International Institute for Applied Systems Analysis
IIASA
www.iiasa.ac.at

A Linear Differential Pursuit Game

Pontryagin, L.S.

IIASA Collaborative Paper
January 1985

A LINEAR DIFFERENTIAL PURSUIT GAME

L.S. Pontryagin

January 1985
CP-85-1

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
2361 Laxenburg, Austria

PREFACE

In this paper the author returns to the differential game $\dot{z} = Cz - u + v$ which he examined in earlier papers. Here z is the phase vector of the game in n-dimensional vector space R, C is a linear mapping of the space R into itself, and u and v are controls, i.e., vector functions of time t which are not known in advance. Vectors u (pursuer control) and v (evader control) satisfy the inclusions $u \in P$, $v \in Q$, where P and Q are convex compact subsets of the space R and have arbitrary dimension. The game is considered finished when the point z enters a given closed convex set M from R.

In pursuit problems the control v is a function of time t and is not known in advance; the problem is to choose the control u in such a way as to finish the game as quickly as possible.

In previous work it was necessary to use knowledge of the function $v(s)$ for $t \leq s \leq t + \varepsilon$, where $\varepsilon > 0$ is any given arbitrary small value, to find control u (*discrimination* of the evader control). This deficiency was overcome in the past by some natural assumptions on the smoothness of certain sets.

In this paper the author makes stronger assumptions which eliminate the discrimination of the control $v(t)$ and make it possible to define the optimal control u more constructively.

This paper was presented by the author at the Workshop on the Dynamics of Macrosystems held at IIASA in September 1984.

Alexander Kurzhanski
Chairman
System and Decision Sciences

A LINEAR DIFFERENTIAL PURSUIT GAME

L.S. Pontryagin

Steklov Institute of Mathematics, USSR Academy of Sciences, ul. Vavilova 42, 117333 Moscow, USSR

The differential game described by the equation

$$\dot{z} = Cz - u + v \tag{1}$$

was studied in [2], where complete proofs of the results given in [1] may be found. Here z is the phase vector of the game in n-dimensional vector space R, C is a linear mapping of the space R into itself, and u and v are controls, i.e., vector functions of time t which are not known in advance. Vectors u and v satisfy the inclusions

$$u \in P \; , \quad v \in Q \; , \tag{2}$$

where P and Q are convex compact subsets of the space R and have arbitrary dimension. The game is considered finished when the point z enters a given closed convex set M from R. Control u is called the ***pursuer control*** and v the ***evader control***.

In pursuit problems the control v is a function of time t, $v = v(t)$, and is not known in advance; the problem is to choose the control u as a function of t in such a way as to finish the game as quickly as possible. This is done at time t using information on $z(s)$ and $v(s)$ for $s \le t$.

The most natural way to solve this problem is to try to choose the control $u(t)$ at any time t in such a way that the distance from the point $z(t)$ to the set M decreases as rapidly as possible. However, this turns out to be impossible. We have to use another method to estimate the rate of approach of the point $z(t)$ to the set M. We shall construct a convex set $W(\tau)$, $\tau \ge 0$, $W(0) = M$, and define the minimal value $\tau = T(z)$ for which a point $e^{\tau C} z$ belongs to the set $W(\tau)$. It is evident that the point $w = e^{\tau C} z$ lies on the boundary of the set $W(\tau)$ and depends on z. Let $\psi(w)$ be a unit exterior normal to the surface $\partial W(\tau)$ at the point w. The resulting function $T(z)$ is an estimation function for the time of approach of the point z to the set M.

If the value of $T(z)$ decreases during the game and finally becomes equal to zero then the game comes to an end. It can be proved that the rate of decrease of the function $T(z)$ during the game is not less than the rate of increase of the time t. Thus a game beginning at the point z_0 will finish at a time not greater than the value $T(z_0)$. It is important that an incorrect choice of evader control $v(t)$ gives an advantage to the pursuer, i.e., will accelerate the end of the game.

An important deficiency of [2] is that we use knowledge of the function $v(s)$ for $t \leq s \leq t + \varepsilon$, where $\varepsilon > 0$ is any given arbitrary small value, to find control $u(t)$. This is called *discrimination* of the evader control.

This deficiency is overcome in [2] under some natural assumptions on the smoothness of certain sets.

Since we use stronger assumptions here, the present paper is not simply a generalization of [2] but eliminates the discrimination of the control $v(t)$ and allows us to define optimal control $u(t)$ more constructively.

