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THE MAXIMUM PRINCIPLE FOR A DIFFERENTIAL INCLUSION PROBLEM

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PREFACE

In this report, the Pontryagin principle is extended to optimal control problems with feedbacks (i.e., in which the controls depend upon the state). New techniques of non-smooth analysis (asymptotic derivatives of set-valued maps and functions) are used to prove this principle for problems with finite and infinite horizons.

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THE MAXIMUM PRINCIPLE FOR A DIFFERENTIAL INCLUSION PROBLEM

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The Pontriagin principle is extended to the case of minimization of solutions to differential inclusions by using a concept of derivative of set-valued maps.

Introduction

Consider a control system with feedbacks

(0.1) $\dot{x}(t) = f(x(t),u(t)) \quad , \quad u(t) \in U(x(t))$

where $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ and $U : \mathbb{R}^n \rightsquigarrow \mathbb{R}^m$ is a set valued map. Let S be the set of all solutions to (0.1) and assume $z \in S$ solves the following problem :

$$\text{minimize } \{g(x(0),x(1)) : x \in S\}$$

g being a function on $\mathbb{R}^{2n}$ taking values in $\mathbb{R} \cup \{+\infty\}$.

If there is no feedback, i.e. if U does not depend on x , and the datas are smooth enough the celebrated maximum principle (see Pontriagin and others [16]) tells us that for some absolutely continuous function $q : [0,1] \to \mathbb{R}^n$ the following holds true :

(0.2)
$$\begin{cases} - \dot{q}(t) = [\frac{\partial f}{\partial x}(z(t),\bar{u}(t))]^{\star} q(t) \\ \langle q(t),f(x(t),\bar{u}(t))\rangle = \max\limits_{u \in U} \langle q(t),f(z(t),u)\rangle \end{cases}$$

(0.3) $(-q(0),q(1)) = g'(z(0),z(1))$

where $\bar{u}$ is the corresponding control, $[\frac{\partial f}{\partial x}(z(t),\bar{u}(t))]^{\star}$ denotes the transpose of the Jacobian matrix of f with respect to x at $(z(t),\bar{u}(t))$, and g' is the derivative of g .

To study the necessary conditions in a more general case we have to consider the set valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ defined by :

$$F(x) := \{f(x,u) : u \in U(x)\}$$

and the associated differential inclusion

$$(0.1)' \qquad \dot{x} \in F(x)$$

Under some measurability assumptions on f and U it can be shown that the solutions to (0.1) and (0.1)' coincide.

This approach to optimal control problem was firstly proposed by Wazewski in [21] who was followed by many authors. (See for example [2], [3], [5], [6], [8], [11], [13], [14], [17], [21]).

For obtaining results similar to (0.2), (0.3) in the set valued case we need a notion generalizing the differential to a set valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and its transpose.

In this paper we use such a generalization, called the asymptotic differential $DF(x,y)$ and asymptotic co-differential $DF(x,y)^{\star}$ of F at $(x,y) \in \text{graph}(F)$. We consider also the related notion of asymptotic gradient $\partial_a g$ of a real valued function g .

The necessary conditions then take the following form :

There exists an absolutely continuous function $q : [0,1] \to \mathbb{R}^n$ satisfying the following conditions :

$$(0.2)' \qquad -\dot{q}(t) \in DF(z(t),\dot{z}(t))^{\star}(q(t))$$

$$(0.3)' \qquad (-q(0),q(1)) \in \partial_a g(z(0),z(1))$$

The outline of the paper is as follows. We devote the first section to some background definitions which we shall use. We state in section 2 the main theorem concerning the necessary conditions satisfied by an optimal solution to a differential inclusion problem. We show also how this problem can be embedded in a class of abstract optimization problems. This general problem is studied in section 3. Section 4 provides an example of application. In particular we extend in this paper to the non convex case some results obtained by Aubin-Clarke [3].

1 . Asymptotic differential and co-differential of a set valued map.

