**. They are set-valued maps $P : Y \rightsquigarrow X$ and $Q : X \rightsquigarrow Y$ which are interpreted in the following way. When the state of Yves is y, Xavier's choice is constrained to belong to $P(y)$. In a symmetric way, the set-valued map Q assigns to each state x the set $Q(x)$ of states y that Yves can implement[3].

[1] See [2, Chapter VII] for an introduction to nonsmooth and set-valued analysis

[2] In the extent where Isaacs are partial differential equations, they have, under adequate assumptions, **unique** "viscosity solutions", which are only continuous (See [3,4,8,9,10,20,21] and the references of theses papers). In the case of control problems, it has been shown in [12,13,14] that any viscosity solution is a solution to an adequate contingent version of Hamilton-Jacobi-Bellman equation. The comparison of the solutions in the case of differential games remains to be done.

[3] We can easily extend the results below to the time-dependent case using the methods of [1].

Hence, the **playability domain** of the game is the subset $K \subset X \times Y$ defined by:

$$(3) \qquad K := \{ (x,y) \in X \times Y \mid x \in P(y) \text{ and } y \in Q(x) \}$$

Naturally, we must begin by providing sufficient conditions implying that the playability domain is non empty. Since the playability domain is the subset of fixed-points (x,y) of the set-valued map $(x,y) \rightsquigarrow P(y) \times Q(x)$, we can use one of the many fixed point theorems to answer this type of questions[4].

From now on, we shall assume that the playability domain associated with the rules P and Q is not empty.

By denoting by $z := (x,y) \in Z := X \times Y$ the global state, by $h(z,u,v) := (f(x,u,v), g(y,u,v))$ the values of the map $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ describing the dynamics of the game, by $L := \text{Graph}(P)$ Xavier's closed domain, by $M := \text{Graph}(Q^{-1})$ Yves's one and by $K := L \cap M$ the playability domain. We shall also identify the set-valued maps U and V with their restrictions to L and M respectively by setting $U(z) := \emptyset$ whenever $z \notin L$ and $V(z) := \emptyset$ when $z \notin M$. $\square$

2 Contingent Isaacs Equations

Since we want to include indicators of subsets among the solutions of Isaacs equations and also, look for smaller lower semicontinuous solutions to such an equation satisfying such or such property, we are led to weaken the concept of usual derivatives involved in these partial differential equations by replacing them by contingent epiderivatives, since any extended function $\Phi : X \to \mathbf{R} \cup \{+\infty\}$ has contingent epiderivative[5], and in particular,

[4] For instance, Kakutani's Fixed Point Theorem furnishes such conditions: *Let $L \subset X$ and $M \subset Y$ be compact convex subsets and $P : M \rightsquigarrow L$ and $Q : L \rightsquigarrow M$ be closed maps with nonempty convex images. Then the playability domain is not empty.*

[5] the contingent epiderivative of Φ at $z \in \text{Dom}(\Phi)$ in the direction v is defined by

$$D_{\uparrow}\Phi(z)(v) := \liminf_{h \to 0+, u \to v} (\Phi(x + hu) - \Phi(x))/h$$

It is characterized by the fact that its epigraph is the contingent cone to the epigraph of Φ at $(z, \Phi(z))$.

indicators, for which we have the relation

$$D_{\uparrow}\Psi_L(z)(v) \;=\; \Psi_{T_L(z)}(v) \;:=\; \begin{cases} 0 & \text{if } v \in T_L(z) \\ +\infty & \text{if } v \notin T_L(z) \end{cases}$$

Theorem 2.1 *Let us assume at least that $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ is continuous, has linear growth , and that the set-valued maps are closed with linear growth.*

We assume that the all extended functions Φ are nonnegative and contingently epidifferentiable[6] *and that their domains are contained in the intersection K of the domains of U and V.*

1 — If the values of the set-valued maps U and V are convex and if h is affine with respect to the controls, Φ is a solution to the contingent equation

$$(4) \qquad \inf_{u\in U(z)} \inf_{v\in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \;=\; 0$$

if and only if

$$\forall\, z \in \Phi,\ \exists\, z(\cdot) \in \mathcal{S}(z) \mid \forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$$

2 — Assume that h is uniformly lipschitzean with respect to x. Then Φ is a solution to the contingent equation

$$(5) \qquad \sup_{u\in U(z)} \sup_{v\in V(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \;=\; 0$$

if and only if

$$\forall\, z \in \mathrm{Dom}(\Phi),\ \forall\ \ z(\cdot) \in \mathcal{S}(z), \forall\, t \geq 0,\ \ \Phi(z(t)) \leq \Phi(z)$$

