



# Controllability of Discontinuous Systems

**Veliov, V.M. and Krastanov, M.**

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# ***WORKING PAPER***

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## Foreword

This report presents an approach to the local controllability problem for a discontinuous system. The approach is based on a concept of tangent vector field to a generalized dynamic system, which makes possible the differential geometry tools to be applied in the discontinuous case. Sufficient controllability conditions are derived.

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# CONTROLLABILITY OF DISCONTINUOUS SYSTEMS

V. Veliov and M. Krastanov

## 1. Introduction

This paper deals with the instantaneous local controllability (ILC) of a discontinuous control system. The differential geometry approach, which is widely used in the local analysis of control systems is not directly applicable to the discontinuous (non-smooth) case. The only results known to the authors which concern the local controllability of a system with non-differentiable right-hand side, are those of Sztajnic and Walczak [13], where, however, specific approximations by linear systems are used. In our paper [12] a necessary and sufficient condition for ILC is given for piece-wise linear systems. Here we present two approaches which give the opportunity to study the ILC property of a discontinuous system (the system model and the corresponding definitions are given in Section 2) by differential geometry tools. The first approach (presented briefly in Section 3) is based on the construction of an appropriate collection of smooth vector fields, whose controllability properties are equivalent to those of the original discontinuous system, and which can be studied by smooth analysis. The second approach (Section 4) is more general and in our opinion can be applied also to other more specific ILC problems (e.g. in the state constrained case). It is based on the concept of "tangent" vector field to a generalized dynamic system, which is not connected with a priori given family of smooth vector fields (cf. Hermes [6], Kunita [9] and Hirshorn [7]). This approach enables us to derive sufficient ILC conditions for a discontinuous system (section 5), where the attainable set may not be generated by a family of smooth vector fields.

For convenience all proofs are collected in Appendix.

## 2. Discontinuous System Model.

Consider a control system described by the following model:

$$\dot{x} = A_1(x) + B_1(x)U_1 \quad \text{if } x \in \pi_1 \subset \mathbb{R}^n, \quad (1a)$$

$$\dot{x} = A_2(x) + B_2(x)U_2 \quad \text{if } x \in \pi_2 \subset \mathbb{R}^n, \quad (1b)$$

where  $x \in \mathbb{R}^n$  is the state vector,

$$u_1 \in U_1 \subset \mathbb{R}^r, \quad u_2 \in U_2 \subset \mathbb{R}^r \quad (2)$$

are control parameters,  $A_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and  $B_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$ ,  $i = 1, 2$ , are given functions. Equation (1a) describes the evolution of the system in the domain  $\pi_1 \subset \mathbb{R}^n$ , while (1b) describes the evolution in  $\pi_2$ . Further we suppose that the sets  $\pi_1$  and  $\pi_2$  are defined by the manifold

$$\pi = \{x \in \mathbb{R}^n ; f(x) = 0\},$$

where  $f(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}$  is a given analytic function, namely

$$\pi_1 = \{x \in \mathbb{R}^n ; f(x) \geq 0\}, \quad \pi_2 = \{x \in \mathbb{R}^n ; f(x) \leq 0\}$$

Since we are interested in the local properties of system (1) around a given point  $x_0$ , we suppose that  $f(x_0) = 0$  and  $f'(x_0) = \partial f / \partial x(x_0) \neq 0$ , the discontinuity set  $\pi$  being then locally an analytic manifold.

Systems of the type of (1) - (2) arise in modeling of physical or economical objects, where some parameters (or the structure of the system) change discontinuously on a given manifold in the state space (depending on some critical speeds, voltages, masses, etc), as well as in case of models, containing maximum of two smooth functions in the right-hand side of the differential equation.

Our concept for a trajectory of (1) is based on the supposition that movements in accordance to both equations in (1) are possible on the manifold  $\pi$ . The definition given below is related to that of Filippov [4] and is natural when  $u_1$  and  $u_2$  are interpreted as uncertain inputs. Nevertheless, under our further assumptions, the main controllability result concerning system (1), (2) would no be affected if the movement on  $\pi$  is governed by one (fixed) of the equations (1a) and (1b), or even (with appropriate changes in the formulations) if the evolution on  $\pi$  is described by a specific equation of the same type.

**Definition 1.** The absolutely continuous function  $x(\cdot) : [0, T] \rightarrow \mathbb{R}^n$  is called trajectory of (1), (2), if

$$\dot{x}(t) \in F(x(t)) \quad \text{for a.e. } t \in [0, T] \quad (3)$$

where the multifunction  $F(\cdot) : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$  is defined by

$$F(x) = \begin{cases} A_1(x) + B_1(x)U_1 & , x \in \pi_1 \setminus \pi \\ A_2(x) + B_2(x)U_2 & , x \in \pi_2 \setminus \pi \\ \text{co}\{A_1(x) + B_1(x)U_1, A_2(x) + B_2(x)U_2\} & , x \in \pi \end{cases} .$$

Denote by

$$R(x, T) = \{y \in \mathbb{R}^n ; \text{there exists a trajectory } x(\cdot) \text{ of (1), (2) on } [0, T], \text{ such that}$$

$$x(0) = x, x(T) = y\}$$

the attainable set of (1), (2) on  $[0, T]$ , starting from the point  $x$  at  $t = 0$ .

**Definition 2.** System (1), (2) is called instantly locally controllable (ILC) at  $x_0$ , if

$$x_0 \in \text{int } R(x_0, T)$$

for every  $T > 0$ .

Our aim in this paper is to develop an appropriate techniques for investigation of the ICL property of (1), (2). However, the approach presented in Section 4 is much more general and can be useful also in the "classical" affine analytic case, as well as in the control or state constrained cases (shown in forthcoming publications).