In what follows E denotes a Banach space, $\overset{\circ}{B}$ denotes the open unit ball in E and $<\ ,\ >$ the duality paring on $E^{\star} \times E$.

The tangent cone of Ursescu to a set $K \subset E$ at a point $x \in K$ is defined by

$$(1.1) \qquad I_K(x) := \bigcap_{\substack{\varepsilon > 0 \\ \delta > 0}} \bigcup_{\delta > 0} \bigcap_{h \in]0,\delta[} [\tfrac{1}{h}(K-x) + \varepsilon \overset{\circ}{B}]$$

The above cone is sometimes called the intermediate tangent cone since it lies between more familiar contingent cone (of Bouligand)

$$T_K(x) := \bigcap_{\varepsilon > 0} \bigcup_{h \in]0,\delta[} [\tfrac{1}{h}(K-x) + \varepsilon \overset{\circ}{B}]$$

and tangent cone (of Clarke)

$$C_K(x) := \bigcap_{\varepsilon > 0} \bigcup_{\substack{\delta > 0 \\ \rho > 0}} \bigcap_{\substack{x' \in B(x,\rho) \cap K \\ h \in]0,\delta[}} [\tfrac{1}{h}(K-x') + \varepsilon \overset{\circ}{B}]$$

Indeed

$$C_K(x) \subset I_K(x) \subset T_K(x)$$

(see [4], [6] for properties of $C_K(x)$, $T_K(x)$). The cone $I_K(x)$ is less known. We only state here

(1.2) Proposition. The following statements are equivalent :

(i) $v \in I_K(x)$

(ii) For all sequence $h_n > 0$ converging to zero there exists a sequence $v_n \in E$ converging to v such that $x + h_n v_n \in K$ for all n.

(iii) $\lim\limits_{h \to 0+} \frac{1}{h} d_K(x+hv) = 0$

In the study of some nonsmooth problems we are often led to deal with convex tangent cones. We define one of them.

(1.3) Definition. The asymptotic tangent cone to a subset K at $x \in K$ is given by

$$I_K^\infty(x) \quad := \quad \{u \in I_K(x) \; : \; u + I_K(x) \subset I_K(x)\}$$

$I_K^\infty(x)$ is closed convex cone. One can easily verify that $C_K(x) \subset I_K^\infty(x) \subset I_K(x) \subset T_K(x)$.

We now define the differential and co-differential of a set valued map F from E to a Banach space E_1 .

(1.4) Definition. The asymptotic differential of F at $(x,y) \in \text{graph}(F)$ is the set valued map $DF(x,y) : E \overset{\rightarrow}{\rightarrow} E_1$ defined by

$$v \in DF(x,y)(u) \quad \text{if and only if} \quad (u,v) \in I^\infty_{\text{graph}(F)}(x,y)$$

The asymptotic co-differential of F at $(x,y) \in \text{graph}(F)$ is the set valued map $DF(x,y)^\star : E_1^\star \overset{\rightarrow}{\rightarrow} E^\star$ defined by

$$q \in DF(x,y)^\star(p) \quad \text{iff} \quad \langle q,u\rangle - \langle p,v\rangle \leqslant 0 \quad \text{for all} \quad v \in DF(x,y)(u)$$

(1.5) Remark. We give in [11] another characterization of $DF(x,y)^\star$. Let us only mention that $q \in F(x,y)^\star(p)$ means that $(q,-p)$ is contained in the negative polar cone to $I^\infty_{\text{graph}(F)}(x,y)$, the asymptotic normal cone to graph(F) at (x,y) .

Let $g : E \to \mathbb{R} \cup \{+\infty\}$, $x \in \text{Dom}(g)$. Define

$$F(y) \quad = \quad \begin{cases} g(y) + \mathbb{R}_+ & \text{when} \quad y \in \text{Dom}(g) \\ \emptyset & \text{when} \quad g(y) = +\infty \end{cases}$$

Then $\text{graph}(F) = \text{Epi}(g)$ (Epigraph of g).

