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FOREWORD

The modeling of stochastic processes is a fundamental tool in the study of models involving uncertainty, a major topic at SDS.

A number of classical convergence results (and extensions) for probability measures are derived by relying on new tools that are particularly useful in stochastic optimization and extremal statistics.

Alexander B. Kurzhanski
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ABSTRACT

The hypo-convergence of upper semicontinuous functions provides a natural framework for the study of the convergence of probability measures. This approach also yields some further characterizations of weak convergence and tightness.

CONTENTS

WEAK CONVERGENCE OF PROBABILITY MEASURES REVISITED

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1. ABOUT CONTINUITY AND MEASURABILITY

A probabilistic structure – a space of possible events, a sigma-field of (observable) subcollections of events, and a probability measure defined on this sigma-field – does not have a built-in topological structure. This is the source of many technical difficulties in the development of Probability Theory, in particular in the theory of stochastic processes. Much progress was made, in reconciling the measure-theoretic and topological viewpoints, by the study of limits in terms of the weak*-convergence of probability measures, also called weak convergence [5],[16]. In this paper, we approach these questions from a fundamentally different point of view, although eventually we show that weak*-convergence and convergence in the sense introduced here, coincide for probability measures defined on separable metric spaces. We proceed by a "direct" construction: it is shown that the spaces of probability measures is in one-to-one correspondence with a certain space of upper semicontinuous functions, called *sc-measures*, for which there is a natural topology, and thus an associated notion of convergence. This means that instead of relying on the pre-dual to generate the notion of convergence, we use the "topological" properties of the space of probability sc-measures itself, and much insight is gained by doing so.

The major tool is the theory of epi- or hypo-convergence that has been developed in Optimization Theory to study the limits of (infinite-valued) semicontinuous functions. Functions are said to hypo-converge if their hypographs converge (as sets); the hypograph of a (extended-)real valued function consists of all points on and below its graph. This "global" view of functions provided by the hypographical approach is particularly appealing when dealing with limit theorems in Probability Theory. Hypo-convergence is not blinded by what happens at single points, a cause of major chagrin when working with standard pointwise convergence, but takes into account the behavior of the (converging) functions in the neighborhood of each single point.

We associate to every probability measure, its restriction to the hyperspace of a given class of closed sets. It is easy, but crucial, to observe that the resulting function, called an sc-measure, is upper semicontinuous with respect a certain topology on this (hyper)space of closed sets. Now, limits can be defined in terms of the hypo-limits of these sc-measures as is done in Section 4. The structural compactness of the space of upper semicontinuous functions with the hypo-topology is the key to a number of limit results, in particular in the study of the role played by tightness, cf. Sections 4 and 5. In Section 3, we show that when the probability measures are defined on the Borel sigma-field of a metric space, hypo-limits of probability sc-measures and weak*-limits of the associated sequences of probability measures coincide, providing us with an indirect proof of a Riesz-type representation theorem for the space of sc-measures. We think that this new characterization of weak*-convergence, that supplements those that have already been studied extensively [13] and which at least conceptually is non-standard, is by its intrinsic nature closely related to the usual notion of convergence of distribution function of random vectors. In fact, one may feel that it would be much more natural to approach convergence in distribution for random vectors in terms of the hypo-convergence of their distribution functions rather than through the equivalent but less meaningful notion of pointwise convergence on the continuity set. The reader can convince himself of this, all what is needed is the definition of hypo-convergence [9, Section 1] and the accompanying geometric cappretation.

We work in general with probability measures defined on a separable metric space, but we reserve a special place to the case when in addition the underlying space is compact. For this, there are some technical and didactic reasons – namely in the compact case the notion of convergence of sets that we introduce corresponds to the usual notion of set-convergence – but in addition the compact case played an important role in our research on the convergence of stochastic infima (extremal processes) that originally motivated this work. This was first elaborated in the presentation of these results in a communication to the European Meeting of Statisticians at Palermo (September 1982), and also at the Third Course in Optimization Theory and Related Fields at Erice (September 1984). The connections to problems in stochastic homogenization and related questions in the Calculus of Variations was brought to our attention by a recent article of De Giorgi [8], whose research is following a path that in some ways, is parallel to ours.

To assume that the domain of definition of the probability measures is compact, or even locally compact, is not a standard assumption in probability theory, it would exclude a large number of functional spaces that are of capest in the study of stochastic processes. However that is only true if we restrict ourselves to a "classical" functional view of stochastic processes. Instead, if we approach the theory of stochastic processes as in [18 (Sections 3 or 6), 19] where paths are viewed as elements of a space of semicontinuous functions which we equip with the epi-topology, or when paths are identified with their graphs and the space of graphs is given the topology of set-convergence, then the underlying space is actually compact. Specific examples of such constructions can be found in [18, 20] and in the work of Dal Maso and Modica [7] on the stochastic homogenization problem (the space of integral functionals on $L^p_{loc}(R^n)$ is given the epi-topology).

Because set-convergence – or the variant that we introduce here to handle the infinite dimensional case, is not yet a familiar tool in probabilistic circles – in the appendices the proofs are given in painstaking detail. The alerted reader can of course skip much or all of this, and devote her or his full attention to the actual substance of the results.

2. CONVERGENCE OF SETS AND SEMICONTINUOUS FUNCTIONS

Let (E,d) be a separable metric space with τ the topology generated by the metric d. By $\mathcal{F}$, or $\mathcal{F}(E)$, we denote the ***hyperspace of τ-closed subsets*** of E. We endow $\mathcal{F}$ with the topology $\mathcal{T}$ that corresponds to the following notion of convergence: for $\{F;\ F^n,\ n \in \mathbb{N}\}$

$$F = \mathcal{T}\text{-}\lim_{n \longrightarrow \infty} F^n$$

if for all $x \in E$,

$$d(x,\ F) = \lim_{n \longrightarrow \infty} d(x,\ F^n) \quad ;$$

i.e. $\mathcal{T}$ is generated by the ***pointwise convergence of the distance functions*** $\{d(\cdot,\ F^n),\ n \in \mathbb{N}\}$ to $d(\cdot,\ F)$. For any nonempty set $D \subset E$,

$$d(x,\ D) := \inf_{y \in D} d(x,\ y)$$

and

$$d(x,\ \emptyset) := \infty \quad .$$

Francaviglia, Lechicki and Levi have extensively studied the properties of $\mathcal{T}$ and related topologies and uniformities [10], [14]† In particular, they point out that $\mathcal{T}$-convergence can

†They call $\mathcal{T}$ the Wijsman topology, but that does not seem to be totally appropriate, since already Choquet in his 1947 paper "Convergences" (Annales de l'Univ. de Grenoble, 23) introduces this notion for set-convergence.

be characterized in the following terms. For any $\eta > 0$, and nonempty set D in E, the *open η- fattening of D* is the set

$$\eta^\circ D := \{x \in E \mid d(x, D) < \eta\} \quad ,$$

with $\eta D := cl_{\mathcal{T}} \eta^\circ D$ the *η- fattening* of D. By definition, for some fixed z in E, usually 0 if E is a linear space, we set

$$\eta^\circ \emptyset := cl(\eta \emptyset := \{x \in E \mid d(x, z) > \eta^{-1}) \quad .$$

We also reserve the notations

$$B^\circ_\eta(x) : \eta^\circ \{x\}, \quad \text{and} \quad B_\eta(x) = \eta\{x\}$$

for the *open* and *closed balls* of radius η and center x.

PROPOSITION 2.1 [9, Propositions 2.1 and 2.2] *Suppose $\{F;\, F^n,\, n \in \mathbb{N}\} \subset \mathcal{F}$. Then $F = \mathcal{T}\text{-}\lim_{n \to \infty} F^n$ if and only if*

(i) *for all $x \in E$, to every pair $0 < \epsilon < \eta$, there corresponds n' such that for all $n \geq n'$,*

$$F \cap B^\circ_\epsilon(x) \neq \emptyset \quad \Longrightarrow \quad F^n \cap B^\circ_\eta(x) \neq \emptyset \quad ; \tag{2.1}$$

(ii) *for all $x \in E$, to every pair $0 < \epsilon < \eta$, there corresponds n' such that for all $n \geq n'$,*

$$F \cap B_\eta(x) = \emptyset \quad \Longrightarrow \quad F^n \cap B_\epsilon(x) = \emptyset \text{'}. \tag{2.2}$$

It follows immediately from this proposition that $\mathcal{T}\text{-}\lim_{n \to \infty} F^n = \emptyset$ if and only if to any bounded set Q, there corresponds n' such that for all $n \leq n'$,

$$F^n \cap Q = \emptyset \quad ; \tag{2.3}$$

if d is a bounded metric, this means that the F^n are empty for n sufficiently large.