Let us recall the construction of convex set $W(\tau)$ given in [2]. First of all we introduce some natural operations over convex sets from the space R.

1. If X and Y are convex sets from the space R, and α and β are real numbers, then we define the convex set

$$Z = \alpha X + \beta Y \tag{3}$$

of all vectors $z = \alpha x + \beta y$, where $x \in X$, $y \in Y$. Hence we can define the Riemann integral

$$\int_{s_0}^{s_1} X(s)\,ds \quad . \tag{4}$$

Here it is assumed that the convex set-valued mapping $X(s)$ is continuous in real parameter s, $s_0 \leq s \leq s_1$. In (3) we consider only non-negative α, β.

2. Define the geometrical difference

$$Z^* = X \stackrel{*}{-} Y \tag{5}$$

of two convex sets X and Y from the space R. The set Z^{*} consists of all vectors $z^{*} \in R$ such that $Y + z^{*} \subset X$. Note that the sets (3–5) are convex and are also compact if X and Y are compact.

3. Define the set $W(\tau)$ in the form of an alternating integral

$$W(\tau) = \int_{M,0}^{\tau} (P(\tau)d\tau \,\underline{*}\, Q(\tau)d\tau) \quad , \tag{6}$$

where $P(\tau) = e^{\tau C}P$, $Q(\tau) = e^{\tau C}Q$. To evaluate this we define an alternating sum of convex sets $(A, X_1, \ldots, X_n, Y_1, \ldots, Y_n)$. We set

$$A_0 = A \ , \ A_i = (A_{i-1} + X_i) \,\underline{*}\, Y_i \ , \ i = 1, \ldots, n \quad . \tag{7}$$

Let $(\tau_0, \tau_1, \ldots, \tau_n)$ be a partition of the interval

$$0 = \tau_0 < \tau_1 < \cdots < \tau_n = \tau \quad . \tag{8}$$

We set (see (4))

$$X_i = \int_{\tau_{i-1}}^{\tau_i} P(\tau)d\tau \ , \quad Y_i = \int_{\tau_{i-1}}^{\tau_i} Q(\tau)d\tau \quad . \tag{9}$$

We consider the alternating sum A_n (see (7)) for set $A = M$, with X_i, Y_i given by formula (9), as an approximate value of alternating integral (6). It can be proved that alternating sum (9) has a limit if the maximal length of intervals from partition (8) tends to zero. This limit is the value of the alternating integral (6).

In [2] it is proved that if a function $v(s)$ is known on the interval $t \le s \le t + \varepsilon$ then we can choose the control $u(t)$ on the same interval in such a way that the inequality

$$T(z(t + \varepsilon) < T(z(t)) - \varepsilon$$

holds. For this we choose the control $u(t)$ in such a way that the difference

$$P(z(t + \varepsilon)) - T(z(t))$$

has its largest absolute value. Hence we solve some nontrivial variational problem with discrimination of evader control on every time interval of length ε.

In the simple case considered in [2] (see §6, p.325), the set M is a linear vector subspace. Consider an orthogonal complement L of dimension ν to the subspace M in the space R. Let π be the orthogonal projection of the space R onto the subspace L, and consider the sets

$$P(\tau) = \pi e^{\tau C} P \ , \quad Q(\tau) = \pi e^{\tau C} Q \ . \tag{10}$$

Suppose that the set

$$S(\tau) = P(\tau) \stackrel{*}{-} Q(\tau) \tag{11}$$

has dimension ν for $0 < \tau < T$. We distinguish between two separate cases:

1. $P(\tau) = Q(\tau) + S(\tau)$ (the exhaustive case)
2. $P(\tau) \neq Q(\tau) + S(\tau)$.

Consider the convex set

$$\overline{W}(\tau) = \int_0^\tau S(t)\mathrm{d}t \ . \tag{12}$$

We define the estimating function $\overline{T}(z)$ as the minimal value of τ for which the inclusion

$$\pi e^{\tau C} z \in \overline{W}(\tau) \tag{13}$$

holds.

In the present paper we give a way of constructing the pursuit control $u(t)$ without discrimination of the evader control $v(t)$ under certain differentiability conditions. In particular, we suppose that the $\overline{W}(\tau)$ are convex sets with smooth boundaries and that the boundaries of the sets $P(\tau)$ and $Q(\tau)$ do not contain linear segments.

