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Set Valued Calculus in Problems of Adaptive Control

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FOREWORD

This paper deals with *feedback control* for a linear nonstationary system whose objective is to reach a preassigned set in the state space while satisfying a certain *state constraint*. The state constraint to be fulfilled cannot be predicted in advance being available only on the basis of observations. It is specified through an *adaptive procedure* of "guaranteed estimation" and the objective of the basic process is to *adapt to this constraint*.

The problems considered in the paper are motivated by some typical applied processes in environmental, technological, economical studies and related topics.

The techniques used for the solution are based on nonlinear analysis for *set-valued maps*.

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CONTENTS

SET-VALUED CALCULUS IN PROBLEMS OF ADAPTIVE CONTROL

A. B. Kurzhanski

Introduction

This paper deals with *feedback control* for a linear nonstationary system whose objective is to reach a preassigned set in the state space while satisfying a certain *state constraint*. The state constraint to be fulfilled cannot be predicted in advance being governed by a second "uncertain" system, with its state space variable unknown and available only on the basis of observations. It is assumed that there is no statistical data for the uncertain parameters of the second system the only information on these being the knowledge of some constraints on their admissible values. Therefore the state constraint to be satisfied by the basic system may be specified only through an *adaptive procedure* of "guaranteed estimation" and the objective of the basic process is to *adapt to this constraint*.

The problems considered in the paper are motivated by some typical applied processes in environmental, technological, economical studies and related topics.

The techniques used for the solution are based on nonlinear analysis for *set-valued maps*. They also serve to illustrate the relevance of set-valued calculus to

- problems of control in devising solutions for the "guaranteed filtering and extrapolation" problems
- constructing set-valued feedback control strategies,
- duality theory for systems with set-valued state space variables,
- approximation techniques for control problems with set-valued solutions, etc.

The research in the field of control and estimation for uncertain systems (in a deterministic setting), in differential games and also in set-valued calculus, that motivated this paper, is mostly due to the publications of [1-10].

1. The Uncertain System

Consider a system modelled by a linear-convex differential inclusion

$$\dot{q} \in A(t)q + \mathbf{P}(t) \tag{1.1}$$

$$t \in \mathbf{T} = \{t : t_0 \leq t \leq t_1\} ,$$

where $q \in \mathbf{R}^n$, $A(t)$ is a continuous matrix function $(A : \mathbf{T} \longrightarrow \mathbf{R}^{n \times m})$, $\mathbf{P}(t)$ is a continuous multivalued map from T into the set ***conv*** $\mathbf{R}^n$ of convex compact subsets of $\mathbf{R}^n$. (Here $\mathbf{R}^n$ will stand for the n-dimensional vector space and $\mathbf{R}^{m \times n}$ for the space of $m \times n$ - matrices.)

The function $\mathbf{P}(t)$ reflects the ***uncertainty in the specification of the system inputs.*** The initial state $q(t_0) = q^{(0)}$ is also taken to be unknown in advance. Namely,

$$q^{(0)} \in Q^{(0)} \tag{1.2}$$

with the set $Q^{(0)} \in$ ***conv*** $\mathbf{R}^n$ being given.

An isolated trajectory of (1.1) generated by point $q^{(\tau)} = q[\tau]$ will be further denoted as $q[t] = q(t \, , \tau \, , q^{(\tau)})$, while the set of all solutions to (1.1) that start at $q^{(\tau)}$ will be denoted as $Q(t \, , \tau \, , q^{(\tau)})$.

We also assume

$$Q(t \, , \tau \, , Q^{(\tau)}) = \bigcup \{ Q(t \, , \tau \, , q^{(\tau)}) \mid q^{(\tau)} \in Q^{(\tau)} \} \, .$$

The sets $Q(t \, , t_0 \, , q^{(0)})$,$Q(t \, , t_0 \, , Q^{(0)})$ are therefore the ***attainability domains*** for (1.1) (from $q(t_0) = q^{(0)}$ and $Q^{(0)}$ respectively).

It is known that the multivalued function

$$Q[t] = Q(t \, , t_0 \, , Q^{(0)})$$

satisfies the "funnel equation", [11]

$$\lim_{\sigma \longrightarrow 0} \sigma^{-1} \, h(Q[t + \sigma] \, , (E + A(t)\sigma) \; Q[t] + \mathbf{P}(t) \; \sigma) = 0 \tag{1.3}$$

where

$$h(Q' \, , Q'') = \max\{h^+(Q' \, , Q'') \, , h^-(Q'' \, , Q')\} \, ,$$

$$h^+(Q' \, , Q'') = \max_p \min_q \, \{ \, || \, p - q \, || \mid p \in Q' \, , q \in Q'' \, \} \, ,$$

$$h^+(Q' \, , Q'') = h^-(Q'' \, , Q')$$

is the ***Hausdorff distance*** between $Q' \in$ ***conv*** $\mathbf{R}^n$, $Q'' \in$ ***conv*** $\mathbf{R}^n$ [12].