This notion of convergence for closed sets is related to, but is more restrictive than, the more standard definition of set-convergence, by which one means that

$$F = \lim_{n \to \infty} F^n \tag{2.4}$$

if

$$\limsup_{n \to \infty} F^n \subset F \subset \liminf_{n \to \infty} F^n \tag{2.5}$$

where

$$\limsup_{n\to\infty} F := \left\{\tau - \text{cluster points of all sequences}\{x^n\}_{n=1}^{\infty} \mid x^n \in F^n\right\} \tag{2.6}$$

$$= \cap_{H\in\mathcal{H}}\ \tau - \text{cl}\ (\cup_{n\in H} F^n), \tag{2.7}$$

and

$$\liminf_{n\to\infty} F^n := \left\{\tau - \text{limit points of all sequences}\{x^n\}_{n=1}^{\infty} \mid x^n \in F^n\right\} \tag{2.8}$$

$$= \cap_{H\in\mathcal{H}^\#}\ \tau - \text{cl}\ (\cup_{n\in H} F^n), \tag{2.9}$$

here $\mathcal{H}$ is the Fréchet *filter* on $\mathbf{N} = \{1,2,...\}$ and $\mathcal{H}^\#$ its *grill*, i.e.

$$\mathcal{H}^\# = \{H \in \mathbf{N} \mid H \cap H' \neq \emptyset \text{ for all } H' \in \mathcal{H}\}.$$

The grill $\mathcal{H}^\#$ consists of all infinite (countable) subsets of $\mathbf{N}$. Since $\mathcal{H} \subset \mathcal{H}^\#$, we always have that

$$\liminf_{n\to\infty} F^n \subset \limsup_{n\to\infty} F^n, \tag{2.10}$$

and thus $F = \lim F^n$ actually means that equality holds in (2.5). From these definitions, we immediately have the following proposition; the details can be found in Appendix A.

PROPOSITION 2.2 [10, Proposition 2.3, Theorem 2.6] *Consider* $\{F^n, n=1,,,\}$ *a sequence in* $\mathcal{F}(E)$. *Then* $F = \mathcal{T}\text{-}\lim_{n\to\infty} F^n$ *implies that* $F = \lim_{n\to\infty} F^n$. *The converse also holds if* (E,d) *is boundedly compact (i.e., every closed ball is compact).*

As a simple example of a case when the converse does not hold, take $E = l^1$, $F^n = \{$ the unit vector on the n-th axis $\}$. Then $\lim F^n = \emptyset$ whereas $\mathcal{T}\text{-}\lim F^n$ does not exist. Some further properties of $\mathcal{T}$ and set-convergence are reviewed in Appendix A.

Francaviglia, Lechicki and Levy prove that $(\mathcal{F}, \mathcal{T})$ is metrizable and separable [10, Theorem 4.6] which of course also implies that it is regular and has a countable base, since singletons are closed. It is these latter properties that play a key role in what follows. (One can also rely on the separability of (E, τ), and the characterization of convergence provided by Proposition 2.1 to obtain a countable base.) For easy reference we record these facts in the next theorem. The remaining assertion is well documented in the literature [11; 15; 9, Proposition 3.2].

THEOREM 2.3 *Suppose (E, d) is a separable metric space. The topological space $(\mathcal{F}(E), \mathcal{T})$ has a countable base and is regular (which also means that it is separable and metrizable, since it is T_1). Moreover it is compact, if in addition (E, d) is locally compact.*

We note that there is no loss of generality in introducing $\mathcal{T}$-convergence in terms of sequences instead of nets (or filters).

Now let us consider $SC^u(\mathcal{F}; 0, 1])$, *the space of $\mathcal{T}$-upper semicontinuous functions (u. sc.) on $\mathcal{F}$ with values in* $[0, 1]$. Recall that a function $v(F) \longrightarrow [0, 1]$ is $\mathcal{T}$ *u. sc. at* F whenever

$$v(F) \geq \limsup_{n \longrightarrow \infty} v(F^n)$$

for all sequences $\{F^n\}_{n=1}^{\infty}$ with $\mathcal{T}\text{-}\lim_{n \longrightarrow \infty} F^n = F$.

Next, we endow $SC^u(\mathcal{F}; [0, 1])$ with a convergence structure based on the sequential convergence of the hypographs. Recall that the ***hypograph*** of $v: \mathcal{F} \longrightarrow [0, 1]$ is the set

$$\text{hypo } v := \{(F, \alpha) \in (\mathcal{F} \times R) \,|\, v(F) \geq \alpha\} \quad , \tag{2.11}$$

i.e. all points that lie on or below the graph of v.

DEFINITION 2.4 *A sequence of functions $\{v_n, n \in \mathbb{N}\}$ in $SC^u(\mathcal{F}; [0, 1])$ hypo-converges to $v \in SC^u$ at $F \in \mathcal{F}$ if*

(i) *whenever $F = \mathcal{T}\text{-}\lim_{n \longrightarrow \infty} F^n$, then*

$$\limsup_{n \longrightarrow \infty} v_n(F^n) \leq v(F) \quad , \tag{2.12}$$

and

(ii) *for some sequence $\{F^n\}_{n=1}^{\infty}$ with $F = \mathcal{T}\text{-}\lim_{n \longrightarrow \infty} F^n$,*

$$\limsup_{n \longrightarrow \infty} v_n(F_n) \geq v(F) \quad . \tag{2.13}$$

Note that the first condition could equivalently be formulated as follows: for any subsequence $H \in \mathcal{H}^{\#}$, and collection $\{F^n, n \in H\}$ with $F = \mathcal{T}\text{-}\lim_{n \in H} F^n$, (2.12) holds when n goes to ∞ on H. This observation is also useful in showing [9, Proposition 1.9] that hypo-convergence (at all F in $\mathcal{F}$) corresponds to the (sequential) set-convergence of the hypographs, i.e.,

$$v = \text{hypo-lim } v_n \text{ if and only if hypo } v = \lim_{n \longrightarrow \infty} \text{hypo } v_n \quad . \tag{2.14}$$

If E is a compact metric space, hypo-convergence of upper semicontinuous real valued functions can be characterized in terms of convergence of the Hausdorff distances between the graphs [4].

Of crucial importance to the ensuing development is the following compactness result. A very elegant proof appears in [2, Theorem 2.22] that relies directly on the definition of hypo-convergence. It can also be derived from a general theorem about set-convergence, for the convenience of the reader, a proof can be found in Appendix B.

THEOREM 2.5 *Suppose (E, d) is a separable metric space, and $\mathcal{F}$ is the hyperspace of closed subsets of E, that we equip with the topology $\mathcal{T}$ of pointwise convergence of the distance functions. Then $(SC^u(\mathcal{F}; [0, 1], \text{hypo})$ is sequentially compact, i.e., any sequence $\{v_n \in SC^u(\mathcal{F}; [0, 1]), n \in \mathbb{N}\}$ contains a subsequence that hypo-converges to a function v in $SC^u(\mathcal{F}; [0, 1])$. If, in addition (E, d) is locally compact, then $(SC^u(\mathcal{F}; [0, 1]), \text{hypo})$ is a regular compact topological space.*

3. SC-MEASURES AND SC-PREMEASURES ON $\mathcal{F}(E)$

An *sc-premeasure* λ is a (set-) function on $\mathcal{F}(E)$ with the following properties:

i. *nonnegativity*: $\lambda(F) \geq 0$ for all $F \in \mathcal{F}$;

ii. *increasing* (nondecreasing): if for any F_1, F_2 in $\mathcal{F}$,

$$\lambda(F_1) \leq \lambda(F_2) \quad \text{whenever} \quad F_1 \subset F_2;$$

iii. λ is $\mathcal{T}$-*u.sc.* (upper semicontinuous) on $\mathcal{F}$.

