# Differential Calculus of Set-Valued Maps. An Update 

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## WORKING PAPER

DIFFERENTIAL CALCULUS OF SET-VALUED MAPS. AN UPDATE

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## FOREWORD

IIASA has played a crucial role in the development of the "graphical approach" to the differential calculus of set-valued maps, around J.-P. Aubin, H. Frankowska, R.T. Rockafellar and allowed to make contacts with Soviet and eastern European mathematicians (C. Olech, B. Pschenichnyiy, E. Polovinkin, V. Tihomirov, ...) who were following analogous approaches. Since 1981, they and their collaborators developed this calculus and applied it to a variety of problems, in mathematical programming (KuhnTucker rules, sensitivity of solutions and Lagrange multipliers), in nonsmooth analysis (Inverse Functions Theorems, local uniqueness), in control theory (controllability of systems with feedbacks, Pontryagin's Maximum Principle, Hamilton-Jacobi-Bellman equations, observability and other issues), in viability theory (regulation of systems, heavy trajectories), ....

The first version of this survey appeared at IIASA in 1982, and constituted the seventh chapter of the book Applied Nonlinear Analysis published in 1984 by I. Ekeland and the author. Since then, many other results have been motivated by the successful applications of this calculus, and, may be unfortunately, other concepts (such the concept of intermediate tangent cone and derivatives introduced and used by H. Frankowska). Infinite-dimensional problems such as control problems or the more classical problems of calculus of variations require the use of adequate adaptations of the same main idea, as well as more technical assumptions.

The time and the place (IIASA) were ripe to update the exposition of this differential calculus. The Russian translation of Applied Nonlinear Analysis triggered this revised version.

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## Introduction

There are few ideas in mathematics, but so many ways to implement them .... At each stage of the development of a branch, when the technical improvements require too much technical skill from the mathematicians, it is time to return to basic ideas.

This is what happened with the basic idea of differential calculus, when, despite a strong reluctance for asing set-valued maps, the pressure of the many contemporary problems arising in systems theory (optimization, regulation, viability and control of evolution systems) forced many applied mathematicians to use set-valued maps as naturally as the familiar singlevalued maps.

Natural inertia (or is it conservatism?) led most of us to consider setvalued maps as ...a maps, and not as a graphs (or representative curves), as it should be ${ }^{1}$, and as it was at the very origin of analytical geometry, when our ancestors used representative curves before the concept of function.

Fortunately, facts are there to force us to return to long forgotten tracks. During the last decades, the "graphical" side of set-valued maps took some preeminence among mathematicians using maximal monotone operators, graphical and epigraphical convergence, and the graphical derivatives we are about to describe. All the tool were there, though, but ignored. The road was paved by French and Polish mathematicians, Baire, Bouligand, Choquet, Kuratowski, Painlevé and many other, but abandoned for some reasons.

There is no doubt that mathematical programming and control theory provided in the sixties the motivation to study again set-valued maps. Let us mention the pioneer roles of C. Berge, A.F. Filippov and T. Wazewski during this crucial period. But at that time, the set-valued maps were mostly regarded as maps.

The beginning of the eighties saw the emergence of the concept of "graphical derivative", which goes back to Pierre de Fermat ${ }^{2}$. The idea

[^0]behind the construction of a differential calculus of set-valued maps is simple and is still the one to which all of us have been first acquainted during our teens. It starts with the concept of tangent to the graph of a function: the derivative is the slope of the tangent to the curve. We should say, now, that the tangent space to the graph of the curve is the graph of the differential. This is this statement that we take as a basis for adapting to the set-valued case the concept of derivative.

Consider a set-valued map $F: X \sim Y$, which is characterized by its graph (the subset of pairs $(x, y)$ such that $y$ belongs to $F(x)$ ).

We need first an appropriate notion of tangent cone to a set in a Banach space at a given point, which coincides with the tangent space when the set is an embedded differentiable manifold and with the tangent cone of convex analysis when the set is convex. At the time, experience shows that three tangent cones seem to be useful:

1. Bouligand's contingent cone ${ }^{8}$
2. Adjacent tangent cone ${ }^{4}$

## 3. Clarke's tangent cone ${ }^{5}$

They correspond to different regularity requirements. The tangent cone of Clarke is always convex. There already exists a sufficiently detailed calculus of these cones, which is exposed below.

Once a concept of tangent cone is chosen, we can associate with it a notion of derivative of a set-valued map $F$ at a point ( $x, y$ ) of its graph:

[^1]it is a set-valued map $F^{\prime}(x, y)$ the graph of which is equal to the tangent cone to the graph of $F$ at the point $(x, y)$.

In this way, we associate with the contingent cone, the adjacent and the Clarke tangent cones the following concepts of derivatives:

1. contingent derivative, corresponding to the Gâteaux derivative,
2. adjacent derivative, corresponding to the Fréchet derivative,
3. circatangent derivative, corresponding to the continuous Fréchet derivative.

For instance, if $d x$ is a direction in the space $X$, a direction $d y$ in the space $Y$ belongs to the contingent derivative $D F(x, y)(d x)$ of $F$ at the point ( $x, y$ ) in the direction $d x$ if and only if the pair ( $d x, d y$ ) belongs to the contingent cone to the graph of $F$ at $(x, y)$.

These derivatives keep enough properties of the derivatives of smooth functions to be quite efficient. They enjoy a pretty rich calculus, and such basic theorems of analysis as the inverse function theorem can be extended to the multivalued case.

Derivatives of set-valued maps (and also of nonsmooth single-valued maps) are set-valued maps, which are positively homogeneous. They are convex (in the sense that their graph is convex) when they depend in a "continuous" way of $(x, y)$. Such maps, , are the set-valued analogues of continuous linear operators.

The chain rule is in particular an example of a property which remains (almost) true.

But what about Newton and Leibnitz, who introduced the derivatives as limits of differential quotients?

Our first duty is to characterize the various graphical limits as adequate limits of differential quotients. Unfortunately, the formulas become very often quite ugly, and nobody in a right frame of mind would have invented them from scratch if they were not derived from the graphical approach.

But all these limits are pointwise limits, which classify all these generalized derivatives in a class different from the class of distributional derivatives introduced by L. Schwartz and S. Sobolev in the fifties, for solving partial differential equations: Their objective was to keep the linearity of
the differential operators, by allowing convergence of the differential quotients in weaker and weaker topologies, the price to be paid being that derivatives may no longer be functions, but distributions.

This survey presents only the definitions, the main properties and the calculus of the graphical derivatives of set-valued maps and epigraphical derivatives of extended real-valued functions, useful whenever the order relation of the real line plays a role, as in mathematical programming or Lyapunov style stability theory of dynamical systems.

The applications to optimization, control theory and viability theory are not described here.

We just provide a small bibliographical complement to the list of references of APPLIED NONLINEAR ANALYSIS

## 1 Tangent Cones

We devote this section to the definitions of some (and may be, too many) of the tangent cones which have been used in applications.

It is difficult to strike the right balance between simplicity (use only the contingent cones) and the needs of more results motivated by further studies.

We have chosen to postpone to the end of this presentation the dual concepts (normal cones, codifferential, generalized gradients) since their properties can be derived from the properties of the tangent cones.

We shall also provide the calculus in infinite-dimensional spaces, since it is required in the framework of control problems and of the calculus of variations, despite ugliness of the technical assumptions which, for the time, have not been simplified.

Definition 1.1 (Tangent cones) Let $K \subset X$ be a subset of a topological vector space $X$ and $x \in \bar{K}$ belong to the closure of $K$. We denote by

$$
\begin{equation*}
S_{K}(x):=\bigcup_{h>0} \frac{K-x}{h} \tag{1}
\end{equation*}
$$

the cone spanned by $K-x$.
We introduce the three following tangent cones

1. the contingent ${ }^{8}$ cone $T_{K}(x):=T_{K}^{\sharp}(x)$, defined by

$$
\begin{equation*}
T_{K}(x):=\left\{v \mid \liminf _{h \rightarrow 0+} d_{K}(x+h v) / h=0\right\} \tag{2}
\end{equation*}
$$

2. the adjacent ${ }^{7}$ cone $T_{K}^{+}(x)$, defined by

$$
\begin{equation*}
T_{K}^{\phi}(x):=\left\{v \mid \lim _{h \rightarrow 0+} d_{K}(x+h v) / h=0\right\} \tag{3}
\end{equation*}
$$

8. the Clarke ${ }^{8}$ tangent cone $C_{K}(x)$, defined by

$$
\begin{equation*}
C_{K}(x):=\left\{\left.v\right|_{h \rightarrow 0+, K \ni \Omega \rightarrow x} d_{K}\left(x^{\prime}+h v\right) / h=0\right\} \tag{4}
\end{equation*}
$$

[^2]We see at once that these three tangent cones are closed, that these tangent cones to $K$ and the closure $\bar{K}$ of $K$ do coincide, that

$$
\begin{equation*}
C_{K}(x) \subset T_{K}^{b}(x) \subset T_{K}(x) \subset \overline{S_{K}(x)} \tag{5}
\end{equation*}
$$

and that

$$
\begin{equation*}
\text { if } x \in \operatorname{Int}(K), \text { then } C_{K}(x)=X \tag{6}
\end{equation*}
$$

It is very convenient to use the following characterization of these cones in terms of sequences.

Proposition 1.1 Let $x$ belong to $K$.

$$
\begin{cases}\text { i) } & v \in T_{K}(x) \text { if and only if } \exists h_{n} \rightarrow 0+, \\ & \exists v_{n} \rightarrow v \text { such that } \forall n, x+h_{n} v_{n} \in K \\ \text { ii) } & v \in T_{K}^{b}(x) \text { if and only if } \forall h_{n} \rightarrow 0+, \\ & \exists v_{n} \rightarrow v \text { such that } \forall n, x+h_{n} v_{n} \in K \\ \text { iii) } v \in C_{K}(x) \text { if and only if } \forall h_{n} \rightarrow 0+, \forall x_{n} \rightarrow x, \\ & \left(x_{n} \in K\right), \exists v_{n} \rightarrow v \text { ouch that } \forall n, x_{n}+h_{n} v_{n} \in K\end{cases}
$$

Remark These tangent cones can be defined in terms of Kuratowski upper and lower limits of $\frac{K-x}{h}$, as the following statement shows:

## Proposition 1.2 Let $x$ belong to $K$. The following equalitics

$$
\begin{cases}\text { i) } & T_{K}(x)=\limsup _{h \rightarrow 0+} \frac{K-x}{h} \\ \text { ii) } & T_{K}^{4}(x)=\liminf _{h \rightarrow 0+} \frac{K-x}{h} \\ \text { iii) } & C_{K}(x)=\liminf _{h \rightarrow 0+, K \ni x \rightarrow x} \frac{K-x^{\prime}}{h}\end{cases}
$$

hold true.
Let us begin by proving an astonishing fact: the Clarke tangent cone $C_{K}(x)$ is always a closed convex cone.

Proposition 1.3 The Clarke tangent cone $C_{K}(x)$ is a closed convex cone actiofying the following propertics

$$
\begin{cases}\text { i) } & C_{K}(x)+T_{K}(x) \subset T_{K}(x) \\ \text { ii) } & C_{K}(x)+T_{K}^{\prime}(x) \subset T_{K}^{\prime}(x)\end{cases}
$$

## Proof

1. Let $v_{1}$ and $v_{2}$ belong to $C_{K}(x)$. For proving that $v_{1}+v_{2}$ belongs to this cone, let us choose any sequence $h_{n}>0$ converging to 0 and any sequence of elements $x_{n} \in K$ converging to $x$. There exists a sequence of elements $v_{1 n}$ converging to $v_{1}$ such that the elements $x_{1 n}:=x_{n}+h_{n} v_{1 n}$ do belong to $K$ for all $n$. But since $x_{1 n}$ does also converge to $x$ in $K$, there exists a sequence of elements $v_{\mathbf{v}_{n}}$ converging to $v_{2}$ such that

$$
\forall n, x_{1 n}+h_{n} v_{2 n}=x_{n}+h_{n}\left(v_{1 n}+v_{2 n}\right) \in K
$$

This implies that $v_{1}+v_{2}$ belongs to $C_{K}(x)$ because the sequence of elements $v_{1 n}+v_{2 n}$ converges to $v_{1}+v_{2}$.
2. Now, let $v_{1}$ belong to $T_{K}(x)$ and $v_{2}$ belong to $C_{K}(x)$. There exists a sequence of elements $h_{n}>0$ converging to 0 and $v_{1 n}$ converging to $v_{1}$ such that the elements $x_{1 n}:=x+h_{n} v_{1 n}$ do belong to $K$ for all $n$. But since $x_{1 n}$ does also converge to $x$ in $K$, there exists a sequence of elements $v_{2 n}$ converging to $v_{2}$ such that

$$
\forall n, x_{1 n}+h_{n} v_{2 n}=x+h_{n}\left(v_{1 n}+v_{2 n}\right) \in K
$$

This implies that $v_{1}+v_{2}$ belongs to $T_{K}(x)$
3. The proof is analogous for the cone $T_{K}^{b}(x)$.

Remark We can interpret the above inclusions by saying that the Clarke tangent cone is contained in the Minkowski difference (or the asymptotic cone, the convex kernel) of the adjacent and contingent cones. Let us recall that the Minkowski difference $K \ominus L$ of two subsets $K$ and $L$ is the subset $\bigcap_{s \in L}(K-x)$ of elements $z$ such that

$$
L+z \subset K
$$

When $P$ is a closed cone, the Minkowski difference $P \ominus P$ is always a convex cone.

Unfortunately, the price to pay for enjoying this convexity property of the Clarke tangent cones is that they may often be reduced to the trivial cone $\{0\}$.

But we shall show in just a moment that the Clarke tangent cone and that the contingent cone do coincide at those points $x$ where the set-valued map $x \sim T_{K}(x)$ is lower semicontinuous.

Definition 1.2 (Sleek Subsets) We shall say that a subset $K \subset X$ is sleek at $x \in K$ if the set-valued map

$$
K \ni x^{\prime} \sim T_{K}\left(x^{\prime}\right) \text { is lower semicontinuous at } x
$$

We shall say that it is sleek if and only if it is sleek at every point of $K$.
We shall prove later that smooth manifolds and convex subsets of finite dimensional vector-spaces are sleek.

But for the time, we just deduce from Theorem 1.1 below this quite important regularity property:

Theorem 1.1 (Tangent Cones of Sleek Subsets) Let $K$ be a weakly closed subset of a reflexive Banach space. If $K$ is sleck at $x \in K$, then the contingent and Clarke tangent cones do coincide, and consequently, are convex.

For that purpose, when $X$ is a normed space, it is quite useful to introduce the following notations:

$$
\left\{\begin{align*}
i) & D_{\uparrow} d_{K}(x)(v) \\
& :=\liminf _{h \rightarrow 0+}\left(d_{K}(x+h v)-d_{K}(x)\right) / h \\
\text { ii) } & D_{\dagger}^{\prime} d_{K}(x)(v) \\
& :=\lim \sup _{h \rightarrow 0+}\left(d_{K}(x+h v)-d_{K}(x)\right) / h  \tag{7}\\
i i i) & C_{\uparrow} d_{K}(x)(v) \\
& :=\lim \sup _{h \rightarrow 0+, x^{-x}}\left(d_{K}\left(x^{\prime}+h v\right)-d_{K}\left(x^{\prime}\right)\right) / h
\end{align*}\right.
$$

which will be justified later ${ }^{9}$.
We need the estimates we provide below to prove our theorem as well as other consequences.

[^3]Theorem 1.2 Let $K$ be a weakly closed subset of a reflexive Banach space and $\pi_{K}(y)$ be the set of projectione of $y$ onto $K$, i.e., the oubeet of $z \in K$ such that $\|y-z\|=d_{K}(y)$. Then we have the following inequalitie $:$

$$
\left\{\begin{array}{lll}
i) & D_{\uparrow} d_{K}(y)(v) & \leq d\left(v, T_{K}\left(\pi_{K}(y)\right)\right)  \tag{8}\\
i i) & D_{\dagger}^{p} d_{K}(y)(v) & \leq d\left(v, T_{K}^{b}\left(\pi_{K}(y)\right)\right) \\
i i i) & C_{\dagger} d_{K}(y)(v) & \leq d\left(v, C_{K}\left(\pi_{K}(y)\right)\right)
\end{array}\right.
$$

## Proof

1. We begin by proving these inequalities when $y$ belongs to $K$. Indeed, for all $w \in X$, inequality $d_{K}(y+h v) \leq d_{K}(y+h w)+h\|v-w\|$ implies that
(i) $\liminf _{h \rightarrow 0+} d_{K}(y+h v) / h \leq\|v-w\|$ when $w \in T_{K}(y)$
(ii) $\limsup \operatorname{sun}_{h \rightarrow 0+} d_{K}(y+h v) / h \leq\|v-w\|$ when $w \in T_{K}^{+}(y)$
2. Assume now that $y \notin K$ and choose $z \in \pi_{K}(y)$. Then

$$
\left\{\begin{array}{l}
\left(d_{K}(y+h v)-d_{K}(y)\right) / h \\
\leq\left(\|y-z\|+d_{K}(z+h v)-d_{K}(y)\right) / h \\
=d_{K}(z+h v) / h
\end{array}\right.
$$

Since $z$ belongs to $K$, intqualities (8) with $y=z$ imply that

$$
\begin{cases}\text { i) } & D_{\dagger} d_{K}(y)(v) \leq d\left(v, T_{K}(z)\right) \\ \text { ii) } & D_{\uparrow}^{b} d_{K}(y)(v) \leq d\left(v, T_{K}^{b}(z)\right)\end{cases}
$$

For proving inequality

$$
C_{\uparrow} d_{K}(y)(v) \leq d\left(v, C_{K}(z)\right)=\inf _{w \in C_{K}(x)}\|v-w\|
$$

we first observe that when $\boldsymbol{y} \notin K$,

$$
\forall z \in \pi_{K}(y), \quad \forall x \in K, \quad\|z-x\| \leq 2\|y-x\|
$$

Hence

$$
\sup _{\substack{k \leq \infty \\ \| y-x \mid \leq \beta}}\left(d_{K}(y+h v)-d_{K}(y)\right) / h \leq \sup _{\substack{x \leq 0 \\\|z-x\| \leq 2 g}} d_{K}(z+h w) / h+\| v-w v
$$

so that our claim is established. We end the proof by taking the infimum when $z$ ranges over $\pi_{K}(y)$.

We shall need actually the following
Corollary 1.1 Let $K$ be a weakly closed oubset of a reflesive Banach space. Then

$$
d_{K}(x+t v)-d_{K}(x) \leq \int_{0}^{t} d\left(v, T_{K}\left(\pi_{K}(x+\tau) v\right)\right) d \tau
$$

Proof We set $g(t):=d_{K}(x+t v)$. Since $g(\cdot)$ is locally lipschitzean, it is almost everywhere differentiable. Theorem 1.2 implies that $g^{\prime}(\tau) \leq$ $d\left(v, T_{K}\left(\pi_{K}(x+\tau) v\right)\right)$. We then integrate from 0 to $t$.

Theorem 1.3 Let $K$ be a weakly closed oubset of a reflesive Banach space. Let us consider a set-valued map $F: K \sim X$ satisfying

$$
\left\{\begin{array}{l}
\text { i) } F \text { is lower semicontinuous }  \tag{9}\\
\text { ii) } \forall x \in K, F(x) \subset T_{K}(x)
\end{array}\right.
$$

Then,

$$
\begin{equation*}
\forall x \in K, F(x) \subset C_{K}(x) \tag{10}
\end{equation*}
$$

Proof Let us take $x \in K$ and $v \in F(x)$. Since
$\forall z \in \pi_{K}(y+t v), \forall x \in K,\|z-x\| \leq 2\|y+t v-x\| \leq 2\|x-y\|+2 t\|v\|$ we infer that, for all $\epsilon>0, y \in K$ close to $x$ and $\tau$ small enough, the lower semicontinuity of $F$ at $x$ implies that

$$
\left\{\begin{aligned}
d\left(v, T_{K}\left(\pi_{K}(y+\tau v)\right)\right) & \leq d_{K}\left(v, F\left(\pi_{K}(y+\tau v)\right)\right) \\
& \leq d(v, F(x))+\epsilon=\epsilon
\end{aligned}\right.
$$

because $v$ belongs to $F(x)$ by assumption. Corollary 1.1 thus implies that, for all $y \in K$ close to $x$ and for all $t \in] 0, h]$ for some positive $h$,

$$
d_{K}(y+t v) \leq \int_{0}^{t} d\left(v, T_{K}\left(\pi_{K}(y+\tau v)\right)\right) d \tau \leq t \epsilon
$$

We have proved that $v$ belongs to $C_{K}(x)$.
Remark In particular, we deduce the following characterization of the tangent cones:

These equalities (11) suggest to extend the definition of these tangent cones to elements which are outside $K$.

Definition 1.3 Let $K$ be a subset of a normed space $X$ and $x$ belong to $X$. We extend the notions of contingent and adjacent cones to $K$ at points outside $K$ in the following way:

$$
\left\{\begin{array}{llll}
i) & T_{K}(x) & :=\{v & \left.D_{\uparrow} d_{K}(x)(v) \leq 0\right\}  \tag{12}\\
i i) & T_{K}^{+}(x) & :=\{v & \left.D_{\dagger}^{\dagger} d_{K}(x)(v) \leq 0\right\} \\
i i i) & C_{K}(x) & :=\{v & \left.C_{\dagger} d_{K}(x)(v) \leq 0\right\}
\end{array}\right.
$$

We deduce at once from Theorem 1.2 the following corollary:
Corollary 1.2 The tangent cones at points outside $K$ are related to the tangent cones at their projections in the following way:

$$
\left\{\begin{array}{lllll}
i) & \forall y \in \pi_{K}(x), & T_{K}(y) & \subset & T_{K}(x) \\
i i) & \forall y \in \pi_{K}(x), & T_{K}^{b}(y) & \subset & T_{K}^{b}(x) \\
i i i) & \forall y \in \pi_{K}(x), & C_{K}(y) & \subset & C_{K}(x)
\end{array}\right.
$$

It will be convenient to name the points $x$ of a subset $K$ where two of the above tangent cones do coincide.