(1.6) Definition. The subset

$$\partial_a g(x) \quad = \quad DF(x,g(x))^\star(1)$$

is called the asymptotic gradient of g at x .

In the case when g is regularly Gâteaux differentiable, i.e. it has the Gâteaux derivative $g'(x) \in E^\star$ and for all $u \in E$

$$\lim_{\substack{u' \to u \\ h \to 0_+}} \frac{g(x+hu') - g(x)}{h} = \langle g'(x), u\rangle ,$$

we have

$$\partial_a g(x) = \{g'(x)\}$$

There is also another way to introduce $\partial_a g(x)$.

Following Rockafellar [19], when a function $\Phi : U \times V \to \mathbb{R} \cup \{+\infty\}$ is given, we define

$$\limsup_{v' \to v} \inf_{u' \to u} \Phi(v',u') := \sup_{\varepsilon > 0} \inf_{\delta > 0} \sup_{v' \in B(v,\delta)} \inf_{u' \in B(u,\varepsilon)} \Phi(v',u')$$

Consider $g : E \to \mathbb{R} \cup \{+\infty\}$, $x \in \mathrm{Dom}(g)$. For all $u \in E$ set

$$i_+ g(x)(u) := \limsup_{h \to 0+} \inf_{u' \to u} \frac{g(x+hu') - g(x)}{h}$$

and

$$i_+^\infty g(x)(u) := \sup_v \left(i_+ g(x)(u+v) - i_+ g(x)(v)\right)$$

The function $i_+^\infty g(x) : E \to \mathbb{R} \cup \{+\infty\}$ is called the asymptotic derivative and enjoys the following nice properties

$$I^\infty_{\mathrm{Epi}(g)}(x,g(x)) = \mathrm{Epi}\ (i_+^\infty g(x))$$

$$\partial_a g(x) = \{q \in E^\star : \langle q,u\rangle \leqslant i_+^\infty g(x)(u) \text{ for all } u \in E\}$$

(see [11]) .

2 . The differential inclusion problem.

Let $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set valued map and, let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a Lipschitzean function, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$. We denote by S the set of all solutions to the differential inclusion

$$\dot{x} \in F(x)$$

i.e.

$$S = \{x \in W^{1,1}(0,1) : \dot{x}(t) \in F(x(t)) \quad \text{a.e.}\}$$

For a function $z \in S$ the contingent cone to S at z is given by

$$T_S(z) = \{w \in W^{1,1}(0,1) : \text{for some sequence } h_n > 0 \text{ converging to zero there exists a sequence } w_n \in S \text{ such that } z + h_n w_n \in S \ , \ \lim_{n \to \infty} w_n = w \}$$

Assume $z \in S$ solves the following problem

$$\text{minimize} \left\{ g(x(0),x(1)) + \int_0^1 \varphi(x(t))dt : x \in S \right\}$$

In order to characterize z we assume the following surjectivity hypothesis

(*H*) For some $p > 1$ and all $u,e \in L^p$ there exists a solution $w \in W^{1,p}(0,1)$ to the "linearized" problem

(i) $(w(0),w(1)) \in \text{Dom}\,(i_+^\infty g(z(0),z(1)))$

(ii) $\dot{w}(t) \in DF(z(t),\dot{z}(t))(w(t)+u(t))+e(t)$ a.e.

and

(iii) if $u = e = 0$ then every w satisfying (i), (ii) belongs to $T_S(z)$.

Remark. The last part of the above hypothesis holds in particular when $z(t) \in \text{Int}(\text{Dom } F)$ and F is Lipschitzean in Hausdorff metric. Indeed if $\dot{w}(t) \in DF(z(t),\dot{z}(t))(w(t))$ then there exists a sequence $(u_k,v_k) \in L^1$ converging to $(w,\dot{w})$ such that $[(z,\dot{z}) + \frac{1}{k}(u_k,v_k)](t) \in \text{graph}(F)$ for all $k > 0$.