Let us now assume that there is some ***additional information*** on the system (1.1), (1.2). Namely, this information arrives through an ***equation of observations***

$$y \in G(t) \; q(t) + \mathbf{R}(t) \tag{1.4}$$

where $y \in \mathbf{R}^m$, $G(t)$ is continuous $(G : \mathbf{R}^n \longrightarrow \mathbf{R}^m)$ and the set-valued function $\mathbf{R}(t)$ from T into ***conv*** $\mathbf{R}^m$ reflects the presence of "noise" in the observations. The realization $y_\tau(\sigma) = y(\tau + \sigma)$, $t_0 - \tau \leq \sigma \leq 0$, of the observation y being given, it is possible to construct an "informational domain" $Q_\tau \, (\bullet \, , t_0 \, , Q^{(0)} \mid y_\tau(\bullet))$ of all trajectories consistent with (1.1)-(1.3) and with the given realization $y_\tau(\bullet)$. The cross-section $\mathbf{Q} \; (\tau \, , t_0 \, , Q^{(0)})$ of this set is the "generalized state" of the "total" system (1.1), (1.2), (1.4), (for convenience we further omit an explicit indication of $y_\tau(\bullet)$ taking it to be fixed).

Clearly, for $\tau' \le \tau''$ we have $\mathbf{Q}(\tau'' , t_0 , Q^{(0)}) = \mathbf{Q}(\tau'' , \tau' , \mathbf{Q}\ (\tau' , t_0 , Q^{(0)}))$ The map $\mathbf{Q}(\tau'' , t_0 , Q^{(0)}) = \mathbf{Q}[\tau]$ thus satisfies a *semigroup property* and defines a *generalized dynamic system.* The function $\mathbf{Q}\ [\tau]$ also satisfies a more complicated version of the funnel equation (1.3), [3].

$$\lim_{\sigma \longrightarrow o} \sigma^{-1}\, h(\mathbf{Q}[\tau + \sigma]\ ,\ (E + A(\tau)\sigma)\ \mathbf{Q}[\tau] + \mathbf{P}(\tau)\sigma)) \bigcap \mathbf{Y}[\tau + \sigma]) = 0$$

$$\mathbf{Q}[t_0] = Q^{(0)} \tag{1.5}$$

where

$$\mathbf{Y}[\tau] = \{q : G(\tau)\ q \in y(\tau) - \mathbf{R}(\tau)\}$$

is taken to be such that its support function

$$\rho(l \mid \mathbf{Y}\ [\tau]\ ) = sup\{(l\ ,\ y) \mid y \in \mathbf{Y}[\tau]\}\ .$$

is continuously differentiable in l and τ. The latter property is true if $\rho(l \mid Y\ [\tau]\ )$ and $y(\tau)$ are continuously differentiable in the respective variables. This in turn is ensured if the measurement $y(t)$ is generated due to equation

$$y(t) = G(t)\ x(t) + \xi(t)\ ,\ \xi(t) \in \mathbf{R}(t)$$

by continuously differentiable functions $\xi(t)$ and $G(t)$.

Consider the inclusion

$$\dot{q}_L \in (A(t) - L(t)\ G(t))\ q_L + L(t)\ (y(t) - \mathbf{R}(t)) + \mathbf{P}(t) \tag{1.7}$$

$$q_L(t_0) = q_L^{(0)}\ ,\ q_L^{(0)} \in Q^{(0)}$$

whose attainability domain is

$$Q_L\ (t\ ,t_0\ ,Q^{(0)}) = Q_L\ [t]$$

Lemma 1.1 [13,14] *The following relation is true*

$$\bigcap Q_L\ (t\ ,t_{(0)}\ ,Q^{(0)}) = \mathbf{Q}\ (t\ ,t_0\ ,Q^{(0)}) = \mathbf{Q}\ [t]\ , \tag{1.8}$$

where the intersection is taken at all continuous matrix-valued functions $L(t)$ with values $L \in \mathbf{R}^{n \times m}$.

The last Lemma allows to decouple the calculation of $\mathbf{Q}[t]$ into the calculation of sets $Q_L[t]$ governed by "ordinary" differential inclusions of type (1.7).

According to [11] each of the multivalued functions $Q_L\ [t]$ satisfies a respective funnel equation

$$\lim_{\sigma \longrightarrow 0} \sigma^{-1}\, h(Q_L[\tau+\sigma], (E+\sigma(A(\tau)-L(\tau)\ G(\tau))\ \mathbf{Q}_L[\tau]+ \tag{1.9}$$

$$+ L(\tau)(y(\tau) - \mathbf{R}(\tau))\ \sigma + \mathbf{P}(\tau)\sigma) = 0$$

$$Q_L\ [t_0] = Q^{(0)}\ .$$

Hence from (1.8) it follows that the solution to (2.5) may be decoupled into the solutions of equations (1.9). The latter relations allow for a respective difference scheme.