### 3. Reduction to ILC Problem for a Smooth System.

In the present section we shall transform system (1), (2) to a more convenient form without changing its ILC property. Then we shall construct a family of analytic vector fields, which can be considered as generating family for a specific projection of the attainable set  $R(x_0, T)$  on the manifold  $\pi$ . Then the ILC property of (1), (2) can be deduced from the ILC property of this family.

Some notations:

$$|\cdot|, \langle \cdot, \cdot \rangle - \text{the norm and the scalar product in } \mathbb{R}^n ;$$

$$O_\epsilon = \{v \in \mathbb{R}^n ; |v| \leq \epsilon\} ;$$



$F(V) = \{w \in \mathbb{R}^n ; \alpha w \in V \text{ for all sufficiently small } |\alpha|\}$

- the facial space of the convex set  $V \subset \mathbb{R}^n$ .

**Assumptions**

A1.  $f(\cdot)$  is analytic,  $f(x_0) = 0$  ,  $f'(x_0) \neq 0$ ;

A2.  $A_i(\cdot)$  ,  $B_i(\cdot)$  are analytic,  $A_i(x_0) = 0$  ,  $i = 1,2$ ;

A3.  $U_i$  is closed and convex and  $f'(x_0)$  is not orthogonal to the set  $B_i(x_0)F(U_i)$  ,  $i = 1,2$ .

Assumptions A1 and A2 are standard, while A3 needs some comment. It is a technical condition, which guarantees that starting from a point being sufficiently closed to  $x_0$ , the state of (1) can reach the manifold  $\pi$ , as well as the interior of any of the "half spaces"  $\pi_1$  and  $\pi_2$ . That is seen from the following assertion: if A3 holds, then there exists  $p_i \in \mathbb{R}^r$  and convex compact  $\tilde{U}_i \subset \mathbb{R}^r$  containing the origin, such that

$$V_i = \{u_0 p_i + \tilde{U}_i ; u_0 \in [-1,1]\} \subset U_i , \quad i = 1,2 , \quad (4)$$

$$\langle B_1(x_0)p_1 , f'(x_0) \rangle = \alpha_1 > 0 \quad (5)$$

$$\langle B_2(x_0)p_2 , f'(x_0) \rangle = \alpha_2 > 0 .$$

Since we are interested in sufficient controllability conditions for (1), (2), we may replace the control constraints  $u_i \in U_i$  by  $u_i \in V_i$ , obtaining thus a system in the following form

$$\dot{x} = A_1(x) + B_1(x)u_1 + \gamma_1(x)u_0 \quad \text{if } x \in \pi_1 \quad (6a)$$

$$\dot{x} = A_2(x) + B_2(x)u_2 + \gamma_2(x)u_0 \quad \text{if } x \in \pi_2 \quad (6b)$$

$$u_i \in \tilde{U}_i , \quad u_0 \in [-1,1] , \quad (7)$$

where

$$\gamma_i(x) = B_i(x)p_i , \quad i = 1,2 .$$

**Remark 1.** The natural question whether the ILC property of (6), (7) not only implies, but is equivalent to the ILC property of (1), (2) is open, but the analogical question is not solved for analytic systems too.

Define the functions  $P_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^n$  ,  $Q_i(\cdot) : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times r}$  and  $R_i(\cdot) : \mathbb{R}^1 \rightarrow \mathbb{R}^n$  ,  $i = 1,2$ , by

$$\begin{aligned}
 P_i(x) &= A_i(x) - \frac{\langle A_i(x), f(x) \rangle}{\langle \gamma_i(x), f(x) \rangle} \gamma_i(x) , \\
 Q_i(x) &= B_i(x) - \frac{\gamma_i(x) f^*(x) B_i(x)}{\langle \gamma_i(x), f(x) \rangle} , \\
 R_i(x) &= \frac{\gamma_i(x)}{\langle \gamma_i(x), f(x) \rangle} ,
 \end{aligned} \tag{8}$$

where \* means the transposition. Now, consider the system

$$\dot{x} = P_1(x) + Q_1(x)u_1 + R_1(x)v \quad \text{if } x \in \pi_1 \tag{9a}$$

$$\dot{x} = P_2(x) + Q_2(x)u_2 + R_2(x)v \quad \text{if } x \in \pi_2 \tag{9b}$$

$$u_i \in 0_\alpha \cap \tilde{U}_i , \quad v \in [-\beta, \beta] , \tag{10}$$

where  $\alpha > 0$  and  $\beta > 0$  are parameters.

**Proposition 1.** If (6), (7) is ILC, then (9), (10) is ILC for  $\alpha$  and  $\beta$  sufficiently large. If (9), (10) is ILC for arbitrarily small positive  $\alpha$  and  $\beta$ , then (6), (7) is also ILC.

The above proposition shows that ILC property of (9), (10) for every  $\alpha > 0$ ,  $\beta > 0$  implies ILC of (1), (2) (a remark analogous to Remark 1 can be made in connection with the equivalence). The advantage of (9), (10) is contained in the following relations:

$$\begin{aligned}
 \langle P_i(x), f(x) \rangle &= 0 \\
 \langle Q_i(x)u, f(x) \rangle &= 0 \quad \text{for every } u \in \mathbb{R}^r \\
 \langle R_i(x), f(x) \rangle &= 1
 \end{aligned} \tag{11}$$

which hold for every  $x$  from a neighborhood of  $x_0$ . Thus if  $v = 0$ , then the right-hand sides of (9) are vector fields on  $\pi$ . To the end of this section we shall outline the idea of passing to a family of vector fields on  $\pi$ , with attainable set - the intersection of the attainable set of (9), (10) with  $\pi$ . The details will be omitted since further we shall use the more general approach developed in the next section. We shall use same notations and results concerning exponential presentations of flows, which are developed in Agrachev and Gamkrelidze [1].