It is finitely ***sub-additive*** if for an F_1, F_2, in $\mathcal{F}$,

$$\lambda(F_1) + \lambda(F_2) \leq \lambda(F_1 \cup F_2) + \lambda(F_1 \cap F_2) \quad . \tag{3.1}$$

If, it actually is ***finitely additive***, i.e. for any F_1, F_2 in $\mathcal{F}$,

$$\lambda(F_1) + \lambda(F_2) = \lambda(F_1 \cup F_2) + \lambda(F_1 \cap F_2) \quad , \tag{3.2}$$

then λ is an *sc-measure*. It is a ***probability sc-measure*** if in addition

$$\lambda(\emptyset) = 0, \quad \lambda(E) = 1 \quad . \tag{3.3}$$

Sc-measures (semicontinuous measures) defined on $\mathcal{F}(E)$ are of course intimately related to measures defined on $\mathcal{B}(E)$, the Borel field generated by $\mathcal{F}$, the τ-closed subsets of E. Conceptually, however, there is a basic difference between measures and sc-measures.

The measure-calculus relies on the underlying sigma-field structure (countable additivity, etc.), the calculus of sc-measures is topological in nature, and consequently provides a richer structure for studying convergence, and other limit questions. It is, however, possible to identify probability measures and sc-measures as we show next.

THEOREM 3.1 *There is a one-to-one correspondence between probability measures on $\mathcal{B}(E)$ and probability sc-measures on $\mathcal{F}(E)$. More precisely, given a probability measure μ on $\mathcal{B}(E)$, then the restriction of μ to $\mathcal{F}(E)$ is a probability sc-measure. And given λ, a probability sc-measure, there exists a unique probability measure μ on $\mathcal{B}(E)$ such that $\mu = \lambda$ on $\mathcal{F}(E)$.*

PROOF Suppose μ is a probability measure on $\mathcal{B}$ and λ is its restriction to $\mathcal{F}$. Clearly, it is enough to check if λ is $\mathcal{T}$-u.sc. Since $\mathcal{T}$ has a countable base, it suffices to show that

$$\limsup_{n\to\infty} \lambda(F^n) \le \lambda(F)$$

whenever $F = \mathcal{T}\text{--}\lim F^n$. First observe that

$$F = \limsup_{n\to\infty} F^n = \cap_{H\in\mathcal{H}} \text{cl}\,(\cup_{n\in H} F^n) \supset \cap_{H\in\mathcal{H}} \cup_{n\in H} F^n := F'$$

where the first equalities follow from Proposition 2.2, and (2.7). Since μ is a probability measure and $\lambda(F^n) = \mu(F^n)$ we have that

$$\limsup_{n\to\infty} \lambda(F^n) = \limsup_{n\to\infty} \mu(F^n) \le \mu(F') \le \mu(F) = \lambda(F).$$

If λ is a probability sc-measure on $\mathcal{F}$, we set $\mu = \lambda$ on $\mathcal{F}$ and define for every open set G, $\mu(G) = 1-\lambda(E\backslash G)$. We see that μ is an increasing finitely additive set-function on $\mathcal{A}$, the field consisting of finite cups of open and closed sets. We can now appeal to the standard argument to extend μ to a probability measure on $\mathcal{B}$, the sigma-field generated by $\mathcal{A}$ [1, Theorem 1.3.10]. This extension is unique. In fact, since (E,τ) is a separable metric space, for every $A \in \mathcal{B}$, we have that $\mu(A) = \sup\,\{\lambda(F) \mid F\subset A\}$. □

As can be expected from the preceding theorem, measures and sc-measures have many common properties. Of immediate capest are certain continuity properties used in the sequel.

By *bdy* D we denote the *τ-boundary* of a set $D \subset E$.

PROPOSITION 3.2 *Suppose λ: $\mathcal{F} \longrightarrow [0,1]$ is a sc-premeasure. Then given any nonempty $F \in \mathcal{F}$, the function*

$$\epsilon \mapsto \lambda(\epsilon F) : R_+ \longrightarrow \bar{R}_+$$

has at most countably many discontinuity points and $\lambda(F) = \lim_{\epsilon\downarrow 0}\lambda(\epsilon F)$. *Thus, if* λ *is a probability sc-measure, the family of sets* $\{\epsilon F \mid \epsilon \geq 0\}$ *contains at most countably many sets such that* $\lambda(bdy\ \epsilon\ F) > 0$.

PROOF First note that $F = \mathcal{T}\text{-}\lim_{\epsilon\downarrow 0} \epsilon\ F$, see A.5. Since λ is $\mathcal{T}$-u.sc at F and increasing, and hence $\lambda(F) \leq \liminf_{\epsilon\downarrow 0} \lambda(\epsilon\ F)$, it follows that $\lambda(F) = \lim_{\epsilon\downarrow 0} \lambda(\epsilon\ F)$. The assertion of at most countable discontinuity points follows directly from the (topological) argument given in [6, iv p.4] that applies to all monotone functions. Finally, if λ is a probability sc-measure, for any $\epsilon_1 > 0$ with $\lambda(bdy\ \epsilon_1\ F) > 0$, we have for any $\epsilon' < \epsilon_1$

$$\lambda(\epsilon' F) + \lambda(bdy\ \epsilon_1 F) = \lambda(\epsilon' F \cup bdy\ \epsilon_1 F) + \lambda(\emptyset) \leq \lambda(\epsilon_1 F)$$

Taking the above into account, it yields

$$\lim_{\epsilon''\downarrow\epsilon_1} \lambda(\epsilon'' F) = \lambda(\epsilon_1 F) \geq \lambda(bdy\ \epsilon_1 F) + \lim_{\epsilon'\uparrow\epsilon_1} \lambda(\epsilon' F)$$

which shows that ϵ_1 is a discontinuity point of $\epsilon \mapsto \lambda(\epsilon F)$ and there are at most a countable number of such points. □

The correspondence between probability measures and sc-measures carries over to the natural convergence notions for probability measures (weak convergence) and sc-measures (hypo-convergence). In those terms it is a "bicontinuous" correspondence, as is demonstrated next. Recall that *weak convergence*, or more precisely weak* convergence, of a sequence of probability measures $\{\mu_n\colon \mathcal{B} \longrightarrow [0,1],\ n \in \mathbb{N}\}$ to a probability measure $\mu\colon \mathcal{B}(E) \longrightarrow [0,1]$, denoted $\mu = \text{weak}^*\text{-}\lim_{n\to\infty}\text{-}\mu_n$ means that

$$\int g\ d\mu = \lim_{n\to\infty} \int g\ d\mu_n \tag{3.4}$$

for any bounded continuous $g\colon E \longrightarrow R$, or equivalently [5, Theorem 2.1]

$$\limsup_{n\to\infty} \mu_n(F) \leq \mu(F) \quad \text{for all closed sets } F, \tag{3.5}$$

$$\liminf_{n\to\infty} \mu_n(G) \geq \mu(G) \quad \text{for all open sets } G, \tag{3.6}$$

$$\lim_{n\to\infty} \mu_n(A) = \mu(A) \quad \text{for all sets } A \in \text{cont } \mu, \tag{3.7}$$

where

$$\text{cont } \mu := \{A \in \mathcal{B} \mid \mu(bdy\ A) = 0\} \tag{3.8}$$

THEOREM 3.3 *Suppose (E, τ) is a separable metric space, and $\{\mu; \mu_n, n \in \mathbb{N}\}$ is a family of probability measures on $\mathcal{B}(E)$, and $\{\lambda; \lambda_n, n \in \mathbb{N}\}$ the corresponding family of probability sc-measures on $\mathcal{F}(E)$. Then*

$$\mu = \text{weak}^*\!-\!\lim_{n\to\infty} \mu_n,$$

whenever

$$\lambda = \text{hypo}\!-\!\lim_{n\to\infty} \lambda_n.$$

This is also a necessary condition if (E,d) is a Polish space, i.e. (E,d) is also complete.

PROOF If $\lambda = \text{hypo}\!-\!\lim_{n\to\infty} \lambda_n$ then (3.5) follows directly from (2.12), since $\mu_n = \lambda_n$ on $\mathcal{F}$, and hence $\mu = \text{weak}^*\!-\!\lim \mu_n$.