Let $y_k(t) = w(0) + \int_0^t v_k(\tau)d\tau$ and $\alpha_k(t) = u_k(t) - y_k(t)$. Clearly $\alpha_k \to 0$ in L^1 when $k \to +\infty$ and

$$\text{dist}\left(\dot{z}(t) + \frac{1}{k}\dot{y}_k(t) , F\left(z(t) + \frac{1}{k}y_k(t)\right)\right) \leq \frac{L}{k}\alpha_k(t)$$

where L denotes the Lipschitz constant of F . Then by Corollary 2.4.1 [2] there exists a constant C and functions $x_k \in S$ such that for all $k \geq 1$

$$|\dot{x}_k(t) - \dot{z}(t) - \frac{1}{k}\dot{y}_k(t)| \leq \frac{C}{k}\left[\alpha_k(t) + \int_0^1 \alpha_k(\tau)d\tau\right]$$

$$|x_k(t) - z(t) - \frac{1}{k}y_k(t)| \leq \frac{C}{k}\int_0^1 \alpha_k(\tau)d\tau$$

and therefore $w \in T_S(z)$.

(2.1) Theorem. Assume that surjectivity hypothesis (H) is verified. Then there exists a solution $q \in W^{1,p_\star}(0,1)$ (where $\frac{1}{p} + \frac{1}{p_\star} = 1$) of the adjoint inclusion

$$-\dot{q}(t) \in \partial_a\varphi(z(t)) + DF(z(t),\dot{z}(t))^\star(q(t)) \qquad \text{a.e.}$$

$$(-q(0),q(1)) \in \partial_a g(z(0),z(1))$$

Proof. We first reduce the above problem to an abstract optimization problem which has many other applications. The reduction is done in two steps. Set $E = L^p(0,1;\mathbb{R}^n)$, $W = W^{1,p}(0,1;\mathbb{R}^n)$, $T = \mathbb{R}^n \times \mathbb{R}^n$, $\gamma(w) = (w(0),w(1))$, $Lw = \dot{w}$ for all $w \in W$.

Step 1. We claim first that if $\dot{w}(t) \in DF(z(t),z(t))(w(t))$ for all $t \in [0,1]$ then

$$i_+^\infty f(z)(w) + i_+^\infty g(\gamma z)(\gamma w) \geq 0$$

Indeed by (H) there exist sequences $h_n > 0$ and $w_n \in W$ converging to zero and w respectively such that $z + h_n w_n \in S$. Since z is a minimiser we have $f(z+h_n w_n) + g(\gamma z + h_n\gamma w_n) \geq f(z) + g(\gamma z)$. Thus

$$\limsup_{\substack{w' \to w \\ h \to 0+}} \frac{f(z+hw') + g(\gamma z+h\gamma w') - f(z) - g(\gamma z)}{h} \geq 0$$

and therefore using Lipschitzeanity of f we obtain

$$0 \leq \limsup_{h \to 0+} \inf_{w' \to w} \frac{g(\gamma z + h\gamma w) - g(\gamma z)}{h} + \limsup_{\substack{w' \to w \\ h \to 0+}} \frac{f(z+hw') - f(z)}{h}$$

$$\leq i_+^\infty g(\gamma z)(\gamma w) + i_+^\infty f(z)(w)$$

<u>Step 2</u>. Let $F : E \overset{\to}{\to} E$ be defined by $F(x) = \{y \in E : y(t) \in F(x(t)) \text{ a.e.}\}$. Thus z solves the following problem

$$\text{minimize } \{f(x)+g(\gamma x) : x \in W , Lx \in F(x)\}$$

Consider the closed convex cone

$$C = \{(x,y) \in E \times E : y(t) \in DF(z(t),\dot{z}(t))(x(t)) \text{ a.e.}\}$$

Using the measurable selection theorems (see for example [20]) one can verify that $C \subset I_{\text{graph}(F)}(z,\dot{z})$. (See [11] for the details of the proof). Let C^- be the negative polar to C . We claim that if a function $q \in W^{1,p_\star}(0,1;\mathbb{R}^n)$ satisfies the following inclusions

$$(-\dot{q},-q) \in \partial_a f(z) \times \{0\} + C^-$$

$$(-q(0),q(1)) \in \partial_a g(\gamma z)$$

then q satisfies also all reauirement of Theorem. This can be directly proved using a contradiction argument (see [11]).