Definition 1.4 We shall say that a subset $K \subset X$

1. is pseudo-convex at $x \in K$ if and only if either one of the equivalent properties

$$
\begin{cases}a / T_{K}(x)=\overline{S_{K}(x)} & =: \overline{U_{h>0} \frac{K-x}{h}}  \tag{13}\\ b / K \subset x+T_{K}(x)\end{cases}
$$

holde true.
2. is derivable at $x \in K$ if and only if

$$
\begin{equation*}
T_{K}^{b}(x)=T_{K}(x) \tag{14}
\end{equation*}
$$

Remark We shall justify later why we are led to introduce this ménagerie of tangent cones. Each of them corresponds to a classical regularity requirement. We shall see that the contingent cone is related to

Gàteaux derivatives, the adjacent cone to the Fréchet derivative and the Clarke tangent cone to the continuous Fréchet derivative.

If $L \subset K$ is a subset of $K$, Bouligand has also introduced the paratingent ${ }^{10}$ cone $P_{K}^{L}(x)$ to $K$ relative to $L$ at $x \in L$, defined by

$$
\begin{equation*}
P_{K}^{L}(x):=\left\{v \mid \limsup _{h \rightarrow 0+, L \ni x^{\prime} \rightarrow x} d_{K}\left(x^{\prime}+h v\right) / h=0\right\} \tag{15}
\end{equation*}
$$

and we observe that

$$
\begin{equation*}
T_{K}(x)-T_{K}^{b}(x) \subset P_{K}^{K}(x) \tag{16}
\end{equation*}
$$

We can also introduce open tangent cones. Let us mention the two following ones:

Definition 1.5 Let $x$ belong to $K$.

1. The cone $D_{K}(x)$ defined by

$$
\left\{\begin{array}{l}
v \in D_{K}(x) \text { if and only if } \forall h_{n} \rightarrow 0+,  \tag{17}\\
\forall v_{n} \rightarrow v, \text { we have } \forall n, x+h_{n} v_{n} \in K
\end{array}\right.
$$

is called the Dubovicki-Miliutin tangent cone
2. The cone $H_{K}(x)$ defined by

$$
\left\{\begin{array}{l}
v \in H_{K}(x) \text { if and only if } \forall h_{n} \rightarrow 0+, \forall x_{n} \rightarrow x,  \tag{18}\\
\left(x_{n} \in L\right), \forall v_{n} \rightarrow v, \text { we have } \forall n, x_{n}+h_{n} v_{n} \in K
\end{array}\right.
$$

is called the hypertangent cone.
We see at once that

$$
\begin{cases}\text { i) } & H_{K}(x) \subset C_{K}(x) \cap D_{K}(x)  \tag{19}\\ i i) & D_{K}(x) \subset T_{K}^{*}(x) \\ i i i) & H_{K}(x)+T_{K}^{*}(x) \subset D_{K}(x)\end{cases}
$$

[^4]
## 2 Tangent Cones to Convex sets

For convex subsets $K$, the situation is dramatically simplified by the fact that the Clarke tangent cones, the adjacent and the contingent cones coincide with the closed cone spanned by $K-x$.

Proposition 2.1 (Tangent Cones to Convex Sets) Let us assume that $K$ is convex. Then the contingent cone $T_{K}(x)$ to $K$ at $x$ is convex and

$$
C_{K}(x)=T_{K}^{\star}(x)=T_{K}(x)=\overline{S_{K}(x)}
$$

In particular,

$$
N_{K}(x)=N_{K}^{\curvearrowright}(x)=\left\{p \in X^{*} \mid \max _{\sin K}<p, y>=<p, x>\right\}
$$

Remark We shall denote by $T_{K}(x)$ the common value of these cones, and call it the tangent cone to the convex subset $K$ at $x$.

Proof We begin by stating the following consequence of convexity:

$$
\begin{equation*}
\forall v \in S_{K}(x), \exists h>0, \text { such that } \forall t \in[0, h], x+t v \in K \tag{20}
\end{equation*}
$$

since we can write that

$$
x+t v=\left(1-\frac{t}{h}\right) x+\frac{t}{h}(x+h v)
$$

is a convex combination of elements of $K$.
It is enough to prove that $S_{K}(x)$ is contained in the Clarke tangent cone. Let $v:=(y-x) / h$ belong to $S_{K}(x)$ (where $y \in K$ and $h>0$ ) and let us consider sequences of elements $h_{n}>0$ and $x_{n} \in K$ converging to 0 and $x$ respectively. We see that $v_{n}:=\left(y-x_{n}\right) / h$ converges to $v$ and that

$$
x_{n}+h_{n} v_{n}=\left(1-\frac{h_{n}}{h}\right) x_{n}+\frac{h_{n}}{h} y \in K
$$

since it is a convex combination of elements of $K$.
Actually, convex subsets of finite dimensional vector-spaces are sleek:
Theorem 2.1 Let $K$ be a closed convex subset of a finite dimensional vector-space $X$. Then $K$ is sleek.

Proof It is equivalent to prove that the graph of the set-valued map $K \ni x \sim N_{K}(x)$ is closed ${ }^{11}$.

But this is obviously the case: let us consider sequences of elements $x_{n} \in$ $K$ and $p_{n} \in N_{K}\left(x_{n}\right)$ converging to $x$ and $p$ respectively. Then inequalities

$$
\forall y \in K,\left\langle p_{n}, y\right\rangle \leq\left\langle p_{n}, x_{n}\right\rangle
$$

implies by passing to the limit inequalities

$$
\forall y \in K,\langle p, y\rangle \leq\langle p, x\rangle
$$

which state that $p$ belongs to $N_{K}(x)$. Hence the graph is closed, so that the set-valued map $T_{K}(\cdot)$ is lower semicontinuous, since the dimension of $X$ is finite.

We observe easily that the normal cones are contained in the barrier cone of a convex subset $K$ :

Proposition 2.2 Let $K$ be convex. Then, for all $x \in K$,

$$
\left\{\begin{array}{l}
i) N_{K}(x) \subset b(K) \\
\text { ii) the asymptotic cone } b(K)^{-} \subset T_{K}(x)
\end{array}\right.
$$

It may be useful to characterize the interior of the tangent cone to a convex subset.

Proposition 2.3 (Interior of a Tangent Cone) Assume that the interior of $K \subset X$ is not empty. Then

$$
\forall x \in K, \quad \operatorname{Int}\left(T_{K}(x)\right)=\bigcup_{h>0}\left(\frac{\operatorname{Int}(K)-x)}{h}\right.
$$

Furthermore, the graph of the set-salued map $K \ni x \leadsto \operatorname{Int}\left(T_{K}(x)\right)$ is open.

## Proof

1. The union of the interiors of $(K-x) / h$ being open, it is contained in the interior of the tangent cone. Since $K$ is convex, so are the

[^5]cones $S_{K}(x)$, and thus, the closures of $S_{K}(x)$ and of their interiors do coincide. Then it is enough to prove that if $v$ belongs to the interior of $S_{K}(x)$, it is interior to one of the $(K-x) / h$.
Let $\eta>0$ such that $v+\eta B \subset S_{K}(x)$. If $x+v$ belongs to the interior of $K$, the proof is completed. If not, let us choose $x_{0} \in \operatorname{Int}(K)$ and set $v_{0}:=x_{0}-x$. Hence $v-\eta v_{0} /\left\|v_{0}\right\|$ belongs to $S_{K}(x)$, and thus, there exists some $h>0$ such that $x+h\left(v-\eta v_{0} /\left\|v_{0}\right\|\right)$ belongs to $K$. By setting
$$
\lambda:=h \eta /\left(h \eta+\left\|v_{0}\right\|\right)
$$
we deduce that
$$
x+(1-\lambda) h v=\lambda x_{0}+(1-\lambda)\left(x+h\left(v-\eta v_{0} /\left\|v_{0}\right\|\right)\right)
$$

Since $x_{0}$ belongs to the interior of $K, x+h\left(v-\eta v_{0} /\left\|v_{0}\right\|\right)$ belongs to $K$ and $\lambda$ is smaller than 1 , we deduce from the convexity of $K$ that $x+(1-\lambda) h v$ belongs to the interior of $K$, i.e., that $v$ belongs to the interior of $(K-x) /(1-\lambda) h$.
2. Let us take a pair ( $x_{0}, v_{0}$ ) in the interior of the graph of $T_{K}(\cdot)$. Then, by the above statement, there exists $h>0$ such that

$$
v_{0} \in\left(\operatorname{Int}(K)-x_{0}\right) / h
$$

Hence there exists $\eta>0$ such that

$$
x_{0}+h v_{0}+\eta B=x_{0}+h\left(v+\frac{\eta}{h} B\right) \subset \operatorname{Int}(K)
$$

Therefore

$$
\left(x_{0}+\frac{\eta}{2} B\right) \times\left(v_{0}+\frac{\eta}{2 h} B\right) \subset \operatorname{Graph}\left(T_{K}(\cdot)\right)
$$

Remark Convex subsets are star-shaped around each of their elements and thus, share with them some properties.

Definition 2.1 (Star-Shaped Subsets) $A$ subset $K$ is said to be starshaped around $x \in K$ if

$$
\forall y \in K, \forall \lambda \in[0,1], \quad x+\lambda(y-x) \in K
$$

## We observe the following

Lemma 2.1 If $K \subset X$ is star-shaped around $x \in K$, then it is pseudo. convex and derivable at this point.

For the convenience of the reader, we list below some useful calculus of tangent cones to convex subsets (see [5, Section 4.1.]). The subsets $K, K_{i}$, $L, M, \ldots$ are assumed to be convex.

## Properties of Tangent and Normal Cones (1)

1. If $K \subset L$, then

$$
\begin{equation*}
T_{K}(x) \subset T_{L}(x) \& N_{L}(x) \subset N_{K}(x) \tag{21}
\end{equation*}
$$

2. If $K_{i} \subset X_{i}, \quad(i=1, \cdots, n)$, then

$$
\left\{\begin{array}{l}
\text { i) } T_{\prod_{i=1}^{n} K_{i}}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n} T_{K_{i}}\left(x_{i}\right)  \tag{22}\\
\text { ii) } N_{\prod_{i=1}^{n} K_{i}}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n} N_{K_{i}}\left(x_{i}\right)
\end{array}\right.
$$

3. If $K_{1}$ and $K_{2}$ are contained in $X$, then

$$
\begin{cases}\text { i) } & T_{K_{1}+K_{2}}\left(x_{1}+x_{2}\right)=\overline{T_{K_{1}}\left(x_{1}\right)+T_{K_{2}}\left(x_{2}\right)}  \tag{23}\\ i i) & N_{K_{1}+K_{2}}\left(x_{1}+x_{2}\right)=N_{K_{1}}\left(x_{1}\right) \cap N_{K_{2}}\left(x_{2}\right)\end{cases}
$$

In particular, if $P$ is a closed vector subspace, then

$$
\begin{cases}\text { i) } & T_{K+P}\left(x_{1}+x_{2}\right)=\overline{T_{K_{1}}\left(x_{1}\right)+P}  \tag{24}\\ \text { ii) } & N_{K_{1}+P}\left(x_{1}+x_{3}\right)=N_{K_{1}}\left(x_{1}\right) \cap P^{\perp}\end{cases}
$$

4. If $B \in \mathcal{L}(X, Y)$, then

$$
\left\{\begin{array}{l}
i) \quad T_{B(K)}(x)=\overline{B\left(T_{K}(x)\right)}  \tag{25}\\
i i) N_{B(K)}(x)=B^{\star-1} N_{K}(x)
\end{array}\right.
$$

5. If $L \subset X$ and $M \subset Y$ are closed convex subsets and $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that the qualification constraint condition

$$
\begin{equation*}
0 \in \operatorname{Int}(M-A(L)) \tag{26}
\end{equation*}
$$

holds true, then

$$
\left\{\begin{array}{l}
i) T_{\left.L \cap A^{-1}(M)\right)}=T_{L}(x) \cap A^{-1} T_{M}(A x)  \tag{27}\\
i i) N_{\left.L \cap A^{-1}(M)\right)}=N_{L}(x)+A^{*} N_{M}(A x)
\end{array}\right.
$$

## Properties of Tangent and Normal Cones (2)

6. If $M \subset Y$ is a clooed convex subset and if $A \in \mathcal{L}(X, Y)$ is a continuous linear operator such that

$$
\begin{equation*}
0 \in \operatorname{Int}(\operatorname{Im}(A)-M) \tag{28}
\end{equation*}
$$

then

$$
\left\{\begin{array}{l}
\text { i) } T_{A^{-1}(M)}(x)=A^{-1} T_{M}(A x)  \tag{29}\\
\text { ii) } N_{A^{-1}(M)}(x)=A^{*} N_{M}(A x)
\end{array}\right.
$$

7. If $K_{1}$ and $K_{2}$ are closed convex subsets contained in $X$ and satisfy ${ }^{12}$

$$
\begin{equation*}
0 \in \operatorname{Int}\left(K_{1}-K_{2}\right) \tag{30}
\end{equation*}
$$

then

$$
\begin{cases}\text { i) } & T_{K_{1} \cap K_{2}}(x)=T_{K_{1}}(x) \cap T_{K_{2}}(x)  \tag{31}\\ \text { ii) } & N_{K_{1} \cap K_{2}}(x)=N_{K_{1}}(x)+N_{K_{2}}(x)\end{cases}
$$

8. If $K_{i} \subset X, \quad(i=1, \ldots, n)$, are closed and convex and if there exists $\gamma>0$ such that

$$
\begin{equation*}
\forall x_{i} \mid\left\|x_{i}\right\| \leq \gamma, \bigcap_{i=1}^{n}\left(K_{i}-x_{i}\right) \neq 0 \tag{32}
\end{equation*}
$$

Then,

$$
\left\{\begin{array}{l}
i) \quad \prod_{\bigcap_{i=1}^{n} K_{i}}(x)=\bigcap_{i=1}^{n} T_{K_{i}}(x)  \tag{33}\\
i i) \quad N_{\bigcap_{i=1}^{n} K_{i}}^{n}(x)=\sum_{i=1}^{n} N_{K_{i}}(x)
\end{array}\right.
$$

[^6]
## 3 Inverse Function Theorems

We derive from the basic Inverse stability Theorem ${ }^{18}$ a series of equivalent results which extend in several ways the Liusternik Inverse Faction Theorem. We refer to [?] for more powerful results based on the concept of "variations" of set-valued map defined on any metric space, and which are related to images of the unit ball by derivatives of set-valued maps when the definition space is normed.

[^7]Theorem 8.1 (Inverse Stabllity Theorem) Let $X$ and $Y$ be two Banach spaces. We introduce a sequence of continuoue linear operators $A_{n} \in \mathcal{L}(X, Y)$, a sequence of closed subsets $K_{n} \subset X$.
Let us consider elements $x_{n}^{\star}$ of the subeets $K_{n}$ such that both $x_{n}^{*}$ converges to $x_{0}^{*}$ and $A_{n} x_{n}$ converges to yo.

We posit the following stablility assumption: there exist conotants $c>0, \alpha \in[0,1]$ and $\eta>0$ asch that

$$
\left\{\begin{array}{l}
\forall x_{n} \in K_{n} \cap B\left(x_{0}, \eta\right),  \tag{34}\\
A_{n} S_{K_{n}}\left(x_{n}\right) \cap B_{Y} \subset A_{n}\left(T_{K_{n}}\left(x_{n}\right) \cap c B_{X}\right)+\alpha B_{Y}
\end{array}\right.
$$

Let us set $1:=c /(1-\alpha), \rho<\eta / 31$ and consider elemente $y_{n}$ and $x_{0 n}$ satiofying:

$$
\left\{\begin{array}{l}
\text { i) } x_{0 n} \in K_{n} \cap B\left(x_{0}, \eta / 3\right), A_{n} x_{0 n} \in B\left(y_{0}, \rho\right)  \tag{35}\\
\text { ii) } y_{n} \in A_{n}\left(K_{n}\right) \cap B\left(y_{0}, \eta / 3\right)
\end{array}\right.
$$

Then, for any $l^{\prime}>1$ and $n>0$, there exist solutione $\widehat{x_{n}}$ satiofying

$$
\begin{cases}i) & \widehat{x_{n}} \in K_{n} \& A_{n} \widehat{x_{n}}=y_{n}  \tag{36}\\ i i) & \left\|\widehat{x_{n}}-x_{0_{n}}\right\| \leq\left\|^{\prime}\right\| y_{n}-A_{n} x_{0_{n}} \|\end{cases}
$$

so that

$$
\left\{\begin{array}{l}
d\left(x_{0}, K_{n} \cap A_{n}^{-1}\left(y_{n}\right)\right) \leq l\left\|y_{n}-A_{n} x_{0_{n}}\right\|  \tag{87}\\
\leq\left\|x_{0}-x_{0 n}\right\|+l\left\|y_{n}-y_{0}\right\|+l\left\|y_{0}-A_{n} x_{0_{n}}\right\|
\end{array}\right.
$$

converges to 0 when $x_{0 n}$ converges to $x_{0}$ and both $A_{n} x_{0_{n}}$ and $y_{n} \in A_{n} K_{n}$ converge to $y_{0}$.

Theorem 3.2 (Criterion of Pseudo-Lipschitzeanity) Let $K$ be a closed subet of a Banach space $X$ and $A \in \mathcal{L}(X, Y)$ be a continuous linear operator from $X$ to another Banach space $Y$. Let ue assume that for some $x_{0} \in K$, there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ ouch that

$$
\left\{\begin{array}{l}
\forall x \in K \cap B\left(x_{0}, \eta\right),  \tag{38}\\
A S_{K}(x) \cap B_{Y} \subset A\left(T_{K}(x) \cap c B_{X}\right)+\alpha B_{Y}
\end{array}\right.
$$

Then the set-valued map

$$
\begin{equation*}
A(K) \ni y \sim A^{-1}(y) \cap K \tag{39}
\end{equation*}
$$

is pseudo-lipschitzean around ( $A x_{0}, x_{0}$ ): For any $x_{1}$ close to $x_{0}$ and $y \in$ $K$ close to $A x_{0}$,

$$
\begin{equation*}
d\left(x_{1}, A^{-1}(y) \cap K\right) \leq l\left\|y-A x_{1}\right\| \tag{40}
\end{equation*}
$$

Remark Assumption (4.3) can be written in the form

$$
\sup _{x \in B_{K}\left(x_{0, \eta}\right)} \sup _{u \in \mathcal{S}_{\mathcal{K}}(x)} \inf _{v \in T_{K}(x), A v \in A v+\alpha\| \| \| B}\|v\|
$$

Observe that when $\alpha=0$, assumption (4.3) implies that $A(K)$ is pseudo-convex on a neighborhood of $A x_{0} \in K$ since for all $x$ in this neighborhood,

$$
S_{A(K)}(A x)=A\left(S_{K}(x)\right) \subset A\left(T_{K}(x)\right) \subset T_{A(K)}(A x)
$$

Hence, we can regard stability assumption (4.3) as a weakened local pseudo-convexity.

In particular, we obtain the following inverse mapping theorem:
Theorem 3.3 (Linear Inverse Function Theorem) Let $K$ be a closed subset of a Banach space $X$ and $A \in \mathcal{L}(X, Y)$ be a continuouo linear operator from $X$ to another Banach space $Y$. Let us assume that for some $x_{0} \in K$, there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ ouch that

$$
\left\{\begin{array}{l}
\forall x \in K \cap B\left(x_{0}, \eta\right),  \tag{41}\\
B_{Y} \subset A\left(T_{K}(x) \cap c B_{X}\right)+\alpha B_{Y}
\end{array}\right.
$$

Then $A x_{0}$ belongs to the interior of $A(K)$ and the set-valued map $y \sim$ $A^{-1}(y) \cap K$ is pseudo-lipschitzean around $\left(A x_{0}, x_{0}\right)$.

It implies the following apparently more general statement:

Theorem 3.4 (Set-Valued Inverse Function Theorem) Let us consider a closed set-valued map $F: X \sim Y$, an element $\left(x_{0}, y_{0}\right)$ of its graph and let us assume that there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall(x, y) \in \operatorname{Graph}(F) \cap B\left(\left(x_{0}, y_{0}\right), \eta\right), \\
\forall v \in Y, \exists u \in X, \exists w \in Y \text { such that } v \in D F(x, y)(u)+w \\
\text { and }\|u\| \leq c\|v\| \quad \& \quad\|w\| \leq \alpha\|v\|
\end{array}\right.
$$

Then $y_{0}$ belongs to the interior of the image of $F$ and $F^{-1}$ is preudolipschitzean around $\left(x_{0}, y_{0}\right)$.
which, actually, is equivalent, because, by taking for $F$ the restriction of $A$ to $K$, or, even more generally, the restriction of a differentiable singlevalued map, we infer that

Theorem 3.5 (Constrained Inverse Function Theorem) Let $X$ and $Y$ be two Banach spaces. We introduce a (single-valued) continuous map $f: X \mapsto Y$, a closed subset $K \subset X$ and an element $x_{0}$ of $K$.

We assume that $f$ is differentiable on a neighborhood of $x_{0}$ and we posit the following stability assumption: the ere exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall x \in K \cap B\left(x_{0}, \eta\right),  \tag{42}\\
B_{Y} \subset f^{\prime}(x)\left(T_{K}(x) \cap c B_{X}\right)+\alpha B_{Y}
\end{array}\right.
$$

Then $f\left(x_{0}\right)$ belongs to the interior of $f(K)$ and the set-valued map $y \sim$ $f^{-1}(y) \cap K$ is pseudo-lipschitzean around ( $\left.f\left(x_{0}\right), x_{0}\right)$.

We obtain as a consequence the Liusternik Inverse Function Theorem:

Corollary 3.1 (Liusternik Theorem) Let $X$ and $Y$ be two Banach spaces. We introduce a (oingle-valued) continuoue map $f$ from $X$ to $Y$. We assume that $f$ is continuously differentiable on a neighborhood of $x_{0}$ and we posit the following surjectivity assumption

$$
f^{\prime}\left(x_{0}\right) \text { is ourjective }
$$

Then the oet-valued map $y \sim f^{-1}(y)$ is pseudo-lipschitzean around ( $\left.f\left(x_{0}\right), x_{0}\right)$.

Proof Since the continuous linear operator $f^{\prime}\left(x_{0}\right)$ is surjective, we deduce from the Banach Theorem that there exists a constant $c$ such that

$$
\forall v \in Y, \exists u \in X \mid f^{\prime}\left(x_{0}\right) u=v \quad \& \quad\|u\| \leq c\|v\|
$$

Since $x \mapsto f^{\prime}(x)$ is continuous at $x_{0}$, we infer that

$$
\begin{aligned}
& \forall v \in Y, \exists u \in X \quad \text { such that } \\
& f^{\prime}(x) u=v+w \quad \& \quad\|u \leq c\| v\|,\| w\|\leq\| f^{\prime}(x)-f^{\prime}\left(x_{0}\right)\| \| u \|
\end{aligned}
$$

so that $\|w\| \leq \alpha\|v\|$ when $x$ is close to $x_{0}$.
We can extend this theorem to the case of set-valued maps by introducing and adequate definition of strongly sleek map.