To prove the converse we start with (2.13), i.e. we show that given any $F \in \mathcal{F}$ there exist a sequence with $F = \mathcal{T}\!-\!\lim_{n\to\infty} F^n$ such that

$$\lambda(F) \leq \liminf_{n\to\infty} \lambda_n(F^n).$$

We take F nonempty, since otherwise the inequality is trivially satisfied. From (3.6) it follows that for every $\epsilon > 0$,

$$\mu(\epsilon^\circ F) \leq \liminf_{n\to\infty} \mu_n(\epsilon^\circ F)$$

and thus to every $\delta > 0$ there corresponds n_δ such that for all $n \geq n_\delta$:

$$\lambda(F) \leq \mu(\epsilon^\circ F) \leq \mu_n(\epsilon^\circ F) + \delta \leq \mu_n(\epsilon F) + \delta = \lambda_n(\epsilon F) + \delta.$$

In particular, this means that to every $k \in \mathbb{N}$ we can associate n_k, with $n_{k+1} > n_k$, such that for all $n \geq n_k$

$$\lambda(F) \leq \lambda_n(k^{-1}F) + k^{-1}$$

With $F^n = k^{-1}F$, $\delta_n = k^{-1}$ whenever $n \in [n_k, n_{k+1}]$, by A.5 we have that $F = \mathcal{T}\!-\!\lim F^n$ and

$$\lambda(F) \leq \liminf_{n\to\infty} [\lambda_n(F^n) + \delta_n] = \liminf_{n\to\infty} \lambda_n(F^n).$$

There remains to show that (2.12) holds when (E,d) is a Polish space. Since (E,d) is complete μ is tight, i.e. given any $\delta > 0$ there exist a compact set K such that $\mu(K) > 1-\delta$. This implies that for any $\epsilon > 0$, $\mu(\text{cl }(E\backslash \epsilon K)) \leq \delta$. For such a compact set

K, ϵ sufficiently small, and for any sequence with $F = \mathcal{T}\text{-}\lim F^n$, we have

$$\limsup_{n\to\infty} \lambda_n(F^n) \le \limsup_{n\to\infty} \mu_n(F^n \cap \epsilon K) + \limsup_{n\to\infty} \mu_n(\text{cl } (E \setminus \epsilon K))$$

and by A.8, (3.5) and the above, this yields

$$\limsup_{n\to\infty} \lambda_n(F^n) \le \limsup_{n\to\infty} \mu_n(\epsilon' F) + \delta \le \mu(\epsilon' F) + \delta = \lambda(\epsilon' F) + \delta$$

where $\epsilon' > 2\epsilon$. Since by A.5, $\mathcal{T}\text{-}\lim_{\epsilon' \downarrow 0} \epsilon' F = F$, λ is $\mathcal{T}$-u.sc. (Theorem 3.1 and the definition of sc-premeasure) and δ is arbitrary, it follows that

$$\limsup_{n\to\infty} \lambda_n(F^n) \le \lambda(F),$$

which completes the proof. □

Theorem 3.3 can be viewed as giving a new characterization of weak convergence for probability measures. The classical results of Prohorov [5, Section 6] could be obtained as a direct consequence of this characterization, and the compactness results of Section 2. However it is more enlightening to derive it as a consequence of the properties of the hypo-limits of sc-measures as is done in the next section.

4. HYPO-LIMITS OF SC-MEASURES AND TIGHTNESS

In this section we are capested in the properties of the limit functions of a sequence of probability sc-measures. In view of Theorem 2.5 we know that there always will be a limit function, at least for some subsequence, what cannot be guaranteed is that this limit function is a probability sc-measure. We being with a general result about sc-premeasures.

LEMMA 4.1 *The space of sc-premeasures is sequentially compact with respect to hypo-convergence. In particular, if* $\{\lambda_n,\ n \in \mathbb{N}\}$ *is a sequence of sc-premeasures such that* $\lambda_n(E) = 1$ *for all* n, *and* $\lambda := \text{hypo-}\lim_{n\to\infty} \lambda_n$, *then* λ *is a sc-premeasure with* $\lambda(E) = 1$.

PROOF Every sequence of sc-premeasures has a hypo-convergent subsequence (Theorem 2.5). The limit function is then $\mathcal{T}$-u.sc., and clearly it is nonnegative. We need to show that it is an increasing function. Suppose $\lambda = \text{hypo-}\lim_{n\to\infty} \lambda_n$, and let us consider any pair $F_1 \subset F_2$ in $\mathcal{F}$. Since λ is the hypo-limit of the λ_n, it follows from (2.13) and (2.12) that there exists a sequence with $F_1 = \mathcal{T}\text{-}\lim F_1^n$ such that $\lambda(F_1) = \lim \lambda_n(F_1^n)$. Since $F_2 = \mathcal{T}\text{-}\lim_{\nu\to\infty} F_2 \cup F_1^n$, see A.3, it follows that

$$\lambda(F_2) \geq \limsup_{n\to\infty} \lambda_n(F_2 \cup F_1^n) \geq \limsup_{n\to\infty} \lambda_n(F_1^n) = \lambda(F_1).$$

Finally, $\lambda(E) = 1$ whenever $\lambda_n(E) = 1$ for all n, since by (2.12)

$$\lambda(E) \geq \limsup_{n\to\infty} \lambda_n(E) = 1 \quad ,$$

and $\lambda(E) \leq \liminf_{n\to\infty} \lambda_n(F^n) \leq 1$ whatever be the sequence $\{F^n,\ n\in \mathbb{N}\}$ $\mathcal{T}$-converging to E. □

Thus the hypo-limit λ of a sequence of probability sc-measures is a sc-premeasure. But not much more can be said except for a super-additivity property, at least not without making some further assumption about (E,τ); it is easy to verify that λ always satisfies:

$$\lambda(F_1) + \lambda(F_2) \geq \lambda(F_1 \cup F_2) \qquad \text{for all} \quad F_1, F_2 \text{ in } \mathcal{F} \ .$$

In general, λ is neither finitely additive, nor is $\lambda(\emptyset) = 0$. As the ensuing development will show these two properties are not unrelated. Let us begin by giving a necessary and sufficient condition for having $\lambda(\emptyset) = 0$.

LEMMA 4.2 *Suppose $\{\lambda_n,\ n\in \mathbb{N}\}$ is a sequence of probability sc-measures on $\mathcal{F}(E)$, and $\lambda = \text{hypo-}\lim_{n\to\infty} \lambda_n$. Then $\lambda(\emptyset) = 0$ if and only if to every $\epsilon > 0$, there corresponds a closed ball C_ϵ and n_ϵ such that for all $n \geq n_\epsilon$,*

$$\lambda_n(C_\epsilon) > 1 - \epsilon \ . \tag{4.1}$$

PROOF Fix any $\epsilon > 0$. Proposition 2.1 and the definition of hypo-convergence yield the existence of a sequence $\{F^n,\ n\in \mathbb{N}\}$ such that for all $n \geq n_\epsilon$,

$$F^n \cap C_\epsilon = \emptyset, \qquad \text{and} \quad \lambda(\emptyset) = \lim_{n\to\infty} \lambda_n(F^n).$$

Since for all $n \geq n_\epsilon$, $\lambda_n(F^n) \leq 1 - \lambda_n(C_\epsilon) \leq \epsilon$, it implies that $0 \leq \lambda(\emptyset) < \epsilon$. This holds for all $\epsilon > 0$, which means that $\lambda(\emptyset) = 0$.

Let us now prove the converse. We argue by contradiction. Suppose $\lambda(\emptyset) = 0$, but for some $\epsilon > 0$ and every closed ball C, there exists $H_C \in \mathcal{H}^{\#}$ such that for all $n \in H_C$, $\lambda_n(C) \leq 1 - \epsilon$. Since

$$\lambda_n(C) + \lambda_n(\text{cl}\,(E \setminus C) \ \geq \ \lambda_n(E) = 1 \quad ,$$

it follows that for all $n\in H_C$, $\lambda_n(\text{cl}\,(E\setminus C)) > \epsilon$. The fact that λ is the hypo-limit of the

λ_n implies that for all closed ball C,

$$\epsilon \leq \limsup_{n\to\infty} \lambda_n(\text{cl }(E\backslash C)) \leq \lambda(\text{cl }(E\backslash C)).$$

Consider any increasing sequence $\{C^n,\ n\in \mathbb{N}\}$ of closed balls that $\mathcal{T}$-converges to E. This means that $\mathcal{T}\text{-}\lim_{n\to\infty} \text{cl }(E\backslash C^n) = \emptyset$, cf. A.6. The preceding inequality and the $\mathcal{T}$-u.sc. of λ (Lemma 4.1) would yield the following contradiction.

$$0 < \epsilon \leq \limsup_{n\to\infty} \lambda(\text{cl }(E\backslash C^n) \leq \lambda(\emptyset) = 0. \qquad \square$$