2. An Inverse Problem

Assume that a square-integrable function $y_{t_1}(\sigma \mid \tau) = y(t_1 + \sigma)$, $\tau - t_1 \leq \sigma \leq 0$ and a set $N \in \text{conv}\ \mathbf{R}^n$ are given. Denote $\mathbf{W}(\tau\ , t_1\ , N)$ to be the variety of all points $w \in \mathbf{R}^n$ for each of which there exists a solution $q(t\ , \tau\ , w)$ that satisfies (1.1), (1.4) for $t \in [\tau\ , t_1]$, and $q(t_1\ , \tau\ , w) \in N$.

We observe that $W(\tau\ , t_1\ , N)$ is of the same nature as $\mathbf{Q}(t\ , t_0\ , Q^{(0)})$ except that it should be treated in backward time.

Hence, we will have to deal with the solutions to the inclusions

$$\dot{q} \in A(t)q + \mathbf{P}(t)\ , \qquad t \leq t_1\ , \tag{2.1}$$

$$t \in T\ , q(t_1) = q^{(1)}\ , q^{(1)} \in N$$

with isolated trajectories $q(t\ , t_1\ , q^{(1)})$ that satisfy the restriction

$$q(t) \in \mathbf{Y}(t)\ \forall\ t \in T \tag{2.2}$$

Following Lemma 1.1, we have a similar

Lemma 2.1. The following equality is true

$$W(t\ , t_1\ , N) = \bigcap_L W_L\ (t\ , t_1\ , N) \tag{2.3}$$

the intersection being taken over all continuous matrix-valued functions $L(t)$ *with* $L \in \mathbf{R}^{m \times n}$*, and* $W_L(t\ , t_1\ , N)$ *is the assembly of all solutions to the inclusion*

$$\dot{w}_L \in (A(t) - L(t)\ G(t))w_L + L(t)(y(t) - \mathbf{R}(t)) + \mathbf{P}(t)\ , \tag{2.4}$$

$$w(t_1) \in N$$

Lemma 2.2 Each of the realizations $W_L\ (t\ , t_1\ , N) = W_L[t]$ *may be achieved as a solution to the funnel equation*

$$\lim_{\sigma \longrightarrow +0} \sigma^{-1}\ h(\ W(t - \sigma)\ , (E - \sigma(A(t) - L(t)\ G(t)))\ W(t) -$$

$$- L(t)(y(t) - \mathbf{R}(t))\ \sigma - \mathbf{P}(t)\sigma) = 0$$

$$W(t_1) = \mathbf{N}$$

The uncertain system and inverse problem of the above will play an essential part in the formulation and the solution of the *adaptive control problem* discussed in this paper.

3. The Adaptive Control Problem

Consider a control process governed by the equation

$$\frac{dp}{dt} = C(t)p + u \ , t \in T \tag{3.1}$$

where $p \in \mathbf{R}^n$, $C(t)$ is a continuous matrix function $(C : \mathbf{R}^n \longrightarrow \mathbf{R}^n)$ and u is restricted by the inclusion

$$u \in \mathbf{V}(t)$$

where $\mathbf{V}(t)$ is a continuous multivalued map from $\mathbf{T}$ into conv $\mathbf{R}$.

The basic problem considered in this paper is to devise a feedback control law that would allow the system to *adapt* to an *uncertain state constraint.*

Assume that an uncertain system (1.1), (1.2), (1.4) is given and a *state constraint* is defined by a continuous multivalued map

$$K(t) \ (\mathbf{K} : \mathbf{T} \longrightarrow \text{conv } \mathbf{R}^n)$$

The objective of the control process for system (3.1) will be to satisfy the constraint

$$p(t) + q(t) \in \mathbf{K}(t) \ , \forall \, t \in T \ , \tag{3.2}$$

and also a terminal inclusion

$$p(t_1) \in \mathbf{M} \ , \mathbf{M} \in \text{conv } \mathbf{R}^n \tag{3.3}$$

The principal difficulty is here caused by the fact that vector $q(t)$ of (3.2) is unknown and that the information on its values is confined to the inclusion

$$q(t) \in \mathbf{Q}(t \ , t_0 \ , Q^{(0)})$$

Therefore the total state constraint on p at instant t will actually be

$$p(t) + \mathbf{Q}(t \ , t_0 \ , Q^{(0)}) \in K(t) \tag{3.4}$$

where the realization

$$\mathbf{Q}[t] = \mathbf{Q}(t \ , t_0 \ , Q^{(0)})$$

cannot be predicted in advance, being governed by the uncertainty

$$\omega_t \, (\bullet) = \{q(t_0) \ , \xi_t \, (\bullet) \ , v_t \, (\bullet)\}$$

Here the notation $f_t\ (\bullet)$ stands for

$$f_t\ (\sigma) = f(t+\sigma)\ ,\ t_0 - t \leq \sigma \leq t\ .$$

In order to pose the *adaptive control* problem it is necessary to introduce the notion of the *state (the position)* of the overall system (3.1)-(3.3).

