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FOREWORD

This paper by Tõnu Puu uses a topological model together with a structural stability principle as a means to identify and characterize long run solutions as regards regional specialization, direction of trade, and spatial organization. Within this setting not too restrictive assumptions are used to deduce results which shed new light on von Thünen's theory of location and spatial interaction.

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AN ATTEMPT AT RESTORING VON THÜNEN
- A TOPOLOGICAL MODEL

by

Tõnu Puu

Assume a production and exchange economy, extended over a region R of the Euclidean plane. We denote the location coordinates by x, y. There are n productive activities, represented by production functions:

$$Q_i = F^i(K_i, L_i, M_i) \quad , \tag{1}$$

which depend on the primary inputs capital, labor, and land. If we assume the functions to be linearly homogeneous, we can divide through by the input of land, to obtain the areal densities of outputs

$$q_i = f^i(k_i, \ell_i) \tag{2}$$

as well defined functions of the areal densities of capital and labor used. Formally $q_i=Q_i/M_i, k_i=K_i/M_i$, and $\ell_i=L_i/M_i$.

We assume no spatial productivity differences, so that (2) apply at all locations. Moreover, the aggregate stocks of capital goods and labor force are supposed to be perfectly mobile.

Considering a long-run equilibrium, we disregard the relocation costs of these stocks. Accordingly we have removed all the reasons for specialization and trade employed by traditional economists. Considering productivity differences and/or relocation costs for inputs would only reinforce the points we are going to make. The single immobile input is land, but its quality is the same everywhere, and its usefulness only varies with its centrality of location. This variation in the usefulness of land will be reflected in the spatial variation of land rent. Capital rent and wage rate will, due to the assumption of costless relocation, become spatial invariants.

Supposing that a given production activity is established at a location, the optimal capital and labor densities will be determined by the conditions

$$p_i f_k^i(k_i, \ell_i) = r \tag{3}$$

and

$$p_i f_\ell^i(k_i, \ell_i) = w \tag{4}$$

By p_i we denote the local price of the commodity, whereas r is capital rent and w wage rate. The latter two are, as we have noted, spatial invariants. Accordingly, (3)-(4) determine k_i and ℓ_i, and, due to (2), q_i everywhere once p_i is known.

The question is which of the n different activities to establish at any given location. To answer this we first derive a set of relations that must hold among the commodity prices, provided the commodities are produced. Profits per unit land area from the production of the i:th commodity are

$$\pi_i = p_i f_i(k_i, \ell_i) - rk_i - w\ell_i. \tag{5}$$

From (3)-(4) we recall that k_i and ℓ_i are completely determined by p_i. Accordingly, π_i as well is determined by p_i. Now, the total profits of the landlord from combining the various activities are

$$\sum_{i=1}^{n} \pi_i m_i \ , \tag{6}$$

where m_i are the fractions of land used for the various activities. As $\pi_i(p_i)$ are independent of m_i, we note that (6) is linear in the latter. Assuming that the fraction of land disposable for production is m, we get

$$\sum_{i=1}^{n} m_i = m \tag{7}$$

as a linear constraint.

Maximizing (6) subject to (7) has the obvious solution: Put m_i=m for that activity for which π_i is maximal and m_i=0 for the rest. If several of the π_i attain this maximum value all these activities are equivalent, and m can be distributed in any way among them.

The maximum profit per unit land area, obtained from putting land to the best possible use obviously determines land rent. We denote it

$$g = \operatorname*{Max}_i \pi_i \tag{8}$$

The condition for a mix of various activities obviously is that

$$g = \pi_i(p_i) \tag{9}$$

for all those activities that are established. This is our first important conclusion. We should recall that the forms of the functions $\pi_i(p_i)$ are completely determined from the production technologies, as represented by the production functions $f^i(k_i, \ell_i)$, and depend on nothing else.

There is one activity that has not yet been introduced, namely the production of transportation services. To vary assumptions a little, let these services be produced by a fixed coefficient technology. This is by no means crucial - any of the activities listed in (1) and (2) could do as well.

Thus, transportation of one unit of commodities, per unit distance traversed, is assumed to require the services of κ units of capital, λ units of labor and μ units of land. For simplicity we define the units of measurement of the commodities so that the input coefficients are the same for all. This in no way restricts the analysis. We are free to choose the units and can hence define suitably small units of goods that are heavy or bulky as compared to those more easily transportable.