Let  $M$  be a  $C^\infty$  - finite dimensional manifold and  $Der(M)$  be the algebra of the vector fields on  $M$ . Given a function  $Y(\cdot) : [t_0, T] \rightarrow Der(M)$  we denote by

$$\exp \int_{t_0}^t Y(\tau) d\tau = K(t) : M \rightarrow M$$

the diffeomorphism (provided  $Y$  is in the some sense locally integrable and bounded) defined by  $K(t)x = x(t)$ ,  $x(\cdot)$  being the solution of the equation

$$\dot{x}(\cdot) = Y(x(\cdot), \cdot) , x(t_0) = x .$$

If  $Y(t) \equiv Y$  we use the notation  $\exp(tY)$  for the same diffeomorphism. Given  $X \in Der(M)$  we denote

$$adX(Y) = [X, Y] = X \circ Y - Y \circ X , Y \in Der(M)$$

If  $A : Der(M) \rightarrow Der(M)$  is linear and  $v(\cdot) : [t_0, T] \rightarrow \mathbb{R}^1$  is integrable, then for  $X \in Der(M)$ .

$$e^{\int_{t_0}^t v(s) ds A} X \in Der(M)$$

can be defined by the expansion of the exponent, using a natural weak topology in  $Der(M)$ .

Now, let us return to system (9),(10). From the last equality in (11) we can conclude that its ILC property is equivalent to

$$x_0 \in \text{int} (R(x_0, t) \cap \pi) , t > 0 ,$$

where the interior is in  $\pi$ . Let  $x(\cdot)$  be a trajectory of (9), (10) on  $[0, T]$ , and suppose that  $[t_1, t_2]$  is an interval in which (9a) is satisfied for some  $u_1(t) = u(t) \in O_\alpha \cap \tilde{U}_1$  and  $v(t) \in [-\beta, \beta]$ , and moreover,  $x(t_1), x(t_2) \in \pi$ . Using the composition formula proved in [1] we obtain that for  $t \in [t_1, t_2]$

$$x(t) = \exp \int_{t_1}^t v(s) R_1 ds \circ \exp \int_{t_1}^t e^{\int_{t_1}^r v(s) ds ad R_1} (P_1 + Q_1 u(t)) dr (x(t_1)) . \quad (12)$$

Denoting  $y(t) = f(x(t))$  we have  $y(t_1) = y(t_2) = 0$  and from (11)

$$\dot{y}(t) = \langle f'(x(t)) , v(t) R_1(x(t)) \rangle = v(t) .$$

Since  $y(t) \geq 0$  in  $[t_1, t_2]$  we get

$$\int_{t_1}^t \dot{y} = y(t) - y(t_1) \geq 0 , t \in [t_1, t_2] .$$

For  $t=t_2$  we obtain

$$\int_{t_1}^{t_2} v(s) ds = 0 \quad \text{and thus} \quad \exp \int_{t_1}^{t_2} v(s) R_1 ds = Id.$$

The last equality, combined with (12) means that the point  $x(t_1)$  is transferred to  $x(t_2)$  on  $[t_1, t_2]$  also along the trajectory of the (nonstationary) vector field

$$X(t) = e^{\int_{t_1}^t v(s) ds ad R_1} (P_1 + Q_1 u(t)) , \quad (13)$$

where  $u(\cdot)$  and  $v(\cdot)$  satisfy

$$u(t) \in 0_\alpha \cap \tilde{U}_1 , \quad v(s) \in [-\beta, \beta] , \quad \int_{t_1}^t v(s) ds \geq 0 , \quad \int_{t_1}^{t_2} v(s) ds = 0 . \quad (14)$$

Repeating backwards the same arguments we see that if a point  $x_1 \in \pi_1$  can be steered to  $x_2 \in \pi$  on  $[t_1, t_2]$  by a vector field of the type (13) with  $u$  and  $v$  satisfying (14), then  $x_2$  can be reached from  $x_1$  on  $[t_1, t_2]$  also according to system (9), (10). Thus, it is possible to investigate the ILC property of (1), (2) by examining the same property of the family of nonstationary vector fields on  $\pi$  described by (13),(14) (the fact that  $x$  given by (13) is a vector field on  $\pi$  follows from (11)). Unfortunately, this family is rather sophisticated, because of the integral control constraints, the time dependence and the nonlinear dependence on the control parameters. However, it is possible to simplify it, considering

$$w(t) = \int_{t_1}^t v(s) ds$$

as a new control variable, constrained by  $0 \leq w \leq M$ , obtaining this way the family

$$\{ e^{w ad R_i} (P_i + Q_i u) ; w \in [0, M] , u \in \tilde{U}_i \cap 0_\alpha \}$$

The technical complications arising in the above treatment will be overcome by combining the same idea with the general approach presented in the next section.