This leads us to the following observations. A collection of probability sc-measures Λ on $\mathcal{F}$ (resp. measures M on $\mathcal{B}$) is said to be *tight*, if to every $\epsilon > 0$ there corresponds a compact set $K_\epsilon \subset E$ such that for all, but a finite number, of λ in Λ (resp. $\mu \in M$)

$$\lambda(K_\epsilon) > 1 - \epsilon, \qquad (\text{resp. } \mu(K_\epsilon) > 1 - \epsilon). \tag{4.2}$$

If the metric space E has compact closed balls, we can rewrite the assertion of Lemma 4.2 as follows: $\lambda(\emptyset) = 0$ if and only if the sequence $\{\lambda_n,\ n\in \mathbb{N}\}$ is tight. But, as it turns out, having $\lambda(\emptyset) = 0$ is all what is needed to obtain the finite additivity of λ when E is locally compact. To show this one first proves that in the locally compact case, the hypo-limit of a sequence of probability sc-measures is always a finitely sub-additive sc-premeasure. Next, it is shown that if in addition this hypo-limit has $\lambda(\emptyset) = 0$, then λ is actually finitely additive and hence λ is a probability sc-measure. This means: *if the* $\{\lambda_n,\ n\in \mathbb{N}\}$ *are probability sc-measures on* $\mathcal{F}(E)$ *with* E *a locally compact separable metrizable space, and* $\lambda = \text{hypo-}\lim_{n\to\infty} \lambda_n$, *then* λ *is a probability sc-measure if and only if the sequence* $\{\lambda_n,\ n\in \mathbb{N}\}$ *is tight.* This follows from the fact that E locally compact, separable and metrizable admits a boundedly compact metric [23]. When E is given this metric, we are in the following situation: $\mathcal{T}$-convergence and set convergence coincide, and $(SC^u(\mathcal{F};\ [0,\ 1]),\ \text{hypo})$ is compact; details can be found in [19].

The same assertion can be made if E is simply Polish – and this is proved below – but tightness plays then a double role. As in the locally compact case, it guarantees that there is no escape of the probability mass "at infinity", but it is also used to generate "relative compactness" in the space of probability measures. Tightness already enters in the proof that the hypo-limit is sub-additive. It essentially allows us to restrict our attention to compact subsets of E where $\mathcal{T}$-limits coincide with the standard set-limits, and up to some technical adjustments we can rely, as in the locally compact case, on the built-in relative compactness of the space of u.sc. functions (with the hypo-topology).

THEOREM 4.3 *Suppose* $\{\lambda_n,\ n\in \mathbb{N}\}$ *is a sequence of probability sc-measures on* $\mathcal{F}(E)$ *where* E *is a separable metric space, and* $\lambda = \text{hypo-}\lim_{n\to\infty} \lambda_n$*. Then*

(i) *if the sequence* $\{\lambda_n,\ n\in \mathbb{N}\}$ *is tight,* λ *is a probability sc-measure;*

(ii) *if* λ *is a probability sc-measure and* E *is a Polish space, the sequence* $\{\lambda_n\ n\in \mathbb{N}\}$ *is tight.*

PROOF If the $\{\lambda_n,\ n\in \mathbb{N}\}$ are tight, then λ is a sc-premeasure with $\lambda(E) = 1$ (Lemma 4.1), but also $\lambda(\emptyset) = 0$ (Lemma 4.2). To show that λ is sub-additive (3.1), observe that since λ is the hypo-limit of the λ_n, for any pair F_1,F_2 in $\mathcal{F}$, there exists sequences such that $\mathcal{T}\text{-}\lim F_1^n = F_1$ and $\mathcal{T}\text{-}\lim_{n\to\infty} F_2^n = F_2$ such that

$$\lim_{n\to\infty} \lambda_n(F_1^n) = \lambda(F_1), \qquad \text{and} \quad \lim_{n\to\infty} \lambda_n(F_2^n) = F_2.$$

Given $\epsilon > 0$, let K_ϵ be a compact set such that for all, but finitely many, n: $\lambda_n(K_\epsilon) > 1 - \epsilon$. Since the λ_n are probability sc-measures:

$$\begin{aligned}\lambda_n(F_1^n) + \lambda_n(F_2^n) &= \lambda_n(F_1^n \cup F_2^n) + \lambda_n(F_1^n \cap F_2^n)\\ &\le \lambda_n(F_1^n \cup F_2^n) + \lambda_n((F_1^n \cap F_2^n) \cup (F_1 \cap F_2))\\ &\le \lambda_n(F_1^n \cup F_2^n) + \lambda_n[((F_1^n \cap F_2^n) \cup (F_1 \cap F_2)) \cap K_\epsilon] + \epsilon.\end{aligned}$$

Taking $\limsup$ on both sides, using the fact that λ is the hypo-limit of the λ_n (2.12), since $F_1 \cup F_2 = \mathcal{T}\text{-}\lim_{n\to\infty} (F_1^n \cup F_2^n)$ by A.3, and $F_1 \cap F_2 \cap K_\epsilon = \mathcal{T}\text{-}\lim$ $[((F_1 \cap F_2) \cup (F_1^n \cap F_2^n)) \cap K_\epsilon]$

(cf. Proposition 2.2, A.2 and A.4), we obtain

$$\begin{aligned}\lambda(F_1) + \lambda(F_2) &\le \lambda(F_1 \cup F_2) + \lambda(F_1 \cap F_2 \cap K_\epsilon) + \epsilon\\ &\le \lambda(F_1 \cup F_2) + \lambda(F_1 \cap F_2) + \epsilon,\end{aligned}$$

where for the last inequality we used the fact that λ is increasing since it is sc-premeasure. This yields (3.1) since $\epsilon > 0$ is arbitrary. To complete the proof of part (i), there remains only to show that

$$\lambda(F_1) + \lambda(F_2) \ge \lambda(F_1 \cup F_2) + \lambda(F_1 \cap F_2). \tag{4.3}$$

Since $\lambda(\emptyset) = 0$, we may as well assume that F_1 and F_2 are nonempty, the inequality being trivially satisfied otherwise. Let $\{R^n,\ n\in \mathbb{N}\}$ and $\{S^n,\ n\in \mathbb{N}\}$ be sequences of sets such that $\mathcal{T}-\lim_{n\to\infty} R^n = F_1 \cup F_2$, $\mathcal{T}-\lim_{n\to\infty} S^n = F_1 \cap F_2$, and

$$\lambda(F_1 \cup F_2) = \lim_{n\to\infty} \lambda_n(R^n), \qquad \lambda(F_1 \cap F_2) = \lim_{n\to\infty} \lambda_n(S^n).$$

The existence of such sequences follows as before from (2.13) and (2.12). Pick any $\eta > 0$. With $F' = \text{cl}\,(E \backslash \eta\, F_2)$, and using the fact that the λ_n are probability sc-measures, we have

$$\begin{aligned}\lambda_n(R^n) + \lambda_n(S^n) &\leq \lambda_n(R^n \cap \eta\, F_2) + \lambda_n(R^n \cap F') + \lambda_n(S^n)\\ &\leq \lambda_n(\eta\, F_2) + \lambda_n((R^n \cap F') \cup S^n) + \lambda_n(R^n \cap F' \cap S^n)\\ &\leq \lambda_n(\eta\, F_2) + \lambda_n((R^n \cap F') \cup S^n \cup F_1) + \lambda_n(R^n \cap F' \cap S^n).\end{aligned}$$

Moreover, the sequence $\{\lambda_n,\ n \in \mathbb{N}\}$ is tight. Let K_ϵ be a compact set such that $\lambda_n(K_\epsilon) > 1 - (\epsilon/2)$ for n sufficiently large. Thus for n sufficiently large

$$\begin{aligned}\lambda_n(R^n) + \lambda_n(S^n) \leq \lambda_n(\eta\, F_2) &+ \lambda_n[((R^n \cap F') \cup S^n \cup F_1) \cap K\epsilon]\\ &+ \lambda_n(R^n \cap F' \cap S^n \cap K_\epsilon) + \epsilon.\end{aligned}$$

Taking lim sup on both sides, using the fact that λ is the hypo-limit of the $\{\lambda_n,\ n \in \mathbb{N}\}$, and since

$$\mathcal{T} - \lim\, [((R^n \cap F') \cup S^n \cup F_1) \cap K_\epsilon] = F_1 \cap K_\epsilon$$

and

$$\emptyset = \mathcal{T} - \lim_{n\to\infty} (R^n \cap S^n \cap F' \cap K_\epsilon)$$

as follows from Proposition 2.2, A.2, A.3 and A.4, we have

$$\begin{aligned}\lambda(F_1 \cup F_2) + \lambda(F_1 \cup F_2) &\leq \lambda(\eta\, F_2) + \lambda(F_1 \cap K_\epsilon) + \epsilon,\\ &\leq \lambda(\eta\, F_2) + \lambda(F_1) + \epsilon.\end{aligned}$$

We obtain (4.3) from the above by observing that λ is $\mathcal{T}$-u.sc at F_2, $\mathcal{T}-\lim_{n\to\infty} \eta F_2 = F_2$ by A.5, and that $\epsilon > 0$ can be chosen arbitrarily small.