The cost of transportation accordingly is

$$(\kappa r + \lambda w + \mu g) = h(g) \tag{10}$$

at any location. Using the Beckmann (1952) continuous model of transportation, where the flows of traded commodities are construed as vector fields, the conditions for optimum transportation are

$$h(g)\frac{\phi^i}{|\phi^i|} = \text{grad } p_i \tag{11}$$

Here, $\phi^i=(\phi_1(x,y),\phi_2(x,y))$ represent the flows of trade, with $|\phi^i| = \sqrt{(\phi_1^{i2} + \phi_2^{i2})}$ being the quantities of traded commodities, and the unit direction fields $\phi^i/|\phi^i|=(\cos\theta_i, \sin\theta_i)$ being the spatial directions of the flows.

We observe that the cost of transportation $h(g)$ only depends on land rent, due to (10), as capital rent and wages are invariants, and that, due to our choice of units, the cost is equal for all commodities.

The condition (11) tells us two things: For goods actually transported, they are shipped in the directions of the price gradients, and, in these directions, prices increase at the rate of transportation costs. Concentrating on the latter fact we take norms of both sides of (11) and get

$$|\text{grad } p_i| = h(g) \tag{12}$$

for all commodities actually transported.

The question is how the conditions (9) and (12) fit together. Taking the gradient of (9) we find

$$\pi_i' |\text{grad } p_i| = |\text{grad } g|. \tag{13}$$

so

$$\pi_i' h(g) = |\text{grad } g| \tag{14}$$

for all goods both produced and transported. As $|\text{grad } g|$ and $h(g)$ are the same for all i we conclude that π_i' must be the same functions for all commodities produced and transported.

But, we saw that π_i, and hence π_i' were independently determined by the production functions f^i and nothing else. Accordingly, one function π_i' could be used in (14), which thus provides a differential equation for determining the function $g(x,y)$. But, it would be most unlikely that any other π_i' would then fit into (14) with g already solved. Using such a simple transversality principle we arrive at the following:

I. Theorem of Specialization:

At each location exactly one commodity is both locally produced and transported. There may be any number of other goods only produced for local consumption. The same is true about goods only shipped but not locally produced.

We will disregard local production and concentrate on the trade pattern. Still, as we are dealing with an arbitrary number of commodities, the pattern might be incomprehensibly complex where all those flows, that actually occur, cross each other. Observe that we have no a priori radial communication with a single "central city" as von Thünen. Accordingly, it might be difficult to speak of any structure of the spatial economic organization.

However, the picture can be very much simplified by a few elementary observations. From (9) we conclude that in a given specialization zone the price gradient of the single actually produced and transported commodity coincides in direction with the land rent gradient. Accordingly, for this commodity, the constant price lines, and the constant rent lines coincide everywhere in the zone of specialization considered, including the boundaries. The latter actually are curves of constant prices of the commodities in the specialization zones on either side of the boundary in question.

This, however, implies that the boundary conditions are the same for *both* differential equations (12) for the commodities on either side of a common boundary.

As the differential equations are the same for all commodity prices, the only way the solution could differ would be by reason of different boundary conditions. Accordingly, the solutions coincide (except for some space-invariant constants), and so do the trajectories of trade.

This reasoning can be repeated, by continuing to the next specialization zone, and we finally establish a:

II. Theorem on Unique Direction of Trade

The flow directions of all traded commodities coincide, and coincide with the land rent gradient.

From the complicated picture of a mess of crossing trajectories of trade for various commodities we have obtained a much simpler picture with one direction of all trade everywhere.

Accordingly, this unique direction field of the land rent gradient, and its orthogonal trajectories, the constant rent lines, some of which become boundaries between specialization zones, makes it meaningful to speak of a structure of the whole space economy with an arbitrary number of commodities involved.

This structure, however, need not have such a simple character as in von Thünen's theory, with the zones being concentric rings and the flows being radial from or to one center. For all we have learned up to now, the structure might take on so many different forms that it could be an impossible task to describe or classify them.

The sequel will be devoted to attempting a topological description of the structures, using only the principle that they should be robust to disturbances due to factors not explicitly accounted for in the model. We will see that the picture becomes surprisingly precise and even refutes some rather basic ideas about the organization of economic space.

The trajectories traced by the land rent gradient field can be defined by the pair of differential equations:

$$\frac{dx}{ds} = \xi_1(x,y),$$

and (15)

$$\frac{dy}{ds} = \xi_2(x,y),$$

where $\xi_1(x,y) = \partial g/\partial x$ and $\xi_2(x,y) = \partial g/\partial y$. Can we say anything in general about what the system of flow lines might look like?