Definition 3.1 (Strongly Sleek Sets and Maps) We ohall say that a closed subset $K$ is strongly sleek at $x_{0} \in K$ if the cone-valued map $K \ni$ $x \sim T_{K}(x)$ is strongly lower semicontinuous at $x_{0}$ in the sense that

$$
\lim _{x \rightarrow x_{0}} \sup _{w \in T_{K}\left(x_{0}\right) \cap B_{X}} d\left(z, T_{K}(x)\right)=0
$$

We shall eay that $F$ is strongly sleek at a point $\left(x_{0}, y_{0}\right)$ of ite graph if its graph is strongly sleek at this point ${ }^{14}$.
With this definition, we can state a natural set-valued version of Liusternik's Theorem

[^8]converges to 0 when ( $x, y$ ) converges to ( $x_{0}, y 0$ ).

Theorem 3.6 (Set-Valued Liusternik Theorem) Let us consider reflesive Banach spaces $X$ and $Y$, a closed set-valued map $F: X \leadsto Y$ and an element $\left(x_{0}, y_{0}\right)$ of its graph. Let us assume that $F$ is strongly sleek at ( $x_{0}, y_{0}$ ). If

$$
D F\left(x_{0}, y_{0}\right) \text { is surjective }
$$

then $y_{0}$ belongs to the interior of the image of $F$ and $F^{-1}$ is pseudolipschitzean around ( $x_{0}, y_{0}$ ). If the dimension of $Y$ is finite, it is oufficient to assume that $F$ is sleek at $\left(x_{0}, y_{0}\right)$.

Actually, this results follows (and thus, is equivalent) to its "constrained linear" version.

Theorem 3.7 (Pointwise Inverse Function Theorem) Let $X$ and $Y$ be reflexive Banach spaces, $K$ be a weakly closed subset of $X$ and $A \in$ $\mathcal{L}(X, Y)$ be a continuous linear operator. If $K$ is strongly sleck at $x_{0}$ and if

$$
A T_{K}\left(x_{0}\right)=Y
$$

then $A x_{0}$ belongs to the interior of $A(K)$ and the set-valued map $y \sim$ $A^{-1}(y) \cap K$ is pseudo- lipschitzean around $\left(A x_{0}, x_{0}\right)$. If the dimenssion of $Y$ is finite, it is sufficient to asume that $K$ is sleek at $x_{0}$.

Proof We have to prove that in both cases, the stability assumption is satisfied. The proof of the first case is easy. There exists a constant $c>0$ such that, for all $v$ in the unit sphere $S_{Y}$, there exists a solution $u_{0}$ to the equation $A u=v$ such that $\left\|u_{0}\right\| \leq c\|v\| \|$, thanks to Robinson-Ursescu's Theorem, because $T_{K}\left(x_{0}\right)$ is a closed convex cone, $K$ being sleek at $x_{0}$.

Since $K$ is actually strongly sleek at $x_{0}$, we can associate with any $\epsilon>0$ an $\eta>0$ such that, for all $u_{0} \in X$ and all $x \in B_{K}\left(x_{0}, \eta\right)$, there exists $u \in T_{K}(x)$ such that $\left\|u-u_{0}\right\| \leq \epsilon\left\|u_{0}\right\|$.

Hence any $v \in S_{Y}$ can be written $v=A z+w$ where $\|z\| \leq(1+\epsilon) c\|v\|$ and $\|w\| \leq\|A\|\left\|u_{0}-u\right\| \leq \alpha\|v\|$ when $\epsilon \leq \alpha /\|A\|$.

When the dimension of $Y$ is finite, the unit sphere $S_{Y}$ is compact. We know that for any $v_{i} \in S_{Y}$, there exits a solution $\boldsymbol{v}_{0 i}$ to the equation $A u=v_{i}$ such that $\left\|u_{0 i}\right\| \leq c\left\|v_{i}\right\| \|$. Hence for any $\epsilon>0$ and $v_{i}$, there exist $\eta_{i}>0$ such that, for all $x \in B_{K}\left(x_{0}, \eta\right)$, there exists $u \in T_{K}(x)$ such that $\left\|u_{i}-u_{0 i}\right\| \leq$ $\epsilon\left\|\boldsymbol{u}_{0 i}\right\| / 2\|A\|$.

We can cover $S_{Y}$ by $p$ balls $B\left(v_{i}, \epsilon / 2\right)$ so that, by taking $\eta:=\min _{i=1, \ldots, p} \eta_{i}$, we obtain that for any $v \in S_{Y}$ and any $x \in B_{K}\left(x_{0}, \eta\right)$, there exist $u_{i} \in T_{K}(x)$ and $w_{i} \in Y$ related by the equation $v=A x_{i}+w_{i}$ where $\left\|u_{i}\right\| \leq c\left\|v_{i}\right\|=c$ and where $\left\|w_{i}\right\| \leq\left\|v-A u_{0 i}\right\|+\|A\|\left\|u_{0 i}-w_{i}\right\| \leq \epsilon$.

We provide now theorems on local nniqueness and injectivity of setvalued maps.

Definition 3.2 Let $F: X \sim Y$ be a set-valued map. We shall say that it enjoys local inverse univocity around an element ( $x^{*}, y^{*}$ ) of its graph if and only if there exists a neighborhood $N\left(x^{*}\right)$ such that

$$
\left\{x \mid \text { such that } y^{\star} \in F(x)\right\} \cap N\left(x^{\star}\right)=\left\{x^{*}\right\}
$$

If the neighborhood $N\left(x^{*}\right)$ coincides with the domain of $F, F$ is said to have (global) inverse univocity.

We shall say that it is locally injective around $x^{*}$ if and only if there exists a neighborhood $N\left(x^{*}\right)$ ouch that, for all $x_{1} \neq x_{2} \in N\left(x^{*}\right)$, we have $F\left(x_{1}\right) \cap F\left(x_{3}\right)=$. It is said to be (globally) injective if we can take for neighborhood $N\left(x^{*}\right)$ the whole domain of $F$.

Since $0 \in D F\left(x^{\star}, y^{\star}\right)(0)$, we observe that to say that the "linearized system" $D F\left(x^{*}, y^{\star}\right)$ enjoys the inverse univocity amounts to saying that the inverse image $D F\left(x^{*}, y^{*}\right)^{-1}(0)$ contains only one element, i.e., that its kernel $\operatorname{Ker} D F\left(x^{\star}, y^{\star}\right)$ is equal to 0 , where the kernel is naturally defined by

$$
\operatorname{Ker} D F\left(x^{\star}, y^{\star}\right):=D F\left(x^{\star}, y^{\star}\right)^{-1}(0)
$$

Theorem 3.8 Let $F$ be a set-valued map from a finite dimensional vectorspace $X$ to a Banach space $Y$ and $\left(x^{*}, y^{*}\right)$ belong to its graph.

If the kernel of the contingent derivative $D F\left(x^{*}, y^{*}\right)$ of $F$ at $\left(x^{*}, y^{*}\right)$ is equal to $\{0\}$, then there existo a neighborhood $N\left(x^{\star}\right)$ such that

$$
\begin{equation*}
\left\{x \text { such that } y^{\star} \in F(x)\right\} \cap N\left(x^{\star}\right)=\left\{x^{\star}\right\} \tag{43}
\end{equation*}
$$

Let us assume that there exits $\gamma>0$ such that $F\left(x^{*}+\gamma B\right)$ is relatively compact and that $F$ has a closed graph. If for all $y \in F\left(x^{*}\right)$ the kernels of the paratingent ${ }^{18}$ derivatives $P F\left(x^{*}, y\right)$ of $F$ at $\left(x^{*}, y\right)$ are equal to $\{0\}$, then $F$ is locally injective around $x^{*}$.

Proof

1. Assume that the conclusion (43) is false. Then there exists a sequence of elements $x_{n} \neq x^{*}$ converging to $x^{*}$ satisfying

$$
\forall n \geq 0, \quad y^{\star} \in F\left(x_{n}\right)
$$

Let us set $h_{n}:=\left\|x_{n}-x^{*}\right\|$, which converges to 0 , and

$$
u_{n}:=\left(x_{n}-x^{*}\right) / h_{n}
$$

The elements $u_{n}$ do belong to the unit sphere, which is compact. Hence a subsequence (again denoted) $u_{n}$ does converge to some $u$ different from 0 . Since the above equation can be written

$$
\forall n \geq 0, \quad y^{*}+h_{n} 0 \in F\left(x^{\star}+h_{n} u_{n}\right)
$$

we deduce that

$$
0 \in D F\left(x^{\star}, y^{\star}\right)(u)
$$

Hence we have proved the existence of a non zero element of the kernel of $D F\left(x^{\star}, y^{\star}\right)$, which is a contradiction.
2. Assume that $F$ is not locally injective. Then there exists a sequence of elements $x_{n}^{1}, x_{n}^{2} \in N\left(x^{*}\right), x_{n}^{1} \neq x_{n}^{2}$, converging to $x^{*}$ and $y_{n}$ satisfying

$$
\forall n \geq 0, \quad y_{n} \in F\left(x_{n}^{1}\right) \cap F\left(x_{n}^{2}\right)
$$

Let us set $h_{n}:=\left\|x_{n}^{1}-x_{n}^{2}\right\|$, which converges to 0 , and

$$
z_{n}:=\left(x_{n}^{1}-x_{n}^{2}\right) / h_{n}
$$

The elements $z_{n}$ do belong to the anit sphere, which is compact. Hence a subsequence (again denoted) $u_{n}$ does converge to some $u$ different from 0 .

[^9]Then for all large $n$

$$
y_{n} \in F\left(x_{n}^{1}\right) \cap F\left(x_{n}^{2}\right):=F\left(x_{n}^{2}+h_{n} u_{n}\right) \cap F\left(x_{n}^{2}\right) \subset F\left(x^{\star}+\gamma B\right)
$$

we deduce that a subsequence (again denoted) $y_{n}$ converges to some $y \in F\left(x^{*}\right)$ (because $\operatorname{Graph}(F)$ is closed).
Since the above equation implies that

$$
\forall n \geq 0, \quad y_{n}+h_{n} 0 \in F\left(x_{n}^{2}+h_{n} u_{n}\right)
$$

and we deduce that

$$
0 \in P F\left(x^{*}, y\right)(u)
$$

Hence we have proved the existence of a non zero element of the kernel of $P F\left(x^{*}, y\right)$, which is a contradiction.

## 4 Calculus of Tangent Cones

We shall present now a calculus of tangent cones, from which we shall deduce a calculus of derivatives of set-valued maps and a calculus of epiderivatives.

### 4.1 Subsets and Products

If $K \subset L$, then

$$
\begin{equation*}
T_{K}(x) \subset T_{L}(x) \& T_{K}^{b}(x) \subset T_{L}^{b}(x) \tag{44}
\end{equation*}
$$

If $K_{i} \subset X, \quad(i=1, \cdots, n)$, then

$$
\left\{\begin{array}{l}
i) \quad T_{\bigcup_{i=1}^{n} K_{i}}^{n}(x)=\bigcup_{i=1}^{n} T_{K_{i}}(x)  \tag{45}\\
i i) \quad T_{\bigcup_{i=1}^{n} K_{i}}^{n}(x)=\bigcup_{i=1}^{n} T_{K_{i}}^{b}(x)
\end{array}\right.
$$

If $K_{i} \subset X_{i}, \quad(i=1, \cdots, n)$, then

$$
\begin{cases}i) & T_{\prod_{i=1}^{n} K_{i}}\left(x_{1}, \ldots, x_{n}\right) \subset \prod_{i=1}^{n} T_{K_{i}}\left(x_{i}\right)  \tag{46}\\ i i) & T_{\prod_{i=1}^{n} K_{i}}^{n}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n} T_{K_{i}}^{*}\left(x_{i}\right) \\ i i i) & \prod_{i=1}^{n} K_{i}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n} C_{K_{i}}\left(x_{i}\right)\end{cases}
$$

### 4.2 Inverse Images

Let us consider now two topological vector spaces $X$ and $Y$, subsets $L \subset X$ and $M \subset Y$ and a differentiable single-valued map $f$ from $X$ to $Y$. The following statement is obvious:

## Proposition 4.1 For any $x \in L \cap f^{-1}(M)$, we always have

$$
\left\{\begin{array}{llll}
\text { i) } & T_{L \cap f^{-1}(M)}(x) & \subset & T_{L}(x) \cap f^{\prime}(x)^{-1} T_{M}(f(x)) \\
\text { ii) } & T_{L \cap f^{-1}(M)}^{*}(x) & \subset & T_{L}^{k}(x) \cap f^{\prime}(x)^{-1} T_{M}^{b}(f(x))
\end{array}\right.
$$

We shall deduce from the Constrained Inverse Function Theorem ${ }^{16}$ converse inclusions.

[^10]Theorem 4.2 Let $X$ and $Y$ be Banach spaces, $L \subset X$ and $M \subset Y$ be closed subsets, $f$ be a continuously differentiable map around an element $x_{0} \in L \cap f^{-1}(M)$. If $X$ and $Y$ are finite dimensional vector-spaces, we posit the pointwise traneversality condition

$$
\begin{equation*}
f^{\prime}\left(x_{0}\right)\left(C_{L}\left(x_{0}\right)\right)-C_{M}\left(f\left(x_{0}\right)\right)=Y \tag{47}
\end{equation*}
$$

Then

$$
\begin{cases}\text { i) } & T_{L}^{\infty}\left(x_{0}\right) \cap f^{\prime}\left(x_{0}\right)^{-1} T_{M}\left(f\left(x_{0}\right)\right) \subset T_{L \cap f^{-1}(M)}\left(f\left(x_{0}\right)\right) \\ \text { ii) } & T_{L}^{*}\left(x_{0}\right) \cap f^{\prime}\left(x_{0}\right)^{-1} T_{M}^{*}\left(f\left(x_{0}\right)\right)=T_{L \cap \mathcal{S}^{-1}(M)}^{\prime}\left(f\left(x_{0}\right)\right) \\ \text { iii) } & C_{L}\left(x_{0}\right) \cap f^{\prime}\left(x_{0}\right)^{-1} C_{M}\left(f\left(x_{0}\right)\right) \subset C_{L \cap f^{-1}(M)}\left(f\left(x_{0}\right)\right)\end{cases}
$$

Otherwise, we have to replace the pointwise transversality condition by the local transversality condition condition : there exist constants $c>0$, $\alpha \in[0,1[$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall x \in L \cap B\left(x_{0}, \eta\right), y \in M \cap B\left(f\left(x_{0}\right), \eta\right)  \tag{48}\\
B_{Y} \subset f^{\prime}(x)\left(T_{L}^{k}(x) \cap c B_{X}\right)-T_{M}(y)+\alpha B_{Y}
\end{array}\right.
$$

As a consequence, we infer that $L \cap f^{-1}(M)$ is sleek (respectively derioable) whenever $L$ and $M$ are sleek (respectively derivable).

Proof Let us prove for instance the inclusion for the Clarke tangent cones. Take any sequence of elements $x_{n} \in L \cap f^{-1}(M)$ which converges to $x$. Let us take any $u \in C_{L}(x)$ such that $f^{\prime}\left(x_{0}\right) u \in C_{M}(f(x))$. Hence for any sequence $h_{n}>0$, there exist sequences $u_{n}$ and $v_{n}$ converging to $u$ and $f^{\prime}\left(x_{0}\right) u$ respectively such that, for all $n \geq 0$,

$$
x_{n}+h_{n} u_{n} \in L \& f\left(x_{n}\right)+h_{n} v_{n} \in M
$$

We apply now the Constrained Inverse Function Theorem 3.5 to the subset $L \times M$ of $X \times Y$ and the continuous map $f \ominus 1$ associating to any $(x, y)$ the element $f(x)-y$, since we can write

$$
K:=L \cap f^{-1}(M)=(f \ominus 1)^{-1}(0) \cap(L \times M)
$$

It is obvious that the transversality condition (48) implies the stability assumption of the Constrained Inverse Function Theorem. The pair ( $x_{n}+$ $\left.h_{n} \boldsymbol{u}_{n}, f\left(x_{n}\right)+h_{n} v_{n}\right)$ belongs to $L \times M$ and

$$
(f \ominus \mathbf{1})\left(x_{n}+h_{n} w_{n}, f\left(x_{n}\right)+h_{n} v_{n}\right) \text { converges to } 0
$$

because $f$ is continuously differentiable at $x_{0}$.

Therefore, by the Constrained Inverse Function Theorem 3.5, there exits a solution $\left(\widehat{x_{n}}, \widehat{y_{n}}\right) \in L \times M$ to the equation $(f \ominus 1)\left(\widehat{x_{n}}, \widehat{y_{n}}\right)=0$ (i.e., $\left.\widehat{y_{n}}=f\left(\widehat{x_{n}}\right)\right)$ such that
$\left\|x_{n}+h_{n} u_{n}-\widehat{x_{n}}\right\|+\left\|f\left(x_{n}\right)+h_{n} v_{n}-\widehat{y_{n}}\right\| \leq l\left\|f\left(x_{n}+h_{n} u_{n}\right)-f\left(x_{n}\right)-h_{n} v_{n}-0\right\|$
Hence $\widehat{\boldsymbol{u}_{n}}:=\left(x_{n}-\widehat{x_{n}}\right) / h_{n}$ converges to $u$, and for all $n \geq 0$, we know that $x_{n}+h_{n} u_{n}$ belongs to $L \cap f^{-1}(M)$ because $x_{n}+h_{n} \widehat{x_{n}}=\widehat{x_{n}}$ belongs to $L$ and $f\left(x_{n}+h_{n} \widehat{u_{n}}\right)=\widehat{y_{n}}$ belongs to $M$.

We list now three useful corollaries of this theorem:
Corollary 4.1 (Tangent Cones to Inverse Images) Asoume that $M \subset$ $Y$ is a closed subset and that $f$ is a continuously differentiable map around an element $x_{0} \in f^{-1}(M)$.

When the dimenoion of $X$ and $Y$ io finite, we ouppose that

$$
\operatorname{Im}\left(f^{\prime}\left(x_{0}\right)\right)+C_{M}\left(f\left(x_{0}\right)\right)=Y
$$

Then

$$
\begin{cases}i) & T_{f^{-1}(M)}\left(x_{0}\right)=f^{\prime}\left(x_{0}\right)^{-1} T_{M}\left(f\left(x_{0}\right)\right) \\ i i) & T_{f^{-1}(M)}^{b}\left(x_{0}\right)=f^{\prime}\left(x_{0}\right)^{-1} T_{M}^{b}\left(f\left(x_{0}\right)\right) \\ i i i) & C_{f^{-1}(M)}\left(x_{0}\right) \subset f^{\prime}\left(x_{0}\right)^{-1} C_{M}\left(f\left(x_{0}\right)\right)\end{cases}
$$

Othervise, we assume that there exist conotants $c>0, \alpha \in[0,1]$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall x \in B\left(x_{0}, \eta\right), y \in B\left(f\left(x_{0}\right), \eta\right) \cap M \\
\left.B_{Y} \subset \operatorname{Im}\left(f^{\prime}(x)\right) \cap c B_{X}\right)+T_{M}(y)+\alpha B_{Y}
\end{array}\right.
$$

Corollary 4.2 (Tangent Cone to an Intersection) If $K_{1}$ and $K_{2}$ are closed subsets contained in $X$. If $X$ is a finite dimensional vector-space, soe asoume that

$$
C_{K_{1}}(x)-C_{K_{2}}(x)=X
$$

Then

$$
\left\{\begin{array}{llll}
\text { i) } & T_{K_{1}}^{b}(x) \cap T_{K_{2}}(x) \subset & T_{K_{1} \cap K_{2}}(x) \\
\text { ii) } & T_{K_{1}}^{\mathrm{b}}(x) \cap T_{K_{2}}^{b}(x)= & T_{K_{1} \cap K_{2}}^{b}(x) \\
\text { iii) } & C_{K_{1}}(x) \cap C_{K_{2}}(x) & \subset & C_{K_{1} \cap K_{2}}(x)
\end{array}\right.
$$

Otherwise, we suppose that there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall x \in K_{1} \cap B\left(x_{0}, \eta\right), y \in B\left(x_{0}, \eta\right) \cap K_{2} \\
\left.B_{Y} \subset A T_{K_{1}}^{*}(x) \cap c B_{X}\right)-T_{K_{2}}(y)+\alpha B_{Y}
\end{array}\right.
$$

Finally, for a finite intersection, we can state:
Corollary 4.3 (Tangent Cone to a Finite Intersection) Let ue consider $n$ closed subsets $K_{i} \subset X$. When the dimension of $X$ is finite, we assume that

$$
\forall v_{1}, \ldots, v_{n} \in X, \quad \bigcap_{i=1}^{n}\left(C_{K_{i}}\left(x_{0}\right)-v_{i}\right) \neq 0
$$

Then

$$
\left\{\begin{array}{l}
\text { i) } \bigcap_{i=1}^{n} T_{K_{i}}^{b}\left(x_{0}\right)=T_{\bigcap_{i=1}^{n} K_{i}}^{n}\left(x_{0}\right) \\
\text { ii) } \bigcap_{i=1}^{n} C_{K_{i}}\left(x_{0}\right) \subset C_{\bigcap_{i=1}^{n} K_{i}}\left(x_{0}\right)
\end{array}\right.
$$

If the dimension of $X$ is infinite, we assume that there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall x_{i} \in K_{i} \cap B\left(x_{0}, \eta\right), \forall v_{i} \in X, \\
\exists w_{i} \in X, \exists u \in \bigcap_{i=1}^{n}\left(T_{K_{i}}\left(x_{i}\right)-v_{i}-w_{i}\right) \\
\text { such that }\|w\| \leq c \max _{i=1, \ldots, n}\left\|v_{i}\right\| \&\left\|w_{i}\right\| \leq \alpha \max _{i=1, \ldots, n}\left\|v_{i}\right\|
\end{array}\right.
$$

### 4.3 Direct Images

Let as consider now two topological vector spaces $X$ and $Y$, a subset $K \subset$ $X$ and a differentiable single-valued map $f$ from $X$ to $Y$. The following statement is obvious:

Proposition 4.2 For any $y \in f(K)$, we have

$$
\left\{\begin{array}{l}
\text { i) } \overline{U_{x \in K \cap f^{-1}(y)} f^{\prime}(x)\left(T_{K}(x)\right)} \\
\text { ii) } \\
\text { U } \\
U_{x \in K \cap f^{-1}(y)} f^{\prime}(x)\left(T_{K}^{b}(x)\right)
\end{array} T_{f(K)}(y)\right.
$$

It is not that easy to find elegant sufficient conditions implying the equality

$$
\begin{equation*}
\bigcup_{x \in K \cap f^{-1}(y)} f^{\prime}(x)\left(T_{K}(x)\right)=T_{f(K)}(y) \tag{49}
\end{equation*}
$$

Let us review some simple ones:
Proposition 4.3 If $A \in \mathcal{L}(X, Y)$ is a continuous linear operator and if $K$ is poeudo-convex at some point $x$ of $K \cap A^{-1}(y)$, then

$$
\overline{A T_{K}(x)}=T_{A(K)}(y)
$$

In particular, when $K$ is convex, we have

$$
\bigcap_{x \in K \cap A^{-1}(y)} \overline{A\left(T_{K}(x)\right)}=T_{A(K)}(y)
$$

The Criterion of Pseudo-Lipschitzeanity ${ }^{17}$ provides a more general sufficient condition:

[^11]\[

\left\{$$
\begin{array}{l}
\forall x \in K \cap B\left(x_{0}, \eta\right), \\
A S_{K}(x) \cap B_{Y} \subset A\left(T_{K}(x) \cap a B_{X}\right)+\alpha B_{Y}
\end{array}
$$\right.
\]

Then the aet-valued map

$$
A(K) \ni y \leadsto A^{-1}(y) \cap K
$$

is pseudo-llpschitsean around $\left(A x_{0}, x_{0}\right)$ : For any $x_{1}$ close to $x_{0}$ and $y \in K$ close to $A x_{0}$,

$$
d\left(x_{1}, A^{-1}(y) \cap K\right) \leq t\left\|y-A x_{1}\right\|
$$

Theorem 4.4 Let $K$ be a elosed oubset of a refleaive Banach space $X$ and $A \in \mathcal{L}(X, Y)$ be a continuous linear operator from $X$ to another Banach opace $Y$. Let ue aosume that for some $x_{0} \in K \cap A^{-1}\left(y_{0}\right)$, there exist conotanto $c>0, \alpha \in[0,1 \mid$ and $\eta>0$ such that

$$
\left\{\begin{array}{l}
\forall x \in K \cap B\left(x_{0}, \eta\right) \\
A S_{K}(x) \cap B_{Y} \subset A\left(T_{K}(x) \cap c B_{X}\right)+\alpha B_{Y}
\end{array}\right.
$$

Then, if $X$ is supplied with the weak topology, we obtain the equality

$$
\begin{equation*}
\overline{A T_{K}\left(x_{0}\right)}=T_{A(K)}\left(y_{0}\right) \tag{50}
\end{equation*}
$$

Proof Let $v$ belong to $T_{A(K)}\left(y_{0}\right)$. Then there exist sequences of elements $h_{n}>0$ and $v_{n}$ converging to 0 and $v$ respectively such that

$$
\begin{equation*}
y_{0}+h_{n} v_{n}=A x_{n} \in A(K) \tag{51}
\end{equation*}
$$

The point is to choose solutions $x_{n} \in K$ to the above equation (51) and a solution $x_{0} \in K$ to the equation $A x_{0}=y_{0}$ such that

$$
\text { a subsequence of } u_{n}:=\left(x_{n}-x_{0}\right) / h_{n} \text { converges to some } u
$$

Such an element $u$ belongs to the contingent cone $T_{K}\left(x_{0}\right)$ and is a solution to the equation $A u=v$.

Since the set-valued map $A(K) \ni y \sim K \in A^{-1}(y)$ is pseudo-lipschitzean around ( $y_{0}, x_{0}$ ) by the Criterion of Pseudo-Lipschitzeanity, there exist a constant $l^{\prime}$ and solutions $x_{n} \in K$ to the equation (51) such that

$$
\left\|x_{0}-x_{n}\right\| \leq\left\|y_{0}-y_{0}-h_{n} v_{n}\right\|=h_{n}\left\|v_{n}\right\|
$$

Therefore, the sequence of elements $u_{n}$ is bounded, so that a subsequence (again denoted) $\boldsymbol{u}_{n}$ converges (weakly when the dimension of $X$ is infinite) to some $\boldsymbol{u}$.

Remark Any sufficient condition implying that for some $x_{0} \in$ $K \cap A^{-1}\left(y_{0}\right)$, the set-valued map $A(K) \ni y \leadsto K \cap A^{-1}(y)$ is pseudolipschitzean will automatically imply the above equality (50) between the contingent cone to the image and the closure of the image of the contingent cone.

The sequence of elements $z_{n}:=\left(x_{n}-x_{0}\right) / h_{n}$ satisfies

$$
\left\{\begin{array}{l}
\text { i) } u_{n} \in S_{K}\left(x_{0}\right)  \tag{52}\\
\text { ii) } A v_{n}=v_{n}
\end{array}\right.
$$

Therefore, any properness criterion ${ }^{18}$ of the map $A$ from the closed cone $\overline{S_{K}\left(x_{0}\right)}$ spanned by $K-x_{0}$ to $Y$ implies equality ( 50 ).

In particular, the Closed Range Theorem provides such a simple criterion. Since the barrier cone of $S_{K}\left(x_{0}\right)$ is its polar cone, which is the subnormal cone $N_{K}^{0}\left(x_{0}\right)$ to $K$ at $x_{0}$, we then obtain the following statement:

Theorem 4.5 Let $K$ be a closed subset of a reflesive Banach space $X$ and $A \in \mathcal{L}(X, Y)$ be a continuous linear operator from $X$ to another Banach space $Y$. Let us assume that for some $x_{0} \in K \cap A^{-1}\left(y_{0}\right)$,

$$
\begin{equation*}
\operatorname{Im}\left(A^{*}\right)+N_{K}^{0}\left(x_{0}\right)=X^{*} \tag{53}
\end{equation*}
$$

Then, if $X$ is supplied with the weak topology, we obtain the equality (50).

Remark Criterion (53), which is easy to use, is much too strong, since it requires implicitly that the inverse $y \sim K \cap A^{-1}(y)$ of the restriction of $A$ to $K$ is actually locally single-valued. Indeed, by taking the polar cones in equality (53), we obtain

$$
\operatorname{ker}(A) \cap \overline{c o}\left(S_{K}\left(x_{0}\right)\right)=\{0\}
$$

Actually, when the dimension of $X$ is finite, we have a stronger criterion:

[^12]Proposition 4.4 If the dimension of $X$ is finite and if for some $x_{0} \in$ $K \cap A^{-1}\left(y_{0}\right)$, then condition

$$
\operatorname{ker}(A) \cap T_{K}\left(x_{0}\right)=\{0\}
$$

implies equality (50).
Proof We have to prove that the sequence of solutions $u_{n}$ to (51) are bounded. If not, $\left\|u_{n}\right\|$ will go to $\infty$. Hence a subsequence of elements $\widehat{u_{n}}:=u_{n} /\left\|u_{n}\right\|$ of the unit sphere, which is compact, converges to some $u \neq 0$. Since the sequence $A \widehat{u_{n}}=v_{n} /\left\|u_{n}\right\|$ converges to 0 , we infer that $u$ belongs to the kernel of $A$. It belongs also to the contingent cone $T_{K}\left(x_{0}\right)$, since we can write

$$
x_{0}+h_{n}\left\|u_{n}\right\| \widehat{u_{n}} \in K
$$

and since $h_{n}\left\|u_{n}\right\|$ converges to 0 .

## 5 Tangent Cones in Lebesgue Spaces

Let ( $\Omega, S, \mu$ ) be a measure space and $X$ be a Banach space. Let us consider a sequence of measurable set-valued maps

$$
K: \omega \in \Omega \leadsto K(\omega) \subset X .
$$

We associate to it the subsets $K$ of $L^{p}(\Omega, X)$ defined by

$$
K:=\left\{x(\cdot) \in L^{p}(\Omega, X) \mid \text { for almost all } \omega \in \Omega, x(\omega) \in K(\omega)\right\}
$$

We shall characterize the adjacent and contingent cones to such a subset in termes of the tangent cones to the subsets $K(\omega)$.

Theorem 5.1 Let ue assume that the set-valued map $K$ is measurable and that the subset $K$ is not empty. Then

$$
\left\{\begin{array}{l}
\left\{v(\cdot) \in L^{p}(\Omega, X) \mid \text { for almost all } \omega, v(\omega) \in T_{K(\omega)}^{b}(x(\omega))\right\} \\
\subset T_{K}^{b}(x(\cdot)) \subset T_{K}(x(\cdot)) \\
\subset\left\{v(\cdot) \in L^{p}(\Omega, X) \mid \text { for almost all } \omega, v(\omega) \in T_{K(\omega)}(x(\omega))\right\}
\end{array}\right.
$$

## Proof

1. Let $v(\cdot)$ belong to the first subset. We have to prove that that when $h>0$ goes to 0 , there exists functions $v_{h}(\cdot) \in L^{p}(\Omega, X)$ converging to $v(\cdot)$ such that

$$
\text { for almost all } \omega \in \Omega, \quad x(\omega)+h v_{h}(\omega) \in K(\omega)
$$

Let us set:

$$
a_{h}(\omega):=d(v(\omega),(K(\omega)-x(\omega)) / h)
$$

The function $a_{h}$ is measurable and converge to 0 almost everywhere because for almost all $\omega \in \Omega, v(\omega)$ belongs to the adjacent cone to $\boldsymbol{K}(\omega)$ at $\boldsymbol{x}(\omega)$.

Since

$$
\text { for almost all } \omega, a(\omega) \leq\|v(\omega)\|
$$

and since the right-hand side of this inequality belongs to $L^{p}(\Omega)$, we deduce from Lebesgue's Theorem that the functions $a_{h}(\cdot)$ do corverge to 0 in $L^{p}(\Omega)$. Let us introduce now the subsets $L_{h}(\omega)$ defined by

$$
L_{h}(\omega):=\left\{z \in \overline{K(\omega)} \mid d(v(\omega),(z-x(\omega)) / h)=a_{h}(\omega)\right\}
$$

It is clear that the set-valued map $L_{h}(\cdot)$ is also measurable. The Measurable Selection Theorem allows us to choose a measurable selection $z_{h}(\cdot)$ of the set-valued map $L_{h}(\cdot)$.
We define now the functions $v_{h}(\cdot)$ by

$$
\left.v_{h}(\omega):=\left(z_{h}(\omega)-x(\omega)\right) / h\right)
$$

They are measurable, satisfy

$$
\left\|v_{h}(\omega)-v(\omega)\right\|=a_{h}(\omega)
$$

and thus, converges to $v(\cdot)$ in $L^{p}(\Omega ; X)$ since $a_{h}(\cdot)$ converges to 0 in $L^{p}(\Omega)$. We infer that $v(\cdot)$ belongs to $T_{k}^{p}(x(\cdot))$ because

$$
\text { for almost all } \Omega \in \Omega, x(\omega)+h v_{h}(\omega) \in K(\omega)
$$

2. Let us choose now some $v(\cdot)$ in the contingent cone to the subset $K$. Then there exist subsequences $h_{n}>0$ and $v_{n}(\cdot)$ converging respectively to 0 and to $x(\cdot)$ in $L^{p}(\Omega ; X)$ and satisfying

$$
\text { for almost all } \omega \in \Omega, \quad x(\omega)+h_{n} v_{n}(\omega) \in K(\omega)
$$

Then a subsequence (again denoted) $v_{n}(\cdot)$ converges almost everywhere to $v(\cdot)$ and consequently, for almost all $\omega, v(\omega)$ belongs to the contingent cone to the subset $K(\omega)$ at $x(\omega)$.
Naturally, we infer that
Corollary 5.1 Let us assume that the set-valued map $K$ is measurable and that the oubset $K$ is not empty. If the subsets $K(\omega)$ are derivable, so is $K$ and

$$
T_{K}(x(\cdot))=\left\{v(\cdot) \in L^{P}(\Omega, X) \mid \text { for almost all } \omega, v(\omega) \in T_{K(\omega)}(x(\omega))\right\}
$$

## 6 Derivatives of Set-Valued maps

We shall derive from each concept of tangent cone to a subset an associated concept of graphical derivative of a set-valued map $F$ from a topological vector space $X$ to another $Y$.

The idea is very simple, and goes back to the protohistory of the differential calculus, when Pierre de Fermat introduced in the first half of the seventeenth century the concept of a tangent to the graph of a function.

The tangent space to the graph of a function $f$ at a point $(x, y)$ of its graph is the line of slope $f^{\prime}(x)$, i.e., the graph of the linear function $u \mapsto f^{\prime}(x) u$.

It is possible to implement this idea for any set-valued map $F$ since we have introduced (unfortunately, several) ways to implement the concept of tangency for any subset of a topological vector space. Therefore, in the framework of a given problem, we can choose the adequate concept of tangent cone, and thas, regard this tangent cone to the graph of the set-valued map $F$ at some point $(x, y)$ of its graph as the graph of the associated "graphical" derivative of $F$ at this point $(x, y)$.

Since the tangent cones are at least ... cones, all these derivatives are at least positively homogeneous set-valued maps (also called processes). This is what remains of the familiar, but luxurious, requirement of linearity.

However, they are closed convex processes, i.e., set-valued analogues of continuous linear operators, when the tangent cones happen to be closed and convex (this is the case when we use the Clarke tangent cone or the Minkowski differences of the contingent or adjacent cones).

Hence, we start with some definitions and notations.

[^13]We introduce the three following graphical derivatives

1. the contingent derivative $D F(x, y):=D^{\sharp} F(x, y)$, defined by

$$
\begin{equation*}
\operatorname{Graph}(D F(x, y)):=T_{\mathrm{Graph}_{(F)}}(x, y) \tag{54}
\end{equation*}
$$

2. the adjacent derivative $D^{b} F(x, y)$ defined by

$$
\begin{equation*}
\operatorname{Graph}\left(D^{\prime} F(x, y)\right):=T_{\operatorname{Graph}_{(F)}}^{b}(x, y) \tag{55}
\end{equation*}
$$

3. the circatangent derivative $C F(x, y)$ defined by

$$
\begin{equation*}
\operatorname{Graph}(C F(x, y)):=C_{\operatorname{Graph}_{(F)}}(x, y) \tag{56}
\end{equation*}
$$

We shall say that $F$ is sleek at $(x, y) \in \operatorname{Graph}(F)$ if and only if

$$
\left(x^{\prime}, y^{\prime}\right) \sim \operatorname{Graph}(D F)\left(x^{\prime}, y^{\prime}\right) \text { is lower semicontinuous at }(x, y)
$$

and it is sleek if it is sleek at every point of its graph.
We shall say that $F$ is derivable at $(x, y) \in \operatorname{Graph}(F)$ if and only if the contingent and adjacent derivatives coincide:

$$
D F(x, y):=D^{b} F(x, y)
$$

and that it is derivable if it is derivable at every point of ito graph.
Finally, we shall say that $F$ is pseudo-convex at $(x, y) \in \operatorname{Graph}(F)$ if and only if

$$
\forall x^{\prime} \in \operatorname{Dom}(F), \quad F\left(x^{\prime}\right) \subset D F(x, y)\left(x^{\prime}-x\right)+y
$$

We see at once that these three graphical derivatives are closed processes and that

$$
\begin{equation*}
\forall u, C F(x, y)(u) \subset D^{\downarrow} F(x, y)(u) \subset D F(x, y)(u) \tag{57}
\end{equation*}
$$

Naturally, the circatangent derivative is always a closed convex process, and the contingent derivative is a closed convex process whenever $F$ is sleek at $(x, y)$.

When $F:=f$ is single-valued, we set

$$
D f(x):=D f(x, f(x)), D^{\prime} f(x):=D^{\prime} f(x, f(x)), \quad C f(x):=C f(x, f(x))
$$

We see easily from Propositions 6.2 and 6.3 below that
$\left\{\begin{array}{lll}i) & D f(x))(u)=f^{\prime}(x) u & \text { if } f \text { is Gâteaux differentiable } \\ i i) & D^{\prime} f(x)=f^{\prime}(x) u & \text { if } f \text { is Fréchet differentiable } \\ i i i) & C f(x)(u)=f^{\prime}(x) u & \text { if } f \text { is continuously differentiable }\end{array}\right.$
Restrictions $F:=\left.f\right|_{K}$ of single-valued maps $f$ to subsets $K \subset X$ provide a wide class of set-valued maps defined by

$$
\left.f\right|_{K}(x):=\left\{\begin{array}{lll}
f(x) & \text { if } x \in K \\
0 & \text { if } x \notin K
\end{array}\right.
$$

for which we obtain the following formulas: if $f$ is continuously differentiable around a point $x \in K$, then the derivative of the restriction is the restriction of the derivative to the corresponding tangent cone:

$$
\left\{\begin{array}{llll}
i) & D\left(\left.f\right|_{K}\right)(x) & :=D\left(\left.f\right|_{K}\right)(x, f(x)) & =\left.f^{\prime}(x)\right|_{T_{K}(x)}  \tag{59}\\
i i) & D^{\prime}\left(\left.f\right|_{K}\right)(x) & :=D^{\prime}\left(\left.f\right|_{K}\right)(x, f(x)) & =\left.f^{\prime}(x)\right|_{T_{K}^{\prime}(x)} \\
i i i) & C\left(\left.f\right|_{K}\right)(x) & :=C\left(\left.f\right|_{K}\right)(x, f(x))=\left.f^{\prime}(x)\right|_{C_{K}(x)}
\end{array}\right.
$$

Actually, this follows from the useful
Proposition 6.1 Let $X$ and $Y$ be normed space,$f$ a continuously differentiable operator from an open subset $\Omega \subset X$ to $Y, L \subset X$ and $M \subset Y$ closed subsets. Let $F: X \sim Y$ be the set-valued map defined by:

$$
F(x):=\left\{\begin{array}{lll}
f(x)-M & \text { when } & x \in L \\
& \text { when } & x \notin L
\end{array}\right.
$$

Let $(x, y)$ belong to the graph of $F$. Then its adjacent derivative is equal to

$$
D^{b} F(x, y)(x):= \begin{cases}f^{\prime}(x) u-T_{M}^{h}(f(x)-y) & \text { when } u \in T_{L}^{b}(x) \\ 0 & \text { when } u \notin T_{L}^{b}(x)\end{cases}
$$

The same formula holds true for the cireatangent derivative and the Clarke tangent cones.

## Proof

Let $v$ belong to $D F(x, y)(u)$ and let us prove that it belongs to $f^{\prime}(x) u-$ $T_{M}(f(x)-y)$. We know that for all $h_{n}>0$ converging to 0 , there exist sequences $u_{n}$ and $v_{n}$ converging to $u$ and $v$ respectively such that $x+h_{n} u_{n}$ belongs to $L$ and $y+h_{n} v_{n}$ belongs to $f\left(x+h_{n} v\right)-M$ for all $n$. This implies that $u$ belongs to to $T_{L}^{l}(x)$ and, since we can write $f\left(x+h_{n} u_{n}\right)=$ $f(x)+h_{n}\left(f^{\prime}(x) u+o\left(h_{n}\right)\right)$, that $f(x)-y+h_{n}\left(f^{\prime}(x) u-v_{n}+o\left(h_{n}\right)\right)$ belongs to $M$. Hence $f^{\prime}(x) w-v$ does belongs to $T_{M}^{b}(f(x)-y)$.

Conversely, assume that $u$ belongs to $T_{L}^{\prime}(x)$ and that $f^{\prime}(x) u-v$ belongs to $T_{M}^{b}(f(x)-y)$. Hence, for all sequence $h_{n}>0$ converging to 0 , there exist sequence $u_{n}$ amd $w_{n}$ converging to $u$ and $f^{\prime}(x) u-v$ such that $x+h_{n} u_{n}$ and $f(x)-y+h_{n} w_{n}$ belongs to $L$ and $M$ respectively. Then $v_{n}:=f^{\prime}(x) u+$ $o\left(h_{n}\right)-w_{n}$ converges to $v$ and satisfies $y+h_{n} v_{n} \in f(x)+h_{n} u_{n}-M$.

Remark For contingent derivatives, we can only prove that

$$
D F(x, y)(u) \subset \begin{cases}f^{\prime}(x) u-T_{M}(f(x)-y) & \text { when } u \in_{L}(x) \\ & \text { when } u \notin T_{L}(x)\end{cases}
$$

and that

$$
D F(x, y)(u) \supset \begin{cases}f^{\prime}(x) u-T_{M}^{b}(f(x)-y) & \text { when } u \in_{L}(x) \\ & \text { when } u \notin T_{L}(x)\end{cases}
$$

Another familiar instance of set-valued maps is the inverse of a setvalued map $F$ (or even of a non injective single-valued map). We can easily compate any of their graphical derivative because a graphical derivative of the inverse of a set-valued map $F$ is the inverse of the derivative:

$$
\begin{cases}i) & D(F)^{-1}(y, x)=D F(x, y)^{-1}  \tag{60}\\ i i) & D^{b}(F)^{-1}(y, x)=D^{b} F(x, y)^{-1} \\ \text { iii) } C(F)^{-1}(y, x)=C F(x, y)^{-1}\end{cases}
$$

The first task is to characterize these derivatives by adequate limits of difference quotients. We begin with the case of contingent derivatives.

Proposition 6.2 Let $(x, y) \in \operatorname{Graph}(F)$ belong to the graph of a set-valued mapF $: X \sim Y$ from a normed space $X$ to another $Y$. Then

$$
\left\{\begin{array}{l}
v \text { belonge to } D F(x, y)(x) \text { if and only if }  \tag{61}\\
\liminf _{h \rightarrow 0+, x^{\prime} \rightarrow x} d\left(v, \frac{F\left(x+h x^{\prime}\right)-y}{h}\right)=0
\end{array}\right.
$$

If $F$ is lipschitzean around an element $x_{0} \in \operatorname{Int}(\operatorname{Dom}(F))$, then

$$
\left\{\begin{array}{l}
v \text { belongs to } D F(x, y)(u) \quad \text { if and only if }  \tag{62}\\
\liminf _{h \rightarrow 0+}, d\left(v, \frac{F(x+h)-y}{h}\right)=0
\end{array}\right.
$$

Furthermore, if the dimension of $Y$ io finite, or if $Y$ is a reflexive Banach opace supplied with the weak topology, then

$$
\begin{equation*}
\operatorname{Dom}(D F(x, y))=X \tag{63}
\end{equation*}
$$

Proof The first two statements being obvious, let us check the last one. Let $\boldsymbol{x}$ belong to $X$. Then, for all $h>0$ small enough and $y \in F(x)$,

$$
y \in F(x) \subset F(x+h u)+l h\|u\| B
$$

Hence there exits $y_{h} \in F(x+h u)$ such that $v_{h}:=\left(y_{h}-y\right) / h$ belongs to $l\|u\| B$, which is compact. Hence a subsequence (again denoted) of $v_{h}$ converges (weakly if the dimension of $Y$ is infinite) to some $v$, which belongs to $D F(x, y)(z)$.

