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Frankowska, H.

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Halina Frankowska

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
A-2361 Laxenburg, Austria

PREFACE

The theory of stochastic differential equations with reflecting boundary conditions leads to the "Skorohod" problem. This report proposes a solution to this problem using techniques from viability theory and non-smooth analysis, allowing very general situations to occur.

The research described here was conducted within the framework of the Dynamics of Macrosystems study in the System and Decision Sciences Program.

ANDRZEJ WIERZBICKI
Chairman
System and Decision Sciences Program

1 . Introduction

The theory of stochastic differential equations with reflecting boundary conditions leads to the "Skorohod" problem, as it was remarked by N. El Karoui and M. Chaleyat-Maurel in [13].

The Skorohod problem is defined as follows. We consider a compact subset K in $\mathbb{R}^n$ with nonempty interior and a vector field n_x on the boundary ∂K , not necessarily single valued, such that $n_x \in S^{n-1}$ for all x in ∂K . Let a function $w \in C(\mathbb{R}_+,\mathbb{R}^n)$ be given, $w(0) \in K$ and let $x \in C(\mathbb{R}_+,K)$, $k \in C(\mathbb{R}_+,\mathbb{R}^n)$. We denote by $|k|_t$ the total variation of k on $[0,t]$ and by $1_{\partial K}$ the characteristic function of ∂K .

The pair (x,k) is called a solution to the Skorohod problem (w,K,n_x) if for all $t \geqslant 0$

(i) $w(t) + k(t) = w(t)$

(ii) $|k|_t < \infty$

(iii) $|k|_t = \int_0^t 1_{\partial K}(x(s)) \, d|k|_s$

(iv) $k(t) = \int_0^t \mathcal{E}(s) \, d|k|_s$, where $\mathcal{E}(s) \in n(x(s))$

The existence and uniqueness of solutions to (w,K,n_x) has been first considered - via explicit formulas - in the particular case when K is some half space (see N. El Karoui, M. Chaleyat-Maurel [13] and N. El Karoui, M. Chaleyat-Maurel, B. Marechal [14]) ; the first general study was done by H. Tanaka [31] in the case when the domain is convex and vector field n_x is normal.

Finally P.L. Lions, A.S. Sznitman [24] studied the case when $K = \overline{\Omega}$ and the domain Ω has some "semi-smoothness" property and when the vector field n_x is smooth : in [24] the existence, and the uniqueness for bounded variation data is proved and these results are applied to the solvability of stochastic differential equations (or more generally for data given by semimartingales).

We want to complete these results, by a different approach. As usual, we have to allocate some smoothness requirement between the function w and the boundary ∂K of K for obtaining existence. We shall provide two types of compromise : one assumes that w has a contingent derivative, and that the vector field n_x has a closed graph (Theorem 3.3) ; the second assumes only that w is continuous but requires more assumptions on the normal cone to K .

We shall follow a direct approach to the Skorohod problem, by looking at it as a viability problem for a differential inclusion. Set

$$\Gamma(x) := \begin{cases} \mathrm{cl} \bigcup_{\lambda \geq 0} \lambda n_x & \text{for } x \in \partial K \\ \{0\} & \text{for } x \in \mathrm{Int}\, K \end{cases}$$

This approach consists in looking for a pair of continuous functions (x,k) satisfying for all $t \geq 0$

(i) $x(t) \in K$

(ii) $x(t) + k(t) = w(t)$

(iii) $\dot{k}(t) \in L^1_{loc}$

(iv) $\dot{k}(t) \in \Gamma(x(t))$ for almost every $t \geq 0$

Or equivalently, by eliminating $x(\cdot)$ in the above, we can look for an absolutely continuous function $k : \mathbb{R}_+ \to \mathbb{R}^n$ satisfying

$$\begin{aligned} &\text{(i)} \quad w(t) - k(t) \in K \\ &\text{(ii)} \quad \dot{k}(t) \in L^1_{loc} \\ &\text{(iii)} \quad \dot{k}(t) \in \Gamma(w(t)-k(t)) \qquad \text{for almost all } t \geqslant 0 . \end{aligned}$$