#### 4. Local Controllability of Generalized Dynamic Systems.

Let  $M$  be a finite dimensional real analytic manifold and  $D(\cdot, \cdot) : \mathbb{R}_+ \times M \rightrightarrows M$  be a multivalued mapping satisfying the weakened semigroup property

$$D(D(x, t_1), t_2) \subset D(x, t_1 + t_2) \quad (15)$$

for all  $x \in M$ ,  $t_1, t_2 \geq 0$ . In the above relation and further we use the notation

$$D(P, t) = \bigcup \{D(p, t) ; p \in P\}$$

Let  $x_0 \in M$  be an equilibrium point for  $D$ , i.e.

$$x_0 \in D(x_0, t) \text{ for every } t \geq 0. \quad (16)$$

**Definition 3.** The generalized dynamic system defined by  $D$  is called ILC, if

$$x_0 \in \text{int } D(x_0, t) \text{ for every } t > 0. \quad (17)$$

**Remark 2.** From (15) it follows that any of the relations (16) and (17) is fulfilled, if it holds for all sufficiently small  $t > 0$ . Moreover,  $D(x_0, \cdot)$  is monotone in  $t$  with respect to inclusion.

We shall study the ILC property of  $D$  under the following conditions:

B1.  $D(x, t)$  is closed for every  $t \in [0, T_0]$ ,  $x \in M$  ( $T_0$  is some given fixed positive number), and satisfies (15) and (16).

Together with  $Der(M)$  defined in Section 3, we shall use

$$Der_0(M) = \{Z \in Der(M) ; Z(x_0) = 0\}$$

(the isotropy subalgebra of  $Der(M)$  at  $x_0$ ) and for  $F \subset Der(M)$

$$F(x) = \{Z(x) : Z \in F\} \subset M_x,$$

$M_x$  being the tangent space to  $M$  at  $x$ . We shall use the norm  $\|\cdot\|$  in  $Der(M)$ , defined as  $\|X\| = \sup \{|X(x)| ; x \in K_0\}$ , where  $K_0$  is a fixed neighborhood of  $x_0$ .

By  $o(t), o'(t), \dots$  we shall indicate any family (parametrized by  $t$ ) of vector fields on  $M$ , which is continuous in  $t$  and for some  $p > 1$  the ratio  $\|o(t) / t^p\|$  is bounded when  $t$  goes to zero. A function of the type

$$\sum_{i=1}^m \alpha_i t^{q_i}$$

with  $\alpha_i \geq 0$ ,  $q_i > 0$  will be called positive polynomial.

Let us define the following subset  $E^+$  of  $Der(M)$ .

**Definition 4.**  $Z \in E^+$  iff there exist  $T \in (0, T_0]$ , compact neighborhood  $K$  of  $x_0$ , finite number of  $Z_i \in Der_0(M)$  ( $i = 1, \dots, l$ ),  $o(t)$  and positive polynomials  $S(t)$  and

$$\sum_{i=1}^l p_i t^{q_i}, \text{ such that}$$

$$\exp (tZ + a(t) + o(t))x \in D(x, S(t)) \text{ for every } x \in K, t \in [0, T], \quad (18)$$

where

$$a(t) = \sum_{i=1}^1 p_i t^{q_i} Z_i .$$

The above definition is related to the one given in Hermes [6] for a specific affine control system. The main difference here is that our definition is not based on a priori given family of vector fields. The set  $D(x, t)$  may not be generated by such a family as in the case of a discontinuous system. As it will be seen from the proofs, an important technical difference is the presence of the sufficiently general term  $a(t)$  in the lefthand side of (18).

On the other hand Definition 4 is related also to the concept of variational cone, introduced by Frankowska [5]. Namely, if  $Z \in E^+$ , then

$$\lim_{t \rightarrow 0} \text{dist} (Z(x_0), \frac{D(x_0, S(t)) - x_0}{t}) = 0 ,$$

where *dist* is defined in an arbitrarily fixed local coordinate chart on  $M$ .

**Theorem 1.** Let

$$0 \in \text{int } E^+(x_0) . \quad (19)$$

then  $x_0 \in \text{int } D(x_0, t)$  for every  $t > 0$ , i.e. the generalized dynamic system  $D$  is ILC at  $x_0$ .

Condition (19) can be replaced by

$$0 \in \text{int } \text{co} \{Z_i(x_0) ; i = 1, \dots, k\} \quad (20)$$

for some  $Z_1, \dots, Z_k \in E^+$ , as it follows from the following lemma.

**Lemma 1.**  $E^+$  is a convex cone.

Lemma 1 throws some light on the structure of the set  $E^+$ , but gives no information about its content, which to help to the verification of (20). For this reason we introduce a subset  $F^+$  of  $E^+$ , which can be used for finding "new" elements of  $E^+$ , provided that  $F^+$  and some other elements of  $E^+$  are already known.

**Definition 5.**  $Z \in F^+$  iff there are  $T \in (0, T_0]$ , a compact neighborhood  $K$  of  $x_0$  and a real number  $s$ , such that

$$\exp (tZ) x \in D(x, st) \text{ for every } t \in (0, T] \text{ and } x \in K. \quad (21)$$

Vector fields with the property (21) are used by Kunita [9] and Hirshorn [7]. As in [9] it can be proved that  $F^+$  is a convex cone.

An obvious but very useful construction of elements of  $E^+$  is given by the next lemma.

**Lemma 2.** Let  $Z_1, \dots, Z_k \in F^+$ . Consider the diffeomorphism

$$K(t) = \exp(tZ_k) \circ \dots \circ \exp(tZ_1).$$

By using Campbell - Hansdorff formula (see (25) in the appendix)  $K(t)$  can always be presented in the form

$$K(t) = \exp\left(\sum_{i=1}^p t^i Y_i + t^{p+1} X + o(t^{p+1})\right),$$

where  $Y_i(x_0) = 0, i=1, \dots, p$ . Then  $X$  belongs to  $E^+$ . The same is true for arbitrary vector fields  $Z_1, \dots, Z_k$  such that  $K(t)x \in D(x, s(t))$  for some positive polynomial  $s(\cdot)$ .