The *position* of the system (3.1)-(3.3) will be defined as the triplet

$$\{t\ ,\ p\ ,\ y_t(\bullet)\}$$

Hence the solution to the problem will be sought for in the *class of multivalued strategies*

$$V = U(t\ ,\ p\ ,\ y_t(\bullet))$$

with $U \in conv\ \mathbf{R}^n$ and with the dependence of U upon $t\ ,\ p\ ,\ y_t(\bullet)$ being such that the joint system

$$\dot{p} \in C(t)p + \mathbf{U}(t\ ,\ p\ ,\ y_t(\bullet)) \tag{3.5}$$

$$\dot{q} \in A(t)q + \mathbf{P}(t) \tag{3.6}$$

$$y - Gq \in \mathbf{R}(t) \tag{3.7}$$

has a solution for any

$$p(t_0) = p^0 \in \mathbf{R}^n\ ,\ q(t_0) = q^0 \in \mathbf{R}^n.$$

For the solution to (3.5)-(3.7) to exist, in the sense that (3.5) - (3.7) are satisfied for almost all $t \in [t_o\ ,\ t_1]$, it suffices that $\mathbf{U}(t\ ,\ p\ ,\ y_t(\bullet))$ is a convex compact valued map, measurable in t and upper semicontinuous in $\{p\ ,\ y_t(\bullet)\} \in \mathbf{R}^n \times \mathbb{L}_2\ (t_0\ ,\ t)$, and that $\mathbf{P}(t)\ ,\ \mathbf{R}(t)$ are of convex compact values and measurable in t, [8]. A strategy $\mathbf{U}(t\ ,\ p\ ,\ y_t(\bullet))$ that ensures the existence of a solution to (3.5) - (3.7) will be further referred to as an *admissible strategy*.

The Basic Problem

With mapping $K(t)$ and set M being given, specify a feedback control strategy

$$\mathbf{U} = \mathbf{U}(t\ ,\ p\ ,\ y_t(\bullet))$$

that would ensure the inclusions (3.2), (3.3) whatever is the realization $q(t)$ of the system (3.6), with $q(t_o) \in Q^{(0)}$ and set $Q^{(0)}$ given.

For $t + \Delta t$ the element $y_t(\bullet)$ to be compared with $y_{t+\Delta t}(\bullet)$ should be modified to $y_t^{\Delta}(\bullet)$ which will be defined for $[t_0\ ,\ t + \Delta t]$ and such that

$$y_t^{\Delta}\ (\sigma) = \begin{cases} y\ (t+\sigma) & t_0 - t \leq \sigma \leq 0 \\ y\ (t)\ , & 0 < \sigma \leq \Delta t \end{cases}$$

Thus the control problem is to adapt the process $p(t)$ to the uncertain state constraint:

$$p(t) \in \mathbf{K}(t) \dot{-} \mathbf{Q}(t , t_o , \mathbf{Q}^{(0)})$$

where $\mathbf{Q}(t , t_0 , Q^{(0)})$ is achieved through a ***guaranteed estimation process*** for the system (3.6), (3.7) and $K \dot{-} Q$ stands for the geometrical (Minkowski) difference of sets K , Q $(K \dot{-} Q = \{p : p + Q \subseteq K\})$

The ***information*** on the ***basic system*** (3.1) is ***complete*** since the exact value of the vector p is assumed to be available.

We shall now proceed with the formal solution schemes for constructing the desired strategy

$$\mathbf{U} = \mathbf{U}(t , p , y_t(\bullet)) .$$

4. The Extrapolation Problem

Assume that at instant τ a realization $\overset{*}{y}_\tau(\bullet)$ is given and therefore, a set $\mathbf{Q}^*[\tau] = \mathbf{Q}(t , t_0 , Q^{(0)} \mid \overset{*}{y}_\tau(\bullet))$ is available. (From now on we will start to vary $y_\tau(\bullet)$ and will therefore include $y_\tau(\bullet)$ into the respective notations, substituting $\mathbf{Q}(\tau , t_0 , Q^{(0)})$ for $\mathbf{Q}(t , t_0 , Q^{(0)} \mid y_\tau(\bullet))$.

Suppose that the realization $\overset{*}{y}_\tau(\bullet)$ may be prolongated onto the interval $(\tau , t_1]$ in the form of a possible future measurement $\overset{*}{y}_\tau(\bullet)$ generated by a triplet

$$\omega_\tau^{o\,*}(\bullet) = \{q^* , v_\tau^{o\,*}(\bullet) , \xi_\tau^{o\,*}(\bullet)\}$$

where our further notation will be taken in the form $\psi_t^o(\sigma) = \psi(t + \sigma)$, $o < \sigma \le t_1 - t$, so that the upper zero index would assign the respective element $\psi_t^o(\bullet)$ to the interval $(t , t_1]$. For a multivalued map $\Psi(t)$ the notation is similar $\Psi_t^o(\sigma) = \Psi(t + \sigma)$, $0 < \sigma \le t_1 - t$.