To prove the converse, part (ii), observe that since (E,τ) is a Polish space every probability measure [5, Theorem 1.4], and thus every probability sc-measure (Theorem 3.1) is tight. In particular, this means that for all $\epsilon > 0$ there exists $K \subset E$, compact such that $\lambda(K) > 1 - \epsilon$. In turn, this implies that for all $\eta > 0$, $\lambda(E \backslash \eta^\circ K) \leq \epsilon$. Since

$\lambda = \text{hypo-lim}\ \lambda_n$,

$$\limsup_{n\to\infty} \lambda_n(E\backslash\ \eta^\circ K) \leq \lambda(E\backslash\eta^\circ K) \leq \epsilon$$

from which it follows that there exists n_ϵ such that for all $n \geq n_\epsilon$ and $\eta > 0$:

$$\lambda_n(\eta K) \geq 1 - \lambda_n(E\backslash\ \eta^\circ K) \geq 1 - 2\epsilon.$$

Now, we use the fact that $\eta > 0$ is arbitrary, that by A.5, $K = \mathcal{T}\text{-lim}\ \eta K$ and that the λ_n are $\mathcal{T}$-u.sc. at K to conclude that for all $n \geq n_\epsilon$, $\lambda_n(K) > 1 - 3\epsilon$. This means that the sequence $\{\lambda_n,\ n \in \mathbb{N}\}$ is tight, since there is such a compact set K for any $\epsilon > 0$. $\square$

The theorems of Prohorov and Varadajaran [5, Section 6; 15, Theorem 6.7] are immediate consequences of the above. As in [5] or [16], we say that a family M of probability measures is ***relatively compact*** (with respect to the weak* topology on M) if any sequence $\{\mu_n,\ n \in \mathbb{N}\} \subset M$ contains a subsequence weak converging to a probability measure.

COROLLARY 4.4 Prohorov's Theorems *Suppose (E,d) is a separable metric space and M is a family of probability measures on $\mathcal{B}(E)$. Then, if M is tight it is relatively compact. Moreover if (E,d) is a Polish space, then relative compactness of M implies tightness.*

PROOF Let $\Lambda = \{\lambda\colon \mathcal{F}(E) \longrightarrow [0,1] \mid \lambda = \mu \text{ on } \mathbf{F},\ \mu \in \mathrm{M}\}$ be the associated family of probability sc-measures, cf. Theorem 3.1. If M is tight, so is Λ. Moreover, any sequence in Λ contains a hypo-convergent subsequence (Theorem 2.5) which is necessarily tight, and hence its hypo-limit is a probability sc-measure (Theorem 4.3. (i)). The first assertion now follows directly from Theorem 3.3.

For the converse, let us first consider the case when $M = \{\mu_n,\ n \in \mathbb{N}\}$. It follows directly from the definition of relative compactness, Theorem 3.3, Theorem 4.3. (ii), the definition of tightness and the fact every probability sc-measure on the Polish space (E,d) is complete, that for all $\epsilon > 0$ there exists K_ϵ such that not only $\mu_n(K_\epsilon) > 1 - \epsilon$ for *all* $n \in \mathbb{N}$, but also $\mu(K_\epsilon) > 1 - \epsilon$ where μ is any (probability) measure in the weak* closure of M. To complete the proof simply observe that the weak* topology (on the space of measures on $\mathcal{B}$) is separable, and thus there exists a dense subset $\{\mu_n,\ n \in \mathrm{N}\} \subset M$ whose weak* closure is (uniformly) tight. $\square$

We note that the separability of weak* topology can be obtained as a consequence of Theorems 3.1 and 2.3.

5. TIGHTNESS AND EQUI-SEMICONTINUITY

Theorem 2.5 can be viewed as a generalization of a version of Helly's Theorem for probability sc-measures. The standard formulation of Helly's Theorem is, however, in terms of a "pointwise" convergence of the distribution. From the results of this section it will follow that hypo-convergence of sc-measures can be given a pointwise characterization, which also leads us to relate tightness to an equi-upper semicontinuity condition "at infinity".

The relationship between pointwise and hypo-convergence has been studied in [9]. Neither one implies the other, unless the collection of functions is equi-u.sc. [9, Theorem 2.9]. A family $U \subset SC_u(\mathcal{F}; [0,1])$ is *equi-upper semicontinuous at* F (with respect to the $\mathcal{T}$-topology) if to every $\epsilon > 0$ there corresponds a $\mathcal{T}$-neighborhood $\mathcal{V}$ of F such that for all $\nu \in U$,

$$\sup_{V \in \mathcal{V}} \nu(V) \le \nu(F) + \epsilon. \tag{5.1}$$

PROPOSITION 5.1 *Suppose* $\{\lambda; \lambda_n, n \in \mathbb{N}\}$ *are probability sc-measures such that* $\lambda = \text{hypo-}\lim_{n\to\infty} \lambda_n$. *Then*

(i) *the sequence is equi-u.sc. on* cont λ,

(ii) $\emptyset \in$ cont λ

where cont $\lambda = \{F \in \mathcal{F} \mid \lambda\ (bdyF) = 0\}$.

PROOF From Theorem 3.3, (3.7) and Theorem 3.1, it follows that for all $F \in cont\, \lambda$, $\lim_{n\to\infty} \lambda_n(F) = \lambda(F)$. This means that on cont λ we have both hypo- and pointwise convergence, and this only occurs if the sequence is equi-u.sc. on cont λ [9, Theorem 2.18]; a direct proof is given in Appendix B.

Part (ii) follows from: *bdy* $\emptyset = \emptyset$ and $\lambda(\emptyset) = 0$. □

THEOREM 5.2 *Suppose* $\{\lambda; \lambda_n, n \in \mathbb{N}\}$ *are probability sc-measures. Then* $\lambda = \text{hypo-}\lim_{n\to\infty} \lambda_n$ *if and only if* $\lambda(F) = \lim_{n\to\infty} \lambda_n(F)$ *for all* $F \in$ cont λ.

PROOF For the "only if" part, see above. For the converse we rely on Proposition 3.2 and the fact that λ is $\mathcal{T}$-u.sc. Indeed they imply that given any $\epsilon_1 > 0$, there always exists $\epsilon \in (0,\epsilon_1)$ such that $\epsilon\, F \in$ cont λ and $\lambda(\epsilon\, F) < \lambda(F) + \epsilon_1$. Because of pointwise convergence on cont λ, we have that

$$\limsup_{n\to\infty} \lambda_n(F) \le \lim_{n\to\infty} \lambda_n(\epsilon\, F) = \lambda(\epsilon\, F) < \lambda(F) + \epsilon_1,$$

whatever be $\epsilon_1 > 0$. Hence $\limsup_{n\to\infty} \lambda_n(F) \leq \lambda(F)$ for all $F \in \mathcal{F}$. But this via Theorem 3.1, (3.5), and Theorem 3.4 yields the hypo-convergence of the λ_n to λ. □

If we are given a sequence $\{\lambda_n,\ n \in \mathbb{N}\}$ of probability sc-measures that hypo-converges to λ, and it also pointwise converges on cont λ, this would not yet imply that λ is a probability sc-measure: finite additivity and $\lambda(\emptyset) = 0$ might still fail to be satisfied. If (E,τ) is boundedly compact however, it does suffice to have $\emptyset \in$ cont λ, as follows from the next proposition.