This problem is a particular case of a viability problem of the following type :

Let K be a closed subset of $\mathbb{R}^n$, $F : K \rightrightarrows \mathbb{R}^n$ be a set valued map, $x_o \in K$. We are looking for a solution of the problem :

$$\text{(VP)} \qquad \begin{cases} \dot{x} \in F(x) \\ x(0) = x_o \quad , \quad x(t) \in K \text{ for all } t \geqslant 0 . \end{cases}$$

Therefore for studying the Skorohod problem, we can use a viability theorem providing necessary and sufficient conditions for the existence of a solution to (VP), which we now explain :

Let $T_K(x)$ be the contingent cone to K at x (see [2], [4]) or section 2 of this paper for a definition). Then under some continuity assumptions on F the problem (VP) has a solution $x(\cdot)$ if and only if the tangential condition

$$T_K(x) \cap F(x) \neq \emptyset$$

holds true.

In this way we obtain an existence theorem for a general set K which, may be, could be used for solving stochastic differential equations.

The outline of this paper is as follows. We shall give in section 2 some background notes and we shall state in section 3 two main theorems. In the fourth section, we specialize the map $\Gamma(x)$ to be the normal cone to K at x and consider also the case of oblique reflecting boundary conditions. We prove the main theorems in the fifth and sixth sections.

The author would like to thank P.L. Lions for raising up questions studied here and many helpful discussions.

2 . Background notes.

We denote here by $\overset{\circ}{B}$ (B) the open (respectively closed) unit ball in $\mathbb{R}^n$, by S^{n-1} its boundary, the unit sphere in $\mathbb{R}^n$. Let K be a subset of $\mathbb{R}^n$.

a) Tangent and normal cones.

The intermediate tangent cone (of Ursescu) $I_K(x)$ to K at x is given by

$$I_K(x) \quad := \quad \bigcap_{\varepsilon > 0} \quad \bigcup_{\delta > 0} \quad \bigcap_{h \in]0,\delta[} \left(\frac{K-x}{h} + \varepsilon \overset{\circ}{B} \right)$$

(see [15], [16] or [26]) .

(2.1) Proposition. The following statements are equivalent

(i) $v \in I_K(x)$

(ii) for all sequence $h_i > 0$ converging to zero there exist a sequence $v_i \in \mathbb{R}^n$ converging to v such that $x + h_i v_i \in K$ for all i .

(iii) $\lim_{h \to 0+} \frac{1}{h} d_K(x+hv) = 0$, where $d_K(y) := \text{dist}(y,K)$

(2.2) Definition. The asymptotic tangent cone to K at x is the recession cone to $I_K(x)$, which is defined by

$$I_K^\infty(x) := \{u \in I_K(x) : u+v \in I_K(x) \text{ for all } v \in I_K(x)\}$$

The asymptotic normal cone to K at x is the negative polar cone of $I_K^\infty(x)$, which is given by

$$N_K^\infty(x) := \{p \in \mathbb{R}^n : \langle p,v\rangle \leqslant 0 \text{ for all } v \in I_K^\infty(x)\}$$

(see [16] for further properties).

(2.3) Remark : The asymptotic tangent cone is a closed convex cone contained in the contingent cone of Bouligand

$$T_K(x) := \bigcap_{\substack{\varepsilon > 0 \\ \delta > 0}} \bigcup_{h \in]0,\delta[} \left(\frac{K-x}{h} + \varepsilon \mathring{B}\right)$$

and containing the tangent cone (of Clarke)

$$C_K(x) := \bigcap_{\varepsilon > 0} \bigcup_{\substack{\delta > 0 \\ \rho > 0}} \bigcap_{\substack{h \in]0,\delta[\\ y \in B(x,\rho) \cap K}} \left(\frac{K-y}{h} + \varepsilon \mathring{B}\right)$$

(see [4], [7], [28]).