3 — Assume that V is lower semicontinuous, that the values of U and V are convex and that h is affine with respect to u. Then Φ is a solution to the contingent equation

$$(6) \qquad \sup_{v\in V(z)} \inf_{u\in U(z)} D_{\uparrow}\Phi(z)(h(z,u,v)) \;=\; 0$$

[6] This means that for all $z \in \mathrm{Dom}(\Phi)$, $\forall\ v \in X$, $D_{\uparrow}\Phi(z)(v) > -\infty$ and that $D_{\uparrow}\Phi(z)(v) < \infty$ for at least a $v \in X$.

if and only if for any continuous closed-loop control $\tilde{v}(z) \in V(z)$ played by Yves and any initial state $z \in \mathrm{Dom}(\Phi)$, there exists a solution $z(\cdot)$ to Xavier's control problem

$$
(7) \qquad \begin{cases} i) & z'(t) \; = \; h(z(t), u(t), \tilde{v}(z(t))) \\ ii) & u(t) \; \in \; U(z(t)) \end{cases}
$$

starting at z and satisfying $\forall\, t \geq 0, \;\; \Phi(z(t)) \leq \Phi(z)$.

4 — *Assume that V is lower semicontinuous with convex values. Then Φ is a solution to the contingent equation*

$$
(8) \qquad \inf_{u \in U(z)} \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z, u, v)) \; = \; 0
$$

if and only if Xavier can play a closed-loop control $\tilde{u}(z) \in U(z)$ such that, for any continuous closed-loop control $\tilde{v}(z) \in V(z)$ played by Yves and for any initial state $z \in \mathrm{Dom}(\Phi)$, there exists a solution $z(\cdot)$ to

$$
(9) \qquad z'(t) \; = \; h(z(t), \tilde{u}(z(t), \tilde{v}(z(t)))
$$

starting at z and satisfying $\forall\, t \geq 0, \;\; \Phi(z(t)) \leq \Phi(z)$. The converse is true if

$$
B_{\Phi} := \{\bar{u} \in U(z) \; : \; \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z, \bar{u}, v)) = \inf_{u \in U(z)} \sup_{v \in V(z)} D_{\uparrow}\Phi(z)(h(z, u, v))\}
$$

is lower semicontinuous with closed convex values.

Proof — It is based on the properties of lower semicontinuous Lyapunov functions and universal Lyapunov functions of a differential inclusion which are stated in the appendix.

— The two first statements are translations of the theorems characterizing Lyapunov and universal Lyapunov functions applied to the differential inclusion $z'(t) \in H(z(t))$ where $H(z) := f(z, U(z), V(z))$.

— Let us prove the third one. Assume that Φ satisfies the stated property. Since V is lower semicontinuous with convex values, Michael's Theorem implies that for all $z_0 \in \mathrm{Dom}(V)$ and $v_0 \in V(z_0)$, there exists a continuous selection $\tilde{v}(\cdot)$ of V such that $v(z_0) = v_0$. Then Φ enjoys the

Lyapunov property for the set-valued map $H_{\tilde{v}}(z) := h(z, U(z), \tilde{v}(z))$, and thus, there exists $u_0 \in U(z_0)$ such that

$$D_\uparrow \Phi(z_0)(h(z_0, u_0, \tilde{v}(z_0))) \leq 0$$

Hence Φ is a solution to (6).

Conversely, assume that Φ is a solution to (6). Then for all closed-loop control $\tilde{v}$, the set-valued map $H_{\tilde{v}}$ satisfies the assumptions of the theorem characterizing Lyapunov functions, so that there exists a solution to the inclusion $z' \in H_{\tilde{v}}(z)$ for all initial state $z \in \mathrm{Dom}(\Phi)$ satisfying $\forall\ t \geq 0,\ \Phi(z(t)) \leq \Phi(z)$.

— Consider finally the fourth statement. Assume that Xavier's can find a continuous closed-loop control $\tilde{u}$ such that for all closed-loop control $\tilde{v}$, Φ enjoys the stated property. Since V is lower semicontinuous with convex values, Michael's Theorem implies that for all $z_0 \in \mathrm{Dom}(V)$ and $v_0 \in V(z_0)$, there exists a continuous selection $\tilde{v}(\cdot)$ of V such that $v(z_0) = v_0$. Since for any continuous closed-loop control $\tilde{v}(\cdot)$, Φ enjoys the Lyapunov property for the single-valued map $z \to h(z, \tilde{u}(z), \tilde{v}(z))$, we deduce that for all $z_0 \in \mathrm{Dom}(\Phi)$, that there exists $u := \tilde{u}(z)$ such that for all $v \in V(z)$, $D_\uparrow \Phi(z)(h(x, u, v)) \leq 0$, so that Φ is a solution to (6).