The specific triplet $\omega_\tau^{o\,*}(\bullet)$ should satisfy the inclusions

$$q^* \in \mathbf{Q}[\tau] , v_\tau^*(\bullet) \in \mathbf{P}_\tau^o(\bullet) , \xi_\tau^*(\bullet) \in \mathbf{R}_\tau^o(\bullet)$$

A triplet of this kind will be further referred to as an ***admissible triplet***, i.e.

$$\omega_\tau^*(\bullet) \in \Omega_\tau(\bullet) \tag{4.1}$$

where

$$\Omega_\tau^o(\bullet) = \mathbf{Q}[\tau] \times \mathbf{P}_\tau^o(\bullet) \times \mathbf{R}_\tau^o(\bullet)$$

and as indicated above

$$\mathbf{P}_\tau^o(\bullet) = \{v_\tau^o(\bullet) : v(t) \in \mathbf{P}(t) , \tau \le t \le t_1\}$$

$$\mathbf{R}_\tau^o(\bullet) = \{\xi_\tau^o(\bullet) : \xi(t) \in \mathbf{R}(t) , \tau \le t \le t_1\}$$

Now obviously it will be possible to devise a related prolongation for the set-valued function $\mathbf{Q}^*[t]$ from $[t_0 , \tau]$ onto the interval $(\tau , t_1]$ in the form of a realization

$$\mathbf{Q}^*[t] = \mathbf{Q}(t , \tau , \mathbf{Q}(t , t_0 , Q^{(0)} \mid y_\tau^*(\bullet)) \mid y_\tau^{o\,*}(\bullet))$$

According to [7] and to the statements of § 1 of this paper, the multivalued map $\mathbf{Q}^*[\bullet]$ may be specified through the system

$$\dot{q} \in (A(t) - L(t)G(t))q + \mathbf{P}(t) + L(t)(y^* - \mathbf{R}(t))$$

$$\dot{q}^* = A(t)\ q^* + v^*(t)$$

$$y^* = G(t)\ q^* + \xi^*(t)\ ,$$

$$q^*(\tau) = q_\tau^*\ ,\ q(\tau) = q_\tau$$

or, in equivalent form, through the system

$$\dot{z}^* \in (A(t) - L(t)G(t)\ z^* + (\mathbf{P}(t) - v^*(t)) - L(t)(\mathbf{R}(t) - \xi^*(t)) \tag{4.2}$$

$$z^*(\tau) = q_\tau - q_\tau^*$$

where

$$z^*(t) = q(t) - q^*(t) \qquad\qquad \tau \leq t \leq t_1$$

Denote $Z_L^*(\bullet , \tau , Z^*[\tau])$ to be the set of all solutions to (4.2) that start from $Z^*[\tau]$ at instant τ.

What follows from [13,14] is

Lemma 4.1. The prolongation $\mathbf{Q}_\tau^{o\,*}[\bullet]$ generated by $\omega_\tau^{o\,*}(\bullet)$ may be given by the relation

$$\mathbf{Q}_\tau^{o\,*}[\bullet] = \bigcap_L \left[q^*(\bullet , \tau , q_\tau^*) + Z_L^*(\bullet , \tau , Q^*[\tau] - q_\tau^*)\right] \tag{4.3}$$

over all constant matrices $L \in \mathbf{R}^{m \times n}$.

It is not difficult to observe that the following relation is true

Lemma 4.2. The union of all possible cross sections $\mathbf{Q}^*\,[t_1]$ of the prolongation $\mathbf{Q}_\tau^{o\,*}[\bullet]$ of $\mathbf{Q}^*[\tau]$ (over all triplets $\omega_\tau^*(\bullet)$ that satisfy (4.1)), is a convex compact set - the attainability domain $Q(t_1 , \tau , \mathbf{Q}^*[\tau])$ at time t_1 for the inclusion (1.1), starting from $\{\tau , \mathbf{Q}^*[\tau]\}$. Namely

$$\bigcup \{\ \mathbf{Q}^*[t_1] \mid \omega_\tau^{o\,*}(\bullet) \in \Omega_\tau^o(\bullet)\} = Q(t_1 , \tau , \mathbf{Q}^*[\tau])\ ,$$

The schemes of the above allow to construct a solution procedure for the *basic problem*.

5. The Solution Scheme

Suppose that the position (the "state") of the overall system is given as

$$\{\tau , p , y_\tau(\bullet)\}$$

or in equivalent form as

$$\{\tau , p , \mathbf{Q}[\tau]\}$$

where

$$\mathbf{Q}[\tau] = \mathbf{Q}(\tau , t_0 , Q^{(0)} \mid y_\tau(\bullet))$$

A possible prolongation for $\mathbf{Q}[\tau]$ onto $(\tau , t_1]$ is the multivalued function $\mathbf{Q}_\tau^{o\,*}[\bullet]$ generated due to a possible "future" measurement $y_\tau^{o\,*}(\bullet)$ (which is uniquely defined by a triplet

$$\omega_\tau^{o\,*}(\bullet) = \{q^* , v_\tau^{o\,*}(\bullet) , \xi_\tau^{o\,*}(\bullet)\} , \omega_\tau^{o\,*}(\bullet) \in \Omega_\tau^{o\,*}(\bullet))$$