If ∂K is smooth (locally a graph of a differentiable function) then $I_K^\infty(x) = T_K(x)$ coincide with the usual tangent space to K at x and if K is convex these three cones coincide with the tangent cone $\text{cl}\left(\bigcup_{h > 0} \frac{1}{h}(K-x)\right)$ of convex analysis. For $x \in \text{Int } K$ they are equal to whole space.

b) Monotone maps.

We recall that a set valued map $F : K \overset{\to}{\to} \mathbb{R}^n$ is called monotone if

$$\langle n_1 - n_2 \, , \, x_1 - x_2 \rangle \geq 0$$

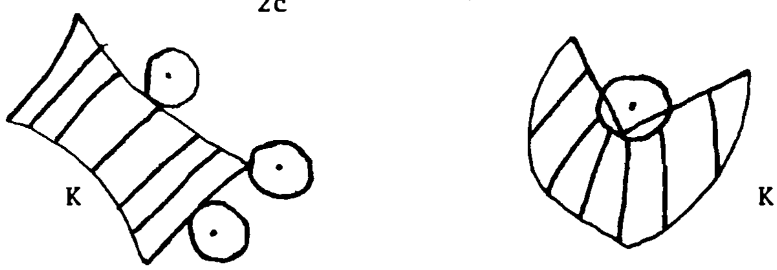
for all $n_i \in F(x_i)$, $x_i \in K$, $i=1,2$.

The set K is called weakly convex if there exists $c > 0$ such that the map $x \to B \cap N_K^\infty(x) \to cx$ is monotone. It can be verified that this condition is equivalent to the following :

for all $x \in \partial K$ there exists $y \in \mathbb{R}^n$ such that

$$K \cap \left(y + \frac{1}{2c}\overset{\circ}{B}\right) = \emptyset \qquad , \qquad K \cap \left(y + \frac{1}{2c} B\right) = \{x\}$$

Geometrically it means that at every point $x \in \partial K$ there exists a supporting ball of radius $\frac{1}{2c}$ (see [8]).



weakly convex   not weakly convex

3 . Main results.

For a function $k : \mathbb{R}_+ \to \mathbb{R}^n$ let $|k|_t$ denote the total variation of k on $[0,t]$.

Let K be a closed subset of $\mathbb{R}^n$ and let ∂K denote its boundary The characteristic function of the boundary ∂K is defined by

$$1_{\partial K}(x) := \begin{cases} 1 & \text{if } x \in \partial K \\ 0 & \text{otherwise} \end{cases}$$

Let $\Gamma : \mathbb{R}^n \to \mathbb{R}^n$ be a set valued map whose values $\Gamma(x)$ are closed convex cones such that $\Gamma(x) = \{0\}$ when $x \in \text{Int } K$.

Consider a function $w : \mathbb{R}_+ \to \mathbb{R}^n$ such that $w(0) \in K$.

(3.1) <u>Definition</u>. A pair (x,k) of continuous functions $x : \mathbb{R}_+ \to K$, $k : \mathbb{R}_+ \to \mathbb{R}^n$ is called a solution to the Skorohod problem (w,K,Γ) if for all $t \geq 0$

$$(3.2) \quad \begin{cases} \text{(i)} & |k|_t < \infty \\ \text{(ii)} & x(t) + k(t) = w(t) \\ \text{(iii)} & |k|_t = \int_0^t 1_{\partial K}(x(s)) \, d|k|_s \\ \text{(iv)} & k(t) = \int_0^t \&(s) \, d|k|_s \end{cases}$$

where $\quad \&(s) \in \Gamma(x(s)) \cap S^{n-1} \quad$ if $\quad x(s) \in \partial K$

First, we do not impose any smoothness assumptions on the boundary ∂K , but we assume that w satisfies a weak smoothness requirement ;

$$(\star) \quad \begin{cases} \text{For some function } \alpha \in L^1_{loc} \\ \liminf\limits_{s \to t+} \dfrac{\|w(s) - w(t)\|}{s - t} \leq \alpha(t) \end{cases}$$

Observe that functions of bounded variation satisfy the above property.