Thus to achieve the proof we have only to verify the existence of $q \in W^{1,p_\star}(0,1;\mathbb{R}^n)$ as above. This will be done in the next section where an abstract problem is treated.

3 . The abstract problem.

Consider reflexive Banach spaces W,H,E,T where W is continuously embedded into H by the canonical injection i . Let $L \in \mathcal{L}(W,E)$, $\gamma \in \mathcal{L}(W,T)$ be continuous linear operators and γ satisfies the

"trace property" γ has a continuous right inverse and the kernel W_o of γ is dense in H

We denote by i_o (L_o) the restriction of i (respectively L) to W_o . Define

$$E_o^* = \{p \in E^* : L_o^* p \in H^*\}$$

Thus L_o^* maps E_o^* to H^* . (For the problem considered in § 2 $H = E$, $E_o^* = W^{1,p*}(0,1;\mathbb{R}^n)$ and $L_o^* q = -\dot{q}$ on E_o^*). We have the following abstract Green formula (see [1]) :

There exists a unique operator $\beta^* \in \mathcal{L}(E_o^*,T^*)$ such that for all $u \in W$, $p \in E_o^*$

$$\langle L_o^* p,u\rangle - \langle p,Lu\rangle = \langle \beta^* p,\gamma u\rangle$$

Let a closed convex cone $C \subset H \times E$ and functions $\pi : W \to R$, $\psi : T \to R \cup \{+\infty\}$ be given. We assume that the epigraphs of π,ψ are closed convex cones and define the closed convex processes $G : H \overset{\to}{\to} E$, $G^* : E^* \overset{\to}{\to} H^*$ by

$$v \in G(u) \quad \text{if and only if} \quad (u,v) \in C$$

$$r \in G^*(q) \quad \text{if and only if} \quad (r,-q) \in C^-$$

We assume that the element $w = 0$ is a solution of the problem

$$\text{minimize} \quad \{\pi(w) + \psi(\gamma w) : Lw \in G(w)\}$$

(3.1) Theorem. Assume that the following surjectivity assumption holds true :

for all $(u,v,e) \in H \times H \times E$ there exists a solution $w \in W$ to the problem :

$$\begin{cases} \text{(i)} \quad Lw \in G(w+u) + e \\ \text{(ii)} \quad w \in \mathrm{Dom}(\pi) \quad , \quad \gamma w \in \mathrm{Dom}(\psi) \end{cases}$$

Then there exists $q \in E_o^\star$ such that

$$L_o^\star q \in \partial_a \pi(0) + G^\star(q)$$

$$-\beta^\star q \in \partial_a \psi(0)$$

<u>Remark</u>. For the problem considered in § 2 we have :

$$\begin{cases} \pi(w) = i_+^\infty f(z)(w) \quad ; \quad \psi(t) = i_+^\infty g(\gamma z)(t) \quad ; \quad \partial_a \pi(0) = \partial_a f(z) \quad ; \\ \partial_a \psi(0) = \partial_a g(\gamma z) \end{cases}$$

The proof of Theorem 3.1 follows immediately from the following Lemmas.

(3.2) <u>Lemma</u>. Under the assumptions of Theorem 3.1 the set A defined by

$$A := i^\star \partial_a \pi(0) + \gamma^\star \partial_a \psi(0) + \{i^\star r - L^\star q \, : \, r \in G^\star(q)\}$$

(where $i^\star$ is the adjoint of i) is closed in $W^\star$.

