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PREFACE

One of the problems that arises in the theory of evolution and control under uncertainty is to specify the set of all the solutions to a differential inclusion that also satisfy a preassigned restriction on the state space variables (the "viability" constraint).

The latter set of "viable" trajectories may be described by either a new differential inclusion whose right-hand side is formed with the aid of a contingent cone to the restriction map or by a variety of parametrized differential inclusions each of which has a relatively simple structure. The second approach is described here for a linear-convex differential inclusion with a convex valued restriction on the state space variables.

CONTENTS

ON VIABLE SOLUTIONS FOR UNCERTAIN SYSTEMS

A.B. Kurzhanski, T.F. Filippova

This paper deals with the description of the set of all those solutions of a linear differential inclusion that emerge from given set X^0 and satisfy a preassigned restriction on the state space variables (the "viability" constraint). This problem leads to the analytical description of the evolution of the attainability domains for the given inclusion under the preassigned "viability" constraint. The solution is then reduced to the treatment of a parametrized variety of new differential inclusions without any state space constraints. These inclusions depend upon a functional parameter. The intersection of the attainability domains for the new inclusions over the variety of all the functional parameters yield the precise solution of the primary problem. For the specific problem of this paper the technique given here.therefore allows to avoid the introduction of tangent cones or other related analytical constructions. It also allows to present the overall solution as an intersection of "parallel" solutions over a variety of ordinary linear differential inclusions without any state constraints.

A similar technique is given for the description of "viable" domains — the sets of all starting points from which there emerges at least one viable solution that reaches a preassigned set M. The available results are useful for the solution of problems of control and observation for uncertain systems [1,2].

1. The Statement of the Problem. The Basic Assumptions.

Consider the following differential inclusion

$$\frac{dx}{dt} \in A(t)x + \mathrm{P}(t), \quad t_0 \le t \le \vartheta, \tag{1.1}$$

where $x \in \mathbf{R}^n$, $A(t)$ is a continuous map from $\mathrm{T} = [t_0, \vartheta]$ into the set $\mathbf{R}^{n \times n}$ of $(n \times n)$-matrices, $\mathrm{P}(t)$ is a continuous multivalued map from T into the set $conv\ \mathbf{R}^n$ of convex compact subsets of $\mathbf{R}^n$, [3].

Assuming set $X^0 \in conv\ \mathbf{R}^n$ to be given, denote $X(\cdot, t_0, X_0)$ to be the "bundle" of all Caratheodory - type solutions $x(\cdot, t_0, x^0)$ to (1.1) that start at

$$x(t_0) = x^0 \in X^0 \tag{1.2}$$

and are defined for $t \in \mathrm{T}$ [4]. The cross-section at instant "t" of $X(\cdot, t_0, X^0)$ will be denoted as $X(t, t_0, X^0)$.

Denote $co\ \mathbf{R}^n$ to be the set of closed convex subsets of $\mathbb{R}^n$, $Y(\cdot)$ to be a continuous multivalued map from T into $co\ \mathbb{R}^n$, [5,6], $X^0 \subseteq Y(t_0)$.

Definition 1.1. A trajectory $x[t] = x(t, t_0, x^0)$, $t \in \mathrm{T}$, of equation (1.1) will be said to be viable on $\mathrm{T}_\tau = [t_0, \tau]$, $\tau \le \vartheta$, if

$$x[t] \in Y(t), \text{ for } all\ t \in \mathrm{T}_\tau, \tag{1.3}$$

We further assume that there exists at least one solution $x[t]$ of (1.1) that satisfies (1.2) and is viable on T_τ. The conditions for the existence of those solutions may be given in terms of generalized duality concepts [2,7].

The subset of $X(\cdot, t_0, X^0)$ that consists of all solutions viable on $\mathbf{T}_\tau$ will be denoted as $X_\tau(\cdot, t_0, X^0)$ and its cross-section at instant $s \in \mathbf{T}_\tau$ as $X_\tau(s, t_0, X^0)$. Our further aim will be to find an analytical description for the evolution of sets $\mathrm{X}[\tau] = \mathbf{X}(\tau, t_0, X^0) = X_\tau(\tau, t_0, X^0)$ which are actually the attainability domains of inclusion (1.1) under the phase constraint (1.3). It is known that $\mathrm{X}[\tau] \in conv\ \mathbb{R}^n$ [2]. (According to our assumption we further have $\mathrm{X}[\tau] \ne \phi$ for $all\ \tau \in \mathbf{T}$).

It is not difficult to observe that sets $\mathrm{X}(t, t_0, X^0)$ satisfy a semigroup property:

$$X(\tau, t_0, X^0) = \mathbf{X}(\tau, s, X(s, t_0, X^0)) .$$

They therefore define a generalized dynamic system. The description of this dynamic system will be given through a variety of new differential inclusions constructed from (1.1), (1.3). (See [8]).

2. The Set $X[\tau]$.

Introducing some notations let us denote the support function of set X as

$$\rho(l \mid X) = sup\{(l,x) \mid x \in X\} , \quad l \in \mathbf{R}^n .$$

(here (l,x) stands for the inner product $l'x$ with the prime as the transpose).

Also denote $C^n(\mathrm{T})$ ($C_k^n(\mathrm{T})$) to be the set of all n-vector-valued continuous functions defined on T (respectively the set of k times continuously differentiable functions with values in $\mathbf{R}^n$, defined on T). Let $\mathrm{M}^n(\mathrm{T})$ stand for the set of all n-vector-valued polynomials of any finite degree, defined on T. Obviously $g(\cdot) \in \mathbf{M}^n(\mathrm{T}_\tau)$ if

$$g(s) = \sum_{i=1}^{k} l^{(i)} s^i , \quad s \in \mathrm{T}_\tau , \quad l^{(i)} \in \mathbf{R}^n$$

and

$$\mathrm{M}^n(\mathrm{T}) \subseteq C_\infty^n(\mathrm{T}) .$$

Applying some duality concepts of infinite dimensional convex analysis [7] as given in the form presented in [2] we come to the following relations. For any $l \in \mathbf{R}^n$, $\lambda(\cdot) \in C^n(\mathrm{T})$ denote

$$\Phi_\tau(l,\lambda(\cdot)) = \rho(l'S(t_0,\tau) - \int_{t_0}^{\tau} \lambda'(\xi)\mathrm{S}(t_0,\xi)d\xi \mid X^0) + \tag{2.1}$$
$$\int_{t_0}^{\tau} \rho(l'\mathbf{S}(\xi,\tau) - \int_{\xi}^{\tau} \lambda'(s)\mathrm{S}(\xi,s)ds \mid \mathbf{P}(\xi))\, d\xi +$$
$$\int_{t_0}^{\tau} \rho(\lambda(\xi) \mid Y(\xi))d\xi$$

Here, in the first variable the function $\mathbf{S}(t,\tau)$ is the matrix solution for the equation

$$\dot{s} = -sA(t), \quad S(\tau,\tau) = E, \quad t \le \tau,$$

the second and third members of the sum (2.1) are Lebesgue-type integrals of multivalued maps $\mathbf{P}(\xi)$, $Y(\xi)$ respectively (see, for example, [4-6]).