Second , we assume only that w is continuous, but the price to pay is to require that the set K is weakly convex (see section 2 for the definition).

(3.3) Theorem. Assume Γ has a closed graph, w satisfies ($\star$) and for some $\nu > 0$, $M > 0$ and all $x \in \partial K$ there exists a symetric positive definite matrix $A(x)$ such that

$$A(x) \geqslant \nu I \quad , \quad \| A(x) \| \leqslant M \quad \text{and} \quad N_K^{\infty}(x) \subset A(x)\ \Gamma(x)$$

Then the problem (w,K,Γ) has a solution. If we assume moreover that A does not depend on x and that there exists $c \geqslant 0$ such the map $x \to A\Gamma(x) \cap N + cx$ is monotone, then there exists a unique solution to (w,K,Γ) .

We shall prove this theorem in section 5.

a) Case when K is smooth.

Assume that the boundary ∂K is smooth (of class C^1) and w satisfies the assumption ($\star$). Let n_x be the unit onter normal to K at $x \in \partial K$ and $N_K(x)$ be the cone generated by n_x , i.e.
$N_K(x) := \bigcup_{\lambda \geqslant 0} \lambda n_x$, for $x \in \partial K$, and $N_K(x) = \{0\}$ for $x \in \text{Int } K$.
Then the graph of $N_K(\cdot)$ is closed and satisfies the assumptions of Theorem 3.3 with $A(x) = \text{Id}$. Thus in this case the problem $(w,K,N_K(\cdot))$ has a solution.

b) Case when K is convex.

Let K be convex and let $N_K(x)$ be the normal cone to K at x in the sense of convex analysis. Then the set valued map $N_K(\cdot)$ has a closed graph and by section 2 , $N_K(x) = N_K^{\infty}(x)$. Moreover $N_K(\cdot)$ is a monotone map. Hence if a function $w \in C(\mathbb{R}_+,\mathbb{R}_n)$ is such that the condition (★) holds by Theorem 3.3 the problem $(w,K,N_K(\cdot))$ has a unique solution.

c) Case when w is of bounded variation.

Assume $w \in C(\mathbb{R}_+,\mathbb{R}^n)$ and $|w|_t < \infty$ for all $t \geqslant 0$, that is the total variation of w on $[0,t]$ is finite. Then $\frac{d}{dt}|w|_t \in L^1_{loc}$ and therefore w verifies the condition (★). Then Theorem 3.3 implies that if Γ satisfies all assumptions of Theorem 3.3 the problem (w,K,Γ) has a solution.

The last case suggests another approach for solving the Skorohod problem. Namely if w is only continuous we can approximate it by smooth functions w_i converging almost uniformly to w . Then if Γ satisfies the requirement of (3.3), the problem (w_i,K,Γ) has a solution (x_i,k_i) . All we need then is the sequential precompactness of $\{(x_i,k_i)\}_{i \geqslant 1}$ in an appropriate topology. To have this precompactness property we shall require a monotonicity condition on the map Γ .

We say that a cone $C \subset \mathbb{R}^n$ has a compact sole if there exists a compact $X \subset \mathbb{R}^n \setminus \{0\}$ such that $C = \bigcup_{\lambda \geqslant 0} \lambda X$ (such a X being a "sole" of the cone C , generating C).

(3.4) Theorem. Assume that Γ has a closed graph and $\Gamma(x)$ has a compact sole for all $s \in \partial K$. Assume further that for some $\nu > 0$, $c > 0$ and all $x \in K$ there exists a symmetric matrix $A(x)$ such that

(i) The set valued map $x \to A(x)\Gamma(x) \cap B + cx$ is monotone

(ii) $A(x) \geqslant \nu I$ for all $x \in \partial K$, $A(\cdot)$ is continuous

(iii) $N^\infty_K(x) \subset A(x)\Gamma(x)$

Then for all $w \in C(\mathbb{R}_+, \mathbb{R}^n)$, $w(0) \in K$ the problem (w,K,Γ) has a solution. Moreover if A does not dpend on x then the solution is unique.