It is a good principle in scientific modelling to use robust assumptions that lead to systems that do not change their qualitative character by the smallest disturbance. Scientific modelling being associated with abstraction from many facts of reality, one must take care that the conclusions do not crucially depend on some assumption that might not hold, see Arnold (1983). So, we must admit for changes in a system like (14)-(15).

A suitable representation of possible small disturbances is by an ε-perturbation, obtained by replacing (14)-(15) by

$$\frac{dx}{ds} = \eta_1(x,y) \tag{16}$$

$$\frac{dy}{ds} = \eta_2(x,y), \tag{17}$$

where

$$|\xi_i - \eta_i| < \varepsilon \tag{18}$$

and

$$|\partial\xi_i/\partial x - \partial\eta_i/\partial x| \ , \ |\partial\xi_i/\partial y - \partial\eta_i/\partial y| < \varepsilon \tag{19}$$

so that (16)-(17) differ little from (14)-(15). We thus consider an abstract metric space of differential equations where "distance" is represented in terms of a C^1 topology.

The land rent gradient thus should be robust, i.e., qualitatively or structurally stable, under ε-perturbations of the kind explained. To make this precise we next have to explain the concept of qualitative equivalence of two sets of trajectories that solve (14)-(15) and (16)-(17) respectively.

If we can find a continuous one-to-one mapping from the solution space of (14)-(15) to the solution space of (16)-(17), such that each trajectory is mapped on another trajectory, and each singularity on another singularity, with directions of trajectories and types of singularities being preserved, then we obviously have an equivalence of the type we are looking for between the sets of trajectories.

The equivalence between the flow portraits can also be understood intuitively if we consider pictures of flows drawn on perfectly elastic rubber sheets. Then all the shapes into which such a picture could be deformed, by stretching without tearing, would constitute a class of qualitatively equivalent flow portraits.

Now, the concept of structural stability is at hand. If perturbations of the differential equations only lead to small, qualitatively equivalent, deformations of the flow portrait then it is stable. If the flow portrait changes qualitatively, so that trajectories are reversed, singularities appear, split, fuse, or disappear, then we deal with structural instability.

This idea was formulated by Andronov and Pontryagin (1937). Later the work was completed by Morse, Smale, and Peixoto. Of great interest are two remarkable theorems.

The *approximation theorem* tells us that, in the abstract space of differential equations, almost all the elements belong to the subset of structurally stable ones. In formal terms the subset is dense and open, and hence the unstable equations are everywhere and in almost all directions surrounded by stable ones.

Accordingly, we not only know that structurally unstable configurations do not persist in a world of change, but also that every time an unstable configuration is destroyed it is transformed into a stable one.

This happy state of affairs only holds in two-dimensional systems, and fortunately this is exactly what we are dealing with. As Smale has demonstrated, with three dimensions already, it is quite likely that unstable systems are turned into other unstable systems.

About this and the even more interesting characterization theorem Peixoto (1977) or Hirsch and Smale (1974) can be consulted.

The *characterization theorem* tells three things: (a) any structurally stable flow is laminar , or topologically equivalent to a set of parallel straight lines, almost everywhere; (b) the exception is a finite set of hyperbolic singularities, i.e., singularities with non-vanishing real parts of the eigenvalues of the linearized differential equations. As we deal with gradient fields the imaginary parts always disappear, and hence nodes and saddles are the only hyperbolic singularities; (c) finally, there is the global result that no trajectory issuing from or going into a saddle point is incident to a saddle at the other end.

As the flow is laminar, except at the singularities, we can use them to organize a basic graph at the stable flow. In particular, we try to draw a regular tessellation or tiling of space in the same spirit as do Christaller and Lösch. The difference is that our picture is only topological, and admits all the distortions of the basic picture attained by stretching it in various ways.

As we are going to organize the picture by the possible singularities, nodes and saddles, the main organizing element consists in the two directions in which trajectories are incident to saddle points. This fact makes the organization quadratic, as demonstrated by Puu (1982), and, in particualr, rules out the hexagonal tilings of Christaller and Lösch. The way the result is obtained is by starting out from a saddle point, where trajectories are incident in the horizontal and vertical directions.

In a regular organization of space these trajectories end up at new singularities, which, according to the condition ruling out saddle connections, must be nodes. Two of them are stable, two unstable.

As the stable and unstable ones are organized in opposite pairs we conclude that the singularities NE, SW, SE, and SW of the saddle points have ingoing as well as outgoing trajectories. In our list of admitted singularities only saddle points have this character, and hence, we identify four new saddle points.