In order to characterize adjacent and circatangent derivatives in terms of limits of difference quotients, we need to introduce the concept of "lim sup inf" of functions of two variables.

Definition 6.2 (Lim sup inf) Let $L$ and $M$ be two metric spaces and $\phi: L \times M \mapsto \mathbf{R}$ be a function. We set

$$
\lim \sup _{x^{\prime} \rightarrow s} \inf _{y^{\prime} \rightarrow y^{\prime}} \phi\left(x^{\prime}, y^{\prime}\right):=\sup _{c>\delta} \inf _{\eta>0} \sup _{\substack{ \\\in(x, y)}} \inf _{y^{\prime} \in B(y, c)} \phi\left(x^{\prime}, y^{\prime}\right)
$$

Hence, by translating the definition of the adjacent and the Clarke tangent cones, we obtain the following characterizations:

Proposition 6.3 Let $(x, y) \in \operatorname{Graph}(F)$ belong to the graph of a set-valued mapF $: X \leadsto Y$ from a normed opace $X$ to another $Y$. Then

$$
\left\{\begin{array}{l}
v \text { belongs to } D^{h} F(x, y)(u) \text { if and only if }  \tag{64}\\
\lim _{\text {sup }}^{h \rightarrow 0+} \\
\inf _{v^{\prime} \rightarrow v} d\left(v, \frac{F\left(x+h u^{\prime}\right)-y}{h}\right)=0
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
v \text { belongs to } C F(x, y)(u) \text { if and only if }  \tag{65}\\
\lim \sup _{h \rightarrow 0+,\left(x^{\prime}, y^{\prime}\right) \rightarrow(x, y)} \text { inf }_{v^{\prime} \rightarrow u^{\prime}} d\left(v, \frac{F\left(x^{\prime}+u^{\prime}\right)-y^{\prime}}{h}\right)=0
\end{array}\right.
$$

If $F$ is lipschitzean around an element $x \in \operatorname{Int}(\operatorname{Dom}(F))$, then the formulas become much simpler:

$$
\left\{\begin{array}{l}
v \text { belongs to } D^{h} F(x, y)(u) \text { if and only if }  \tag{66}\\
\lim _{h \rightarrow 0+} d\left(v, \frac{F\left(x+h k^{\prime}\right)-y}{h}\right)=0
\end{array}\right.
$$

and

$$
\left\{\begin{array}{l}
v \text { belongs to } C F(x, y)(u) \quad \text { if and only if }  \tag{67}\\
\lim _{h \rightarrow 0+,\left(x^{\prime}, y^{\prime}\right) \rightarrow(x, y)} d\left(v, \frac{F\left(x^{\prime}+h v^{\prime}\right)-y^{\prime}}{h}\right)=0
\end{array}\right.
$$

Let us mention the following property:
Proposition 6.4 Let us assume that the images of $F$ are convex and that $F$ is lipschitzean around $x$. Then the images of the adjacent derivative $D F^{b}(x, y)$ are convex.

Proof Let $v^{1}$ and $v^{2}$ belong to $D F(x, y)(u)$. Then, for any sequence $h_{n}>0$, there exist sequences $u_{n}^{1}$ and $u_{n}^{2}$ converging to $u$ and sequences $v_{n}^{1}$ and $v_{n}^{2}$ converging to $v^{1}$ and $v^{2}$ respectively such that

$$
\forall n, y+h_{n} v_{n}^{i} \in F\left(x+h_{n} u_{n}^{i}\right) \quad(i=1,2)
$$

Since $F$ is lipschitzean around $x$,

$$
y+h_{n} v_{n}^{2} \in F\left(x+h_{n} u_{n}^{1}\right)+l h_{n}\left\|u_{n}^{2}-u_{n}^{1}\right\|
$$

so that there exists another sequence $\overline{v_{n}^{2}}$ converging to $v^{2}$ such that

$$
y+h_{n} \overline{v_{n}^{2}} \subset F\left(x+h_{n} u_{n}^{1}\right)
$$

Now, $F\left(x+h_{n} u_{n}^{1}\right)$ being convex, we deduce that for all $\lambda \in[0,1]$,

$$
y+h_{n}\left(\lambda v_{n}^{1}+(1-\lambda) \overline{v_{n}^{2}}\right) \in F\left(x+h_{n} u_{n}^{1}\right)
$$

Since $\lambda v_{n}^{1}+(1-\lambda) \overline{v_{n}^{2}}$ converges to $\lambda v^{1}+(1-\lambda) v^{2}$, we deduce that this element belongs to $D F^{b}(x, y)(z)$.

Remark: Kernel of the Derivative The kernels of the various derivatives characterize the associated tangent cones to the inverse image.

Proposition 6.5 Let $F: X \sim Y$ be a set-valued map and $(x, y)$ belong to its graph. Then

$$
\left\{\begin{array}{lll}
i) & T_{F-1}(y) & \subset \\
\text { (x) } & \subset \operatorname{ker} D(x, y) \\
i i) & T_{F^{-1}(y)}^{*}(x) & \subset \operatorname{ker} D^{b} F(x, y)
\end{array}\right.
$$

If $F^{-1}$ is pseudo-lipschitzean around $(y, x)$, we have

$$
\begin{cases}i) & \operatorname{ker} D F(x, y)=T_{F^{-1}(y)}(x) \\ i i) & \operatorname{ker} D^{4} F(x, y)=T_{F^{-1}(y)}(x) \\ i i i) & \text { ker } C F(x, y) \subset C_{F^{-1}(y)}(x)\end{cases}
$$

Proof The first inclusions are obvious. To prove the converse inclusions, let $u$ belong to the kernel of $C F(x, y)$ for instance: for all sequence $x_{n} \in F^{-1}(y)$ converging to $x$ and sequence $h_{n}>0$ converging to 0 , there exist sequences $u_{n}$ and $v_{n}$ converging respectively to $u$ and 0 satisfying

$$
\forall n, y+h_{n} v_{n} \in F\left(x_{n}+h_{n} u_{n}\right)
$$

Since $F^{-1}$ is pseudo-lipschitzean around ( $y, x$ ), there exists an element $x_{n}^{1} \in F^{-1}(y)$ such that

$$
\left\|x_{n}^{1}-\left(x_{n}+h_{n} u_{n}\right)\right\| \leq l\left\|y_{n}-\left(y_{n}+h_{n} v_{n}\right)\right\|
$$

Hence, by setting $u_{n}^{1}:=\left(x_{n}^{1}-x_{n}\right) / h_{n}$, we see that $x_{n}+h_{n} u_{n}^{1}=x_{n}^{1}$ belongs to $F^{-1}(y)$ and that $u_{n}^{1}$ converges to $u$ because $\left\|u_{n}^{1}-u_{n}\right\| \leq l\left\|v_{n}\right\|$ and because $v_{n}$ converges to 0 .

Therefore we have proved that $u$ belongs to the Clarke tangent cone to $F^{-1}(y)$ at $x$. The proofs for the other tangent cones are the same.

Example: Derivatives of monotone operators
Let $X$ be a Hilbert space (identified with its dual). We recall that a set-valued map $F: X \sim X$ is monotone if and only if

$$
\begin{equation*}
\forall(x, p),(y, q) \in \operatorname{Graph}(F),<p-q, x-y>\geq 0 \tag{68}
\end{equation*}
$$

Subdifferentials of convex functions are monotone ${ }^{19}$ maps. We also recall that when $F$ is monotone, its resolvent $J:=(1+A)^{-1}$ is single-valued and lipschitzean (with constant equal to 1) on its domain. Therefore, we can easily compute derivatives of $F$ in terms of the derivatives of its resolvent.

Proposition 6.6 Let $X$ be a Hilbert space. We identify its dual with $X$ and we supply it with the weak toplogy. If $F$ io a monotone set-valued map, then its adjacent derivative $D^{b} F(x, p)$ at some pair $(x, p)$ of its graph is semi positive-definite in the sense that

$$
\forall(u, r) \in \operatorname{Graph}\left(D^{\prime} F(x, p)\right),<r, u>\geq 0
$$

The same is true for the circatangent derivative. Furthermore, the following statemento are equivalent:

$$
\begin{cases}a) & r \in D F(x, p)(u) \\ b) & r \in D J(x+p)(r+u)\end{cases}
$$

This last statement remains true for the adjacent and circatangent derivatives.

Proof The first statement is obvious, since $\langle r, u\rangle$ is the limit of the sequence $<r_{n}, x_{n}>$ (because $\boldsymbol{a}_{n}$ converges to $u$ strongly and $r_{n}$ converges to $v$ weakly) and since

$$
\left\langle r_{n}, u_{n}\right\rangle=\left\langle x+h_{n} u_{n}-x, p+h_{n} r_{n}-p\right\rangle \geq 0
$$

For proving the second statement, we observe that $p$ belongs to $F(x)$ if and only if $x=J(x+p)$. Since $J$ is the inverse of $(1+A)$, we deduce that

[^14]$$
s \in D(1+A)(x, q)(u) \Longleftrightarrow u \in D J(q)(s)
$$

Hence, by setting $q:=x+p$ and $\varepsilon:=r+u$, we obtain the formula we were looking for.

Since the cone-valued map $N_{K}$ associating with any $x \in K$ the normal cone $N_{K}(x)$ to a closed convex subset is maximal monotone (because the normal cone is the subdifferential of the indicator of $K$ ), and since its resolvaent is the best approximation projector, we deduce the following corollary:

Corollary 6.1 Let $K$ be a closed convex subset of a Hilbert space, and let $p$ belong to the normal cone $N_{K}(x)$ to $K$ at some $x \in K$. Let $\pi_{K}$ denote the best approximation projector onto $K$. Then, the two following satatements are equivalent:

$$
\begin{cases}i) & q \in D^{b} N_{K}(x, p)(u) \\ i i) & u \in D^{b} \pi_{K}(x+p)(u+q)\end{cases}
$$

## 7 Calculus of Derivatives

We derive from the calculus of tangent cones the associated calculus of derivatives of set-valued maps. We begin naturally by the chain rule for computing the composition product of a set-valued map $G: X \sim Y$ and a set-valued map $H: Y \sim Z$.

One can conceive two dual ways for defining composition products of set-valued maps (which coincide when $G$ is single-valued):

Definition 7.1 Let $X, Y, Z$ be Banach spaces and $G: X \sim Y, H: Y \sim$ $Z$ be set-valued maps:

1. the ubual composition product (called simply the product) $H \circ G$ : $X \sim Z$ of $H$ and $G$ at $x$ is defined by

$$
(H \circ G)(x):=\bigcup_{y \in G(x)} H(y)
$$

2. the square product ${ }^{30} H \square G: X \sim Z$ of $H$ and $G$ at $x$ is defined $b y$

$$
(H \square G)(x):=\bigcap_{y \in G(x)} H(y)
$$

[^15]Let us recall that there are two manners to define the inverse image ${ }^{21}$ by a set-valued map $G$ of a subset $M$ :

$$
\left\{\begin{array}{ll}
\text { a) } & G^{-}(M) \\
\text { b) } & G^{+}(M)
\end{array}:=\{x \mid G(x) \cap M \neq 0\}\right.
$$

We deduce the following formulas

$$
\left\{\begin{array}{l}
\text { i) } \quad \operatorname{Graph}(F \circ G)=(G \times 1)^{-} \operatorname{Graph}(H) \\
\text { ii) } \operatorname{Graph}(F \square G)=(G \times 1)^{+} \operatorname{Graph}(H) \tag{69}
\end{array}\right.
$$

as well as the formulas which state that the inverse of a product is the product of the inverses (in reverse order):

$$
\left\{\begin{array}{l}
\text { i) }(H \circ G)^{-1}(y)=G^{-}\left(H^{-1}(y)\right) \\
\text { ii) }(H \square G)^{-1}(y)=G^{+}\left(H^{-1}(y)\right)
\end{array}\right.
$$

We also point aout the following relation:

$$
\begin{equation*}
\operatorname{Graph}(H \circ G)=(1 \times H) \operatorname{Graph}(G) \tag{70}
\end{equation*}
$$

We shall need the following result:
Proposition 7.1 Let $F$ be a set-valued map from $X$ to $Y$ and $K$ be a subset of $X$. Assume that $F$ is lipschitzean around some $x \in K$. Then, for any $y \in F(x)$, we have

$$
\begin{equation*}
D^{b} F(x, y) T_{K}(x) \subset T_{F(K)}(y) \tag{71}
\end{equation*}
$$

As a consequence, we deduce that if $M$ is a subset of $Y$, then

$$
\begin{equation*}
T_{P+(M)}(x) \subset D^{b}(x, y)^{+} T_{M}(y) \tag{72}
\end{equation*}
$$

Proof

[^16]Take in $T_{K}(x)$ and $v \in D^{b} F(x, y)(u)$. Then there exist sequences $h_{n}>0$ converging to $0, \boldsymbol{u}_{n}^{1}$ and $u_{n}^{2}$ conveging to $u$ and $v_{n}$ conveging to $v$ such that

$$
x_{n}+h_{n} u_{n}^{1} \in K \& y+h_{n} v_{n} \in F\left(x+h_{n} u_{n}^{2}\right)
$$

Since $F$ is lipschitzean around $x$, we deduce that

$$
y+h_{n} v_{n} \in F\left(x+h_{n} u_{n}^{1}\right)+l h_{n}\left\|u_{n}^{1}-u_{n}^{2}\right\|
$$

so that there exists another sequence $v_{n}^{*}$ converging to $v$ such that

$$
y+h_{n} v_{n}^{*} \in F\left(x+h_{n} u_{n}^{1}\right) \subset F(K)
$$

This implies that $v$ belongs to the contingent cone to $F(K)$ at $y$.
Consider now $K:=F^{+}(M)$. Since $F\left(F^{+}(M)\right)$ is contained in $M$, we deduce that

$$
D^{\downarrow} F(x, y) T_{F^{+}|M|}(x) \subset T_{F\left(F^{+}(M) \mid\right.}(y) \subset T_{M}(y)
$$

from which formula (72) follows.
Remark Naturally, the formula

$$
D F(x, y) T_{K}^{\prime}(x) \subset T_{F(K)}(y)
$$

is also true.
We begin by this simple result:
Theorem 7.1 Let us consider a set-valued map $G: X \sim Y$ and a setoalued $\operatorname{map} H: Y \sim Z$.

Let us assume that $H$ is lipschitzean around $y$ where $y$ belongs to $G(x)$. Then, for any $z \in H(y)$, we have

$$
\begin{equation*}
D^{\prime} H(y, z) \circ D G(x, y) \subset D(H \circ G)(x, z) \tag{73}
\end{equation*}
$$

Let us asoume that $G$ is lipschitzean around $x$. Then, for all $y \in G(x)$ and $z \in(H \square G)(x)$, we have

$$
\begin{equation*}
D(H \square G)(x, z) \subset D H(y, z) \square D^{\downarrow} G(x, y) \tag{74}
\end{equation*}
$$

In particular, if $G:=g$ is single-valued and lipschitzean around $x$, we obtain

$$
D(H g)(x, z)(u) \subset D H(g(x), z)\left(g^{\prime}(x) u\right)
$$

and the equality holdo true when $H$ is lipochitzean around $g(x)$.
Proof We apply Proposition 7.1 to formulas (70) and (69)ii) respectively for proving the two first formalas. When $G:=g$ is single-valued, we use the fact that both composition products coincide.

We state now a more powerful result:
Theorem 7.2 Let us consider a set-valued map $G: X \sim Y$ and a setoalued map $H: Y \sim Z$.

If the dimension of $Y$ is finite, we supose that

$$
\operatorname{Im}\left(C G\left(x_{0}, y_{0}\right)\right)-\operatorname{Dom}\left(C H\left(y_{0}, z_{0}\right)\right)=Y
$$

Then

$$
\begin{cases}\text { i) } & D^{b} H\left(y_{0}, z_{0}\right) \circ D G\left(x_{0}, y_{0}\right) \subset D(H \circ G)\left(x_{0}, z_{0}\right) \\ \text { ii) } & D^{b} H\left(y_{0}, z_{0}\right) \circ D^{\natural} G\left(x_{0}, y_{0}\right)=D^{\natural}(H \circ G)\left(x_{0}, z_{0}\right) \\ \text { iii) } & C H\left(y_{0}, z_{0}\right) \circ C G\left(x_{0}, y_{0}\right) \subset C(H \circ G)\left(x_{0}, z_{0}\right)\end{cases}
$$

If $Y$ is any Banach space, we assume that there exist constants $c>0$, $\alpha \in[0,1[$ and $\eta>0$ ouch that

$$
\begin{cases} & \forall\left(x, y_{1}\right) \in \operatorname{Graph}(G) \cap B\left(\left(x_{0}, y_{0}\right), \eta\right),  \tag{75}\\ & \forall\left(y_{2}, z\right) \in \operatorname{Graph}(H) \cap B\left(\left(y_{0}, z_{0}\right), \eta\right), \\ \text { i) } & B_{Y} \subset \operatorname{Im}\left(D^{\forall} G\left(x, y_{1}\right)\right) \cap c B_{X}-\operatorname{Dom}\left(D H\left(y_{2}, z\right)\right)+\alpha B_{Y} \\ \text { ii) } & \left\|D G^{\downarrow}\left(x, y_{1}\right)\right\| \leq c \\ \text { iii) } & \left\|D H^{-1}\left(z, y_{2}\right)\right\| \leq c\end{cases}
$$

Proof If we denote by $\omega$ the continuous linear operator from $X \times$ $Y \times Y \times Z$ to $Y$ associating to ( $x, y_{1}, y_{2}, z$ ) the element $y_{1}-y_{2}$ and by $\pi_{X \times z}$ the canonical projection from $X \times Y \times Y \times Z$ onto $X \times Z$, we observe that

$$
\begin{equation*}
\operatorname{Graph}(H \circ G)=\pi_{X \times Z}\left((\operatorname{Graph}(G) \times \operatorname{Graph}(H)) \cap \omega^{-1}(0)\right) \tag{76}
\end{equation*}
$$

Therefore, we apply Theorem 4.2 with $A=\omega, L=\operatorname{Graph}(G) \times$ $\operatorname{Graph}(H)$ and $M=\{0\}$.

Assumption (75) implies the transversality condition. Indeed, for any $v \in Y$, we can find $v_{1} \in \operatorname{Im}\left(D^{\prime} G\left(x, y_{1}\right)\right), v_{2} \in \operatorname{Dom}\left(D H\left(y_{2}, z\right)\right)$ and $w \in Y$ such that $v=v_{1}-v_{2}+e$ with $\left\|v_{1}\right\| \leq c\|v\|$ and $\|e\| \leq \alpha\|v\|$. Hence $\left\|v_{2}\right\| \leq$ $(c+1+\alpha)\|v\|$ and there exist $u \in D G^{b}\left(x, y_{1}\right)^{-1}\left(v_{1}\right)$ and $w \in D H\left(y_{2}, z\right)\left(v_{2}\right)$ such that $\|w\| \leq c\left\|v_{1}\right\|$ and $\|w\| \leq c\left\|v_{2}\right\|$.

Therefore, $v=\omega\left(u, v_{1}, v_{2}, w\right)+e$ where ( $\left.u, v_{1}, v_{2}, w\right)$ belongs to the contingent cone to the product of the graphs of $G$ and $H$ and $e \in \alpha B_{Y}$.

Consequently, we deduce that, for instance,

$$
\left\{\begin{array}{l}
\left(\operatorname{Graph}^{=}\left(D^{b} G\left(x_{0}, y_{0}\right)\right) \times \operatorname{Graph}\left(D^{b} H\left(y_{0}, z_{0}\right)\right)\right) \cap \omega^{-1}(0) \\
\left.=T_{\operatorname{Graph}(G)}^{b}\left(x_{0}, y_{0}\right) \times T_{\operatorname{Graph}}^{\mathrm{Graph}(G) \times}\left(\operatorname{Graph}_{(H)}\left(y_{0}, z_{0}\right)\right) \cap \omega_{0}, y_{0}, z_{0}\right) \cap \omega^{-1}(0) \\
=T_{\operatorname{Graph}(G) \times \operatorname{Graph}(H) \cap \omega^{-1}(0)}^{b}\left(x_{0}, y_{0}, y_{0}, z_{0}\right)
\end{array}\right.
$$

By applying the projection $\pi_{X \times 2}$ to both sides of these equalities, we deduce that

$$
\left\{\begin{array}{l}
\operatorname{Graph}\left(D^{b} H\left(y_{0}, z_{0}\right) \circ D^{b} G\left(x_{0}, y_{0}\right)\right) \\
\left.=\pi_{X \times Z}\left({ }^{(G r a p h}\left(D^{b} G\left(x_{0}, y_{0}\right)\right) \times \operatorname{Graph}\left(D^{b} H\left(y_{0}, z_{0}\right)\right)\right) \cap \omega^{-1}(0)\right) \\
=\pi_{X \times Z}\left(T_{\operatorname{Graph}}^{b}(G) \times \operatorname{Graph}(H) \cap \omega^{-1}(0)\right. \\
\left.\left.\subset x_{0}, y_{0}, y_{0}, z_{0}\right)\right) \\
=T_{\pi \times z( }^{b} \operatorname{Graph}_{(G) \times} \operatorname{Graph}_{\left.(H) \cap \omega^{-1}(0)\right)}^{b}\left(x_{0}, z_{0}\right) \\
=\operatorname{Graph}(H \circ G)\left(x_{0}, y_{0}\right) \\
=\operatorname{Graph}\left(D^{b}(H \circ G)\left(x_{0}, z_{0}\right)\right) \quad \square
\end{array}\right.
$$

We provide now some examples of situations where we have equality in chain rule formulas.

We denote by $\bar{F}$ the set-valued map whose graph is the closure of the graph of $F$.