The next lemma is related to first order ILC conditions and if  $D(x_0, t)$  is the attainable set of an analytic affine control system it directly leads to the result of Hermes [6].

**Lemma 3.** Let  $Y \in F^+, Z_1, Z_2 \in E^+$  and

$$Z_1(x_0) + Z_2(x_0) = 0, Y(x_0) = 0.$$

then  $[Y, Z_1], [Y, Z_2] \in E^+$ .

**Lemma 4.** Let  $Z_i \in F^+, Z_i(x_0) \neq 0, i=1, \dots, k$  and

$$\sum_{i=1}^k Z_i(x_0) = 0$$

Then  $\pm [Z_i, Z_j] \in E^+$  for  $i, j = 1, \dots, k$ .

The above result is an analogy to a result of Sussmann [10] and can be proved by the same idea.

**Lemma 5.** Let  $Z_1, Z_2 \in F^+$  and

$$Z_1(x_0) + Z_2(x_0) = 0.$$

Then

$$[Z_1, [Z_1, Z_2]] + [Z_2, [Z_2, Z_1]] \in E^+.$$

One can derive also other rules for constructing "new" elements of the set  $E^+$ . Every such construction, together with Theorem 1 gives a sufficient condition for ILC. That is obvious for affine systems

$$\dot{x} = A(x) + \sum_{i=1}^k B_i(x)u_i \quad , \quad u_i \in [-1,1] \quad , \quad A(x_0) = 0 \quad ,$$

since  $A + B_i u \in F^+$  and  $B_i U \in E^+$  for every  $u \in [-1,1]$ . In the next section we shall show how one can use the above results in case of a discontinuous system.

### 5. Sufficient ILC Conditions for a Discontinuous System.

Let us return again to the system (1), (2). Instead of it, using Proposition 1, we shall deal with system (9), (10) where  $\alpha$  and  $\beta$  will be arbitrarily fixed. Denote by  $R(x,t)$  its attainable set on  $[0,t]$ , starting from  $x$  and consider the mapping

$$D(x,t) = R(x,t) \cap \pi$$

From the last equality in (11) it follows that the ILC property of (9), (10) is equivalent to  $x_0 \in \text{int } D(x_0,t)$ ,  $t > 0$ . The weakened semigroup inclusion (15) is obviously satisfied (but not as equality!). Condition B1 is also satisfied because of the convexity and the upper semicontinuity of the mapping  $F$  defined for (9), (10) analogously as in (3). Thus we can apply the results from the previous section to investigate the ILC property of  $D$ .

First of all we shall prove that

$$P_i \in F^+ \quad , \quad Q_i u_i \quad , \quad [P_i, R_i] \in E^+ \quad \text{for } u_i \in \tilde{U}_i \quad , \quad i=1,2 \quad .$$

The first inclusion is obvious, since thanks of (11) Definition 5 is satisfied for  $s = 1$ . Let  $u \in \tilde{U}_i$  and let for convenience  $i = 1$ . Denote by  $x(\cdot)$  the solution of (9a) corresponding to  $u_1 = \alpha u/|u|$  and  $v = 0$ , which starts at some  $x \in \pi$ . From (11) it follows that  $x(\cdot)$  is a solution of (9), (10) and  $x(t) \in D(x,t)$ . Since

$$D(x,t|u/|\alpha) \ni x(t|u/|\alpha) = \exp\left(\frac{t|u|}{\alpha}\right) (P_1 + \alpha Q_1 u/|u|) x$$

and  $P_1(x_0) = 0$ , we get  $Q_1 u \in E^+$ .

Now, take  $u = 0$ ,  $v(s) = 1$  on  $[0,t]$  and  $v(s) = -1$  on  $[t,2t]$ . Denote by  $x(\cdot)$  the corresponding solution of (9a) initiating from  $x \in \pi$ . From (11) it follows that



$$f(x(s)) = f(x) + \int_0^s \langle f'(x(\tau)), \dot{x}(\tau) \rangle d\tau = \int_0^s v(\tau) d\tau \begin{cases} > 0, & s \in (0, 2t) \\ = 0, & s = 2t \end{cases}$$

Thus  $x(\cdot)$  is a trajectory of (9), (10) and  $x(2t) \in \pi$ , which implies  $x(2t) \in D(x, 2t)$ .  
On the other hand

$$x(2t) = \exp(t(P_1 - R_1)) \circ \exp(t(P_1 + R_1))x = \exp(2tP_1 + t^2[P_1, R_1] + o(t^2))x,$$

which together with  $P_1(x_0) = 0$  yields  $[P_1, R_1] \in E^+$ .

Thus we already know two elements  $P_1, P_2 \in F^+$  and the set  $S_0 = \{Q_1u_1, Q_2u_2, [P_1, R_1], [P_2, R_2] : u_1 \in \tilde{U}_1, u_2 \in \tilde{U}_2\}$  of elements of  $E^+$ . Given an integer  $k$ , define a multivalued mapping

$$A_k : Der(M) \rightrightarrows Der(M) \quad \text{by}$$

$$A_k(X) = \{[P_{i_1}, [P_{i_2}, \dots, [P_{i_q}, X] \dots]] ; 0 \leq q \leq k, i_j \in \{1, 2\}\}.$$