Conversely, assume that the set-valued map B_Φ is lower semicontinuous with closed convex values. Hence Michael's Theorem implies that there exists a continuous selection $\tilde{u}$ of B_Φ. Then for any continuous closed-loop control $\tilde{v}(\cdot) \in V(\cdot)$, we deduce from (8) that Φ is a Lyapunov function for the single-valued map $z \to h(z, \tilde{u}(z), \tilde{v}(z))$, so that, for all $z \in \mathrm{Dom}(\Phi)$, there exists a solution $z(\cdot)$ to the system (9) satisfying $\forall\, t \geq 0,\ \Phi(z(t)) \leq \Phi(z)$. □

3 Characterization of some behavioral properties

Let L be a closed subset of the intersection K of the domains of U and V. The problem we investigate is to find that a (or all) solution(s) $z(\cdot)$ of the game which is (are) viable in L. There are several ways to achieve that purpose, according to the cooperative or non cooperative behavior of the players. We shall examine here six of them.

Definition 3.1 *We shall say the a subset L enjoys:*

1 — the "playability property" if and only if

$$\forall z \in L, \exists z(\cdot) \in S(z) \mid \forall t \geq 0, \; z(t) \in L$$

2 — the "winability property" if and only if

$$\forall z \in L, \forall \; z(\cdot) \in S(z), \forall t \geq 0, \; z(t) \in L$$

3 — "Xavier's discriminating property" if and only if for any continuous closed-loop control $\tilde{v}(z) \in V(z)$ played by Yves and any initial state $z \in L$, there exists a solution $z(\cdot)$ to Xavier's control problem

$$(10) \qquad \begin{cases} i) \quad z'(t) \; = \; h(z(t), u(t), \tilde{v}(z(t))) \\ ii) \; u(t) \; \in \; U(z(t)) \end{cases}$$

starting at z and which is viable in L.

4 — "Xavier's leading property" if and only if Xavier can play a closed-loop control $\tilde{u}(z) \in U(z)$ such that, for any continuous closed-loop control $\tilde{v}(z) \in V(z)$ played by Yves and for any initial state $z \in L$, there exists a solution $z(\cdot)$ to (9) starting at z and viable in L.

We shall characterize these properties: for that purpose we associate with L the following set-valued maps:

— The regulation map R_L defined by

$$\forall z \in L, \; R_L(z) := \{ \; (u,v) \in U(z) \times V(z) \mid h(z,u,v) \in T_L(z) \; \}$$

— Xavier's discriminating map A_L defined by

$$\forall z \in L, \; A_L(z,v) := \{ \; u \in U(z) \mid (u,v) \in R_L(z) \; \}$$

— Xavier's leading map B_L defined by

$$\forall z \in L, \; B(z) \; := \; \bigcap_{v \in V(z)} A_L(z,v)$$

Definition 3.2 *We shall say that*

— *L is playability domain if* $\forall z \in L, \; R_L(z) \; \neq \; \emptyset$

— *L is a winability domain if*

$\forall\, z \in L,\ R_L(z) := U(z) \times V(z)$

— *L is a Xavier's discriminating domain if*

$$\forall\, z \in L,\ \forall\, v \in V(z),\ A_L(z,v) \neq \emptyset \tag{11}$$

— *L is a Xavier's leading domain if* $\forall\, z \in L,\ B_L(z) \neq \emptyset$

We begin by translating these properties in terms of contingent Isaacs equations:

Proposition 3.3 *Let us assume that $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ is continuous, has linear growth , and that the set-valued maps are closed with linear growth.*

— *L is playability domain if and only if Ψ_L is a solution to (4)*

— *L is a winability domain if and only if Ψ_L is a solution to (5)*

— *L is a discriminating domain for Xavier if and only if Ψ_L is a solution to (6)*

— *L is a leading domain for Xavier if and only if Ψ_L is a solution to (8)*

Therefore, Theorem 2.1 implies the following characterization of these domains:

Corollary 3.4 *Let us assume at least that $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ is continuous, has linear growth , and that the set-valued maps are closed with linear growth.*

1 — If the values of the set-valued maps U and V are convex and if h is affine with respect to the controls, then L enjoys the playability property if and only if it is a playability domain.

2 — Assume that h is uniformly lipschitzean with respect to x. Then L enjoys the winability property if and only if it is a winability domain.

3 — Assume that V is lower semicontinuous, that the values of U and V are convex and that h is affine with respect to u. Then L enjoys Xavier's discriminating property if and only if it is a discriminating domain for Xavier.