Accordingly, the whole basic graph can be oriented, as shown in Figure 1a, by starting anew from the saddles and continuing as before. We obtain the:

III. Theorem of Spatial Organization

The only regular land rent surface, possessing structural stability, is quadratic (up to topological equivalence), where each node is surrounded by four saddles and *vice versa*.

It then is a trivial matter to fill in the whole picture with trajectories as in Figure 1b.

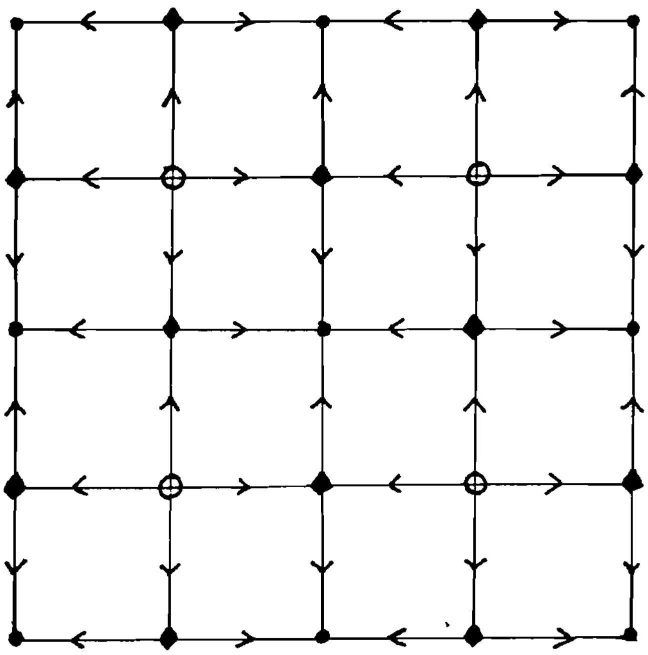

Figure 1a. Basic graph.

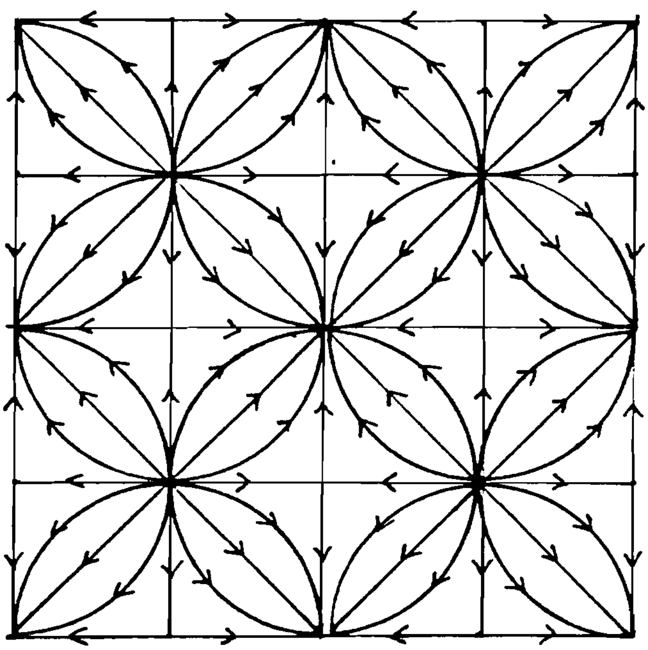

Figure 1b. Complete flow portrait.

We also note that, associated with the gradient field of land rent is a land rent "landscape" with hills, corresponding to stable nodes, and bottoms, corresponding to unstable nodes, equally spaced in a lattice with saddle points in the corners inbetween. This is shown in Figure 2.