Then using lemmas 1 and 3 we obtain that for each  $k_1$

$$A_{k_1}(F(S_0)) \subset E^+,$$

where

$$F(S_0) = \{Y \in S_0 ; Y(x_0) \in F(\text{co}S_0(x_0))\}$$

and as in Section 2,  $F(K)$  is the facial space of the convex set  $K$  at the origin. Using again Lemma 1 we conclude that

$$S_1 = \text{co} \{A_{k_1}(F(S_0)) \cup S_0\} \subset E^+$$

and so on. Thus given a sequence of integers  $k_1, k_2$  we can define the sequence of convex sets  $S_p$  by

$$S_{p+1} = \text{co} \{A_{k_p}(F(S_p)) \cup S_p\},$$

where

$$F(S_p) = \{Y \in S_p ; Y(x_0) \in F(S_p(x_0))\}.$$

Since by induction  $S_p \subset E^+$  for every  $p$ , we obtain

**Theorem 2.** Let conditions A1-A3 hold and  $P_i, Q_i, P_i$  and  $\tilde{U}_i$  be defined by (8) and (4), respectively. If for some integers  $p, k_1, \dots, k_p$  it is fulfilled.

$$\dim F(S_p)(x_0) = n-1, \quad (22)$$

then system (1) is ILC at  $x_0$ .

To prove the theorem it remains only to observe that condition (22) does not depend on  $\alpha$  and  $\beta$  and thus guarantees ICL of (9), (10) for every  $\alpha, \beta > 0$ .

The condition (22) can be simplified if  $Q_i(x_0)\tilde{U}_i$  have non-empty relative interiors.

In this case obviously  $Q_i\tilde{U}_i \subset F(S_0)$ . If we denote

$$A_\infty(X) = \{[P_{i_1}, [P_{i_2}, \dots [P_{i_k}, X] \dots]] ; k \geq 0, i_j \in \{1, 2\}\}$$

**Theorem 3.** Let in addition to the assumptions of Theorem 2 also  $\text{reint}Q_i(x_0)\tilde{U}_i \neq \emptyset$  which holds if  $\text{reint}U_i \neq \emptyset$ . Then each of the following conditions implies ICL of (1), (2):

(i)  $\dim H_0(x_0) = n-1,$

where

$$H_0 = \{A_\infty(Q_i^j); i = 1, 2, j = 1, \dots, \tau\}, Q_i^j - \text{the } j\text{-th column of } Q_i;$$

(ii)  $\lambda [P_1, R_1](x_0) + \mu [P_2, R_2](x_0) \in H_0$  for some  $\lambda, \mu > 0$

and

$$\dim \text{Lin} \{A_\infty(Q_i^j)(x_0), A_\infty([P_i, R_i])(x_0), i = 1, 2, j = 1, \dots, \tau\} = n-1.$$

It is proved in [12] that the sufficient condition given by Theorem 3 is also necessary, when the original system (1) is linear and  $\pi$  is a hyperplane

$$A_i(x) = A_i x, B_i(x) = B_i, f(x) = \langle l, x \rangle. \quad (23)$$

Using the ideas of Brammer [3], Bianchini [2] or Veliov [12] it is possible to prove that condition (22) is also necessary in the linear case (23) without supposing  $\text{reint}U_i \neq \emptyset$ , but replacing the operator  $co$  in the definition of  $S_p$  by  $\overline{cocone}$  - the closed convex cone hull.

Condition (22) is not necessary in the general nonlinear case, since it is of first order only. Using Lemmas 2, 4, 5 and other similar results based on Lemma 2 one can obtain also higher order sufficient conditions. This can be done by replacing the "initial" set  $S_0$  in the construction of the sets  $S_p$  by a richer subset of  $E^+$ , which can be obtained making use of the above mentioned lemmas. This is shown in the following example.

**Example 1.** Consider the system

$$\begin{cases} \dot{x}_1 = -x_2^2 + v & \dot{x}_1 = -x_4^2 + x_3 \\ \dot{x}_2 = -\alpha x_4^2 + v & \text{if } x_4 \geq 0, \quad \dot{x}_2 = -\beta x_4^2 & \text{if } x_4 < 0, \\ \dot{x}_3 = -x_4^2 + v & \dot{x}_3 = -x_2^2 \\ x_4 = u & \dot{x}_4 = u \end{cases} \quad (24)$$

$$u \in [-1,1], v \in [0,1]$$

The above system is just in the form (9), (10) with

$$\begin{aligned} P_1(x) &= -x_2^2 \frac{\partial}{\partial x_1} - \alpha x_4^2 \frac{\partial}{\partial x_2} - x_4^2 \frac{\partial}{\partial x_3} , \\ Q_1(x) &= v \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) , \quad R_1(x) = u \frac{\partial}{\partial x_4} \\ P_2(x) &= (x_3 - x_4^2) \frac{\partial}{\partial x_1} + (x_3 - \beta x_4^2) \frac{\partial}{\partial x_2} - x_2^2 \frac{\partial}{\partial x_3} , \\ Q_2(x) &= 0 , \quad R_2(x) = u \frac{\partial}{\partial x_4} \end{aligned}$$

In this example  $S_0 = \{Z = \partial/\partial x_1 + \partial/\partial x_2 + \partial/\partial x_3\}$  and  $S_0(0) = \{(1,1,1,0)^*\}$ , so that  $F(S_0)(0) = \{0\}$ . But we can obtain other elements of  $E^+$  using Lemma 2 and then to apply the construction of Theorem 2. Namely, we have obviously

$$\exp \left( t \left( P_1 - u \frac{\partial}{\partial x_4} \right) \right) \circ \exp \left( t \left( P_1 + u \frac{\partial}{\partial x_4} \right) \right) x \in D(x, 2t) .$$