In [2], §6, it was proved that

$$\max \{(l,x) \mid x \in \mathbf{X}[\tau]\} = \rho(l \mid \mathbf{X}[\tau]) = \inf \{\Phi_\tau(l,\lambda(\cdot)) \mid \lambda(\cdot) \in C^n[\mathbf{T}_\tau]\}. \tag{2.2}$$

A slight modification of the respective proof shows that the class of functions $C^n(\mathbf{T}_\tau)$ in the last formula may be substituted by either $C^n_\infty(\mathbf{T}_\tau)$ or even $\mathbf{M}^n(\mathbf{T}_\tau)$. Hence

$$\begin{aligned} &\inf \{\Phi_\tau(l,\lambda(\cdot)) \mid \lambda(\cdot) \in C^n(\mathbf{T}_\tau)\} = \\ &\inf \{\Phi_\tau(l,\lambda(\cdot)) \mid \lambda(\cdot) \in C^n_\infty(\mathbf{T}_\tau)\} = \\ &\inf \{\Phi_\tau(l,\lambda(\cdot)) \mid \lambda(\cdot) \in \mathbf{M}^n(\mathbf{T}_\tau)\} \end{aligned} \tag{2.3}$$

From relations (2.2) it is possible to derive the following assertion

Lemma 2.1. The following equality is true

$$\begin{aligned} \mathbf{X}[\tau] &= \bigcap \{R(\tau,M(\cdot)) \mid M(\cdot) \in C^{n\times n}(\mathbf{T}_\tau)\} = \\ &= \bigcap \{R(\tau,M(\cdot)) \mid M(\cdot) \in C^{n\times n}_\infty(\mathbf{T}_\tau)\} = \\ &= \bigcap \{R(\tau,M(\cdot)) \mid M(\cdot) \in \mathbf{M}^{n\times n}(\mathbf{T}_\tau)\}. \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} R(\tau,M(\cdot)) = {} & (\mathbf{S}(t_0,\tau) - \int_{t_0}^{\tau} M(\xi)\mathbf{S}(t_0,\xi)d\xi)X^0 + \\ & + \int_{t_0}^{\tau} (\mathbf{S}(\tau,\xi) - \int_{\xi}^{\tau} M(s)S(\xi, s)\,ds)\,\mathbf{P}(\xi)d\xi \\ & + \int_{t_0}^{\tau} M(s)Y(s)ds \end{aligned}$$

and $C^{n\times n}_k(\mathbf{T})$, $(0 \le k \le \infty)$, $\mathbf{M}^{n\times n}(\mathbf{T})$ stand for the respective spaces of $(n\times n)$-matrix-valued functions defined on $\mathbf{T}$.

The proof of Lemma 2.1 follows immediately from (2.2), (2.3) after a substitution $\lambda'(\cdot) = l'M(\cdot)$ for $l \ne 0$. The infimum over $\lambda(\cdot)$ in (2.2) is then substituted by an infimum over $M(\cdot)$. Hence for every $l \ne 0$ we have

$$\rho(l \mid \mathbf{X}[\tau]) \leq \Phi_\tau(l, M'(\cdot)l) \tag{2.5}$$

for any $M(\cdot) \in C^{n \times n}(\mathrm{T}_\tau)$ (or $C_\infty^{n \times n}(\mathrm{T}_\tau)$ or $\mathbf{M}^{n \times n}(\mathrm{T}_\tau)$). From (2.1) - (2.5) it now follows that

$$\mathrm{X}[\tau] \subseteq R(\tau, M(\cdot))$$

for any $M(\cdot)$.

Hence

$$\mathrm{X}[\tau] \subseteq \bigcap \{R(\tau, M(\cdot)) \mid M(\cdot) \in C^{n \times n}(\mathrm{T}_\tau)\} \tag{2.6}$$

(or over $C_\infty^{n \times n}(\mathrm{T}_\tau)$ or $\mathbf{M}^{n \times n}(\mathrm{T}_\tau)$).

Equalities (2.4) now follow from (2.6) and (2.2), (2.3).

3. A Generalized "Lagrangian" Formulation

The assertions of the above yield the "standard" duality formulations for calculating $\gamma_0(l) = \rho(l \mid X[\tau])$, (see [2, 8, 9]).

Denoting

$$P(\cdot) = \{p(\cdot) : p(t) \in P(t), \ t \in \mathrm{T}_\tau\}$$

we come to the following "standard"

Primary Problem

$$\text{maximize}(l, x[\tau]) \tag{3.1}$$

over all

$$u(\cdot) \in P(\cdot), \ x^0 \in X_0 \tag{3.2}$$

where $x[t]$ is the solution to the equation

$$\dot{x}[t] = A(t)x[t] + u(t), \ x[t_0] = x^0 \tag{3.3}$$

In other words

$$\gamma_0(l) = \max\{\Psi(x^0, u(\cdot) \mid x^0 \in \mathbf{R}^n, \ u(\cdot) \in \mathbf{L}_2^n(T_\tau)\} \tag{3.4}$$

under restriction (3.2) where

$$\Psi(x^0, u(\cdot)) = (l, x[\tau]) + \delta(x^0 \mid X^0) +$$
$$+ \int_{t_0}^{\tau} (\delta(x[t] \mid Y(t)) + \delta(u(t) \mid P(t)))\, dt$$

Here

$$\delta(x \mid Y) = \begin{cases} 0 & \text{if } x \in Y \\ +\infty & \text{if } x \overline{\in} Y \end{cases}$$

The primary problem generates a corresponding "standard"

Dual Problem:

Determine

$$\gamma^0(l) = \inf \{\Phi_\tau(l, \lambda(\cdot)) \mid \lambda(\cdot) \in C^n(T_\tau)\} \tag{3.5}$$

along the solutions $s[t]$ to the equation

$$\dot{s}[t] = -s[t]A(t) + \lambda(t), \; s[\tau] = l \tag{3.6}$$

where $\Phi_\tau(l, \lambda(\cdot))$ may be rewritten as

$$\Phi_\tau(l, \lambda(\cdot)) = \rho(s[t_0] \mid X^0) +$$

$$+ \int_{t_0}^{\tau} (\rho(s[t] \mid P(t)) + \rho(\lambda(t) \mid Y(t)) \, dt$$

Relations (2.2), (2.3) indicate that $\gamma_0(l) = \gamma^0(l)$ and that $\lambda(\cdot)$ in (3.5) may be selected from $C^n_\infty(T_\tau)$ or even from $M^n(T_\tau)$.