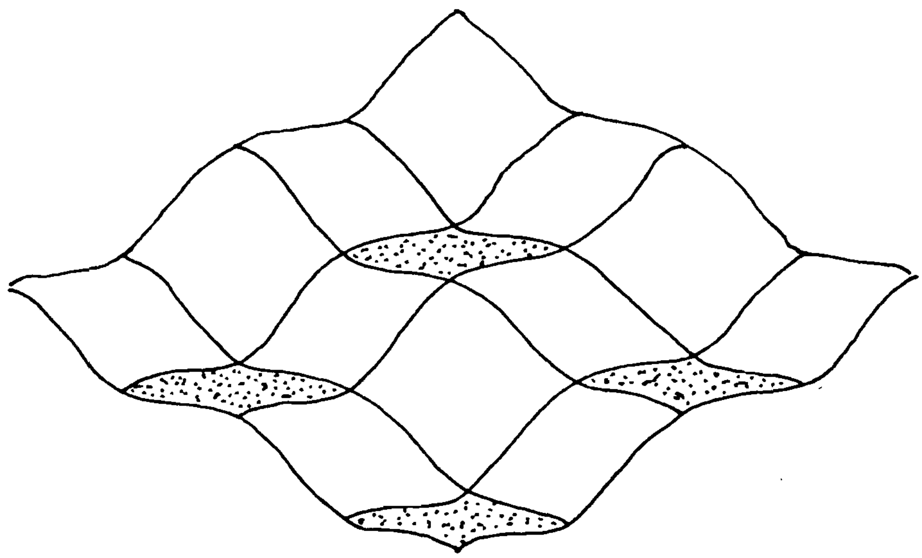

Figure 2. Land rent surface.

But we noted that the specialization pattern was defined by contours of constant land rent. Accordingly, the orthogonal trajectories to the rent gradient lines are the boundaries of specialization areas and we arrive at von Thünen "rings" of specialization as shown in Figure 3. Each hill top corresponds to a von Thünen "central city", whereas the bottoms represent the "wilderness". This "wilderness" no longer completely surrounds the most peripheral ring, because we are not concerned with one "isolated state", but with a set of "central cities" with their hinterlands fused together into one single pattern. For this reason "wilderness" recedes to the corners between the various non-isolated "states".

We have interpreted the node singularities as the most central and the most peripheral points in the system. What about the saddle points? There is an obvious interpretation at hand if we consider the fact that all trajectories seem to

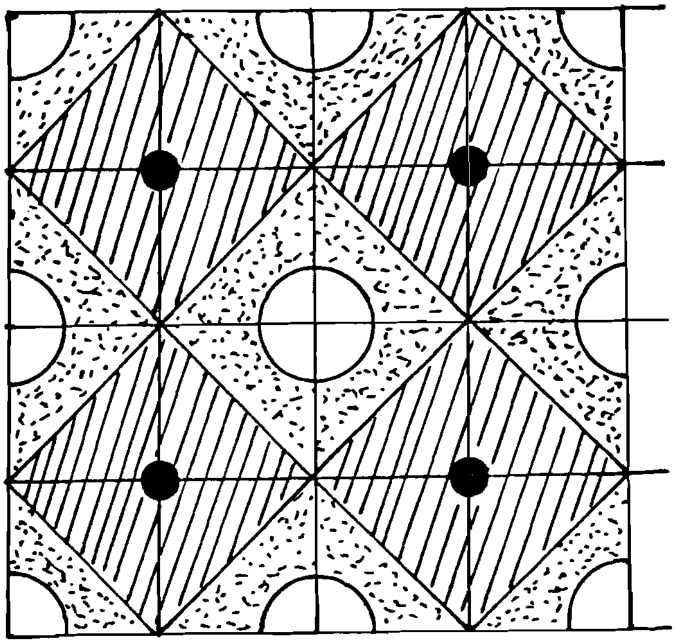

Figure 3. Specialization pattern.

be attracted to them. Land rent was seen to be the main variable item in the cost of transportation. We could interpret this in terms of general congestion or competition for land. Along with congestion a main cost factor, which we have not brought into the picture, is the density and capacity of the road network. Obviously it should be better close to the central locations and worse in "wilderness".

Considering both factors, transportation would be easiest in points where density and capacity of roads is good, but congestion and competition for land are not too bad, i.e., at locations far back from "central cities" and from "wilderness". Such locations above all are the saddle points, and we can therefore understand why the trajectories would be attracted to them.

By summary, we have, using no more than general assumptions of structural stability, established a pattern of specialization and seen it to be like the von Thünen pattern of concentric rings.

The picture differs in two ways from the original. First, von Thünen's picture is geometrical, whereas the present one is topological, admitting a wide class of distorted pictures with only the same qualitative features. Second, we not only consider one isolated economy of the von Thünen type, but a set of any number of such economies put together.

The question is how reasonable structural stability is as a modelling instrument. Obviously, it only is a safeguard that in model construction we have not brought in any assumptions that would destroy the whole system if there were some error in their precise formulation.

This certainly is a non-restrictive principle, often used implicitly in prudent modelling, but seldom explicitly recognized. In economics it seems only to have been applied once, for simpler systems, by Samuelson (1947) in the "correspondence principle".

The conclusions, however, seem astonishingly rich in view of how little restrictive the assumption is.

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