PROPOSITION 5.3 *Suppose (E,τ) is a boundedly compact separable metric space, and $\{\lambda_n,\ n \in \mathbb{N}\}$ are probability sc-measures on $\mathcal{F}(E)$. Then the following statements are equivalent:*

(i) *the sequence $\{\lambda_n,\ n \in \mathbb{N}\}$ is tight;*

(ii) *the sequence $\{\lambda_n,\ n \in \mathbb{N}\}$ is equi - u.sc. at $\emptyset$;*

(iii) *for any sequence $\{F^n,\ n \in \mathbb{N}\}$ with $\emptyset = \lim_{n\to\infty} F^n$, we have*

$$\limsup_{n\to\infty} \lambda_n(F^n) = 0.$$

PROOF Using the characterization of basic neighborhood systems of $\emptyset$, and recalling that the closed balls are compact (E,τ) are compact, we see that the $\{\lambda_n,\ n \in \mathbb{N}\}$ are equi - u.sc. at $\emptyset$ if and only if to every $\epsilon > 0$, there corresponds a compact ball K_ϵ such that for all n

$$\lambda_n(F) \leq \lambda_n(\emptyset) + \epsilon = \epsilon \qquad \text{whenever} \qquad F \cap K_\epsilon = \emptyset,$$

or equivalently $\lambda_n(K_\epsilon) > 1 - \epsilon$. In other words, if and only if the sequence is tight.

The equivalence of (i) and (iii) follows from Theorem 4.3 and Proposition 5.1. □

Proposition 5.3 and Theorem 5.2 allow us, in the compact case, to rephrase Theorem 4.3 as follows: *Suppose (E,τ) is a compact metric space, $\{\lambda_n,\ n \in \mathbb{N}\}$ is a sequence of probability sc-measures on $\mathcal{F}(E)$ such that $\lambda = \lim_{n\to\infty} \lambda_n$ on* cont λ. *Then λ is a probability sc-measure if and only if the $\{\lambda_n,\ n \in \mathbb{N}\}$ are equi - u.sc. at $\emptyset$.*

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APPENDIX A

We begin by a collection of facts about set- and $\mathcal{T}$-convergence that are used in the text. We always assume that (E,d) is a separable metric space.

A.1 $\mathcal{T}$-convergence implies set-convergence

PROOF To being with we show that if $F = \mathcal{T}\text{--}\lim F^n$, then $F \subset \liminf_{n\to\infty} F^n$. There is nothing to prove if $F = \emptyset$, so let us assume that F is nonempty. By Proposition 2.1, if $x \in F$ and $\epsilon > 0$, then $B^\circ_\epsilon(x) \cap F^n \neq \emptyset$ for all n sufficiently large. This means that for all $H \in \mathbf{H}^\#$, $x \in \tau\text{--cl}(\cup_n F^n)$ and hence, by (2.9), $x \in \liminf F^n$.

Next, let us show that $F \supset F' := \limsup_{n\to\infty} F^n$, or equivalently that $E\backslash F \subset E\backslash F'$. There is nothing to prove if $F = E$, so let us assume $E\backslash F \neq \emptyset$. Suppose $x \in E\backslash F$, then there exists $\eta > 0$ such that $F \cap Bub\eta(x) = \emptyset$. By Proposition 2.1 this implies that for any $\epsilon \in (0, \eta)$, for n sufficiently large $F^n \cap B_\epsilon(x) = \emptyset$. Thus there exists $H \in \mathcal{H}$ such that $x \ \tau\text{--cl}(\cup_{n\in H} F^n)$, and by (2.7) this implies that $x \ \limsup F^n$, i.e., $x \in E\backslash F'$. □

The fact that $\mathcal{T}$-convergence and set convergence are the same when (E,d) is a Euclidean space is well know, see [17, Theorem 2.2] for example. Beer [4] pointed out that this equivalence can only be obtained for spaces whose closed balls are compact.

A.2 $F_1 = \lim F_1^n$, $F_2 = \lim F_2^n$ implies $F_1 \cap F_2 = \lim[(F_1 \cap F_2) \cup (F_1^n \cap F_2^n)]$

PROOF Set $F = F_1 \cap F_2$, $F^n = F_1^n \cap F_2^n$. Since $F \subset F \cup F^n$, $F \subset \liminf(F \cup F^n)$. If $x \in \limsup(F \cup F^n)$ this means that there exists $H \in \mathcal{H}^\#$, $x^n \in F \cup F^n$ for all $n \in H$ such that $x = \lim x^n$. If $x^n \in F$ infinitely often then $x \in F$, otherwise $x \in \limsup F^n \subset \limsup F_1^n \cap \limsup F_2^n = F_1 \cap F_2 = F$; the inclusion can be proved from the definition (2.6) of lim sup, or one can consult [12, §25. II]. Hence $\limsup(F \cup F^n) \subset F$ and, with the above, this yields (2.5). □

A.3 $F_1 = \mathcal{T}-\lim F_1^n$, $F_2 = \mathcal{T}-\lim F_2^n$ *implies* $F_1 \cup F_2 = \mathcal{T}-\lim(F_1^n \cup F_2^n)$.

PROOF Use the definition of $\mathcal{T}$-convergence □

A.4 $F = \lim F^n$, $F \subset F^n$. *Then* $F \cap K = \mathcal{T}-\lim F^n \cap K$

where K is any compact set in E.

PROOF We use the characterization of $\mathcal{T}$-convergence given by Proposition 2.1. For any pair $0 < \epsilon < \eta$, $F \cap K \cap B^\circ_\epsilon(x) \neq \emptyset$ implies that $F^n \cap K \cap B^\circ_\eta(x) \neq \emptyset$ for all n, since $F \subset F^n$. Now if for some $\eta > 0$, $F \cap K \cap B_\eta(x) = \emptyset$, but there exists $\epsilon > 0$, and $h \in \mathcal{H}^\#$ such that $F^n \cap K \cap B_\epsilon(x) = \emptyset$ for all $n \in \mathbf{N}$, it would mean that $\limsup_{n \to \infty}(F^n \cap K \cap B_\epsilon(x))$ is nonempty, contradicting the assumption that $F \cap (K \cap B_\eta(x)) = \emptyset$ since

$$\limsup_{n \to \infty}(F^n \cap K \cap B_\epsilon(x)) \subset (\limsup_{n \to \infty} F^n) \cap (K \cap B_\epsilon(x)) \subset F \cap (K \cap B_\eta(x)) \quad . \quad \square$$

A.5 $F = \mathcal{T}-\lim_{\epsilon \downarrow 0} \epsilon F$.

PROOF Since $(\mathcal{F}, \mathcal{T})$ is separable (Theorem 2.3) we only need to consider this limit in terms of a sequence $\{\epsilon_n, n \in \mathbf{N}\}$ with $\lim \epsilon_n = 0$. If $F = \emptyset$ then the result is a direct consequence of the definition of ϵ-fattening of the empty set and the comment that follows Proposition 2.1. Now suppose F is nonempty, then for all $\eta > 0$ such that $F \cap B^\circ_\eta(x) \neq \emptyset$ it follows that $\epsilon_n F \cap B_\eta(x) \neq \emptyset$ for all n, since $F \subset \epsilon_n F$. If $F \cap B_\eta(x) = \emptyset$ then for any $\epsilon \in (0\ \eta)$, for n sufficiently large $\epsilon_n\ F \cap B_\epsilon(x) = \emptyset$ as follows from the fact that E is normal.
□

A.6 $\{C^n, n \in \mathbf{N}\}$ *is an increasing sequence of closed balls with* $E = \mathcal{T}-\lim C^n$.
Then $\{\mathrm{cl}(E \backslash C^n), n \in \mathbf{N}\}$ *is decreasing and* $\emptyset = \mathcal{T}-\lim \mathrm{cl}(E \backslash C^n)$.

PROOF The sequence is clearly decreasing. Suppose D is any bounded set, then for n sufficiently large $D \subset \mathrm{int}\ C^n$ and hence $D \cap \mathrm{cl}(E \backslash C^n) = \emptyset$. Since this holds for any bounded set D, from Proposition 2.1 it follows that $\{\mathrm{cl}\ (E \backslash C^n),\ n \in \mathbf{N}\}$ $\mathcal{T}$-converges to $\emptyset$.
□

A.7 $F = \mathcal{T}\text{-}\lim F^n$, *or more generally* $F = \lim F^n$, *and* $K \subset E$ *compact.*

Then for all $\epsilon>0$, *there exists* n_ϵ *such that for all* $n \geq n_\epsilon$, $F^n \cap K \subset \epsilon F$.