The proof of this theorem, which is related in many aspects to the one of [24] is given in section 6.

4 . Examples of applications.

Let K be a closed subset of $\mathbb{R}^n$, $C_K(x)$ be the tangent cone (of Clarke) to K at $x \in K$ (see Remark 2.3 for definition and [4], [7] for an exposition). Let $N_K(x)$ be the negative polar cone of $C_K(x)$. When the boundary ∂K is of class C^1 then $N_K(x)$ is spanned by the unit outer normal to K at x .

(4.1) Lemma. The set valued function $x \to N_K(x)$ has a closed graph if either one of the following conditions holds

(i) For all $x \in \partial K$, $N_K(x)$ has a compact sole

(ii) For all $x \in K$, $C_K(x) = T_K(x)$

Proof. (i) is equivalent to Int $C_K(x) \neq \emptyset$. Thus by [28] (i) implies that the set valued map $x \to C_K(x)$ is lower semicontinuous. If (ii) holds then by [8] also $C_K(\cdot)$ is lower semi continuous. This is equivalent to say that the map $x \to N_K(x)$ has a closed graph (see [3]).

a) Case when $\Gamma(x)$ is the normal cone to K at x .

(4.2) Corollary. Assume that either condition (i) or (ii) of Lemma 3.1 holds and that $w \in C(\mathbb{R}_+, \mathbb{R}^n)$, $w(0) \in K$, $|w|_t < \infty$ for all $t > 0$ (i.e. w is of bounded variation on finite intervals). Then the problem

$(w,K,N_K(\cdot))$ has a solution. Moreover if the set K is weakly convex then there exists a unique solution to $(w,K,N_K(\cdot))$.

Proof. The first claim follows directly from the case c) of section 3 and Lemma 4.1. The weak convexity of K means the monotonicity of map $x \to N_K(x) \cap B + cx$ for some $c > 0$. By [8], if K is weakly convex, then $T_K(x) = C_K(x)$ for all $x \in K$. Thus $N_K(x) = N_K^\infty(x)$ and therefore the map $x \to N_K(x) \cap B + cx$ is monotone. By Theorem 3.3 then there exists a unique solution to $(w,K,N_K(\cdot))$.

(4.3) Remark. Assumptions (1), (5) from [24] imply that the vector field n_x considered there is the compact sole of $N_K(x)$ and that for some $c > 0$ the set valued map $x \to N_K(x) \cap B + cx$ is monotone. Hence $N_K(\cdot)$ satisfies assumptions of Corollary 4.2.

We shall give next another application of Theorem 3.3 :

b) Case of oblique reflecting boundary conditions :

(4.4) Corollary. Assume ∂K is locally the graph of a differentiable function and let n_x be the unit outer normal to K at $x \in \partial K$. Let $\gamma : \partial K \to S^{n-1}$ be a continuous function such that for some $\nu > 0$ and all $x \in \partial K$

$$\langle \gamma(x), n_x \rangle \geqslant \nu$$

Set $\Gamma(x) = \{\lambda\gamma(x) : \lambda \geqslant 0\}$ and let $w \in C(\mathbb{R}_+,\mathbb{R}^n)$, $w(0) \in K$ be such that the condition (★) from section 3 is satisfied. Then the problem (w,K,Γ) has a solution.