A "standard" Lagrangian formulation is also possible here.

Lemma 3.1 The value $\gamma_0(l) = \gamma(l)$ may be achieved as the solution to the problem

$$\gamma(l) = \inf_{\lambda(\cdot)} \max_{u(\cdot), x^0} L(\lambda(\cdot), u(\cdot), x^0) \tag{3.7}$$

where

$$L(\lambda(\cdot), u(\cdot), x^0) = (s[t_0], x^0) + \int_{t_0}^{\tau} ((s[t], u(t)) + \tag{3.8}$$

$$\rho(\lambda(t) \mid Y(t))) \, dt$$

and

$$\lambda(\cdot) \in C^n(T_\tau), \; u(\cdot) \in P(\cdot), \; x^0 \in X^0 .$$

The passage from (2.2), (2.3) to (2.4) yields another form of presenting $X[\tau]$. Namely, denote $S[t]$ to be the solution to the matrix differential equation

$$\dot{S}[t] = -S[t]A(t) + M(t), \; S[\tau] = E$$

Also denote

$$\wedge (x^0 , u(\cdot) , M(\cdot)) =$$

$$S[t_0]x^0 + \int_{t_0}^{\tau} (S[t]u(t) + M(t) Y(t))dt$$

Obviously

$$R(\tau M(\cdot)) = \cup \{\wedge(x^0 , u(\cdot) , M(\cdot)) \mid x^0 \in X^0 , u(\cdot) \in P(\cdot)\} =$$

$$= S[t_0]X^0 + \int_{t_0}^{\tau} (S[t] P(t) + M(t) Y(t))dt$$

Lemma 2.1 may now be reformulated as

Lemma 3.2. The set $X[\tau]$ may be determined as

$$X[\tau] = \bigcap_{M(\cdot)} \bigcup_{x^0 , u(\cdot)} \wedge(x^0 , u(\cdot) , M(\cdot))$$

over all

$$M(\cdot) \in C^{n \times n} (\mathrm{T}_\tau) , x^0 \in X^0 , u(\cdot) \in \mathrm{P} .$$

This result may be treated as a generalization of the standard Lagrangian formulation. However here one deals with set $X[\tau]$ as a whole rather than with its projections $\rho(l \mid X[\tau])$ on the elements $l \in \mathbb{R}^n$. The results of the above indicate that the description of set $X[\tau]$ may be "decoupled" into the specification of sets $R(\tau , M(\cdot))$, the variety of which describes the generalized dynamic system $X(t , t_0 X^0)$.

However it should be clear that the mapping $R(\tau , M(\cdot))$ may not always be an adequate element for the decoupling procedure, especially for the description of the evolution of $X(t , t_0 , X^0)$ in t. The reasons for this are the following.

Assuming function $M(\cdot)$ to be fixed, redenote $R(\tau , M(\cdot))$ as $\mathbb{R}_M(\tau , t_0 , X^0)$. Then, in general, for any fixed M, we have

$$\mathbf{R}_M(\tau , t_0 , X^0) \neq \mathbb{R}_M(\tau , s , \mathbf{R}_M(s , t_0 , X^0)) .$$

Therefore the map $\mathbf{R}_M(\tau , t_0 , X^0)$ does not generate a semigroup of transformations that may define a generalized dynamic system. The necessary properties may be however achieved for an alternative variety of mappings, each of the elements of which will possess both the property of type (2.4) and the "semigroup" property, [10].

4. An Alternative Presentation of X $[\tau]$

Denote $C_{*}^{n\times n}(T_\tau)$ to be the subclass of $C^{n\times n}$ (T) that consists of all continuous matrix functions $M(\cdot)$ that satisfy the condition;

Assumption 4.1 For any $\zeta \in T_\tau$ we have

$$\det\left(S(\zeta , \tau) - \int_{\zeta}^{\tau} M(s)\, S(\zeta , s)\, ds\right) \neq 0$$

In other words, if $K[t]$ is the solution of the equation

$$\dot{K}(t) = -K(t)\, A(t) + M(t) ,\ K(\tau) = E ,\ (t_0 \leq t \leq \tau) \tag{4.1}$$

then $M(t)$ must be such that $\det K[t] \neq 0$ for all $t \in [t_0 , \tau]$.

We will further denote $K[t] = K(t , \tau , M(\cdot))$ for a given function $M(\cdot)$ in (3.1).

Consider the equation

$$\dot{Z} = (A(t) - L(t))\, Z , \qquad t_0 \leq t \leq \tau \tag{4.2}$$

whose matrix solution $Z[t]$ $(Z[\tau] = E)$ will be also denoted as $Z[t] = Z(t , \tau ; L(\cdot))$ $(Z'(t , \tau , \{0\}) \equiv S(\tau , t))$