PROOF If $F = \emptyset$, then $F^n \cap K = \emptyset$ for n sufficiently large (Proposition 2.1), in which case the inclusion is obviously satisfied. If F is nonempty and $F^n \cap K$ is not included in ϵF, it means that there exists $H \in \mathcal{H}^{\#}$ such that for all $n \in H$, $F^n \cap K \cap (E \setminus \epsilon F) \neq \emptyset$. Passing to a subsequence if necessary, it means that there exists $\{x^n, n \in H\}$ such that $x^n \in F^n \cap K \cap (E \setminus \epsilon F)$, and by (2.6) we have

$$\lim x^n = x \in (\limsup F^n) \cap K \cap \text{cl}\,(E \setminus \epsilon F) = F \cap \text{cl}\,(E \setminus \epsilon F) \cap K$$

But this latter set is empty, contradicting the possibility that $F^n \cap K$ is not included in ϵF for n sufficiently large. □

A.8 $F = \mathcal{T}\text{-}\lim F^n$, $\emptyset \neq K$ *compact and* $\epsilon > 0$. *Then for* n *sufficiently large* $F^n \cap \epsilon K \subset \epsilon' F$ *where* $\epsilon' > 2\epsilon$.

PROOF The case $F = \emptyset$ is argued as in A.7. Otherwise, for contradiction purposes, suppose that there exists $H \in \mathcal{H}^{\#}$ such that for all n in H, dist $(x^n, F^n \setminus \epsilon' F) \leq \epsilon$ for some $x_n \in K$. Passing to a subsequence, if necessary, we have that $\lim x^n = x \in K \cap \epsilon F$, since K is compact, $\epsilon F = \mathcal{T}\text{-}\lim \epsilon F^n$ and every $x_n \in \epsilon F^n$. On the other hand $d(x^n, F) > \epsilon' - \epsilon$ and thus $d(x, F) > \epsilon$, again by $\mathcal{T}$-convergence, contradicting the possibility that $x \in \epsilon F$. □

APPENDIX B

B.1 *Proof of* Theorem 2.5

This theorem, an application of a general result about convergence of sets to the space of hypographs, has a long history that starts with a result of Zoretti [22] in the complex plane; Hausdorff, Lubben, Urysohn, Blaschke and Marczewski, having contributed in bringing the theorem in its present form. We give a direct proof patterned after the argument used in [12] for sequences of sets (see also [21] for sequences of functions).

To being with let us observe that to any sequence $\{\lambda_n \in SC^u(\mathcal{F}; [0, 1]).\ n \in \mathbf{N}\}$ we can associate an *upper hypo-limit* hypo-ls λ_n defined by

$$(\text{hypo}-\text{ls }\lambda_n)(F) := \sup_{H\in\mathcal{H}^\#} \sup_{\{F^n, n\in H \mid F=\mathcal{T}-\lim F^n\}} \limsup_{n\in H} \lambda_n(F^n)$$

and a *lower hypo-limit* hypo-li λ_n defined by

$$(\text{hypo}-\text{li }\lambda_n)(F) := \sup_{\{F^n, n\in\mathbf{N} \mid F=\mathcal{T}-\lim F^\nu\}} \liminf_{n\to\infty} \lambda_n(F^n).$$

As a direct consequence of the definition of hypo-limit, we have that $\lambda = \text{hypo}-\lim \lambda_n$ if and only if

$$\text{hypo}-\text{ls }\lambda_n \leq \lambda \leq \text{hypo}-\text{li }\lambda_n \tag{B.1}$$

If we denote by $\mathcal{N}(F)$, the $\mathcal{T}$-neighborhood system of F, the upper and lower hypo-limit can also be expressed as

$$\text{hypo}-\text{ls }\lambda_n(F) = \inf_{\mathcal{A}\in\mathcal{N}(F)} \limsup_{n\to\infty} \sup_{F'\in\mathcal{A}} \lambda_n(F').$$

and

$$\text{hypo}-\text{li }\lambda_n(F) = \inf_{\mathcal{A}\in\mathcal{N}(F)} \liminf_{n\to\infty} \sup_{F'\in\mathcal{A}} \lambda_n(F').$$

Let $\{\mathcal{A}_l, l=1,...\}$ be a countable open base for $\mathcal{T}$, see Theorem 2.3. Note that for each l, the sequence

$$\{\sup_{F\in\mathcal{A}_l} \lambda_n(F),\ \ n\in\mathbf{N}\}$$

has at least one cluster point in (the compact space) $\bar{R}$. Let $N_1\in\mathcal{H}^\#$ determine a subsequence such that

$$\lim_{n\in N_1} (\sup_{\mathcal{A}1} \lambda_n) \textit{ exists.}$$

Define recursively $N_l \subset N_{l-1}$ such that for all l

$$\lim_{n\in N_l} (\sup_{\mathcal{A}l} \lambda_n) \text{ exists.}$$

By diagonalization, construct $N' \subset \mathbf{N}$ as follows

$$N' := \{n_l \mid n_l \text{ is the } l\text{-th member of } N_l\}\ \ .$$

Since for all $l, \{n_l,\ l = 1,...\} \subset N_l$, we have that for all l:

$$\lim_{n\in N'} \sup_{F\in\mathcal{A}l} \lambda_n(F) \text{ exists.}$$

Now for any $F \in \mathcal{F}$, we have

$$(\text{hypo}-\text{ls}_{n\in N'}\ \lambda_n)(F)$$

$$= \inf_{\mathcal{A}\in\mathcal{N}(F)} \limsup_{n\in N'} \sup_{F'\in\mathcal{A}} \lambda_n(F')$$

$$= \inf_{\mathcal{A}l\in\mathcal{N}(F)} \lim_{n\in N'} \sup_{F'\in\mathcal{A}l} \lambda_n(F')$$

$$= \inf_{\mathcal{A}\in\mathcal{N}(F)} \liminf_{n\in N'} \sup_{F'\in\mathcal{A}} \lambda_n(F')$$

$$= (\text{hypo}-\text{li}_{n\in N'}\ \lambda_n)(F).$$

Since this holds for any $F \in \mathcal{F}$, we have that (B.1) is satisfied with $\lambda = \text{hypo}-\text{ls}\ \lambda_n = \text{hypo}-\text{li}\ \lambda_n$ for the subsequence $N' \subset \mathbf{N}$.

The second assertion follows from the fact that $\mathcal{T}$-convergence coincides with the standard set-convergence (Proposition 2.2) and the second part of Theorem 2.3, if one observes that the hyperspace of hypographs is a closed subset of the hyperspace of closed subsets of $\mathcal{F}(E)\ X\ R$ where $\mathcal{F}$ is $\mathcal{F}$-compact since E is locally compact; for details see [9, Corollary 4.2]. □

B.2 Direct proof of Proposition 5.1. (i).

Arguing by contradiction, let us assume that the $\{\lambda_n,\ n\in \mathbf{N}\}$ are not equi-upper semicontinuous at $F \in \text{cont}\ \lambda$. This means, there exists $\epsilon > 0$ such that to every $\mathcal{T}$-neighborhood $\mathcal{V}$ of F there corresponds $N_V \in\mathcal{H}^\#$ such that for all $n \in N_V$

$$\sup_{F'\in\mathcal{V}} \lambda_n(F') > \lambda_n(F) + \epsilon. \tag{B.2}$$

Now, let $\{\mathcal{V}_k,\ k{=}1,...\}$ a (countable) fundamental neighborhood system of F, the existence of such a system follows from Theorem 2.3. For every k, let $H_k\in \mathbf{H}^\#$ be such that (B.2) holds with $U = U_k$. Pick $n_k\in H_k\backslash\{n_1,\dots,n_{k-1}\}$, and choose $F^{n_k} \in \mathcal{V}_k$ such that

$$\lambda_{n_k}(F^{n_k}) > \lambda_{n_k}(F) + \epsilon$$

Let $N' = \{n_k,\ k{=}1,...\}$ and define the collection $\{F^n,\ n{\in}N\}$ as follows:

$$F^n := F \quad \text{if } n \in N\backslash N',$$

$$F^n := F^{n_k} \quad \text{if } n \in N' \text{ and } n = n_k.$$

then $F = \mathcal{T}-\lim F^n$, and

$$\limsup \lambda_n(F^n) \geq \limsup_{k\to\infty} \lambda_{n_k}(F^{n_k})$$

$$\geq \epsilon + \limsup_{k\to\infty} \lambda_{n_k}(F) = \epsilon + \lambda(F)$$

where the last equality follows from Theorems 3.1 and 3.3, and (3.7) since $F \in \text{cont}\ \lambda$. But this is in contradiction with the hypo-convergence of the λ_n to λ, in particular with (2.12). □

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