Proof. Let $\{\ell_i\}$ $i=1,2,\dots,n$ be an orthonormal basis of $\mathbb{R}^n$ and fix $x \in \partial K$. By assumptions $N_K^\infty(x) = \bigcup_{\lambda \geqslant 0} \lambda n_x$. Let p_i, q_i be orthogonal projections of ℓ_i on $N_K^\infty(x)$ and $\Gamma^-(x) := \{v \in \mathbb{R}^n : \langle v,\gamma(x)\rangle \leq 0\}$ respectively. Set $a_{ij}(x) = \langle\gamma(x),n_x\rangle^{-1} (\langle p_i,p_j\rangle + \langle q_i,q_j\rangle)$. The matrix $A(x) = (a_{ij}(x))$ is symmetric. Let $v \in \mathbb{R}^n$ and $\pi_1 v$, $\pi_2 v$ be orthogonal projections of v onto $N_K^\infty(x)$, $\Gamma^{-1}(x)$ respectively. Then $A(x)v = \langle\gamma(x),n_x\rangle^{-1}(\pi_1 v + \pi_2 v)$. It implies that $A(x)\gamma(x) = n_x$ and for some $\nu' > 0$, $A(x) > \nu' I$, $\|A(x)\| \leq 2/\nu$, where ν' does not depend on x . Hence Γ satisfies the assumptions of Theorem 3.3 and therefore the problem (w,K,Γ) has a solution.

5 . Proof of Theorem 3.3.

We set $w(t) = w(0)$ for all $t < 0$. It is enough to prove the Theorem under the additional assumption that K is bounded.

From now on we assume that K is compact. Clearly the proof of existence will be completed if we show that for all $T > 0$ there exists $(x,k) \in C([0,T], K) \times C([0,T], \mathbb{R}^n)$ such that the relations (3.2) hold for all $t \in [0,T]$. Fix $T > 0$. We shall prove the Theorem in several steps.

Step 1. Assume first that there exists a constant $b > 0$ such that

$$\liminf_{t' \to t+} \frac{\|w(t') - w(t)\|}{t' - t} \leq b \qquad \text{for all } t \in [0,T],$$

Consider the set

$$\mathcal{K} := \{(t,w(t)-x) : t \in [-1,T], x \in K\}$$

and the set valued function G from $\mathcal{K}$ into the subsets of $\mathbb{R}^n$ defined by :

$$G(t,k) = \Gamma(w(t)-k) \cap \frac{Mb}{\nu} B$$

Since K is closed, w is continuous and Γ has a closed graph the multifunction G is upper semicontinuous on its domain of definition $\mathcal{K}$.

Step 2. We claim that for all $t \in [-1,T]$, $k \in w(t) - K$ there exists $a(t) \in bB$ such that

$$\{1\} \times (a(t) - I_K^\infty(w(t)-k)) \subset T_{\mathcal{K}}(t,k)$$

Indeed by the assumption ($\star$) for all $t \in [-1,T]$ the contingent derivative $Dw(t)$ of w at t, defined by

$$Dw(t) := \{p \in \mathbb{R}^n : (1,p) \in T_{\text{graph}(w)}(t,w(t))\} \qquad \text{(see [4])}$$

is nonempty. Then by assumption of step 1 there exists $a(t) \in Dw(t) \cap bB$. By definition of $Dw(t)$ there exists a sequence $h_i > 0$ converging to zero such that

$$\lim_{i \to \infty} \frac{w(t+h_i) - w(t)}{h_i} = a(t)$$

Let $v \in I_K^\infty(w(t)-k)$. By section 2 there exists a sequence $v_i \in \mathbb{R}^n$ converging to v such that $w(t) - k + h_i v_i \in K$. It implies that $(t+h_i, w(t+h_i)-w(t)+k-h_i v_i) \in \mathcal{K}$ and therefore

$$\lim_{i \to +\infty} \frac{(t+h_i, w(t+h_i)-w(t)+k-h_i v_i) - (t,k)}{h_i} = (1,a(t)-v)$$

Thus $\qquad (1,a(t)-v) \in T_{\mathcal{K}}(t,k)$.

Step 3. We claim that the following tangential condition holds

(5.1) $(\{1\} \times G(t,k)) \cap T_{\mathcal{K}}(t,k) \neq \emptyset$

It is enough to consider the case $(t,k) \in \partial\mathcal{K}$ or equivalently $x := w(t)-k \in \partial K$.