Proposition 7.2 Let us consider a set-valued map $G: X \sim Y$ and a set-valued map $H: Y \sim Z$. The following inclusion

$$
D(H \circ G)\left(x_{0}, z_{0}\right) \subset \overline{D H\left(y_{0}, z_{0}\right) \circ D G\left(x_{0}, y_{0}\right)}
$$

holds true under one of the following assumptions:

1. $G$ is peeudo-convex at $\left(x_{0}, y_{0}\right) \in \operatorname{Graph}(G)$ and $H$ is pseudo-convex at $\left(y_{0}, z_{0}\right) \in \operatorname{Graph}(H)$, (and in particular, $G$ and $H$ are convex),
2. The dimension of $X$ and $Y$ is finite and

$$
\begin{equation*}
\operatorname{ker}\left(D H\left(x_{0}, y_{0}\right)\right) \cap D G\left(x_{0}, y_{0}\right)(0)=\{0\} \tag{77}
\end{equation*}
$$

Proof We deduce these statements from the criteria implying that the contingent cones of the images by $\pi_{X \times Z}$ are the closures of the images of the contingent cones:

When the graph of $G$ is pseudo-convex at $(x, y)$ and the graph of $H$ is pseudo-convex at ( $y, z$ ), we derive the above property from Proposition 4.3.

The second case follows from Proposition 4.4 because condition (77) implies obviously that

$$
\text { ker } \left.\pi_{X \times Z} \cap T_{\operatorname{Graph}(G) \times}{\operatorname{Graph}(H) \pi \omega^{-1}(0)}\left(x_{0}, y_{0}, y_{0}, z_{0}\right)\right)=0
$$

Remark It is quite useful to relate the tangent cones to the domain (or the image) of a set-valued map to the domain (or the image) of its derivative.

Proposition 7.3 Let ue consider two topological vector apaces $X$ and $Y$, a set-valued map $F: X \sim Y$ from $X$ to $Y$ and a point $\left(x_{0}, y_{0}\right)$ of its graph. We alvays have

$$
\begin{equation*}
\overline{\operatorname{Dom}\left(D F\left(x_{0}, y\right)\right)} \subset T_{\operatorname{Dom}(F)}\left(x_{0}\right) \tag{78}
\end{equation*}
$$

Equality

$$
\begin{equation*}
\overline{\operatorname{Dom}\left(D F\left(x_{0}, y\right)\right)}=T_{\operatorname{Dom}(F)}\left(x_{0}\right) \tag{79}
\end{equation*}
$$

holds true under one of the following assumptions:

1. $F$ is poeudo-convex at $\left(x_{0}, y_{0}\right) \in \operatorname{Graph}(F)$, (and in particular, $F$ is convex),
2. $X$ and are reflexive Banach spaces and the cosubdifferential of $F$ at $\left(x_{0}, y_{0}\right)$ is surjective:

$$
\begin{equation*}
\operatorname{Im}\left(D F\left(x_{0}, y_{0}\right)^{0^{*}}\right)=Y^{*} \tag{80}
\end{equation*}
$$

9. The dimension of $X$ and $Y$ is finite and

$$
\begin{equation*}
\operatorname{ker}\left(D F\left(x_{0}, y_{0}\right)\right)=\{0\} \tag{81}
\end{equation*}
$$

Proof We deduce these statements from the criteria (Proposition 4.3, Proposition 4.5 and Proposition 4.4) implying that the contingent cones of the images are the closures of the images of the contingent cones, since

$$
\begin{equation*}
\operatorname{Dom}(F)=\pi_{X} \operatorname{Graph}(F) \tag{82}
\end{equation*}
$$

where $\pi_{X}$ is the projection from $X \times Y$ onto $X$. We thus deduce that

$$
\left\{\begin{array}{l}
\overline{\operatorname{Dom}\left(D F\left(\left(x_{0}, y_{0}\right)\right)\right)}=\overline{\pi_{X} \operatorname{Graph}(D F)\left(\left(x_{0}, y_{0}\right)\right)}  \tag{83}\\
=\overline{\pi_{X} T_{\mathrm{Graph}_{(F)}}\left(\left(x_{0}, y_{0}\right)\right)} \subset T_{\pi x(\operatorname{Graph}(F))}\left(x_{0}\right) \\
=T_{\operatorname{Dom}(F)\left(x_{0}\right)}
\end{array}\right.
$$

Remark By using the paratingent derivatives ${ }^{32}$, we obtain upper estimates of the usual composition product.

[^17]Theorem 7.3 Assume that $G$ is lipochitzean around $x$. Then

1. $Y$ is a finste dimensional vector-space and $G(x)$ is bounded, then

$$
D(H \circ G)(x, z) \subset \bigcup_{y \in G(x)} P_{1} H(y, z) \circ P_{2} G(x, y)
$$

2. and

$$
P(H \square G)(x, z) \subset \bigcap_{y \in G(x)} P H(y, z) \square C G(x, y)
$$

Proof Let $w$ belong to $D(H \circ G)(x, z)(u)$ : there exist sequences $h_{n}>0, u_{n}$ and $v_{n}$ converging to $0, u$ and $v$ such that

$$
\forall n, z+h_{n} v_{n} \in H\left(y_{n}\right) \text { where } y_{n} \in G\left(x+h_{n} u_{n}\right)
$$

Since $G$ is lipschitzean around $x$, there exist elements $y_{n}^{0} \in G(x)$ such that

$$
v_{n}:=\frac{y_{n}-y_{n}^{0}}{h_{n}} \text { satisfies }\left\|v_{n}\right\| \leq l\left\|u_{n}\right\|
$$

Furthermore, $G(x)$ being relatively compact, a subsequence (again denoted) $y_{n}^{0}$ converges to some $y$. We can also extract a subsequence (again denoted) $v_{n}$ which converges to some $v$, since this sequence is bounded and the dimension of $Y$ is finite.

Since

$$
y_{n}^{0}+h_{n} v_{n} \in G\left(x+h_{n} u_{n}\right) \& z+h_{n} w_{n} \in H\left(y_{n}^{0}+h_{n} v_{n}\right)
$$

we infer that $w$ belongs to $P H(y, z)(v)$ and $v$ belongs to $P G(x, y)(u)$
Let $u \in \operatorname{Dom} C G(x, y)$ and $w$ belong to $P(H \square G)(x, z)(u)$. Hence there exist a sequence $h_{n}>0$ converging to 0 and sequences of elements $\left(x_{n}, z_{n}\right) \in$ $\operatorname{Graph}(H \square G), u_{n}$ and $w_{n}$ converging to $(x, z), u$ and $w$ respectively such that

$$
\forall n \geq 0, z_{n}+h_{n} w_{n} \in \bigcap_{y \in G\left(x_{n}+h_{n} x_{n}\right)} H(y)
$$

The set-valued map $G$ being lipschitzean, there exists a sequence of elements $y_{n} \in G\left(x_{n}\right)$ converging to $y$. By definition of the square product, we know that $z_{n} \in H\left(y_{n}\right)$ (because $z_{n} \in(H \square G)\left(x_{n}\right)$ ).

Take now any $v$ in $C G(x, y)(u)$. Since $G$ is lipschitzean around $x$, there exists a sequence of elements $v_{n}$ converging to $v$ such that

$$
\forall n \geq 0, \quad y_{n}+h_{n} v_{n} \in G\left(x_{n}+h_{n} u_{n}\right)
$$

Therefore,

$$
\forall n \geq 0, z_{n}+h_{n} w_{n} \in H\left(y_{n}+h_{n} v_{n}\right)
$$

so that we infer that

$$
w \in P H(y, z)(v)
$$

Since this is true for every element $v$ of $C G(x, y)(u)$, we deduce that

$$
w \in \bigcap_{v \in C G(x, y)(u)} P H(y, z)(v)=P H(y, z) \square C G(x, y)(u)
$$

## 8 Epiderivatives

Let us consider an extended real-valued function $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ whose domain

$$
\operatorname{Dom}(V):=\{x \in X \mid V(x)<+\infty\}
$$

is not empty. (Such a function is said to be proper in convex and non smooth analysis. We shall rather say that it is nontrivial for avoiding confusion with proper maps).

We can naturally regard it as the set-valued map $\mathbf{V}: X \sim \mathbf{R}$ defined by

$$
\mathbf{V}(x):=\left\{\begin{array}{lll}
V(x) & \text { if } & x \in \operatorname{Dom}(V) \\
& \text { if } & x \notin \operatorname{Dom}(V)
\end{array}\right.
$$

so that we can define in the usual way contingent, adjacent and circatangent derivatives of $V$ at $x \in \operatorname{Dom}(V)$. We shall set:

```
i) \(D V(x)(u):=D V(x, V(x))(u)\)
\[
=\left\{v \mid \liminf _{h \rightarrow 0+, v^{\prime} \rightarrow \mathbf{w}}\left\|V\left(x+h u^{\prime}\right)-V(x)-h v\right\| / h=0\right\}
\]
ii) \(D^{b} V(x)(u):=D^{b} V(x, V(x))(u)\)
```

```
iii) \(C V(x)(u):=C V(x, V(x))(u)\)
\[
=\left\{v\left\|\lim \sup _{h \rightarrow 0+, x^{\prime} \rightarrow x} \inf _{v^{\prime} \rightarrow v}\right\| V\left(x+h u^{\prime}\right)-V(x)-h v \| / h=0\right\}
\]
```

However, minimization problems and Lyapunov functions involve obviously the order relation of $\boldsymbol{R}$. Hence, when dealing with such problems, we associate with the extended real-valued function $V$ two new set-valued maps $V_{1}$ and $V_{\downarrow}$ defined in the following way:

$$
\begin{cases}\text { i) } & \mathbf{V}_{\uparrow}:=\left\{\begin{array}{lll}
V(x)+\mathbf{R}_{+} & \text {if } & x \in \operatorname{Dom}(V) \\
\text { ii) } & \mathbf{V}_{\downarrow}:= \begin{cases} & \text { if } \\
V(x)-\mathbf{R}_{+} & \text {if } \\
x \in \operatorname{Dom}(V) \\
& \text { if } \\
x \notin \operatorname{Dom}(V)\end{cases}
\end{array}\right. \text { (V)}  \tag{84}\\
& \end{cases}
$$

We see at once that

$$
\left\{\begin{array}{ll}
\text { i) } & \operatorname{Graph}\left(\mathbf{V}_{t}\right) \tag{85}
\end{array}=\mathrm{Ep}_{\mathrm{p}}(V),\right.
$$

Therefore, we are led naturally to associate with these two set-valued maps $\mathbf{V}_{\dagger}$ and $\mathbf{V}_{\downarrow}$ their contingent, adjacent and circatangent derivatives at points ( $x, V(x)$ ) and thus, unfortunately, to introduce still new definitions.

Definition 8.1 (Epiderivatives) Let $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ be a nontrivial extended real-valued function and $x$ belong to its domain. We shall say that the functions $D_{\dagger}(V)(x), D_{\dagger}^{f}(V)(x)$ and $C_{\dagger}(V)(x)$ from $X$ to $\mathbf{R} \cup\{+\infty\} \cup$ $\{-\infty\}$ defined respectively by

```
i) \(\quad D_{f}(V)(x)(x)\)
    \(:=\liminf _{h \rightarrow 0+v^{\prime} \rightarrow v}\left(V\left(x+h u^{\prime}\right)-V(x)\right) / h\)
    ii) \(\quad D_{\dagger}^{b}(V)(x)(u)\)
    \(:=\lim \sup _{h \rightarrow 0+} \inf _{\mathbf{v}^{\prime} \rightarrow \mathbf{x}}\left(V\left(x+h u^{\prime}\right)-V(x)\right) / h\)
    iiii) \(C_{\dagger}(V)(x)(u)\)
        \(:=\lim \sup _{h \rightarrow 0+, x^{\prime} \rightarrow x, V\left(x^{\prime}\right) \leq x^{\prime} \rightarrow V(x)} \inf _{v^{\prime} \rightarrow x}\left(V\left(x+h u^{\prime}\right)-\lambda^{\prime}\right) / h\)
```

are the contingent, adjacent and circatangent epiderivatives of $V$ at $x$ in the direction $u$.

We define in a symmetric way the contingent, adjacent and circatangent hypoderivatives $D_{\downarrow}(V)(x), D_{\downarrow}^{b}(V)(x)$ and $C_{\downarrow}(V)(x)$ from $X$ to $\mathbf{R} \cup\{+\infty\} \cup\{-\infty\}$

$$
\begin{aligned}
& \text { i) } \quad D_{\downarrow}(V)(x)(u)=-\left(D_{\dagger}(-V)\left(x_{0}\right)(u)\right) \\
& \text { ii) } \quad:=\lim _{D_{l}^{b}}^{b}(V)(x)(u) \stackrel{\sup _{h \rightarrow 0+, x^{\prime} \rightarrow \boldsymbol{v}}}{=}\left(V\left(x+h u^{\prime}\right)-V(x)\right) / h \\
& :=\lim _{\inf }^{h \rightarrow 0+} \sup _{\mathbf{x}^{\prime} \rightarrow \mathbf{v}}\left(V\left(x+h u^{\prime}\right)-V(x)\right) / h \\
& \text { iii) } \quad C_{\downarrow}(V)(x)(u)=-\left(C_{\dagger}(-V)\left(x_{0}\right)(u)\right) \\
& :=\lim _{\inf _{h \rightarrow 0+, x^{\prime} \rightarrow x, V\left(x^{\prime}\right) \geq \lambda^{\prime} \rightarrow V(x)} \sup _{\mathbf{v}^{\prime} \rightarrow \mathbf{x}}\left(V\left(x+h u^{\prime}\right)-\lambda^{\prime}\right) / h}
\end{aligned}
$$

Such an epi or hypo derivative is said to be nontrivial if and only if it never takes the value $+\infty$ and has at least one finite value.

Naturally, they coincide with the directional derivatives $\left\langle V^{\prime}(x), u\right\rangle$ when $V$ is respectively Gâteaux, Fréchet and continuously differentiable.

Remark Definition (8.1) provide another interpretation of the epiderivatives in terms of epilimits of the difference quotients

$$
\nabla_{h} V(x):=u \mapsto(V(x+h u)-V(x)) / h
$$

1. the contingent epiderivative is the epi-lower limit of the difference quotients $\nabla_{h} V(x)$ when $h \rightarrow 0+2$. the adjacent epiderivative is the epiupper limit of the difference quotients $\nabla_{h} V(x)$ when $h \rightarrow 0+3$. the circatangent epiderivative is the epi-upper limit of the difference quotients $u \mapsto\left(V\left(x^{\prime}+h u\right)-\lambda^{\prime}\right) / h$ when $h \rightarrow 0+$ and $\left(x^{\prime}, \lambda^{\prime}\right) \in \operatorname{Ep}(V) \rightarrow(x, V(x))$

If $V$ is continuously differentiable around a point $x \in K$, then the epiderivative of the restriction is the restriction of the epiderivative to the corresponding tangent cone:

The formulas become much more simpler when $V$ is lipschitzean.
Proposition 8.1 Let us assume that $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ is lipschitzean around a point $x$ of ite domain. Then
$\left\{\begin{aligned} \text { i) } & D_{\dagger}(V)(x)(u) \\ & :=\liminf _{h \rightarrow 0+}\left(V\left(x+h u^{\prime}\right)-V(x)\right) / h \text { and } \\ \text { ii) } & D_{\dagger}^{\prime}(V)(x)(u) \\ & :=\limsup _{h \rightarrow 0+}\left(V\left(x+h u^{\prime}\right)-V(x)\right) / h \\ & \text { are Dini derivatives } \\ \text { iii) } & C_{\dagger}(V)(x)(u)=: V^{\prime}(x, u) \\ & :=\limsup _{h \rightarrow 0+, s^{\prime} \rightarrow x}\left(V\left(x+h u^{\prime}\right)-V\left(x^{\prime}\right)\right) / h \\ & \text { is the Clarke directional derivative. }\end{aligned}\right.$

Furthermore,

$$
\begin{cases}\text { i) } & (x, u) \in X \times \operatorname{Int}(\operatorname{Dom}(V)) \mapsto C_{\dagger} V(x)(u) \\ & \text { is upper semicontinuous } \\ \text { ii) } & \text { u } \mapsto C_{\dagger} V(x)(u) \text { is lipschitzean } \\ \text { iii) } & C_{\dagger}(-V)(x)(u):=C_{\uparrow} V(x)(-u)\end{cases}
$$

Proposition 8.2 When the function $V$ is convex, all these epiderivatives are equal and the formula becomes:

$$
D_{\uparrow}(V)(x)(u)=\liminf _{u^{\prime} \rightarrow \boldsymbol{u}}\left(\inf _{h>0} \frac{V\left(x+h u^{\prime}\right)-V(x)}{h}\right)
$$

Furthermore, when $x$ belongs to the interior of the domain of $V$, i.e., when $V$ is lipschitzean around $x$, we obtain

$$
D_{\uparrow}(V)(x)(u)=\inf _{h>0} \frac{V\left(x+h u^{\prime}\right)-V(x)}{h}
$$

We first observe that
Proposition 8.3 Let $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ be a nontrivial extended realvalued function and $x$ belong to ito domain.

Then the tangent cones to the epigraph of $V$ at $(x, V(x))$ are epigraphs of the corresponding epiderivatives of $V$ at $x$ :

$$
\left\{\begin{array}{l}
i) \quad \mathrm{Ep}_{\mathrm{p}} D_{\uparrow} V(x)=T_{\mathrm{Ep}(V)}(x, V(x)) \\
i i) \\
\mathrm{Ep}_{\mathrm{p}} D_{\uparrow}^{b} V(x)=T_{\mathrm{Ep}(V)}^{b}(x, V(x)) \\
i i i) \\
\mathrm{Ep}_{\mathrm{p}} C_{\uparrow} V(x)=C_{\mathrm{Ep}(V)}(x, V(x))
\end{array}\right.
$$

Then the circatangent epiderivative is always lower semicontinuous convex and positively homogeneous.

Proof We shall check this fact only for the contingent case, the proof being similar for the adjacent and circatangent cases.

Actually, we shall prove that for all $w \geq V(x)$, we have

$$
\begin{equation*}
\mathrm{Ep}_{\mathrm{p}} D_{\uparrow} V(x)=T_{\mathrm{E} p(V)}(x, V(x)) \subset T_{\mathrm{Ep}(V)}(x, w) \tag{87}
\end{equation*}
$$

Let $v \geq D_{\dagger} V(x)(u)$ where $u$ belongs to the domain of the contingent epiderivative of $V$ at $x$. Then there exist sequences $u_{n}, v_{n}$ and $h_{n}>0$ converging to $u, v$ and 0 such that $h_{n} v_{n} \geq V\left(x+h_{n} u_{n}\right)-V(x)$. Since $w_{0}-V(x) \geq 0$, we deduce that $w-V(x)+h_{n} v_{n} \geq V\left(x+h_{n} u_{n}\right)-V(x)$, i.e., that the pair $(u, v)$ belongs to $T_{\mathrm{Ep}(V)}(x, w)$.

Let us assume that ( $u, v$ ) belongs to $T_{\mathrm{Ep}_{\mathrm{p}}(V)}(x, w)$. We infer that there exists sequences $u_{n}, v_{n}$ and $h_{n}>0$ converging to $u, v$ and 0 such that

$$
\begin{equation*}
w+h_{n} v_{n} \geq V\left(x+h_{n} u_{n}\right) \tag{88}
\end{equation*}
$$

If $w=V(x)$, we deduce that $v \geq D_{\uparrow}(V)(x)(u)$. Hence the equality between the contingent cone to the epigraph of $V$ at $(x, V(x))$ and the epigraph of the contingent epiderivative holds true.

This result leads us to introduce the following definitions.
Definition 8.2 We shall say that an extended function $V$ is pseudoconvex at $x$ if

$$
\forall y \in X, \quad D_{\uparrow} V(x)(y-x) \leq V(y)-V(x)
$$

We shall say that it is epi-derivable at $x$ if the contingent and adjacent epiderivatives do coincide and that it is epi-sleek at $x$ if the epigraph of $V$ is sleek at $(x, V x)$.

Proposition 8.4 Let $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ be a nontrivial extended realvalued function and $x$ belong to its domain. Then

$$
\operatorname{Dom}\left(D_{\uparrow}(V)(x)\right) \times \mathbf{R} \subset T_{\mathrm{E}(V)}(x, w) \subset T_{\operatorname{Dom}(V)}(x) \times \mathbf{R}
$$

If the restriction of $V$ to its domain is upper semicontinuous, then, for all $w>V(x)$,

$$
T_{\mathrm{Ep}_{( }(V)}(x, w)=T_{\operatorname{Dom}(v)}(x) \times \mathbf{R}
$$

Proof If $u$ belongs to the domain of the contingent epiderivative of $V$ at $x$, if $w \geq V(x)$ and if $u$ is any real number, we check that ( $u, v$ ) belongs to the epigraph of $D_{\uparrow}(V)(x)$.

Indeed, there exist sequences of elements $h_{n}>0, u_{n}$ and $v_{n}$ converging to $0, u$, and $D_{\uparrow}(V)(x)$ respectively such that

$$
\left(x+h_{n} u_{n}, V(x)+h_{n} v_{n}\right) \in \mathrm{Ep}_{p}(V)
$$

But we can write
$\left(x+h_{n} w_{n}, w_{n} v\right)=\left(x+h_{n} u_{n}, V(x)+h_{n} v_{n}\right)+\left(0, w-V(x)+h_{n}\left(v-v_{n}\right)\right)$ Since $w-V(x)+h_{n}\left(v-v_{n}\right)$ is strictly positive when $w>V(x)$ and $h_{n}$ is small enough, we deduce that ( $x+h_{n} u_{n}, w+h_{n} v$ ) belongs to the epigraph of $V$, i.e., that $(w, v)$ belongs to the epigraph of $D_{\uparrow}(V)(x)$.

If ( $u, v$ ) belongs to $T_{\mathrm{Ep}_{(V)}}(x, w)$, we deduce from (88) that $x+h_{n} u_{n}$ belongs to the domain of $V$ for all $n$, i.e., that $u$ belongs to the contingent cone to the domain of $V$ at $x$.

Let $w$ be strictly larger than $x$ and $u$ belong to $T_{\operatorname{Dom}(V)}(x)$. Then there exists sequences $u_{n}$ and $h_{n}>0$ converging to $u$ and 0 such that $V\left(x+h_{n} u_{n}\right)<+\infty$ for all $n$.