Since

$$P_1(0) = [P_1, \frac{\partial}{\partial x_4}](0) = [P_1, [P_1, \frac{\partial}{\partial x_4}]](0) = 0 ,$$

and

$$[[P_1, \frac{\partial}{\partial x_4}], \frac{\partial}{\partial x_4}] = -2\alpha \frac{\partial}{\partial x_2} - 2 \frac{\partial}{\partial x_3}$$

applying Campbell - Hausdorff formula (25) we get

$$\exp \left( 2tP_1 + t^2 a - \frac{t^3}{3} \left( \alpha \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) + o(t^4) \right) x \in D(x, 2t)$$

with  $a(0) = 0$ , which yields

$$Y_1 = -\alpha \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} \in E^+$$

Similarly we obtain that

$$Y_2 = -\frac{\partial}{\partial x_1} - \beta \frac{\partial}{\partial x_2} \in E^+$$

and thus we can take  $S_0 = \{Z, Y_1, Y_2\}$ . Then

$$F(S_0(0)) = \{0\} \text{ if } \alpha + \beta \neq 1 \text{ and}$$

$$F(S_0(0)) = \text{Lin}\{(1,1,1,0)^*, (-1,-\beta,0,0)^*\} \text{ if } \alpha + \beta = 1 .$$

Since

$$P_2(0)Z(0) = (1,1,0,0)^* \in S_1$$

and is independent from  $F(S_0(0))$  if  $\beta \neq 1$  we obtain that the system (24) is ILC at  $x = 0$ , if

$$\alpha + \beta = 1 , \alpha \neq 0 .$$

It is easy to verify that the above condition is not only sufficient, but also necessary for ILC of (24). If  $\alpha + \beta \neq 1$ , then points of the type  $c(-1,-1,-1,0), c > 0$  can not be reached for small  $t$ . If  $\alpha + \beta = 1$ , but  $\alpha = 0$ , then  $\beta = 1$  and the first two equations in the lower halfspace  $x_4 \leq 0$  coincide. But in the upper halfspace  $\dot{x}_2$  is always greater than or equal to  $\dot{x}_1$ , which in combination gives  $x_2(t) \geq x_1(t)$  for every  $t$  and (24) can not be ILC.

## Appendix

**Proof of Proposition 1.** Equation (6i) can be rewritten as

$$\dot{x} = P_i(x) + Q_i(x)u_i + \frac{\langle A_i(x), f'(x) \rangle}{\langle \gamma_i(x), f'(x) \rangle} \gamma_i(x) + \frac{\gamma_i(x)f''(x)B_i(x)u_i}{\langle \gamma_i(x), f'(x) \rangle} + \gamma_i(x)u_0$$

and denoting

$$v_i(x, u_i, u_0) = \langle A_i(x), \gamma_i(x) \rangle + \langle B_i(x)u_i, f'(x) \rangle + \langle \gamma_i(x), f'(x) \rangle u_0 ,$$

as

$$\dot{x} = P_i(x) + Q_i(x)u_i + R_i(x)v_i(x, u_i, u_0) .$$

Taking into account A2 and (5) we get  $|v_i(x, u_i, u_0)| \leq c = \text{const}$ , if only  $|x| \leq 1, u_i \in \tilde{U}_i$  and  $|u_0| \leq 1$ . Then obviously, if  $x(\cdot)$  is a trajectory of (6), (7), it is a trajectory of (9), (10) for  $\alpha = \max\{|u|; u \in \tilde{U}_i\}$  and  $\beta = c$  which implies the first assertion of the lemma.

Solving with respect to  $u_0$  the equation

$$v_i(x, u_i, u_0) = v$$

and using A2 and (5) we obtain

$$|u_0| \leq 2(|v| + C|u| + \tau(|x|)) / \gamma$$

where  $C$  is an appropriate constant,  $\gamma = \min \{ \alpha_1, \alpha_2 \}$  and  $\tau(s)$  goes to zero with  $s$ . Thus every trajectory of (9), (10) with  $\alpha \leq \gamma/6C$ ,  $\beta \leq \gamma/6$ , on a sufficiently short interval  $[0, T]$  (such that  $\tau(|x(t)|) \leq \gamma/6$ ) is a trajectory of (6), (7), Q.E.D.

Before starting with the results in Section 3 we shall remind the Campbell - Hansdorff formula up to order three:

$$\begin{aligned} \exp(X) \circ \exp(Y) &= \exp\left(X + Y + \frac{1}{2}[X, Y] + \right. \\ &\quad \left. \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \dots\right). \end{aligned} \quad (25)$$

First we shall prove the following technical proposition.

**Proposition 2.** Let  $Z_i \in E^+$ ,  $i = 1, \dots, k$ . Then there exist numbers  $q > 1$ ,  $c$  and  $T > 0$  and a family  $\phi(t_1, \dots, t_k) \in \text{Der}(M)$  parametrized continuously by  $t_1, \dots, t_k \in [0, T]$  such that

$$\sup \{ \|\phi(st_1, \dots, st_k)\| ; t_i \in [0, T] \} \leq cs^q, \quad s \in (0, 1], \quad (26)$$

$$\exp\left(\sum_{i=1}^k t_i Z_i + A(t_1, \dots, t_k) + \phi(t_1, \dots, t_k)\right)x \in D\left(x, \sum_{i=1}^k S_i(t_i)\right), \quad (27)$$

where  $A$  is a polynomial of  $t_1, \dots, t_k$  with coefficients from  $\text{Der}(M)$  and  $(S_i(\cdot))$  come from the definition of the relation  $Z_i \in E^+$ .