Under Assumption 2.1 there exists a function $L(\cdot) \in C^{n\times n}(T_\tau)$ such that

$$K[t] \equiv Z(t , \tau , L(\cdot)) ,\ \forall t \in T_\tau , \tag{4.3}$$

Indeed, if for $t \in T_\tau$ we select $L(t)$ according to the equation

$$\begin{aligned} L(t) &= A(t) - K^{-1}(t)\, \dot{K}(t) = \\ & A(t) - K^{-1}(t)\, (-K(t)\, A(t) + M(t)) = \\ & -K^{-1}(t)\, M(t) + 2A(t) \end{aligned} \tag{4.4}$$

then, obviously, equation (4.3) will be satisfied. From (2.4), (4.3), (4.4) it now follows $(M(\cdot) \in C_{*}^{n\times n}(T_\tau))$

$$R(\tau , M(\cdot)) = Z(\tau , t_0 ; L(\cdot))\, X^0 + \tag{4.5}$$

$$\int_{t_0}^{\tau} Z(\tau , t ; L(\cdot)) (P(t) + L(t) Y(t)) dt$$

However it is not difficult to observe that the right-hand part of (4.5) is $X_{L(\cdot)}(\tau , t_0 , X^0) = X [\tau \mid L(\cdot)]$ which is the cross-section at instant τ of the set $X_{L(\cdot)} (\cdot , t_0 X^0) = X [\cdot \mid L(\cdot)]$ of all solutions to the differential inclusion

$$\dot{x}(t) \in (A(t) - L(t))x + \mathrm{P} + L(t) Y(t) \qquad (4.6)$$
$$x(t_0) \in X^0 , t \in \mathrm{T}_\tau ,$$

Since the class of all functions $L(\cdot) \in C^{n \times n} (\mathrm{T}_\tau)$ generates a subclass of functions $M(\cdot) \in C^{n \times n} (\mathrm{T}_\tau)$ we now come to the following assertion in view of (2.3), (4.5), (4.6).

Lemma 4.1 The following inclusion is true

$$\mathrm{X}[\tau] \subseteq \cap \{X[\tau \mid L(\cdot)] \mid L(\cdot) \in C^{n \times n} (\mathrm{T}_\tau)\} \qquad (4.7)$$

Therefore $\mathrm{X}[\tau]$ is contained in the attainability domains at instant τ for the inclusion (4.6), whatever is the function $L(t)$.

The objective is now to prove that (4.7) turns to be an equality. We will therefore pursue the proof that an inclusion opposite to (4.7) is to be true.

5. The Exact Formula for $\mathrm{X}[\tau]$.

In order to prove the equality in (4.7) we shall start by some preliminary results.

Lemma 5.1 Consider the system

$$\dot{x} \in \mathrm{P}^*(t) , (\mathrm{P}^*(t) = S(t , \tau) \mathrm{P}(t)) \qquad (5.1)$$

$$x(t_0) = x^0 \in X^{0*} \quad (X^0_* = S(t_0 , \tau)X^0) \qquad (5.2)$$

$$x(t) \in Y^*(t) , t \in \mathrm{T}_\tau , \qquad (5.3)$$
$$(Y^*(t) \equiv S(t , \tau) Y(t)) ,$$

Denote the set of its solutions viable on T_τ with respect to $Y^*(t)$ as $X^*_\tau (\cdot , t_0 , X^0_*)$ and the cross-section of the latter at instant τ as

$$X_\tau^*(\tau, t_0\, X_*^0) = \mathbf{X}^*(\tau, t_0, X_*^0) = \mathrm{X}^*[\tau]$$

Then $X[\tau] = X^*[\tau]$.

The proof of this Lemma follows from definition 1.1 and from the properties of linear systems (1.1), (5.1).

Assume $x^*(\cdot)$ to be a viable trajectory of (5.1) due to constraints (5.2), (5.3), $t \in \mathrm{T}_\tau$. (The existence of at least one viable trajectory $x^*(\cdot)$ was presumed earlier.)

Definition 5.1 Denote $\mathrm{X}^{**}[\tau] = \mathbf{X}^{**}(\tau, t_0, X_{**}^0) = X_\tau^{**}(\tau, t_0, X_{**}^0)$ to be the cross-section at instant τ of the set $X^{**}(\cdot, t_0, \mathrm{X}_{**}^0)$ of solutions of system

$$\dot{x} \in \mathbf{P}^{**}(t)\ (\mathbf{P}^{**}(t) = (\mathbf{P}^* - \dot{x}^*(t)) \tag{5.5}$$

$$x(t_0) \in X_{**}^0 (X_{**}^0 = X_*^0 - x^*(t_0)),$$

$$x(t) \in Y^{**}(t)\ (Y^{**}(t) = Y^*(t) - x^*(t)), \quad t \in \mathrm{T}_\tau,$$

Lemma 5.1 *The following equality is true*

$$\mathrm{X}[\tau] = \mathbf{X}^{**}[\tau]$$

The proof follows from the definition of viable trajectories. Note that sets $\mathrm{P}^{**}(t)$, X_{**}^0, $Y^{**}(t)$ - all contain the origin as an interior point. Their support functions are therefore all nonnegative.

The principal result of this paragraph is given by the proposition:

Theorem 5.1. The following equality is true

$$\mathrm{X}[\tau] = \bigcap\{X[\tau \mid L(\cdot)] \mid L(\cdot) \in C^{n \times n}(\mathrm{T}_\tau)\}$$

Before passing to the proof of this theorem, denote $X^*[\tau \mid L(\cdot)] = X_{L(\cdot)}^*(\tau, t_0, X^0)$ to be the cross-section at instant τ of the set $X_{L(\cdot)}^*(\cdot, t_0, X^0)$ of the solutions to the inclusion.

$$\dot{x} \in -L(t)x + \mathbf{P}^*(t) + L(t)Y^*(t) \tag{5.6}$$

$$x^0 = x(t_0) \in X_*^0, \ t \in \mathrm{T}_\tau \tag{5.7}$$

Hence for any matrix function $L(\cdot) \in C^{n \times n}(\mathrm{T}_\tau)$ we have

$$X^*[\tau \mid L(\cdot)] = X[\tau \mid S(\tau, \cdot) L(\cdot) S(\cdot, \tau)]$$

Therefore it suffices to prove the following equality

$$X^*[\tau] = \cap\{X^*[\tau \mid L(\cdot)] \mid L(\cdot) \in C_\infty^{n \times n}(\mathrm{T}_\tau)\} \tag{5.8}$$

In other words, theorem 5.1 will be already true if it is proved for $A(t) \equiv 0$ and for arbitrary X^0, $\mathbf{P}(t)$, $Y(t)$ from the respective classes of sets and set-valued maps introduced in § 1. We will therefore follow the proof of equality (5.8) omitting the stars in the notations for X^*, X_*^0, P^*, Y^*.