Since $V$ is upper semicontinuous on its domain, for all $\epsilon \in] 0, \frac{w-V(x)}{2}[$, there exists $\eta>0$ such that, for all $h_{n}\left\|u_{n}\right\|<\eta$, we obtain

$$
V\left(x+h_{n} u_{n}\right) \leq V(x)+\epsilon<w-\epsilon
$$

Let $v$ given arbitrarily in $\mathbf{R}$. Then, for any $h_{n}>0$ when $v \geq 0$ or for any $\left.h_{n} \in\right] 0, \left.\frac{f}{-v} \right\rvert\,$ when $v<0$, inequality $w-\epsilon \leq w+h_{n} v$ implies that $V\left(x+h_{n} u_{n}\right) \leq w+h_{n} v$, i.e., that the pair $(u, v)$ belongs to $T_{\left.\mathrm{Ep}_{\mathrm{p}}\right)}(x, w)$.

Proposition 8.5 Let $V: \mapsto \mathbf{R} \cup\{+\infty\}$ be an extended function and $x$ belong to its domain. Then

$$
\left\{\begin{array}{l}
\left\{D_{\uparrow} V(x)(u), D_{1} V(x)(u)\right\} \\
\subset D V(x)(u) \\
\subset\left[D_{\uparrow} V(x)(u), D_{\downarrow} V(x)(u)\right]
\end{array}\right.
$$

These subsets are equal when the images of the contingent derivative are connected.

Proof Since

$$
\operatorname{Graph}(V) \subset \operatorname{Ep}(V) \cap \operatorname{Hp}_{p}(V)
$$

we deduce that the inclusions

$$
T_{\mathrm{Graph}(V)}(x, V(x)) \subset T_{\mathrm{E}_{\mathrm{p}(V)}}(x, V(x)) \cap T_{\mathrm{H} \mathbf{p}(V)}(x, V(x))
$$

can be translated in the following inclusions:

$$
\operatorname{Graph}(D V(x)) \subset \operatorname{Ep}_{p} D_{\uparrow} V(x) \cap \operatorname{Hp}_{p} D_{\downarrow} V(x)
$$

from which inclusion $D V(x)(u) \subset\left[D_{\uparrow} V(x)(u), D_{\downarrow} V(x)(u)\right]$ follows.

Since the contingent epiderivative of $V$ at $x$ in the direction $u$ is equal to

$$
D_{1} V(x)(\mathfrak{u}):=\liminf _{h \rightarrow 0+, \boldsymbol{v}^{\prime} \rightarrow \mathbf{x}} \frac{V\left(x+h \mathfrak{u}^{\prime}\right)-V(x)}{h}
$$

we see that $D_{1} V(x)(a)$ is the limit of a subsequence of $\frac{V\left(x+h x^{\prime}\right)-V(x)}{h}$, and thus, the pair (u, $D_{1} V(x)(u)$ ) belongs to the contingent cone to the graph of $V$ at $(x, V(x))$. The same is true with the contingent hypoderivative.

Remark There is another intimate connection between tangent cones and their corresponding epiderivatives than Proposition 11 linking the tangent cones to the lower sections of the corresponding epiderivatives of the function $d_{K}(\cdot)$ and Proposition 8.3 linking the tangent cone to the epigraph to the epigraph of the epiderivative.

Let $\psi_{K}$ be the indicator of a subset $K$, defined by

$$
\psi_{K}(x):=\left\{\begin{array}{lll}
0 & \text { if } & x \in K \\
& \text { if } & x \notin K
\end{array}\right.
$$

It is easy to observe that

$$
\begin{cases}i) & D_{\dagger}\left(\psi_{K}\right)(x)=\psi_{T_{K}(x)}  \tag{89}\\ i i) & D_{\dagger}^{\dagger}\left(\psi_{K}\right)(x)=\psi_{T_{K}^{\prime}(x)} \\ i i i) & C_{\dagger}\left(\psi_{K}\right)(x)=\psi_{C_{K}(x)}\end{cases}
$$

Hence we can either derive properties of the epiderivatives from properties of the tangent cones or take the opposite approach by using the above formulas.

## 9 Calculus of Epiderivatives

We present below some useful formulas concerning the epiderivatives.
Theorem 9.1 Let us consider two Banach spaces $X$ and $Y$, a continuous single-valued map $f: X \mapsto Y$, and two extended real-valued functions $V$ and $W$ from $X$ and $Y$ to $\mathbf{R} \cup\{+\infty\}$ respectively. Let $x_{0}$ belong to the Kuratowsid lower limit of the domains of the functions $U:=V+W \circ f$. We asume that $f$ is continuously differentiable around $x_{0}$. 1 . We always have:

$$
D_{\uparrow} V\left(x_{0}\right)(u)+D_{\uparrow} W\left(f\left(x_{0}\right)\right)\left(f^{\prime}\left(x_{0}\right) u\right) \leq D_{\dagger}(V+W \circ f)\left(x_{0}\right)(u)
$$

2. If $X$ and $Y$ are finite dimensional, we suppose that

$$
\operatorname{Dom}\left(C_{\uparrow} W\left(f\left(x_{0}\right)\right)\right)-f^{\prime}\left(x_{0}\right) \operatorname{Dom}\left(C_{\uparrow} V\left(f\left(x_{0}\right)\right)\right)=Y
$$

Then, the epiderivatives of the function $U:=V+W \circ f$ satisfy the estimates:
$(90)\left\{\begin{array}{lll}\text { i }) & D_{\dagger}(U)\left(x_{0}\right)(u) \leq D_{\dagger}^{b}(V)\left(x_{0}\right)(u)+D_{\dagger}(W)\left(f\left(x_{0}\right)\right)\left(f^{\prime}\left(x_{0}\right) u\right) \\ i i) & D_{\dagger}^{\}}(U)\left(x_{0}\right)(u) \leq D_{\dagger}^{\dagger}(V)\left(x_{0}\right)(u)+D_{\dagger}^{b}(W)\left(f\left(x_{0}\right)\right)\left(f^{\prime}\left(x_{0}\right) u\right) \\ i i i) & C_{\dagger}(U)\left(x_{0}\right)(u) \leq C_{\dagger}(V)\left(x_{0}\right)(u)+C_{\dagger}(W)\left(f\left(x_{0}\right)\right)\left(f^{\prime}\left(x_{0}\right) u\right)\end{array}\right.$
3. If $X$ or $Y$ is a Banach space, we posit the following stability assumption: there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ such that, for all $n$,

$$
\begin{cases}\text { i) } & \forall x \in \operatorname{Dom}(V) \cap B\left(x_{0}, \eta\right), \forall y \in \operatorname{Dom}(W) \cap B\left(f\left(x_{0}\right), \eta\right) \\ & B_{Y} \subset f^{\prime}\left(x_{0}\right)\left(\operatorname{Dom}\left(D_{\dagger}^{b}(V)(x)\right) \cap c B_{X}\right) \\ & -\operatorname{Dom}\left(D_{\uparrow}(W)(y)\right)+\alpha B_{Y}  \tag{91}\\ \text { ii) } & \sup _{v \in \operatorname{Dom}\left(D_{\dagger}^{\prime}(V)(x)\right)}\left|D_{\dagger}^{\prime}(V)(x)(u)\right| /\|u\| \leq c \\ \text { iii) } & \sup _{v \in \operatorname{Dom}\left(D_{\uparrow}(W)(y)\right)}\left|D_{\dagger}(W)(y)(v)\right| /\|v\| \leq c\end{cases}
$$

Proof We shall prove the formula only in the case of circatangent epiderivatives.

If we set

$$
\begin{cases}i) & K:=\operatorname{Ep}(V) \times \operatorname{Ep}(W) \times \mathbf{R} \subset X \times \mathbf{R} \times Y \times \mathbf{R} \times \mathbf{R} \\ i i) & G(x, a, y, b, c):=(f(x)-y, a+b-c) \\ i i i) & H(x, a, y, b, c):=(x, c)\end{cases}
$$

we can write

$$
\mathrm{Ep}(U)=H\left(K \cap G^{-1}(0,0)\right)
$$

Therefore, we shall ns Theorem 4.2 for estimating tangent cones to $K \cap G^{-1}(0,0)$. We first observe that the transversality conditions of our theorem imply the corresponding stability assumptions of Theorem 4.2.

This is obvious when $X$ and $Y$ are finite dimensional spaces. Otherwise, we have to check that there exists a constant $c^{\prime}>0$ such that, for all $n$, for all

$$
(x, a, y, b, c) \in K \text { close to }\left(x_{0}, V_{\uparrow}\left(x_{0}\right), y_{0}, W_{\uparrow}\left(f\left(x_{0}\right)\right), 0\right)
$$

for all $(z, \lambda) \in X \times \mathbf{R}$, there exist $(u, \mu, v, \nu, \delta)$ and $e$ such that

$$
\begin{cases}i) & z=f^{\prime}(x) u-v+e \& \lambda=\mu+\nu-\delta \\ i i) & \|e\| \leq \alpha(\|z\|+|\lambda|) \\ i i i) & \|u\|+\|v\|+|\mu|+|\nu|+|\delta| \leq c^{\prime}(\|z\|+|\lambda|)\end{cases}
$$

Assumptions (91)i) \& ii) imply right away that

$$
\left\{\begin{array}{l}
i) \quad z=f^{\prime}(x) u-v+e \\
\text { ii) }\|e\| \leq \alpha\|z\| \&\|u\| \leq c\|z\| \\
i i i) \\
i v\left\|+\leq \leq\left(1+\alpha+c\left\|f^{\prime}(x)\right\|\right)\right\| z \|
\end{array}\right.
$$

Let us take now $\mu:=c\|u\|, \nu:=c\|v\|$ and $\delta:=c(\|u\|+\|v\|)-\lambda$. We deduce from (91)iii) that $(t, \mu)$ belongs to $\operatorname{Ep}\left(D_{\uparrow}^{\dagger} V\right)(x)$, that $(v, \nu)$ belongs to $\operatorname{Ep}\left(D_{\dagger} W\right)(y)$ and that

$$
D_{\uparrow}^{b}(V)(x)(\mathfrak{u})+D_{\dagger}(W)(y)(v) \leq c(\|u\|+\|v\|) \leq \mu+\nu=\lambda+\delta
$$

and that $|\delta| \leq(|\lambda|+c(\|u\|+\|v\|)) \leq c^{\prime}(\|z\|+|\lambda|)$.
Let us set $z_{0}=\left(x_{0}, V_{\uparrow}\left(x_{0}\right), f\left(x_{0}\right), W_{\uparrow}\left(f\left(x_{0}\right)\right), U_{\uparrow}\left(x_{0}\right)\right)$ Hence, we deduce that

$$
C_{K}\left(z_{0}\right) \cap G^{\prime}\left(z_{0}\right)^{-1}(0,0) \subset C_{K \cap G^{-1}(0)}\left(z_{0}\right)
$$

It remains to show that this inclusion implies inequality (90).
Let us set $\lambda=C_{\dagger}(V)\left(x_{0}\right)(\mathfrak{u}), \mu=C_{\dagger}(W)\left(f\left(x_{0}\right)\right)\left(f^{\prime}\left(x_{0}\right) \mathfrak{u}\right)$ and $\nu:=\lambda+\mu$.
Hence the element $\left(u, \lambda, f^{\prime}\left(x_{0}\right) \mathfrak{u}, \mu, \nu\right)$ belongs to $C_{K}\left(z_{0}\right) \cap G^{\prime}\left(z_{0}\right)^{-1}(0,0)$ and thus, to the Clarke tangent cone to the subset $K \cap G^{-1}(0,0)$ at $z_{0}$. Then, for all sequence $h_{n}>0$, there exist elements ( $u_{n}, \lambda_{n}, \mu_{n}$ ) converging to ( $u, \lambda, \mu$ ) such that, for all $n \geq N$,

$$
\left(x_{n}+h_{n} u_{n}, a_{n}+h_{n} \lambda_{n}, f\left(x_{n}\right)+h_{n} f^{\prime}\left(x_{0}\right) u_{n}, b_{n}+h_{n} \mu_{n}, a_{n}+b_{n}+h_{n} \nu_{n}\right) \in K
$$

Therefore, the pairs $\left(x_{n}+h_{n} u_{n}, a_{n}+b_{n}+h_{n}\left(\lambda_{n}+\mu_{n}\right)\right.$ ) belong to the epigraph of $U$. Since ( $u_{n}, \lambda_{n}+\mu_{n}$ ) converges to $(u, \nu)$, we deduce that

$$
C_{\dagger}(U)\left(x_{0}\right)(u) \leq \nu=C_{\dagger}(V)\left(x_{0}\right)(u)+C_{\dagger}(W)\left(f\left(x_{0}\right)\right)\left(f^{\prime}\left(x_{0}\right) u\right)
$$

Let us state explicitely useful formulas of the epiderivatives of the restriction of a function $V$ to a closed subset, in the finite-dimensional case for sinmplicity.

Corollary 9.1 Let $X$ be a finite-dimensional space, $V$ be an extended function defined on $X$ and $K$ be a closed subset of $X$. Let $x_{0}$ belong to $K \cap \operatorname{Dom}(V)$.

We always have:

$$
\forall u \in T_{K}\left(x_{0}\right), \quad D_{\uparrow} V\left(x_{0}\right)(u) \leq D_{\uparrow}\left(\left.V\right|_{K}\right)\left(x_{0}\right)(u)
$$

If we assume that

$$
\operatorname{Dom}\left(C_{\dagger}\left(V\left(x_{0}\right)\right)\right)-C_{K}\left(x_{0}\right)=X
$$

then

$$
\left\{\begin{array}{lll}
i) & \forall u \in T_{K}^{b}\left(x_{0}\right), & D_{\uparrow}\left(\left.V\right|_{K}\right)\left(x_{0}\right) \leq D_{\uparrow} V\left(x_{0}\right)(u) \\
i i) & \forall z \in T_{K}^{\prime}\left(x_{0}\right), & D_{\dagger}^{\prime}\left(\left.V\right|_{K}\right)\left(x_{0}\right) \leq D_{\dagger}^{\bullet} V\left(x_{0}\right)(u) \\
i i i) & \forall u \in C_{K}\left(x_{0}\right), & C_{\dagger}\left(\left.V\right|_{K}\right)\left(x_{0}\right) \leq C_{\uparrow} V\left(x_{0}\right)(u)
\end{array}\right.
$$

Let us consider now a family of functions $V_{i}: X \mapsto \mathbf{R} \cup\{+\infty\}, \quad(i \in I)$ and let us associate with it the function $U$ defined by

$$
U(x):=\max _{i \in I} V_{i}(x)
$$

We set $I(x):=\left\{i \in I \mid V_{i}(x)=U(x)\right\}$. The following estimate is obvious:

$$
\begin{equation*}
\forall u \in X, \sup _{i \in I(x)} D_{\uparrow} V_{i}(x)(u) \leq D_{\uparrow} U(x)(u) \tag{92}
\end{equation*}
$$

because, for all $i \in I(x)$,

$$
\left(V_{i}(x+h u)-V_{i}(x)\right) / h \leq(U(x+h u)-U(x)) / h
$$

Conversely, we can obtain the following result:
Proposition 9.1 Let us consider $n$ extended real-valued functions $V_{i}$ : $X \mapsto \mathbf{R} \cup\{+\infty\}$ and the function $U$ defined by

$$
U(x):=\max _{i=1, \ldots, n} V_{i}(x)
$$

If the dimension of $X$ is finite and if we posit the transversality assump. tion

$$
\forall u_{i} \in X, \bigcap_{i=1}^{n}\left(\operatorname{Dom}\left(C_{\uparrow} V_{i}\left(x_{0}\right)\right)-u_{i}\right) \neq 0
$$

then
$\left\{\begin{array}{l}i) \quad \forall u \in \bigcap_{i=1}^{n} \operatorname{Dom}\left(D_{\dagger}^{b} V_{i}\left(x_{0}\right)\right), D^{b} U\left(x_{0}\right)(u) \leq \max _{i \in I\left(x_{0}\right)} D_{\uparrow}^{b} V_{i}\left(x_{0}\right)(u)\end{array}\right.$
(ii) $\forall u \in \bigcap_{i=1}^{n} \operatorname{Dom}\left(C_{\dagger} V_{i}\left(x_{0}\right)\right), C U\left(x_{0}\right)(u) \leq \max _{i \in I\left(x_{0}\right)} C_{\uparrow} V_{i}\left(x_{0}\right)(u)$

Proof Since the epigraph of $U$ is the intersection of the epigraphs of the $n$ functions $V_{i}$, we shall us Corollary 4.3, stating that under convenient assumptions we shall check in a moment,

$$
T_{\mathrm{E}(U)}^{\dagger}\left(x_{0}, U\left(x_{0}\right)\right) \supset \bigcap_{i=1}^{n} T_{\mathrm{Ep}\left(V_{i}\right)}^{\dot{b}}\left(x_{0}, U\left(x_{0}\right)\right)
$$

The left-hand side of this formula is the epigraph of the adjacent epiderivative of $U$ at $x_{0}$. For the right-hand side, either $i$ belongs to $I\left(x_{0}\right)$, and thus the adjacent tangent cone to the epigraph of $V_{i}$ at $U\left(x_{0}\right)$ is equal to the epigraph of the adjacent derivative of $V_{i}$ at $x_{0}$, or $V_{i}\left(x_{0}\right)<U\left(x_{0}\right)$, and we deduce from Proposition 8.4 that the adjacent tangent cone contains $\operatorname{Dom}\left(D_{\uparrow}^{\upharpoonright} V_{i}\left(x_{0}\right)\right) \times \mathbf{R}$.

Then, we deduce from the above relation that

$$
\forall u \in \bigcap_{i=1}^{n} \operatorname{Dom}\left(D_{\uparrow}^{b} V_{i}\left(x_{0}\right)\right), D^{b} U\left(x_{0}\right)(u) \leq \max _{i \in I\left(x_{0}\right)} D_{\uparrow}^{b} V_{i}\left(x_{0}\right)(u)
$$

Since the dimension of $X$ is finite, we have to check that for all pairs $\left(u_{i}, \lambda_{i}\right)$,

$$
\bigcap_{i=1}^{n}\left(C_{\mathrm{Ep}\left(V_{i}\right)}\left(x_{0}, U\left(x_{0}\right)\right)-\left(u_{i}, \lambda_{i}\right)\right) \neq 0
$$

It is clear that this property follows from

$$
\bigcap_{i=1}^{n}\left(\operatorname{Dom}\left(C_{\uparrow} V_{i}\left(x_{0}\right)\right)-u_{i}\right) \neq 0
$$

Let us consider two topological vector spaces $X$ and $Y$ and an extended real-valued function $U: X \times Y \mapsto \mathbf{R} \cup\{+\infty\}$.

We associate with it the marginal function $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ defined by

$$
V(x):=\inf _{y \in \mathcal{Y}} U(x, y)
$$

Let $\pi$ denote the projection from $X \times Y \times \mathbf{R}$ to $X \times \mathbf{R}$.
We observe that:

$$
\pi \mathrm{Ep}(U) \subset \mathrm{Ep}(V) \subset \overline{\pi \mathrm{Ep}(U)}
$$

The first inclusion is obvious. The very definition of the infimum implies that for every $\epsilon>0$ and every $(x, \lambda) \in \operatorname{Ep}(V)$, there exists $y_{\epsilon} \in Y$ such that $\left(x, y_{c}, \lambda+\epsilon\right)$ belongs to $\operatorname{Ep}(U)$.

Proposition 9.2 Let us consider two topological vector spaces $X$ and $Y$, an extended real-valued function $U: X \times Y \mapsto \mathbf{R} \cup\{+\infty\}$, and its marginal function $V$. Suppose that there exists $y_{0} \in Y$ which achieves the minimum of $U\left(x_{0}, \cdot\right)$ on $Y$ :

$$
V\left(x_{0}\right)=U\left(x_{0}, y_{0}\right)
$$

## The inclusion

$$
\forall * \in X, \quad D_{\uparrow}(V)\left(x_{0}\right)(u) \leq \liminf _{x^{\prime} \rightarrow *}\left(\inf _{v \in Y} D_{\uparrow}(U)\left(x_{0}, y_{0}\right)\left(u^{\prime}, v\right)\right)
$$

is always true.

## Equality

$$
\forall u \in X, \quad D_{\uparrow}(V)\left(x_{0}\right)(u)=\liminf _{v^{\prime} \rightarrow \mathfrak{u}}\left(\inf _{v \in Y} D_{\uparrow}(U)\left(x_{0}, y_{0}\right)\left(u^{\prime}, v\right)\right)
$$

holde true under one of the following assumptions:

1. $U$ is poeudo-convex at $\left(x_{0}, y_{0}\right) \in \operatorname{Dom}(U)$, (and in particular, $U$ is conves),
2. $X$ and $Y$ are reflexive Banach space and the subdifferential of $U$ at ( $x_{0}, y_{0}$ ) satiofies
$(93) \mathcal{N}(p, q) \in X^{*} \times Y^{\star}, \exists p_{0} \in X^{*}$ ouch that $\left(p-p_{0}, q\right) \in \partial^{0} U(x, y)$
3. The dimension of $X$ and $Y$ io finite and

$$
\begin{equation*}
\left\{v \mid D_{\uparrow}(U)\left(x_{0}, y_{0}\right)(0, v) \leq 0\right\}:=\{0\} \tag{94}
\end{equation*}
$$

Proof We deduce these statements from the criteria implying that the contingent cones of the images are the closures of the images of the contingent cones since

$$
\left\{\begin{array}{l}
\left(\pi \mathrm{Ep}_{\mathrm{p}} D_{\uparrow}(U)\left(x_{0}, y_{0}\right)\right)=\left(\overline{\pi T_{\mathrm{Ep}(U)}}\left(x_{0}, y_{0}, U\left(x_{0}, y_{0}\right)\right)\right) \\
\subset T_{\pi(\mathrm{Ep}(1)}\left(x_{0}, V\left(x_{0}\right)\right)=T_{\mathrm{Ep}(V)}\left(x_{0}, V\left(x_{0}\right)\right) \\
=\mathrm{E}_{\mathrm{p}}\left(D_{\uparrow}(V)\left(x_{0}\right)\right)
\end{array}\right.
$$

which can be easily translated into the first inequality.
The equality is obtained when $U$ is pseudo-convex because its epigraph is then pseudo-convex, (see Proposition 4.3), when condition (93) holds true because it implies that

$$
\operatorname{Im}\left(\pi^{\star}\right)+N_{\mathrm{Ep}(U)}^{0}\left(x_{0}, y_{0}, U\left(x_{0}, y_{0}\right)\right)=X^{\star} \times Y^{\star} \times \mathbf{R}
$$

(see Proposition 4.5), and when condition (94) is satisfied because it is equivalent to

$$
\operatorname{ker}(\pi) \cap T_{\mathrm{Ep}(U)}\left(x_{0}, y_{0}, U\left(x_{0}, y_{0}\right)\right)=\{0\}
$$

thanks to Proposition 4.4.