<u>Proof</u>. Let $a_n = i^\star \alpha_n + \gamma^\star \alpha_n' + i^\star r_n - L^\star q_n$, where $\alpha_n \in \partial_a \pi(0)$, $\alpha_n' \in \partial_a \psi(0)$, $(r_n, -q_n) \in C^-$, n=1,2,... . Assume $\lim_{n \to \infty} a_n = a$ in $W^\star$. We claim that $\{(\alpha_n, r_n, -q_n)\}_{n \geq 1}$ is bounded. This will be proved if we show that for all $(u,v,e) \in H \times H \times E$

$$\sup_{n \geq 1} (\langle \alpha_n, v\rangle + \langle r_n, u\rangle + \langle q_n, e\rangle) < +\infty \tag{3.3}$$

Let w be such that $Lw \in G(w+u) + e$, $w \in \mathrm{Dom}(\pi)$, $\gamma w \in \mathrm{Dom}(\psi)$. Then $e = Lw - y$, where $(w+u,y) \in C$. Therefore $\langle \alpha_n, v\rangle + \langle r_n, u\rangle + \langle q_n, e\rangle = \langle \alpha_n, v\rangle + \langle r_n, u\rangle + \langle L^\star q_n, w\rangle - \langle q_n, y\rangle = \langle \alpha_n, v+w\rangle + \langle \alpha_n', \gamma w\rangle + \langle (r_n, -q_n), (u+w,y)\rangle - \langle a_n, w\rangle \leq \pi(v+w) + \psi(\gamma w) - \langle a_n, w\rangle$ and (3.3) follows. Thus by reflexivity we may assume that $(\alpha_n, r_n, q_n) \rightharpoonup (\alpha, r, q)$ weakly in $H^\star \times H^\star \times E^\star$. By Mazur lemma [9] and convexity of $\partial_a \pi(0)$, C^- we have $\alpha \in \partial_a \pi(0)$, $(r,-q) \in C^-$. Let σ be the continuous right inverse of γ . Then $\alpha'_n = \sigma^\star\gamma^\star\alpha'_n = \sigma^\star(a_n - i^\star\alpha_n - i^\star r_n + L^\star q_n)$ is weakly convergent to some $\alpha' \in \partial_a\psi(0)$. Hence $a \in A$.

(3.4) Lemma. The following statements are equivalent :

(1) $\pi(w) + \psi(\gamma w) \geq 0$ for all $Lw \in G(w)$

(2) There is $q \in E^\star_o$ such that

$$L^\star_o q \in \partial_a\pi(0) + G^\star(q)$$

$$-\beta^\star q \in \partial_a\psi(0)$$

Proof. If (1) holds, then using the separation theorem we show that

$$0 \in i^\star\partial_a\pi(0) + \gamma^\star\partial_a\psi(0) + i^\star G^\star(q) - L^\star q$$

Let $q \in E^\star$, $\alpha \in \partial_a\pi(0)$, $\alpha' \in \partial_a\psi(0)$, $r \in G^\star(q)$ be such that $i^\star\alpha + \gamma^\star\alpha' + i^\star r - L^\star q = 0$. Thus $L^\star_o q = i^\star_o\alpha + i^\star_o r$. Since W_o is dense in H it implies that $L^\star_o q \in H^\star$ and by consequence $q \in E^\star_o$. Moreover the Green formula implies $0 = \langle\alpha,w\rangle + \langle\alpha',\gamma w\rangle + \langle(r,-q),(w,Lw)\rangle = \langle\alpha'+\beta^\star q,\gamma w\rangle$ for all $w \in W$. Since $\gamma W = T$ we proved $\alpha' + \beta^\star q = 0$ and thus (2) .