According to (2.2), (2.1) we now have $(A(t) \equiv 0)$

$$\rho(l \mid X[\tau]) = \tag{5.9}$$
$$= \inf\{\Phi(l, \lambda(\cdot)) \mid \lambda(\cdot) \in C^n(\mathrm{T}_\tau)\} = \inf\{\Phi(l, \lambda(\cdot)) \mid \lambda(\cdot) \in \mathbf{M}^n(\mathrm{T}_\tau)\}$$

where

$$\Phi(l, \lambda(\cdot)) = \rho(l - \int_{t_0}^{\tau} \lambda(s)\, ds \mid X^0) + \tag{5.10}$$
$$+ \int_{t_0}^{\tau} \rho(l - \int_{s}^{\tau} \lambda(\xi)\, d\xi \mid \mathbf{P}(s))ds + \int_{t_0}^{\tau} \rho(\lambda(s) \mid Y(s))ds$$

Denoting

$$g(s) = l - \int_{s}^{\tau} \lambda(\xi)\, d\xi, \quad s \in \mathbf{T}_\tau,$$

we may substitute (5.9), (5.10) for

$$\rho(l \mid \mathbf{X}[\tau]) = \inf\{\Psi(g(\cdot)) \mid g(\cdot) \in C_\infty^n(T_\tau), g(\tau) = l\} = \tag{5.11}$$
$$= \inf\{\Psi(g(\cdot)) \mid g(\cdot) \in \mathbf{M}^n(\mathrm{T}_\tau), g(\tau) = l\}$$

where

$$\Psi(g(\cdot)) = \rho(g(t_0) \mid X^0) + \int_{t_0}^{\tau} \rho(g(s) \mid \mathrm{P}(s))\, ds +$$
$$+ \int_{t_0}^{\tau} \rho(\dot{g}(s) \mid Y(s))\, ds \tag{5.12}$$

Let us further assume that the vector $l \in \mathbf{R}$ in (5.9), (5.10) and (5.11), (5.12) is such that its coordinates $l_i \neq 0$ for all $i = 1, \ldots, n$ Let us demonstrate that if we substitute the class of functions $g(\cdot)$ that appears in (5.11) for a "narrower" class $\mathbf{M}_l^{n\times n}(\mathrm{T}_\tau)$ then the value of the infimum in (5.11) will not change. The class $\mathbf{M}_l^{n\times n}(\mathrm{T}_\tau)$ which we will consider consists of all functions $g(t)$ of the form

$$g'(t) = l' M(t)\ ,\ M(\tau) = E\ ,$$
$$M(\cdot) \in C_\infty^{n\times n}(\mathrm{T}_\tau) \quad (\text{or } M(\cdot) \in \mathrm{M}^{n\times n}(\mathrm{T}_\tau))$$

and

$$\det M(t) \neq 0\ ,\ \forall t \in [t_0\ ,\ \tau]$$

Hence the following lemma is true.

Lemma 5.2 The support function $\rho(l \mid \mathbf{X}[\tau])$ satisfies the condition

$$\rho(l \mid \mathbf{X}[\tau]) = = inf\,\{\Psi(g(\cdot)) \mid g(\cdot) \in \mathrm{M}_l^{n\times n}(\mathbf{T}_\tau)\}$$

For the proof of this property we will distinguish the cases of n being an even number and n being odd.

Suppose $n = 2$. First of all, note that for calculating the infimum in (5.11) it suffices to restrict ourselves to the class of such functions $g(\cdot) = (g_1(\cdot)\ ,\ g_2(\cdot))$ that $g(\cdot) \in \mathrm{M}^2(\mathrm{T}_\tau)\ ,\ g(\tau) = l\ ,\quad g'(t)\,g(t) \neq 0$ for any $t \in \mathrm{T}_\tau$. Indeed, for any $g(\cdot) \in \mathrm{M}^2(\mathrm{T}_\tau)$ $(g(\tau) = l)$ it is possible to construct a sequence of functions $g^{(\varepsilon)}(\cdot) \in \mathbf{M}^2(\mathrm{T}_\tau)\ ,\ g^{(\varepsilon)}(\tau) = l,\ \ \varepsilon \to +0$, for which

$$\lim_{\varepsilon \to +0} \Psi(g^{(\varepsilon)}(\cdot)) = \Psi(g(\cdot)) \tag{5.13}$$

and

$$g^{(\varepsilon)\prime}(t)\,g^{(\varepsilon)}(t) > 0\ ,\ \forall t \in \mathrm{T}_\tau\,.$$

For example, assume

$$g^{(\varepsilon)}(\cdot) = (g_1(\cdot)\ ,\ g_2^{(\varepsilon)}(\cdot)\,)$$

where

$$g_2^{(\varepsilon)}(t) = l_2\,g_2(t+\varepsilon)\ /\ g_2(\tau+\varepsilon)\ ,\quad t \in \mathrm{T}_\tau\,.$$

Since it is assumed that $g(\tau) = l\ ,\quad l_1 \neq 0\ ,\quad l_2 \neq 0$, the function $g_2^{(\varepsilon)}(\tau)$ is well-defined for minor values of ε (i.e. $g^{(\varepsilon)}(\cdot) \in \mathrm{M}^2(\mathrm{T}_\tau)\ ,\ g^{(\varepsilon)}(\tau) = l)$.

Since the number of nulls of the polynomials $g_1(\cdot)$, $g_2(\cdot)$ is finite, it is possible to select the "shift" $\varepsilon = \varepsilon^0$ in $g_2(t + \varepsilon)$ so that the nulls of $g_1(t)$ and $g_2(t + \varepsilon)$ will not coincide for all $\varepsilon \in (0 , \varepsilon^0]$. Now for each $t \in T_\tau$, $g^{(\varepsilon)}(t) \to g(t)$ and $\dot g^{(\varepsilon)}(t) \to \dot g(t)$ with $\varepsilon \to +0$. The sequence $g^{(\varepsilon)}(t)$, $\dot g^{(\varepsilon)}(t)$ is equibounded in t for $\varepsilon \in (0 , \varepsilon^0]$. Therefore (5.12) is true.

It is now possible to demonstrate that any function $g(\cdot) \in M^2(T_\tau)$, with $g(\tau) = l$, $l' l \neq 0$, $g'(t)\, g(t) \neq 0$, $\forall t \in T_\tau$, may be presented in the form $g(\cdot) = l'\, M(\cdot)$ where $\det M(t) \neq 0$, $\forall t \in T_\tau$, $M(\tau) = E$.