## 10 Normal Cones and Generalized Gradients

We devote this section to dual concepts of tangent cones, derivatives of setvalued maps and epiderivatives of extended real-valued functions. There are three reasons to do so. The first one is familiarity with more classical concepts. For usual functions on Hilbert spaces, there is a canonical identification between, say, a derivative of a differentiable function and its gradient, and it became traditional to formulate many results in terms of gradients, transposes of derivatives and normal cones. The second reason is that the first attempts to generalize the concept of gradients was by limiting procedures. Since it seems easier to take limits of elements (the gradients, for instance) than functionals (the associated directional derivatives, for example), many generalizations of concept of gradient dealt with set of limits of cluster points taken in a variety of ways. The third, and, from our view point, the most important justification for dealing with dual concepts, is the availability of the one to one correspondences between closed convex cones and their polar cones, continuous linear operators and their transposes, lower semicontinuous convex functions and their conjugates. This is why we should use only those concepts which can be "dualized". Unfortunately, this is just a paradisiac wish, since many problems which are not smooth by nature, force us to use naturally concepts as contingent cones, contingent derivatives and contingent epiderivatives. The price to pay in terms of loss of information for playing with duality just to be able to conserve some familiar classical formulation is indeed too high in many situations. Therefore, the dual concepts we are about to present are recommended only in convex, or more generally, sleek situations.

Since the Clarke tangent cone is convex, it can be characterized by its polar cone, which, by analogy with the case of smooth manifolds, will be regarded as the normal cone. On the other hand, we wish to adapt to the nonsmooth case the concept of a normal to a set at a given point, which is orthogonal to all vectors starting from this point and pointing into this set. Except the convex and (more generally), the sleek case, these two concepts are different.

Definition 10.1 Let $x$ belongs to $K \subset X$. We shall say that the (negative) polar cone

$$
\begin{equation*}
N_{K}(x):=C_{K}(x)^{-}=\left\{p \in X^{\star} \mid \forall v \in T_{K}(x),<p, v>\leq 0\right\} \tag{95}
\end{equation*}
$$

is the normal cone to $K$ at $x$. We also say that the polar cone

$$
\begin{equation*}
N_{K}^{\circ}(x):=S_{K}(x)^{-}=\left\{p \in X^{\star} \mid \max _{y \in K}<p, y>=<p, x>\right\} \tag{96}
\end{equation*}
$$

to the cone opanned by $K-x$ is the subnormal cone to $K$ at $x$.
The normal cone is pretty big since it contains the polar cones of the adjacent and contingent cones and the subnormal cone:

$$
\begin{equation*}
N_{K}^{\circ}(x) \subset T_{K}(x)^{-} \subset T_{K}^{b}(x)^{-} \subset N_{K}(x) \tag{97}
\end{equation*}
$$

and is equal to the whole space whenever the Clarke tangent cone is reduced to 0 .

Let us point out the following property:
Proposition 10.1 Let $K$ be a subset of a Hilbert space. Then

$$
\forall y \notin K, \forall x \in \pi_{K}(y), y-x \in T_{K}(x)^{-} \subset N_{K}(x)
$$

Proof Let $v$ belong to the contingent cone $T_{K}(x)$ : there exists a sequence $h_{n}>0$ converging to 0 and a sequence $v_{n}$ converging to $v$ such that $x+h_{n} v_{n}$ belongs to $K$ for all $n$. Since $\|y-x\| \leq\left\|y-x-h_{n} v_{n}\right\|$, we deduce that $\langle x-y, v\rangle \geq 0$ for all $v \in T_{K}(x)$.

Remark When $X$ is a Banach space, we may consider the subdiferential $J(x)$ of its norm at $x$. then the above proposition can be extended to

$$
\begin{equation*}
0 \in J(x-y)+N_{K}(x) \tag{98}
\end{equation*}
$$

The transpose of the derivatives of differentiable maps are often used, in writing chain rule formulas, just to quote an instance. The circatangent derivative being a closed convex process can be transposed. This brings us to introduce the following definitions

Definition 10.2 Let $F: X \sim Y$ be a set-valued map from a topological vector space $X$ to another $Y$. The transpose $C F(x, y)^{*}$ of the circatangent derivative $C F(x, y)$ of $F$ at $a(x, y) \in \operatorname{Graph}(F)$ defined by

$$
\left\{\begin{array}{l}
p \in C F(x, y)^{*}(q) \text { if and only if }  \tag{99}\\
\forall u \in X, \forall v \in C F(x, y)(u),<p, u>\leq<p, v>
\end{array}\right.
$$

is the codifferential of $F$ at $(x, y)$. We shall say that the closed convex process from $Y^{*}$ to $X^{*}$ defined by

$$
\left\{\begin{array}{l}
p \in D F(x, y)^{\circ *}(q) \quad \text { if and only if }  \tag{100}\\
\left.\forall\left(x^{\prime}, y^{\prime}\right) \in \operatorname{Graph}(F),<p, x^{\prime}-x\right\rangle \leq\left\langle q, y^{\prime}-y\right\rangle
\end{array}\right.
$$

is the cosubdifferential of $F$ at $(x, y)$.
When a real-valued function $V$ is continuously differentiable at $x$, its gradient $V^{\prime}(x)$ being a continuous linear functional, is the refore an element $V^{\prime}(x) \in X^{*}$ of the dual of $X$.

Since the circatangent epiderivative of a nontrivial extended real-valued function $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ at a point $x$ of its domain is always lower semicontinuons convex and positively homogeneous, it is the support function of a closed convex subset, which is the generalized gradient of $V$ at $x$. In the same time, we shall deal with the subdifferentials introduced by Moreau and Rockafellar for convex functions.

Definition 10.3 Let $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ be a nontrivial extended realvalued function and $x$ belong to its domain. We shall say that the

$$
\begin{equation*}
\partial V(x):=\left\{p \in X^{\star} \mid \forall u \in X,<p, u>\leq C_{\uparrow} V(x)(u)\right\} \tag{101}
\end{equation*}
$$

is the generalized gradient of $V$ at $x$.

```
We shall say that the closed convex subset of the dual \(X^{*}\) of \(X\) defined by
\[
\begin{equation*}
\partial^{0} V(x):=\left\{p \in X^{\star} \mid \forall y \in X,<p, y-x>\leq V(y)-V(x)\right\} \tag{102}
\end{equation*}
\]
```

is the subdiferential of $V$ at $x$.
Naturally, when $V$ is continuously differentiable at $x$, the circatangent epiderivative coincides with the gradient $V^{\prime}(x)$, so that the generalized gradient is reduced to the only gradient:

$$
\partial V(x)=\left\{V^{\prime}(x)\right\} \text { when } V \text { is continuously differentiable at } x
$$

We observe that

$$
\partial^{0} V(x) \subset \partial V(x)
$$

When $V$ is convex, both the generalized gradient and the subdiferential are equal:

$$
\partial V(x)=\partial^{0} V(x) \text { when } V \text { is convex }
$$

More generally, this also happens when $V$ is sleek and pseudo-convex at $x$.
If $V$ is continuously differentiable around a point $x \in K$, then the generalized gradient of the restriction is the sum of the gradient and the normal cone:

$$
\partial\left(\left.V\right|_{K}\right)(x)=V^{\prime}(x)+N_{K}(x)
$$

We also note that the generalized gradient of the indicator of a subset is the normal cone:

$$
\partial \psi_{K}(x)=N_{K}(x)
$$

Remark When a real-valued function $V$ is continuously differentiable at $x$, its gradient $V^{\prime}(x)$ being a continuous linear functional, it is both an element $V^{\prime}(x) \in X^{*}$ of the dual and the image $V^{\prime}(x)^{\star}(+1)$ of +1 by the transpose $V^{\prime *}$ of $V^{\prime}(x)$, a linear operator from $\mathbf{R}$ to $X^{*}$.

When $V$ is no longer continuously differentiable, the generalized gradient remains intimately connected with the transpose of the circatangent derivative $C \mathbf{V}_{\dagger}(x, V(x)$ ), (which, shall we recall, is the codifferential $\left.C V_{\uparrow}(x, V(x))^{\star}\right)$ of the set-valued map $\mathbf{V}_{\uparrow}$ at $(x, V(x))$.
the generalized gradient is the value at 1 of the codifferential of $\mathbf{V}_{\dagger}$ at $(x, V(x))$.

Indeed, the codifferential is a closed convex process from $\mathbf{R}$ to $X^{\star}$ which, being positively homogeneous, needs to be defined only at the points -1 , 0 and +1 .

We obtain:

$$
\begin{cases}\text { i) } & C V_{\uparrow}(x, V(x))^{*}(-1)=  \tag{103}\\ \text { ii) } & C V_{\uparrow}(x, V(x))^{*}(0)=\left(\operatorname{Dom}\left(C_{\uparrow} V\right)(x)\right)^{-} \\ \text {iii) } & C V_{\uparrow}(x, V(x))^{*}(+1)=\partial V(x)\end{cases}
$$

The table of formulas on support functions allows to translate the properties of the circatangent epiderivatives into corresponding properties of the generalized gradient and vice-versa.

For instance, Proposition 8.1 can be restated in the following form.
Proposition 10.2 When $V$ is locally lipschitzean on the interior of its domain, then the generalized gradient satisfies:

$$
\begin{cases}\text { i) } & (x, u) \in X \times \operatorname{Int}(\operatorname{Dom}(V)) \mapsto \sigma(\partial V(x), u) \\ & \text { is upper semicontinuous, } \\ & \text { and thus, } \partial V(\cdot) \text { is upper hemicontinuous } \\ \text { ii) } & \partial V(x) \text { are nonempty bounded closed convex } \\ \text { iii) } \partial V(x)=-\partial(-V(x))\end{cases}
$$

In the same way, the Fermat and Eleland rules can be presented in the following fashion:

Theorem 10.1 (Fermat and Ekeland Rules) Let $V: X \mapsto \mathbf{R} \cup\{+\infty\}$ be a nontrivial extended real-valued function.

1. Let $x \in \operatorname{Dom}(V)$ achieve the minimum of $V$ on $X$.

Then $x$ is a solution to the inclusion:

$$
0 \in \partial V(x)
$$

the converse being true when $V$ is convex or. more generally, pseudo. convex at $x$.
2. Let $X$ be a Banach space, $V: X \mapsto \mathbf{R}_{+} \cup\{+\infty\}$ be a nontrivial nonnegative extended real-valued function and $x_{0} \in \operatorname{Dom}(V)$ be a given point of its domain. Then, for any $\epsilon>0$, there exists a solution $x_{t} \in \operatorname{Dom}(V)$ to:

$$
\left\{\begin{array}{l}
\text { i) } V\left(x_{\epsilon}\right)+\epsilon\left\|x_{\epsilon}-x_{0}\right\| \leq V\left(x_{0}\right)  \tag{104}\\
\text { ii) } \forall u \in X, 0 \in \partial V\left(x_{t}\right)+\epsilon B
\end{array}\right.
$$

## Remark

When the functions are not sleek, the use of generalized gradients and normal cones involves some loss of information since we have to replace the contingent epiderivative by the larger, but convex, circatangent epiderivative.

We could save part of the information using subsets of the form

$$
\left\{p \in X^{\star} \mid \forall v \in X, \quad<p, u>\leq D_{\uparrow} V(x)(u)\right\}
$$

and the polar of the contingent cones.
This will lead us to increase the population of our ménagerie with species doomed to disappear through Darwinian evolution, since, their use do not allow to recover the original information for lack of duality. The use of generalized gradients for functions is then recommended for functions or cones which are sleek.

For the convenience of the reader, we list below a summary of some formulas dealing with the adjacent ${ }^{28}$ and normal cones to subsets. The subsets $K, K_{i}, L, M, \ldots$ are assumed to be nonempty.

## Properties of Adjacent and Normal Cones (1)

1. If $K \subset L$, then

$$
T_{K}^{b}(x) \subset T_{L}(x)
$$

2. If $K_{i} \subset X_{i}, \quad(i=1, \cdots, n)$, then

$$
\left\{\begin{array}{l}
\text { i) } T_{\prod_{i=1}^{b} K_{i}}^{b}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n} T_{K_{i}}^{p}\left(x_{i}\right) \\
\text { ii) } N_{i=1}^{n} k_{i}\left(x_{1}, \ldots, x_{n}\right)=\prod_{i=1}^{n} N_{K_{i}}\left(x_{i}\right)
\end{array}\right.
$$

3. If $K_{1}$ and $K_{2}$ are contained in $X$, then

$$
T_{K_{1}+K_{2}}\left(x_{1}+x_{2}\right) \supset \overline{T_{K_{1}}\left(x_{1}\right)+T_{K_{2}}\left(x_{2}\right)}
$$

4. If $f: X: \mapsto Y$, is differentiable at $x$, then

$$
T_{f(K)}(x) \supset \overline{f^{\prime}(x)\left(T_{K}(x)\right)}
$$

5. If $X$ and $Y$ are finite dimensional vector-spaces, if $L \subset X$ and $M \subset Y$ are closed subsets, if $f: X \mapsto Y$ is continuously differentiable at $x$ and satisfy the transversality condition

$$
f^{\prime}(x) C_{L}(x)-C_{M}(f(x))=Y
$$

then

$$
\begin{cases}\text { i) } & T_{\left.L \cap A^{-1}(M)\right)}^{b}=T_{L}^{\dagger}(x) \cap A^{-1} T_{M}^{\dagger}(A x) \\ \text { ii) } & N_{\left.L \cap A^{-1}(M)\right)} \subset N_{L}(x)+A^{*} N_{M}(A x)\end{cases}
$$

[^18]
## Properties of Adjacent and Normal Cones (2)

6. If $X$ and $Y$ are finite dimensional vector-spaces, if $M \subset Y$ is a closed subset and if $f: X \mapsto Y$ is continuously differentiable at $x$ such that the transversality condition

$$
\operatorname{Im}\left(f^{\prime}(x)\right)-C_{M}(x)=Y
$$

holds true, then

$$
\begin{cases}\text { i) } & T_{A^{-1}(M)}^{b}(x)=A^{-1} T_{M}^{b}(A x) \\ \text { ii) } & N_{A^{-1}(M)}(x) \leq A^{\star} N_{M}(A x)\end{cases}
$$

7. If $X$ is a finite dimensional vector-space, if $K_{1}$ and $K_{2}$ are closed subsets contained in $X$ and satisfy

$$
C_{K_{1}}(x)-C_{K_{2}}(x)=X
$$

then

$$
\begin{cases}i) & T_{K_{1} \cap K_{2}}^{b}(x)=T_{K_{1}}^{b}(x) \cap T_{K_{2}}^{b}(x) \\ i i) & N_{K_{1} \cap K_{2}}(x) \subset N_{K_{1}}(x)+N_{K_{2}}(x)\end{cases}
$$

8. If $X$ is a finite dimensional vector-space, if $K_{i} \subset X, \quad(i=1, \ldots, n)$, are closed and if

$$
\forall v_{i} \in X, \quad \bigcap_{i=1}^{n}\left(C_{K_{i}}(x)-v_{i}\right) \neq 0
$$

then,

$$
\left\{\begin{array}{l}
\text { i) } T_{\bigcap_{i=1}^{n} K_{i}}^{b}(x)=\bigcap_{i=1}^{n} T_{K_{i}}^{b}(x) \\
\text { ii) } N_{\bigcap_{i=1}^{n} K_{i}}^{n}(x) \subset \sum_{i=1}^{n} N_{K_{i}}(x)
\end{array}\right.
$$

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[^0]:    ${ }^{1}$ at least in the instances of interest for us.
    ${ }^{2}$ Fermat was one of the most important originator in the history of mathematics. Even Newton did recognize explicitly that he received the hint of the differential calculus from Fermat's method of building tangents devised half a century earlier. Fermat was also

[^1]:    the one who discovered that the derivative of a (polynomial) function vanishes when it reaches an extremum (Euler-Lagrange equations, Pontryagin's maximum principle are just implementations to infinite-dimensional problems of what should be called the FERMAT RULE). He also was the first to discover the "principle of least time" in optics, the prototype of the variational principles governing so many physical and mechanical laws. He shared with Descartes the independent invention of analytic geometry and with Pascal the creation of the mathematical theory of probability. He was on top of that a poet, a linguist, a lawyer and, if it has to be recalled, the author of the Fermat Theorems, .... consequences of a revolutionary treatment of number theory
    ${ }^{3}$ introduced in the thirties
    ${ }^{4}$ used by H. Frankowska under the name of intermediate tangent cone
    ${ }^{8}$ introduced in 1975

[^2]:    ${ }^{6}$ from Latin contingere, to touch on all sides, introduced by G. Bouligand
    ${ }^{7}$ from Latin adjacere, to lie near, recently introduced and applied under the name intermediate cone by H. Frankowska and the name of derivable cone by R. T. Rockafellar
    ${ }^{8}$ from Canadian Frank H. Clarke; we shall use the adjective clrcatangent to mention properties derived from this tangent cone, for instance, circatangent derivatives and epiderivatives

[^3]:    ${ }^{9}$ they are the contingent, adjacent and circatangent epiderivatives of the distance functions $d_{K}$.

[^4]:    ${ }^{10}$ Shi Shuzhong showed that when $K$ is the closure of its interior, the contingent cones and the paratingent cones (relative to the boundary) are generically equal (they coincide on a $G_{6}$ dense of the boundary), a consequence of Choquet's theorem.

[^5]:    ${ }^{11}$ see [5, Proposition 8.1.18., p.117].

[^6]:    ${ }^{12}$ This property is false when assumption (30)is not eatisfied. Take for instance two balls $K_{1}$ and $K_{2}$ tangent at a point $x$. The tangent cone to the intersection $\{x\}$ is reduced to $\{0\}$, whereas the intersection of the tangent cones is a hyperplane. This shows that we cannot dispense of the constralnt qualification assumptions in the calculus of tangent cones to inverse images and intersections

[^7]:    ${ }^{13}$ See [4, Theorem 3.1]:

[^8]:    ${ }^{14} \mathrm{Namely}$, if

[^9]:    ${ }^{15}$ by definition, the graph of the paratingent derivative $P F(x, y)$ of $F$ at $(x, y)$ is the paratingent cone to the graph of $F$ at $(x, y)$.

[^10]:    ${ }^{16}$ Let us recall this statement
    Theorem 4.1 (Constrained Inverse Function Theorem) Let $X$ and $Y$ be two Banach spaces. We introduce a (single-valued) continuous map $f: X \mapsto Y$, a closed subset $K \subset X$ and an element $x_{0}$ of $K$.

    We assume that $f$ is differentiable on a neighborhood of $x_{0}$ and we posit the following stablilty asoumption: there exist constants $c>0, \alpha \in[0,1[$ and $\eta>0$ such that

    $$
    \left\{\begin{array}{l}
    \forall x \in K \cap B\left(x_{0}, \eta\right), \\
    B_{Y} \subset f^{\prime}(x)\left(T_{K}(x) \cap c B_{X}\right)+\alpha B_{Y}
    \end{array}\right.
    $$

    Then $f\left(x_{0}\right)$ belongs to the interior of $f(K)$ and the set-valued map $y \sim f^{-1}(y) \cap K$ is pseudo-lipschltzean around $\left(f\left(x_{0}\right), x_{0}\right)$.

[^11]:    ${ }^{17}$ Let us recall this statement
    Theorem 4.8 (Criterion of Pseudo-Lipschitseanity) Let $K$ be a cloeed subset of a Banach apace $X$ and $A \in \mathcal{L}(X, Y)$ be a continuous linear operator from $X$ to another Banach space $Y$. Let we acsume that for some $x_{0} \in K$, there exist eonstanto $c>0$, $\alpha \in[0,1 \mid$ and $\eta>0$ auch that

[^12]:    ${ }^{18}$ Banach's Closed Graph Theorem allows to assume that $A$ is surjective: It is sufficient to decompose $\boldsymbol{A}$ as the product $\hat{\boldsymbol{A}} \circ \phi$ of the canonical surjection $\phi$ from $\boldsymbol{X}$ onto its factor space $\boldsymbol{X} / \operatorname{ker}(\boldsymbol{A})$ and the associated bijective map $\tilde{\boldsymbol{A}}$, which is an isomorphism. Then the properness of $A$ is equivalent to the properness of $\phi$.

[^13]:    Definition 6.1 Let $F: X \sim Y$ be a set-valued map from a topological - ector space $X$ to another $Y$.

[^14]:    ${ }^{19}$ we refer to $[5$, Sections $6.6 \& 6.7]$ for an introduction to monotone and maximal monotone maps.

[^15]:    ${ }^{20}$ Observe that square products are implicitely involved in the factorization of maps. Let $X$ be a subset, $R$ be an equivalence relation on $X$ and $\phi$ denote the canonical surjection from $X$ onto the factor space $X / R$. If $f$ is a single-valued map from $X$ to $Y$, its factorization $\tilde{f}: X / R \mapsto Y$ is defined by

    $$
    \tilde{f}(\xi):=\left(f \square \phi^{-1}\right)(\xi)
    $$

    It is non trivial if and only if $f$ is consistent with the equivalence relation $R$, i.e., if and only if $f(x)=f(y)$ whenever $\phi(x)=\phi(y)$. When $F: X \leadsto Y$ is a set-valued map, we can define its factorization $\tilde{F}: X / R \leadsto Y$ by

    $$
    \tilde{F}(\xi):=\left(F \square \phi^{-1}\right)(\xi)
    $$

[^16]:    ${ }^{21}$ We recall also that a set-valued map $G$ is upper semicontinuous if and only if the inverse images $G^{-}$of open subsets are open and that it is lower semicontinuous if and only if the inverse images $G^{+}$of open subset are open.

[^17]:    ${ }^{22}$ by definition, the graph of the paratingent derivative $P F(x, y)$ of $F$ at $(x, y)$ is the paratingent cone to the graph of $F$ at $(x, y)$.
    We can also define the lop-sided paratingent derivatives $P_{s} F(x, y)$ and $P_{y} F(x, y)$ in the following way:

    $$
    \operatorname{Graph}\left(P_{1} F(x, y)\right):=P_{\operatorname{Graph}(F)}^{\operatorname{Dom}(F)}(x, y) \& \operatorname{Graph}\left(P_{2} F(x, y)\right):=P_{\operatorname{Graph}(F)}^{\operatorname{Im}(F)}(x, y)
    $$

[^18]:    ${ }^{23}$ we chose the adjacent cone rather than the contingent or the Clarke tangent cones because they enjoy more often equalities in the formulas. Formulas for the Clarke tangent cones can be deduced from polarity from the formulas on normal cones.