Returning to an inverse problem of the type described in § 2, (except that system (2.1) is changed to (4.1) and sets $\mathbf{N}$, $\mathbf{Y}(t)$ to $\mathbf{M}$ and $\mathbf{K}(t) \dot{-} \mathbf{Q}^*[\tau]$, respectively), we observe that the set

$$W(\tau , t_1\, M , \mathbf{Q}[\tau] \mid \omega_\tau^{o\,*}(\bullet)) = W(\tau , t_1\, \mathbf{M} , \bullet \mid \omega_\tau^{o\,*}(\bullet))$$

consists of states $\{\tau , p\}$ such that for each of these there exists and "open-loop" control $u(t)$ that steers $\{\tau , p\}$ into M under the constraints

$$u(t) \in \mathbf{V}(t) , \quad p(t) + \mathbf{Q}[t] \in \mathbf{K}(t)$$
$$\tau \le t \le t_1$$

In view of Lemma 2.1 we come to

Lemma 5.1. The set $W(\tau , t_1\, \mathbf{M} , \mathbf{Q}[\tau] \mid \omega_\tau^{o\,*}(\bullet))$ may be described as

$$W(\tau , t_1 , \mathbf{M} , \mathbf{Q}[\tau] \mid \omega_\tau^{o\,*}(\bullet)) =$$
$$= \bigcap \{ W_L(\tau , t_1 , M , \mathbf{Q}[\tau] \mid \omega_\tau^{o\,*}(\bullet)) \mid L_\tau(\bullet)\} \tag{5.1}$$

the intersection being taken over all continuous $(n \times n)$- matrix-valued functions $L(t)$ defined for $[\tau , t_1]$.

Here $W_L[\tau] = W(\tau , t_1 , \mathbf{M} , \mathbf{Q}[\tau] \mid \omega_\tau^{o\,*}(\bullet)) = W(\tau , t_1 , \mathbf{M} , \bullet \mid \omega_\tau^{o\,*}(\bullet))$ is the solution set to the system

$$\dot{w}_L \in (C(t) - L(t))\, w_L + L(K(t) \dot{-} \mathbf{Q}^*[\tau]) + \mathbf{V}(t) \tag{5.2}$$
$$w_L(t_1) \in \mathbf{M}$$

or to the funnel equation

$$\lim_{\sigma \longrightarrow o} \sigma^{-1} h^{+} \left(W[t - \sigma] - L\mathbf{Q}[t]\sigma \,, (E - \sigma(C(t) - L(t)) \; W[t] - L\mathbf{K}(t)\sigma - \mathbf{V}(t)\sigma)\right) = 0$$

$$W_L[t_1] = \mathbf{M}$$

The next step is to construct a set $W(\tau \,, t_1 \,, \mathbf{M} \,, \bullet)$ of such states $\{\tau \,, p\}$ that for every possible prolongation $\mathbf{Q}^*[t]$ (generated by $\omega_\tau^{o\,*}(\bullet)$) there exists an "open-loop" control $u(t)$ that steers $\{\tau \,, p\}$ into $\mathbf{M}$ under the constraints (5.1).

Lemma 5.2. The set $W(\tau \,, t_1 \,, \mathbf{M} \,, \bullet \,)$ *may be described as*

$$W(\tau \,, t_1 \,, \mathbf{M} \,, \bullet) = \bigcap \{ \; W(\tau \,, t_1 \,, M \,, \bullet \mid \omega_\tau^{o\,*}(\bullet)) \mid \omega_\tau^{o\,*} \in \Omega_\tau(\bullet)\}$$

over all admissible triplets $\omega_\tau^{o\,*}(\bullet) \in \Omega_\tau^o(\bullet)$

The graph of each of the multivalued maps $W_\tau^{o\,*}[\bullet]$ over the interval $[\tau \,, t_1]$ is closed, with convex cross-sections $W^*[t] = W(t \,, t_1 \,, M \,, \bullet \mid \omega_\tau^{o\,*}(\bullet))$, [7]. Therefore we come to

Lemma 5.3. The graph of the multivalued map $W_\tau[\bullet]$ *is a closed set with convex cross-sections* $\mathbf{W}[t] = W(t \,, t_1 \,, M \,, \bullet \,) \,, t \in [\tau \,, t_1]$.

With $W[\tau]$ given, the regular *extremal strategy* that follows the scheme of [1,3] is constructed through the relation

$$U(\tau \,, p \,, y_\tau(\bullet) = \begin{cases} \mathbf{V}(\tau) & \text{if } p \in \mathbf{W}[\tau] \\ \partial\rho(l \mid \mathbf{V}(\tau)) \;\;,\;\; l \in \partial \; d(p \,, \; W[\tau]) \,, & \text{if } p \,\overline{\in}\, \mathbf{W}[\tau] \end{cases} \tag{5.4}$$

where

$$d(p \,, \; W[\tau]) = \min\{|| \; p - w|| \mid w \in \mathbf{W}[\tau]\}$$

is the Euclidean distance from p to $W[\tau]$, and $\partial f(l)$ is the *subdifferential* of the function f at point l.