It may be verified directly that with l given

$$M(t) = \begin{bmatrix} g_2(t)\, l_2^{-1} & , \ (g_2(t) - l_2\, l_1^{-1}\, g_1(t)) l_1^{-1} \\ (g_1(t) - l_1\, l_2^{-1}\, g_2(t)) l_2^{-1} \ , & g_1(t)\, l_1^{-1} \end{bmatrix} \tag{5.14}$$

satisfies these conditions, namely

$$\det M(t) = g_1^2(t)\, l_1^{-2} + g_2^2(t)\, l_2^{-2} \neq 0 .$$

Let us now assume that the dimension of $\mathbf{R}^n$ is even: $n = 2k$, $k \geq 2$. Then following the scheme for $n = 2$, it is possible to verify that it suffices to calculate the infimum in (5.11), (5.12) over such functions $g(\cdot) = (g_1(\cdot), \ldots, g(2k))$, $g(\tau) = l$, $l' l \neq 0$, that $g_{2i-1}^2(t) + g_{2i}^2(t) > 0$, $\forall t \in T_\tau$, $\forall i \in [1 , k]$.

Any function $g(\cdot)$ of the given type may be presented in the form $g(t) = l'\, M(t)$ where the $(2k \times 2k)$ - dimensional matrix $M(t)$ is block-diagonal:

$$M(t) = \begin{bmatrix} M_1(t) & & 0 \\ & \ddots & \\ 0 & & M_k(t) \end{bmatrix} \tag{5.15}$$

and each of the matrices $M_i(t)$, $i = 1, \ldots, k$, is (2×2) - dimensional and may be calculated due to formula (5.14) where in the place of $g_1(\cdot)$, $g_2(\cdot)$ we should substitute $(g_{2i-1}(\cdot) , g_{2i}(\cdot))$, $(i = 1, \ldots, k)$. The function $M(\cdot)$ belongs to the class $\mathbf{M}^{n \times n}(T_\tau)$, $M(\tau) = E$ and for any $t \in T_\tau$ we have

$$\det M(t) = \bigcap_{i=1}^{k} \det M_i(t) > 0 \tag{5.16}$$

Assume now that n is odd: $n = 2k + 1$. Then again we may calculate the infimum in (5.11), (5.12) over the class of functions.

$$g(\cdot) = (g_1(\cdot), \ldots, g_{2k}(\cdot), g_{2k+1}(\cdot)) \in \mathbf{M}^n(T_\tau), \; g(\tau) = l$$

such that

$$g_{2i-1}^2(t) + g_{2i}^2(t) > 0, \; \forall t \in [t_0 \; \tau]; \; i = 1, \ldots, k,$$

Each of such functions may be presented in the form $g(t) = l' M(t)$ where

$$M(t) = \begin{bmatrix} M_1(t), & 0, \ldots & 0, & 0, & m(t) \\ 0, & M_2(t), \ldots & 0, & 0, & 0, \\ \cdots & \cdots & & \cdots & \\ 0, & \cdots & 0, & M_k(t), & 0, \\ 0, & \cdots & 0, & 0, & 1 \end{bmatrix}$$

$$m(t) = (g_{2k+1}(t) - l_{2k+1}) / l_1 .$$

Here $M_i(t)$ is determined similarly to (5.14) where $g_1(\cdot)$, $g_2(\cdot)$ are to be substituted by g_{2i-1}, g_{2i} $(i = 1, \ldots, k)$. Obviously

$$M(\tau) = E, \; M(\cdot) \in \mathbf{M}^{n \times n}(T_\tau)$$

and

$$\det M(t) = \prod_{i=1}^{k} \det M_i(t) > 0$$

for all $t \in T_\tau$.

In order to finalize the proof of lemma 5.2 we have to consider the case when $n = 1$. For $n = 1$ the class $\mathbf{M}_l^{1\times 1}(T_\tau)$ may be substituted by all positive functions $m(\cdot) \in C_\infty^1(T_\tau)$. However, due to (2.3) we will be able to confine ourselves to the case when $m(\cdot) \in C^1(T_\tau)$.

As before, let $C_*^1(T_\tau)$ stand for the set of such functions $m(t)$, that $m(\tau) = 1$; $m(t) > 0$; $\forall t \in T_\tau$. We also assume that

$$0 \in X_0 \cap Y(s) \cap P(0) \tag{5.17}$$

where obviously X_0, $Y(s)$, $P(s)$ turn to be compact intervals in R^1.

Recall that in view of (5.12) the function

$$\Psi\,(l\;m(\cdot)) = l\,(\rho(m(0)\mid X^0) + \int_0^\tau \rho(\dot m(s)\mid P(s))ds \;+$$

$$+\int_0^\tau \rho(m(s)\mid Y(s)))$$

where

$$m(t) = 1 - \int_t^\tau \bar\lambda(\zeta)\,d\zeta\,,\ \bar\lambda(\cdot)\in C^1(\mathrm{T}_\tau)$$

We shall demonstrate that

$$inf\,\{\Psi(l\;m(\cdot))\mid m(\cdot)\in C^1_\bullet\,(\mathrm{T}_\tau)\} = inf\,\{\Psi(l\;m(\cdot))\mid m(\cdot)\in C^1\,(\mathrm{T}_\tau)\}$$

With t decreasing from the value τ, denote $\tau*$ to be first instant of time where $m(t)$ turns to zero $(m(\tau^*)=0)$. Therefore $m(\tau^*)=0\,,\ m(t)=1$ and $m(t)>0\,,\ \tau^*<t\le\tau$.

Denote

$$\bar m(t) = m(t) \quad \text{for } \tau^* \le t \le \tau$$
$$\bar m(t) \equiv 0 \quad \text{for } t_0 \le t < \tau^*$$

In view of (5.17) we have

$$\rho(l\;m\;(t_0)\mid X^0) + \int_{t_0}^{\tau_*} \rho(l\;m\;(t)\mid P(t))\,dt$$

$$+\int_{t_0}^{\tau_*} \rho(l\;\dot m\;(t)\mid Y(t))dt \ge \rho(l\;\bar m\;(t_0)\mid X_0)$$

Hence

$$\Psi(l\;m(\cdot)) \ge \Psi(l\;\bar m(\cdot))$$

whatever is the function $m(\cdot)\in C^1_\bullet\,(\mathrm{T}_\tau)$

A number $\varepsilon>0$ being given it is possible for every $m(\cdot)$ to select a $\delta=\delta(\varepsilon\,,\,m(\cdot))>0$ such that the function $m_\delta(t)$ defined as

$$m_\delta(t) \equiv \bar m(t) \quad \text{for } \tau^*(\delta) \le t \le \tau\,,$$
$$m_\delta(t) \equiv \delta \quad \text{for } t_0 \le t \le \tau^*(\delta)\,,$$

satisfies the inequality

$$| \Psi(l \; \bar{m}(\cdot)) - \Psi(l \; m_\delta(\cdot)) | \le \varepsilon$$

Here $\tau^*(\delta)$ is the first instant of time where $m \; (\tau^*(\delta)) = \delta$ with t decreasing from τ to $\tau^*(\delta)$, so that $\tau^*(0) = \tau^*$.