Consider the support function $c(\overline{W}(\tau) - \pi e^{\tau C} z, \psi)$ of convex set $\overline{W}(\tau) - \pi e^{\tau C} z$, where ψ is a unit vector. This support function is greater than or equal to zero for any ψ if

$$\pi e^{\tau C} z \in \overline{W}(\tau) \ , \tag{14}$$

and has negative values for some ψ if inclusion (14) does not hold. We denote the minimum of this function by

$$-\overline{F}(z,\tau) = \min_{\psi} c(\overline{W}(\tau) - \pi e^{\tau C} z, \psi) \ . \tag{15}$$

When point $\pi e^{\tau C} z$ reaches the set $\overline{W}(\tau)$ the function $\overline{F}(z,\tau)$ changes sign from positive to negative. The value of $\overline{T}(z)$ is the smallest positive root of the equation

$$\overline{F}(z,\tau) = 0 \quad . \tag{16}$$

The derivative

$$\overline{G}(z,\tau) = \frac{\partial \overline{F}}{\partial \tau}(z,\tau) \tag{17}$$

is nonpositive when the point $\pi e^{\tau C} z$ reaches the set $\overline{W}(\tau)$. If the inequality $\overline{G}(z,\tau) \neq 0$ holds at this time then $\overline{T}(z)$ is a smooth function of z in a neighborhood of this point. If $\overline{G}(z,\tau) = 0$ then function $\overline{T}(z)$ may be discontinuous.

If u and v are known functions then z is a function of parameter t and $\tau = \overline{T}(z)$ is also a function of t. This means that relation (16) is an identity with respect to t. Differentiating the identity (16) in t we get the relation

$$\dot{\tau}\overline{G}(z,\tau) + \dot{z}\frac{\partial \overline{F}}{\partial z} = 0 \quad .$$

Hence for $\overline{G} \neq 0$ we have

$$\dot{\tau} = \frac{\left(\frac{\partial \overline{F}}{\partial z}, \dot{z}\right)}{-\overline{G}(z,\tau)} \quad . \tag{18}$$

Let $\overline{\psi}(t)$ be the unit vector which minimizes the support function (15) and $s(\psi,\tau)$ be the point on the boundary of the convex set $S(\tau)$ which maximizes the scalar product

$$(s,\psi) \ , \quad s \in S(\tau) \quad .$$

Then function $\overline{G}$ has the form

$$\overline{G}(z,\tau) = (\pi e^{\tau C} Cz - s(\psi,\tau),\psi) \quad , \tag{19}$$

and formula (18) becomes

$$\dot{\tau} = \frac{(\pi e^{\tau C}(Cz - u + \gamma),\psi)}{-(\pi e^{\tau C} Cz - s(\psi,\tau),\psi)} \quad . \tag{20}$$

It is clear from formula (20) that we can choose the control u in such a way that $\dot{\tau} \leq -1$. Take the value of u which minimizes $\dot{\tau}$. The corresponding value of $\dot{\tau}$ is less than or equal to -1. It is evident that $u(t)$ maximizes the scalar product $(\pi e^{\tau C}u,\psi)$. This value of $u = u_{\text{opt}}$ is said to be optimal and is the value of the control chosen during the pursuit process if $\overline{G} \neq 0$.

If we choose control $u(t)$ according to this rule and function $\overline{G}$ tends to zero then the value of $\dot{\tau}$ is defined by the same relation (18). Here we have to consider two different cases. The control v_{opt} is said to be optimal if it maximizes the scalar product $(\pi e^{\tau C}v,\psi)$. Consider the exhaustive case. If the control v is optimal on some time interval and $\overline{G} = 0$ at the initial time t_0, then $\dot{\tau} = -1$ and $\overline{G} = 0$ for all t from this interval. If $v \neq v_{\text{opt}}$ and $\overline{G} = 0$ then the point $z(t)$ leaves the surface $\overline{G}(z,\tau) = 0$ in a small neighborhood of t_0. Moreover, the function τ displays the following behavior:

$$\tau_0 - \tau = A(t - t_0)^{1/k+1} + O((t - t_0)^{1/k+1}) \quad . \tag{21}$$

where k is the multiplicity of the root τ_0 of equation (16). Two cases can arise if the point $z(t)$ arrives at the surface $\overline{C}(z,\tau) = 0$: τ changes continuously or displays a jump. In the first case the behavior of τ has the following form:

$$\tau - \tau_0 = A(t_0 - t)^{1/k+1} + O((t_0 - t)^{1/k+1}) \quad . \tag{22}$$

In the non-exhaustive case the behavior of the trajectory may be considered in a similar way with some small differences.