For the function $\psi(p) = d(p \,, \; W)$, the subdifferential

$$\partial \; \Psi(p) = \partial d(p \,, \; W)$$

consists of a single point $w^* = \arg\min \{||p - w \; || \; |w \in \; W[\tau]\}$,

The *regular extremal strategy* of (5.4) yields the solution to the basic problem under some *additional assumptions*.

Consider the support function

$$\rho(l \mid \; W(\tau \,, t_1 \,, \mathbf{M} \,, \bullet \mid \omega_\tau^{o\,*}(\bullet)))$$

and further on, the function

$$f(l \mid \tau \,, t_1 \,, M \,, \mathbf{Q}[\tau]) = f(l \mid \tau \,, t_1 \,, M \,, \bullet) =$$

$$= \inf \{\rho(l \mid \; W(\tau \,, t_1 \,, \mathbf{M} \,, \bullet \mid \omega_\tau^{o\,*}(\bullet))) \mid \omega_\tau^{o\,*}(\bullet) \in \Omega_\tau^o(\bullet)\}$$

Lemma 5.4. The function $f(l \mid \tau , t_1 , M , \bullet)$ is a closed positively homogeneous function.

Assumption 5.1. Whatever the realization $\mathbf{Q}[\tau]$, the following relation is true

$$f(l \mid \tau , t_1 , \mathbf{M} , \bullet) = f^{**}(l \mid \tau , t_1 , \mathbf{M} , \bullet) > -\infty \tag{5.5}$$

where $f^{**}(l \mid \tau , t_1 , M , \bullet)$ is the second conjugate to $f(l \mid \tau , t_1 , \mathbf{M} , \bullet)$ in the variable l.

The second conjugate ([15]) to a function $f(l)$ is defined as $(f^*)^*(l)$ where $f^*(p) = \sup\{(p , l) - f(l) \mid l \in \mathbf{R}^n\}$

In other words, Assumption 5.1 requires that $f(l \mid \tau , t_1 , \mathbf{M} , \bullet)$ would be convex and lower semi-continuous in l.

This yields

$$f(l \mid \tau , t_1 , \mathbf{M} , \bullet) = \rho(l \mid W(\tau , t_1 , \mathbf{M} , \bullet))$$

Hence, under Assumption 5.1, the support function $\rho(l \mid \mathbf{W}(\tau , t_1 , \mathbf{M} , \bullet))$ of the intersection of sets $\mathbf{W}(\tau , t_1 , \mathbf{M} , \bullet) \mid \omega_\tau^{o\,*}(\bullet))$ (over $\omega_\tau^{o\,*}(\bullet) \in \Omega_\tau^o(\bullet)$) should coincide with

$$\inf\{\rho(l \mid W(\tau , t_1 , M , \bullet \mid \omega_\tau^{o\,*}(\bullet)) \mid \omega_\tau^{o\,*}(\bullet) \in \Omega\}$$

This is a requirement which does not hold in the general case where the support function of the intersection of sets requires an *infimal convolution* of the respective supports rather than their infimum, [15].

Lemma 5.5. Under Assumption 5.1., the multivalued map $W_\tau^o[\bullet]$ has a closed graph with convex compact cross-sections $W[t] = W(t , t_1 , M , \bullet)$.

Lemma 5.6. Under Assumption 5.1., the strategy $\mathbf{U}(\tau , p , y_\tau(\bullet))$ of (5.4) is an admissible strategy.

Theorem 5.2. Suppose the vector $p^0 = p(t_0)$ and the set $Q(t_0) = Q^{(0)}$ are such that Assumption 5.1 is true and that

$$p^0 \in W(t_0\, t_1 , \mathbf{M} , Q^{(0)})$$

Then the respective strategy $U(t , p , y_t(\bullet))$ of (5.4) will ensure the restrictions (3.2), (3.3) whatever are the solutions to the inclusions (3.5)-(3.7).

The regular case described here does not cover all the possible situations that may arise in the basic problem. We will therefore give a short description of two other "extremal" cases for the solution.

6. The "Blunt" Solution

Consider the attainability domain $Q(t\ ,\ t_0\ ,\ Q^{(0)})$ for system (1.1) in the absence of any state constraints.

Assumption 6.1. The set $\mathbf{S}(t) = \mathbf{K}(t) \dot{-} \mathbf{Q}(t\ ,\ t_0\ ,\ Q^{(0)}) \neq \emptyset$ *for any* $t \in [t_0\ ,\ t_1]$.