Hence for any $m(\cdot) \in C^1(T_\tau)$ and any $\varepsilon > 0$ there exists a function $m_\delta(\cdot) \in C^1_*(T_\tau)$ such that

$$\Psi(l \; m(\cdot)) \ge \Psi(l \; m_\delta(\cdot)) - \varepsilon \tag{5.19}$$

Comparing (5.19) with the obvious relation

$$\inf \{\Psi(l \; m(\cdot)) \mid m(\cdot) \in C^1(T_\tau)\} \le \inf \{\Psi(l \; m(\cdot)) \mid m(\cdot) \in C^1_*(T_\tau)\}$$

we arrive at the equality (5.18).

Note that the class $C^1_*(T_\tau)$ in (5.18) might well be substituted by $C^1_{*\infty}(T_\tau)$ where

$$C^n_{*\infty}(T_\tau) = \{M(\cdot) : | M(\cdot) \in C^{n \times n}_\infty(T_\tau) \; ; \; M(\tau) = E \; , \; \det M(t) > 0 \; \forall t \in (T_\tau)\}$$

From the proof of the above we came to the assertion:

Lemma 5.2 The set $X[\tau]$ may be described as

$$\rho(l \mid \mathbf{X}[\tau]) = \inf \{\Psi(l' \; M(\cdot)) \mid M(\cdot) \in C^n_{*\infty}(T_\tau)\} \tag{5.20}$$

for any $l \in \Lambda = \{l : l_i \ne 0 \; , \quad i = 1, \ldots, n\}$

Following the suggestions that led to Lemma 2.2 we may deduce

Corollary 5.1 Relation (5.20) is equivalent to

$$\rho(l \mid \mathbf{X}[\tau] = \inf \; \{\rho(l \mid X[\tau \mid L(\cdot)]) \mid L(\cdot) \in C^{n \times n}_\infty(T_\tau)\} \; . \tag{5.21}$$

In order to finalize the proof we will make use of the following lemma.

Lemma 5.3. Assume $\{X_\alpha\}$ to be a variety of convex compact sets that depend upon the index $\alpha \in A$ with $X = \bigcap \{X_\alpha \mid \alpha \in \mathbf{A}\} \ne \phi$. Denote

$$f(l) = \inf \; \{\rho(l \mid X_\alpha) \mid \alpha \in \mathbf{A}\} \; .$$

Then

$$\rho(l \mid X) = f^{**}(l)$$

where $f^{**}(l)$ is the Fenchel second conjugate to $f(l)$.

In other words

$$f^{**}(l) = (co\ f)\ (l)$$

where $(co\ f)\ (l)$ stands for the function whose epigraph is the closed convex hull for the epigraph of $f\,(l)$ $(l\ \in \mathbb{R}^n)$, [7].

Applying this lemma to $X[\tau\,|L\,(\cdot)]$ with $L\,(\cdot)$ acting as the parameter we find that

$$\rho(l \mid \cap\{X[\tau\,|L\,(\cdot)]\,|L\,(\cdot) \in C_{\infty}^{n \times n}\ (\mathrm{T}_{\tau})\}) = (co\ h)\ (l) \tag{5.21}$$

where

$$h\,(l) = inf\ \{\rho(l\ |X[\tau\,|L\,(\cdot)])\,|L\,(\cdot) \in C_{\infty}^{n \times n}(T_{\tau})\} \tag{5.22}$$

and

$$h\,(l) = \rho(l\ |\mathbf{X}[\tau])\ \ \text{for}\ l\ \in \wedge\,. \tag{5.23}$$

From (5.21) - (5.23) it now follows that

$$\mathrm{X}[\tau] = \cap\ \{X[\tau\,|L\,(\cdot)]\,|L\,(\cdot) \in C_{\infty}^{n \times n}(T_{\tau})\}\,.$$

Indeed, since always

$$\mathrm{X}[\tau] \subseteq X[\tau\,|L\,(\cdot)]\,,\ \ L\ (\cdot) \in C^{n \times n}(T_{\tau})\,. \tag{5.24}$$

assume that there exists a point $x^* \bar{\in} X[\tau]$ such that

$$x^* \in \cap\ \{X[\tau\,|L\,(\cdot)]\,|L\,(\cdot) \in C_{\infty}^{n \times n}(T_{\tau})\}\,.$$

Then there exists a vector l^* that ensures the inequality

$$(l^*, x^*) > \rho(l^*\ |X[\tau])$$

$(\mathrm{X}[\tau]$ being a convex compact set we may always assume $l^* \in \wedge)$. Hence there exists a vector $l^* \in \wedge$ such that

$$\rho(l^* \mid \cap\ \{X[\tau\,|L\,(\cdot)]\,|L\,(\cdot) \in C_{\infty}^{n \times n}(T_{\tau})\}) > \rho(l^*(X[\tau])\,.$$

However, this is in contradiction with (5.23), (5.22).

Thus (5.21) is true and in view of (5.24) Theorem 5.1 is now fully proved. Moreover we have established

Lemma 5.3. The following equality is true

$$\rho(l\ |\mathbf{X}[\tau]) \equiv h^{**}(l)\,,\quad l\ \in R^n\ .$$

A direct consequence of the relations of the above is

Lemma 5.4. Assume that in (1.1) the matrix $A(\cdot) \in C_k^{n \times n}(\mathrm{T})$. Then

$$\mathrm{X}[\tau] = \bigcap \{X(\tau \mid L(\cdot)) \mid L(\cdot) \in C_k^{n \times n}(\mathrm{T})\} .$$

6. The Viable Domains.

Consider system (1.1), (1.3) for $t \in [s, \vartheta]$, with set $M \in comp\ \mathbf{R}^n$.