Hence for an optimal choice of $u(t)$ the solution $z(t)$ of the differential game always satisfies the following condition:

$$\frac{\mathrm{d}}{\mathrm{d}t} T(z(t)) \leq -1 \quad .$$

In the case of the alternating integral we set $P(\tau) = e^{\tau C}P$, $Q(\tau) = e^{\tau C}Q$. Let $L(P(\tau))$ be the affine support of convex set $P(\tau)$. If the vector $\psi(w)$ is not orthogonal to the space $L(P(\tau))$ then we choose the control $u(t)$ which maximizes the function

$$(e^{\tau C}u,\psi(w)) \ , \quad u \in P \quad . \tag{23}$$

This relation defines a unique control u which is the best pursuit strategy. If the vector $\psi(w)$ is orthogonal to the space $L(P(\tau))$ at the time t_0 then rule (23) does not give us the opportunity to choose control $u(t)$ and it must be selected in some other way.

In the general case consider the support function

$$c\,(W(\tau)-e^{\tau C}z,\psi)\ ,\quad |\psi| = 1\ . \tag{24}$$

It is clear that this support function is greater than or equal to zero if $e^{\tau C}z \in W(t)$ and has a negative value if this inclusion does not hold. Define

$$-F(z,\tau) = \min_{\psi} c\,(W(\tau)-e^{\tau C}z,\psi)\ . \tag{25}$$

Hence the value $T(z)$ is the smallest positive root of the equation

$$F(z,\tau) = 0 \tag{26}$$

with respect to τ. Set $G(z,\tau) = \partial F/\partial z$.

We choose the optimal control $u(t)$ in the following way. Since τ is a root of equation (26) we differentiate it in t and obtain the relation

$$\dot{\tau} = -\frac{\left(\dfrac{\partial F}{\partial z},\dot{z}\right)}{G(z,\tau)}\ , \tag{27}$$

which is similar to (18). We choose the control $u(t)$ in such a way that the value of $\dot{\tau}$ given by relation (27) is minimal. This approach is similar to the choice of optimal control $u(t) = u_{\text{opt}}(t)$ given previously.

It can be proved that $\dot{\tau} \leq 1$ if we use this rule. Hence the estimating function $T(z(t))$ decreases more quickly than t increases.

The control $v(t)$ which maximizes $\dot{\tau}$ (see (27)) for any given $u(t)$ is called the optimal evader control and is denoted by $v_{\text{opt}}(t)$. This optimal control $v_{\text{opt}}(t)$ does not depend on the choice of control u.

Relation (27) is meaningful only if $G \neq 0$. It can be proved that

$$G = (e^{\tau C}Cz,\psi(w)) - (e^{\tau C}(u_{\text{opt}}-v_{\text{opt}},\psi(w)))\ . \tag{28}$$

If $G \neq 0$ then formula (27) has the form

$$\dot{\tau} = \frac{(e^{\tau C}(Cz - u_{\text{opt}} + v),\psi)}{-(e^{\tau C}(Cz - u_{\text{opt}} + v_{\text{opt}}),\psi)}\ . \tag{29}$$

Hence $\dot{\tau} \leq -1$ and $\dot{\tau} = -1$ if $v = v_{opt}$.

It can be proved that $\dot{\tau} = -1$ if $G = 0$ and $v = v_{opt}$. This fact does not follow from (29). If $v = v_{opt}$ on some time interval and $G = 0$ at the initial time t_0 then $G = 0$, $\dot{\tau} = -1$ and ψ = const. all over this interval.

If $v \neq v_{opt}$ and $G = 0$ then point $z(t)$ leaves the surface $G = 0$ in a small neighborhood of t_0. Moreover, the behavior of function τ is described by formula (21).

When vector $\psi(w)$ becomes orthogonal to the subspace $L(P(\tau))$ the control u displays a jump. We would therefore have to choose it in a different way were it not for the fact that it can be proved that this orthogonality disappears and we can take the rule for choosing the optimal control u given earlier.

The relation

$$\frac{\mathrm{d}}{\mathrm{d}t} T(z(t)) \leq -1$$

holds for all of the methods of choosing the pursuit control $u(t)$ mentioned here, i.e., the rate of decrease of function $T(z(t))$ is not less than the rate of increase of t.

REFERENCES

1. L.S. Pontryagin. *Doklady Akademii Nauk SSSR* 175 (1967) 764–766.
2. L.S. Pontryagin. *Mathematichesky Sbornik* 112 (1980) 307–330.