**Proof.** Since  $Z_i \in E^+$

$$L_i x = \exp(t_i Z_i + a_i(t_i) + o_i(t_i))x \in D(x, S_i(t_i))$$

for  $x \in K_i$ ,  $t_i \in [0, T]$  and  $a_i$ ,  $o_i$  and  $S_i$  as in (18). Then from the semigroup property of  $D$

$$L_1 \circ \dots \circ L_k x \in D\left(x, \sum_{i=1}^k S_i(t_i)\right).$$

On the other hand from (25), applied for sufficiently small  $t_i$ ,  $L_1 \circ \dots \circ L_k$  can be presented in the form of the left-hand side of (27), where  $A$  summarizes all the terms from  $Der_0(M)$  with degrees with respect to  $t_1, \dots, t_k$  not greater than one. The reminder  $\phi$  is continuous in  $t_1, \dots, t_k$  and satisfies (27), if only  $T > 0$  is sufficiently small.

**Proof of Theorem 1.** Inclusion (19) implies the existence of  $Z_1, \dots, Z_k \in E^+$  such that

$$0 \in co\{Z_1(x_0), \dots, Z_k(x_0)\} .$$

Denoting the left-hand side of (27) by  $L(t_1, \dots, t_k)x$  and using Proposition 2 for  $x = x_0$ , we obtain in view of the property  $A \in Der_0(M)$  that

$$L(t_1, \dots, t_k) \circ \exp\left(\sum_{i=1}^k t_i(Z_i(x_0) - Z_i) - A(t_1, \dots, t_k)\right)x_0 \in D(x_0, \sum_{i=1}^k S_i(t_i)) .$$

Applying (25) we get

$$\exp\left(\sum_{i=1}^k t_i Z_i(x_0) + \tilde{\phi}(t_1, \dots, t_k)\right)x_0 \in D(x_0, \sum_{i=1}^k S_i(t_i)) .$$

with  $\tilde{\phi}$  satisfying the inequality (26). The left hand side of the above inclusion can be presented as

$$\sum_{i=1}^k t_i Z_i(x_0) + \mu(t_1, \dots, t_k)$$

with a continuous function  $\mu : \mathbb{R}_+^k \rightarrow M_{x_0}$  also satisfying (26) (possibly with another constant  $c$ ). Since  $S_i(t_i)$  tends to zero with  $t_i$ , the assertion of the theorem follows from Lemma 4 [10].

**Proof of Lemma 1.** The invariance of  $E^+$  with respect to positive scalar multiplication follows directly from Definition 4. Applying Proposition 2 for  $k = 2$ ,  $t_1 = t_2 = t$  we also get the additive invariance of  $E^+$ .

**Proof of Lemma 3.** In this proof we exploit an idea used by Hermes [6]. Since  $Z_1, Z_2 \in E^+$  and  $Y \in F^+$ , we have

$$\exp(tZ_i + a_i(t) + o_i(t))x \in D(x, S_i(t)) , \quad i = 1, 2$$

$$\exp(tY)x \in D(x, st)$$

for  $t$  – sufficiently small and  $x$  from a neighborhood of  $x_0$ . Let  $d$  be the minimal degree of  $t$  in  $a_i(\cdot)$ . There is  $p > 1$  such that  $\|o_i(t) / t^p\|$  is bounded when  $t$  tends to zero. Let

$q \in (0.5, 1)$  be such that  $pq > 1$  and  $q(1 + d) > 1$ . From the semigroup property of  $D$

$$K(t)x = \exp(t^q Z_1 + a_1(t^q) + o_1(t^q)) \circ \exp(t^{1-q} Y) \circ \exp(t^q Z_2 + a_2(t^q) + o_2(t^q))x \\ \in D(x, S_1(t^q) + S_2(t^q) + st^{1-q}).$$

Using twice (25) we obtain that

$$K(t) = \exp(t^q(Z_1 + Z_2) + t^{1-q}Y + 0.5t([Y, Z_2] + [Z_1, Y]) + \tilde{a}(t) + \tilde{o}(t)).$$

Since  $(Z_1 + Z_2)(x_0) = Y(x_0) = \tilde{a}(t)(x_0) = 0$  we get from Definition 4

$$X = [Y, Z_2] + [Z_1, Y] \in E^+.$$

But

$$[Y, Z_2] = 0.5(X + ([Y, Z_2] + [Y, Z_1]))$$

and the second term in the right-hand side also belongs to  $E^+$ , since  $[Y, Z_1 + Z_2](x_0) = 0$ . Thus from Lemma 1  $[Y, Z_2] \in E^+$ . *Q.E.D.*

**Proof of Lemma 5.** (Krastanov [8]). Let  $s_1$  and  $s_2$  be the numbers from the definition of the relations  $Z_1, Z_2 \in F^+$ . Then

$$K(t)x = \exp(t^{1/3}Z_1) \circ \exp(2t^{1/3}Z_2) \circ \exp(t^{1/3}Z_1)x \in D(x, 2(S_1 + S_2)t^{1/3}).$$

Denoting  $Y = [Z_1, [Z_1, Z_2]] + [Z_2, [Z_2, Z_1]]$  and using (25) we obtain

$$K(t) = \exp(t^{1/3}(Z_1 + Z_2) + 0.5t^{2/3}[Z_1, Z_2] + t/12Y + o'(t)) \\ \circ \exp(t^{1/3}(Z_2 + Z_1)) + 0.5t^{2/3}([Z_2, Z_1] + t/12Y + o''(t)) \\ = \exp(2t^{1/3}(Z_1 + Z_2)) + t/6(Y + 3[[Z_1, Z_2], Z_1 + Z_2]) + o(t)$$

which implies

$$Y + 3[[Z_1, Z_2], Z_1 + Z_2] \in E^+.$$

Lemma 4 gives us  $\pm[Z_1, Z_2] \in E^+$  and from Lemma 3 and Lemma 1 we conclude that  $Y \in E^+$ . *Q.E.D.*

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