4 — Assume that V is lower semicontinuous with convex values. If L enjoys Xavier's leading property, then it is a leading domain for him. The converse is true if B_L is lower semicontinuous with closed convex values.

The existence theorems of the viability and invariance kernels imply the following consequence:

Proposition 3.5 *Let us assume that $h : \mathbf{R}^n \times \mathbf{R}^p \times \mathbf{R}^q \to \mathbf{R}^n$ is continuous, has linear growth , and that the set-valued maps are closed with linear growth.*

1 — If the values of the set-valued maps U and V are convex and if h is affine with respect to the controls, then there exists a largest closed playability domain contained in L, whose indicator is the smallest lower semicontinuous solution to (4) larger than or equal to the indicator Ψ_L of L.

2 — Assume that h is uniformly lipschitzean with respect to x. Then there exists a largest closed winability domain contained in L, whose indicator is the smallest lower semicontinuous solution to (5) larger than or equal to the indicator Ψ_L of L.

Remark — The question whether there are largest **closed** discriminating and leading domains remains an open question. □

4 Appendix: Lower Semicontinuous Lyapunov Functions

We consider now a differential inclusion

$$(12) \qquad \text{for almost all } t \geq 0, \;\; x'(t) \;\in\; F(x(t))$$

and time-dependent functions $w(\cdot)$ defined as solutions to a differential equation

$$(13) \qquad w'(t) = -\phi(w(t)), \;\; w(0) \;=\; V(x(0))$$

where $\phi : \mathbf{R}_+ \to \mathbf{R}$ is a given continuous function with linear growth. This function ϕ is used as a parameter in what follows.

The main instance of such a function ϕ is the affine function $\phi(w) := aw - b$, the solutions of which are $w(t) = (w(0) - \frac{b}{a})e^{-at} + \frac{b}{a}$.

Our problem is to characterize either ϕ -Lyapunov functions, i.e., nonnegative extended functions $V : X \to \mathbf{R}_+ \cup \{+\infty\}$ satisfying

$$(14) \qquad \forall\, t \geq 0,\ \ V(x(t)) \ \leq\ w(t),\quad w(0) = V(x(0))$$

along at least a solution to the differential inclusion (12) or ϕ- universal Lyapunov functions, wich satisfy property (14) along all solutions to (12).

Definition 4.1 *We shall say that a nonnegative contingently epidifferentiable*[7] *extended function V is a* **Lyapunov function** *of F associated with a function $\phi(\cdot) : \mathbf{R}_+ \mapsto \mathbf{R}$ if and only if V is a solution to the contingent Hamilton-Jacobi inequalities*

$$(15) \qquad \forall\, x \in \mathrm{Dom}(V),\ \ \inf_{v \in F(x)} D_\uparrow V(x)(v) + \phi(V(x)) \ \leq\ 0$$

and a **universal Lyapunov function** *of F associated with a function ϕ if and only if V is a solution to the upper contingent Hamilton-Jacobi inequalities*

$$(16) \qquad \forall\, x \in \mathrm{Dom}(V),\ \ \sup_{v \in F(x)} D_\uparrow V(x)(v) + \phi(V(x)) \ \leq\ 0$$

Theorem 4.2 *Let V be an nonnegative contingently epidifferentiable extended function and $F : X \rightsquigarrow X$ be a nontrivial set-valued map.*

— Let us assume that F is upper semicontinuous with compact convex images and linear growth. Then V is a Lyapunov function of F associated with $\phi(\cdot)$ if and only if for all initial state $x_0 \in Dom(V)$, there exist solutions $x(\cdot)$ to differential inclusion (12) and $w(\cdot)$ to differential equation (13) satisfying property (14).

— If F is lipschitzean on the interior of its domain with compact values, then V is a universal Lyapunov function associated with ϕ if and only if for all initial state $x_0 \in Dom(V)$, all solutions $x(\cdot)$ to differential inclusion (12) and $w(\cdot)$ to differential equation (13) do satisfy property (14).