Denote $W_b[t] = W_b(\tau\ ,\ t_1\ ,\ M)$ to be the solution of an inverse problem of the type given in § 2 - the set of all states $p_\tau = p(\tau)$ of system (3.1) such that for each of these there exists an open-loop control $u(t)$ $(u_\tau^o(\bullet) \in \mathbf{V}_\tau^o(\bullet))$ that ensures the inclusions

$$p(t_1\ ,\tau\ ,\ p_\tau) \in \mathbf{M} \tag{6.1}$$

$$p(t\ ,\tau\ ,\ p_\tau) \in Q(t\ ,\tau\ ,\mathbf{Q}(\tau\ ,t_0\ ,\ Q^{(0)}))\ ,\quad \tau \le t \le t_1$$

Denote the "blunt" strategy to be

$$U_2(t\ ,\ p) = \begin{cases} \mathbf{V}(t) & \text{if } p \in W_b(t\ ,\ t_1\ ,\ M) \\ \partial\rho(l \mid \mathbf{V}(t))\ ,\ l \in \partial\, d(p\ ,\ W_b[\tau]) & \text{if } p \bar{\in} W_b(t\ ,\ t_1\ ,\ M) \end{cases} \tag{6.2}$$

Lemma 6.1. The strategy $U_b(t\ ,\ p)$ *ensures the solution to the inclusion*

$$p \in C(t)\ p + \mathbf{U}_b(t\ ,\ p)\ ,\ t_0 \le t \le t_1 \tag{6.3}$$

for any initial state $p(t_0) = p^0$.

The solution is here understood in the sense of Caratheodory [9].

Theorem 6.1. Under Assumption 6.1 suppose $p(t_0) \in W(t_0\ ,\ t_1\ ,\ \mathbf{M})$. *Then the strategy* $\mathbf{U}_b(t\ ,\ p)$ *of (6.2) ensures that any solution* $p(t\ ,\ t_0\ ,\ p^0)$ *to the differential inclusion (6.3) would satisfy the restrictions (6.1).*

The "blunt" solution does not require any on-line measurements for the uncertain system (1.1). It implements an "open-loop" feedback solution under a given state constraint and it may work only if the sets $\mathbf{S}(t)$ are nonvoid, which is a rather strong restriction on the parameters of the problem.

7. The General Approach

The general approach leads to a complicated scheme that follows the constructions of [2], [3] and [7].

Suppose a set $\mathbf{Q}(\tau)$ is given and

$$\mathbf{Q}(\bullet\ ,\ t_0\ ,\ \mathbf{Q}[\tau]\ ,\ \omega_\tau^{o\,*}(\bullet))\ ,\ \omega_\tau^{o\,*}(\bullet) \in \Omega_\tau^o(\bullet)\ ,$$

are the possible realizations of the informational sets (due to possible "future" measurements).

The sequence of operations is as follows. Divide the interval $[\tau , t_1]$ into s subintervals

$$\tau = t^0 , t^1 ,..., t^s = t_1 ,$$

$$\max \mid t^i - t^{i-1} \mid = \epsilon_s$$

For the interval $(t^s , t_1]$ find the set

$$W_s(t^{s-1} , t_1 , \mathbf{M} , \mathbf{Q}[t^{s-1}] \mid \omega^{o*}_{t^s-1}(\bullet)) .$$

Take

$$\mathbf{W}_s(t^{s-1} , t_1 , \mathbf{M}) = \bigcap \{\bigcap W_s(t^{s-1} , t_1 , \mathbf{M} , \mathbf{Q}[t^{s-1}] \mid \omega^{o*}_{t^s-1}(\bullet)) \mid$$

$$\mid \omega^{o*}_{t^s-1}(\bullet)) \in \Omega^o_{t^s-1}(\bullet)\} \mid \mathbf{Q}[t^{s-1}] = \mathbf{Q}(t^{s-1} , t_0 , Q^{(0)} \mid y^*_\tau(\bullet)) : \omega^*_{t^s-1}(\bullet) \in \Omega_{t^s-1}(\bullet)\}$$

Repeat this procedure for $(t^{s-2} , t^{s-1}]$, taking $\mathbf{W}_s(t^{s-1} , t_1 , M)$ instead of M.

In a similar way continue to repeat this procedure for $(t^{s-3} , t^{s-2}]$ taking $\mathbf{W}_s(t^{s-2} , t^{s-1} , \mathbf{W}_s(t^{s-1} , t_1 , M)$ instead of M and so on, finally arriving at

$$\mathbf{W}_s(\tau , t_1 , \mathbf{M}) = \mathbf{W}_s(\tau , t^1 , \mathbf{W}_s \ (t^1 , t^2 , ...\mathbf{W}_s(t^{s-1} , t_1 , \mathbf{M}))...)$$

Under rather conventional conditions with $s \longrightarrow \infty , \epsilon_s \longrightarrow 0$, the set $\mathbf{W}_s(\tau , t_1 , \mathbf{M})$ will converge

$$\mathbf{W}_s(\tau , t_1 , \mathbf{M}) \longrightarrow \mathbf{W}(\tau , t_1 , \mathbf{M})$$

$$s \longrightarrow \infty , \epsilon_s \longrightarrow 0$$

in the Hausdorff metric, and the set-valued function $\mathbf{W} = \mathbf{W}(\tau , t_1 , \mathbf{M})$ may then serve as a basis for a strategy similar to $\mathbf{U}(t , p , y_t(\bullet))$. The detailed treatment of this situation will be the subject of another paper.

A final remark is that the numerical implementation of this scheme requires an appropriate *approximation theory for set-valued maps*. Therefore an approximative scheme that traces the basic solutions in terms of ellipsoidal valued functions seems to be a relevant subject for investigation.

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