To prove the converse, assume (2) holds. Then for some $q \in E^\star_o$, $\alpha \in \partial_a\pi(0)$, $\alpha' \in \partial_a\psi(0)$

$$L^\star_o q = \alpha + r \quad , \quad -\beta^\star q = \alpha'$$

and by Green formula $\alpha + r = \gamma^\star\beta^\star q + L^\star q = L^\star q - \gamma^\star\alpha'$, $\alpha + \gamma^\star\alpha' = L^\star q - r$. Thus if $Lw \in G(w)$ we have $\pi(w) + \psi(\gamma w) \geq \langle\alpha,w\rangle + \langle\alpha',\gamma w\rangle = \langle\alpha+\gamma^\star\alpha',w\rangle = \langle L^\star q-r,w\rangle = -\langle(r,-q),(w,Lw)\rangle \geq 0$, which proves (1) and achieves the proof of Lemma 3.4.

Thus the proof of Theorem 3.1 is completed.

4 . An example.

Let U be a compact subset in $\mathbb{R}^n$, A be n x n matrix, B be n x m matrix and let two lipschitzean functions $\varphi : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be given.

Consider the following problem :

(4.1) $$\text{minimize } [g(x(0),x(1)) + \int_0^1 \varphi(x(t))dt]$$

over the set of solutions to the control system

(4.2) $$\dot{x}(t) = Ax(t) + Bu(t) \quad , \quad u(t) \in U$$

The corresponding differential inclusion then has the form

$$\dot{x} \in F(x) \quad , \quad F(x) = Ax + BU$$

Assume a trajectory-control pair $(z,\bar{u})$ solves (4.1), (4.2).

(4.3) <u>Theorem</u>. There exists an absolutely continuous function q such that

$$\dot{q}(t) \in \partial_a \varphi(z(t)) - A^\star q(t) \qquad \text{a.e. in } [0,1]$$

$$\langle q(t),s\rangle \leqslant 0 \qquad \text{for all } s \in I^\infty_{BU}(B\bar{u}(t))$$

$$(-q(0),q(1)) \in \partial_a g(\dot{z}(0),z(1))$$

<u>Proof</u>. To use Theorem 2.1 we verify directly that $DF(z(t),\dot{z}(t))(v) = Av + I^\infty_{BU}(B\bar{u}(t))$. Fix any $s > 1$ and let $p > 1$ be defined from the equation $\frac{1}{p} + \frac{1}{s} = 1$. Clearly for all $u,e \in L^p$ there exists $w \in W^{1,p}(0,1)$ solving the problem

$$\dot{w}(t) \in Aw(t) + Au(t) + e(t) + I^\infty_{BU}(B\bar{u}(t))$$

On the other hand if w is such that

$$\dot{w}(t) \in Aw(t) + I^\infty_{BU}(B\bar{u}(t))$$

then we can find a sequence $Bu_k \in L^1$ converging to $w(t) - Aw(t)$ such that $B\bar{u}(t) + \frac{1}{k} Bu_k(t)) \in BU$ a.e.. Let w_k be defined from the equation

$$\dot{w}_k(t) = Aw_k(t) + Bu_k(t) \quad , \quad w_k(0) = w(0) \quad .$$

Then $z + h_k w_k$ is a solution to (4.2) and it implies that the hypothesis (H) from § 2 is verified. On the other hand if $r \in DF(z(t),z(t))^{\star}(-\bar{q})$ then for all $v \in \mathbb{R}^n$, $s \in I^{\infty}_{BU}(B\bar{u}(t))$ we have $<(v,Av+s),(r,\bar{q})> \leqslant 0$ and hence $<v,r+A^{\star}\bar{q}> + <s,\bar{q}> \leqslant 0$. It implies that

$$DF(z(t),\dot{z}(t))^{\star}(-\bar{q}) = - A^{\star}\bar{q} \quad ; \quad \bar{q} \in \left(I^{\infty}_{BU} \; (B\bar{u}(t))\right)^{-}$$

and by Theorem 2.1 the proof is complete.

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