The proof is based on the viability and invariance theorems of the closed subset EpV for the differential inclusion:

$$(17) \qquad \begin{cases} i) & x'(t) \in F(x(t)) \\ ii) & w'(t) = -\phi(w(t)) \end{cases}$$

[7] This means that for all $x \in \mathrm{Dom}(V)$, $\forall\ v \in X$, $D_\uparrow V(x)(v) > -\infty$ and that $D_\uparrow V(x)(v) < \infty$ for at least a $v \in X$.

and these viability and invariance theorems can be reformulated the in the following way:

Corollary 4.3 *Let $F : X \rightsquigarrow X$ be a nontrivial set-valued map.*

— *Let us assume that F is upper semicontinuous with compact convex images and linear growth.*

A closed subset K enjoys the viability property if and only if its indicator Ψ_K is a solution to the contingent equation

$$\inf_{v \in F(x)} D_\uparrow \Psi_K(x)(v) \; = \; 0$$

— *If F is lipschitzean on the interior of its domain with compact values, then K is invariant by F if and only if its indicator Ψ_K is a solution to the contingent equation*

$$\sup_{v \in F(x)} D_\uparrow \Psi_K(x)(v) \; = \; 0$$

The functions ϕ and $U : X \to \mathbf{R}_+ \cup \{+\infty\}$ being given, can we construct the smallest lower semicontinuous Lyapunov function of a set-valued map F associated to ϕ larger than or equal to U, i.e., the smallest nonnegative lower semicontinuous solution U_ϕ to the contingent Hamilton-Jacobi inequalities (15) larger than or equal to U?

Theorem 4.4 *Let us consider a nontrivial set-valued map $F : X \rightsquigarrow X$, a continuous function $\phi : \mathbf{R}_+ \to \mathbf{R}$ with linear growth and a proper nonnegative extended function U.*

— *Let us assume that F is upper semicontinuous with compact convex images and linear growth. Then there exists a smallest nonnegative lower semicontinuous solution $U_\phi : Dom(F) \mapsto \mathbf{R} \cup \{+\infty\}$ to the contingent Hamilton-Jacobi inequalities (15) larger than or equal to U (which can be the constant $+\infty$), which then enjoys the property:*

$$\begin{array}{l} \forall\, x \in \mathrm{Dom}(U_\phi), \quad \text{there exists solutions to (13) and (14) satisfying} \\ \qquad\qquad \forall\, t \geq 0, \;\; U(x(t)) \; \leq \; U_\phi(x(t)) \; \leq \; w(t) \end{array}$$

— *If F is lipschitzean on the interior of its domain with compact values and ϕ is lipschitzean, then there exists a smallest nonnegative lower* *semicontinuous solution* $\widetilde{U}_\phi : Dom(F) \mapsto \mathbf{R} \cup \{+\infty\}$ *to the upper contingent Hamilton-Jacobi inequalities (13) larger than or equal to* U *(which can be the constant* $+\infty$*), which then enjoys the property:*

$$\begin{array}{c} \forall\, x \in \mathrm{Dom}(U_\phi), \quad \text{all solutions to (13) and (14) satisfy} \\ \forall\, t \geq 0, \;\; U(x(t)) \;\leq\; U_\phi(x(t)) \;\leq\; w(t) \end{array}$$

In particular, for $\phi(w) := aw$, we deduce that

$$\forall\, x \in \mathrm{Dom}(U_a),\; U(x(t)) \leq U_a(x_0)e^{-at} \text{ and thus, converges to } 0$$

The proof amounts to show that the largest closed viability domain (invariance domain) contained in the epigraph of U, called the viability kernel (invariance kernel) of $\mathrm{Ep}(U)$, which does exist under the assumptions of the first (second) part of the theorem, is actually an epigraph, and thus, the one of the smallest lower semicontinuous (universal) Lyapunov function. Actually, the existence theorems of these kernels are equivalent to the above theorem, since it implies the following

Corollary 4.5 *We posit the assumptions of Theorem 4.4.*

— *Let us assume that* F *is upper semicontinuous with compact convex images and linear growth.*

The indicator $\Psi_{\mathrm{Viab}(K)}$ *of the viability kernel* $\mathrm{Viab}(K)$ *of a closed subset* K *(i.e., the largest closed viability domain of* F *contained in* K*) is the smallest nonnegative lower semicontinuous solution to*

$$\forall\, x \in \mathrm{Dom}(V), \;\; \inf_{v \in F(x)} D_\uparrow V(x)(v) \;\leq\; 0 \tag{18}$$

larger than or equal to Ψ_K.

— *Assume that* F *is lipschitzean on the interior of its domain with compact values.*

The indicator $\Psi_{\mathrm{Inv}(K)}$ *of the invariant kernel* $\mathrm{Inv}(K)$ *of a closed subset* K *(i.e., the largest closed invariance domain of* F *contained in* K*) is the smallest nonnegative lower semicontinuous solution to*

$$\forall\, x \in \mathrm{Dom}(V), \;\; \sup_{v \in F(x)} D_\uparrow V(x)(v) \;\leq\; 0 \tag{19}$$

larger than or equal to Ψ_K.

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