Definition 6.1. The viable domain for system (1.1), (1.3) at time s is the set $W(s, \vartheta)$ that consists of all vectors $w \in \mathbb{R}^n$ such that

$$x(t, \tau, w) \subseteq Y(t), \quad s \le t \le \vartheta , \tag{6.1}$$

$$x(t_1, \tau, w) \subseteq \mathbf{M} . \tag{6.2}$$

Using the duality relations of convex analysis as given in [2] it is possible to observe that

$$W(s, \vartheta) \subseteq \mathbf{R}_-(s, M(\cdot)) , \quad \forall M(\cdot) \in C^{n \times n}[T_s^\vartheta] ,$$

where

$$T_s^\vartheta = \{t : s \le t \le \vartheta\} .$$

$$\mathbf{R}_-(s, M(\cdot) = (\mathrm{S}(\vartheta, \tau) - \int_\vartheta^\tau M(t)\mathrm{S}(\vartheta, t)dt)\, \mathrm{M} +$$
$$+ \int_\vartheta^\tau (\mathrm{S}(\xi, \tau) - \int_\xi^\tau M(t)\mathrm{S}(t, \xi)dt)\mathrm{P}(\xi)d\xi +$$
$$+ \int_\vartheta^\tau \rho(M(t) \mid Y(t))dt .$$

Similar to §2 we come to

Lemma 5.1. The set $W(\tau, \vartheta)$ may be determined as

$$W(\tau, \vartheta) = \bigcap \{\mathrm{R}_-(s, M(\cdot)) \mid M(\cdot) \in C^{n \times n}[T_s^\vartheta]\} .$$

Returning to equation (3.6) denote

$$X_-[s, \mid L(\cdot)] = X_{L(\cdot)}^-(s, \vartheta, \mathrm{M})$$

to be the cross-section at instant s of the set $X_{L(\cdot)}^-(\cdot, \vartheta, \mathrm{M})$ of all the solutions $x_L(t, \vartheta, x_\vartheta)$ to the inclusion (3.6) that are generated at instant ϑ by point $x_\vartheta \in \mathbf{M}$ and evolve in backward time until the instant $\tau < \vartheta$, $(\tau \le t \le \vartheta)$. Along the schemes of §§4, 5 it is possible to arrive at the analogy of Theorem 5.1:

Theorem 6.1. The following relations are true

$$W(\tau,\vartheta) = \bigcap \{X^-_{L(\cdot)}(\tau,\vartheta,\mathbf{M}) \mid L(\cdot) \in C^{n\times n}(T^{\vartheta}_{\tau})\} .$$

7. The State Estimation Problem

Assume inclusion (1.1), (1.2) is considered together with a measurement equation

$$y \in G(t)x + Q(t) , \quad t_0 \le t \le \tau , \tag{7.1}$$

where $y \in \mathbb{R}^m$, $G(t)$ is a continuous matrix and $Q(t)$ a continuous multivalued map from T_τ into *conv* $\mathbb{R}^m$.

Suppose that due to equations (1.1), (1.2) and (6.1) (that substitutes for (1.3)) an "observation" $y^*(t)$, $t \in T$, has appeared. (The function $y^*(t)$ is obviously generated due to equations

$$\dot{x} = A(t)x + u , \quad y = G(t)x + \xi \tag{7.2}$$

by triplet x^0 , $u(\cdot)$, $\xi(\cdot)$, where $x^0 \in X^0$, $u(t) \in P(t)$, $\xi(t) \in Q(t)$ and $u(t)$, $\xi(t)$ are measurable functions.)

The estimation problem will consist in specifying the set $X(\cdot;y^*(\cdot))$ of all the solutions $x(\cdot,t_0,x^0)$ of inclusion (1.1) that start at t_0 from points $x^0 \in X^0$ and satisfy both (1.1) and (7.1) for $y(\cdot) = y^*(\cdot)$, $t_0 \le t \le \tau$, (being therefore consistent with both the system equation (1.1) and the measurement equation (7.1), $y(\cdot) = y^*(\cdot)$). The latter problem then reduces to the one of §§1 - 4: the specification of set $X[\tau]$ and its evolution in τ where the set-valued map $Y(t)$ of (1.3) appears in the form

$$Y(t) = \{x : G(t)x \in Q^*(t)\}$$

and

$$Q^*(t) = y^*(t) - Q(t) .$$

This specific type of set $Y(t)$ may be treated along the schemes of the above.

The results reduce to the following relations. Consider the inclusion

$$\dot{x} = (A(t) - L(t)G(t))x + \mathrm{P}(t) + L(t)(y^*(t) - Q(t)) \tag{7.3}$$
$$x(t_0) = x^0, \quad x^0 \in X^0, \quad t_0 \le t \le \tau$$

denoting its solution as

$$x^*_{L(\cdot)}(t, t_0, x^0)$$

and taking

$$X^*_{L(\cdot)}(t, t_0, X) = \bigcup \{x^*_L(t, t_0, x^0) \mid x^0 \in X^0\} .$$

Along the schemes of §§2-4 we arrive at the proposition.

Theorem 7.1. The cross-section $X^*[\tau]$ at time τ of the set $X(\cdot, y^*(\cdot))$ of all solutions to the system (1.1), (7.1), $y(t) \equiv y^*(t)$, $t_0 \le t \le \tau$, may be described as

$$X^*[\tau] = \bigcap \{X^*_L(t, t_0, X^0) \mid L(\cdot) \in C^{n \times n}(T_\tau)\} . \tag{7.4}$$

Thus if the information on an uncertain trajectory $x(t, t_0, x^0)$ of (1.1), (1.2) is reduced to the knowledge of the function $y^*(t)$, $t \in [t_0, \tau]$, then the set $X^*[\tau]$ gives a "guaranteed" estimate for $x[\tau] = x(\tau, t_0, x^0)$.

Remark. From the assumption that the function $\xi(t)$ in (7.2) is measurable, it follows that set $Q^*(t)$ is measurable in t (with values in *comp* $\mathbf{R}^n$). This leads to the fact that the respective set

$$Y(t) = \{x : G(t)x \in Q^*(t)\}$$

may be also measurable rather than continuous in t as required by the assumptions for Theorem 5.1. The proof of Theorem 5.1 however allows a modification that ensures Theorem 7.1 to be true.

The scheme presented here is other than those suggested in either [2] or [11].

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