Let $A(x)$ be as in assumptions of Theorem and define the scalar product $< , >_x$ on $\mathbb{R}^n$ setting $<p,q>_x = <A(x)p,q>$. Let

$$P(x) := \{p \in \mathbb{R}^n : <p,v>_x \leqslant 0 \quad \text{for all} \quad v \in I_K^\infty(x)\}$$

that is $P(x)$ is the negative polar cone to $I_K(x)$ for the scalar product $< , >_x$, which is equal to $A(x)^{-1}N_K^\infty(x)$. Then, by assumption,

(5.2) $P(x) \subset \Gamma(x)$

By a Theorem of Moreau (see for example [3]), every $y \in \mathbb{R}^n$ has a unique decomposition as $y = y_1+y_2$, $y_1 \in I_K^\infty(x)$, $y_2 \in P(x)$, $<y_1,y_2>_x = 0$, $\|y_1\|_x \leqslant \|y\|_x$; $\|y_2\|_x \leqslant \|y\|_x$.

The properties of $A(x)$ imply the following estimates :

$$\|y_1\| \leqslant \frac{1}{\nu}\|y_1\|_x \leqslant \frac{1}{\nu}\|y\|_x \leqslant \frac{\mu}{\nu}\|y\|$$

and similarly

$$\|y_2\| \leqslant \frac{\mu}{\nu}\|y\|$$

Furthermore inclusion (5.2) implies that for all $x \in \partial K$ the following holds :

For all $y \in \mathbb{R}^n$ there exist $y_1 \in I_K^\infty(x)$, $y_2 \in \Gamma(x)$ such that $y = y_1+y_2$, $\|y_i\| \leqslant \frac{M}{\nu}\|y\|$, $i=1,2$.

In particular it implies the existence of $a_1(t) \in I_K^\infty(w(t)-k)$, $a_2(t) \in \Gamma(w(t)-k)$ such that $a(t) = a_1(t) + a_2(t)$ and $\|a_1(t)\| \leq \frac{M}{\nu} \|a(t)\|$. Hence by Step 2, the tangential assumption (5.1) is satisfied.

Step 4. We claim that the problem (w,K,Γ) has a solution. Indeed consider the differential inclusion

$$(5.3) \qquad \begin{cases} \dot{y} \in \{1\} \times G(y) \\ \\ y(0) = 0 \quad , \quad y(t) \in K \quad \text{for} \quad t \in [0,T] \end{cases}$$

By the viability theorem, (see Haddad [18]), (5.3) has a solution, i.e. there exists an absolutely continuous function $y : [0,T] \to K$ such that

$$\dot{y}(t) \in \{1\} \times G(y(t)) \qquad \text{for all} \quad t \in [0,T]$$

It implies the existence of absolutely continuous function $k : [0,T] \to \mathbb{R}^n$ satisfying for all $t \in [0,T]$:

$$\begin{cases} \text{(i)} & \dot{k}(t) \in \Gamma(w(t)-k(t)) \cap \frac{Mb}{\nu} B \\ \text{(ii)} & k(t) \in w(t) - K \\ \text{(iii)} & k(0) = 0 \end{cases}$$

Let $x(t) := w(t) - k(t)$. Then (3.2) (i)-(iii) are satisfied and moreover

$$(5.4) \qquad |\dot{k}(t)| \leq \frac{Mb}{\nu}$$

Since the multifunction $t \to \Gamma(x(t))$ is measurable, there exists a measurable selection σ on $\{t : x(t) \in \partial K\}$ such that $\sigma(t) \in \Gamma(x(t)) \cap S^{n-1}$ (see [33]).

Set

$$\mathcal{E}(t) = \begin{cases} \dot{k}(t)\,\|\dot{k}(t)\|^{-1} & \text{if } \dot{k}(t) \neq 0 \\ \sigma(t) & \text{if } x(t) \in \partial K \text{ , } \dot{k}(t) = 0 \end{cases}$$

Then $d|k|_t = \|\dot{k}(t)\|dt$ and thus (3.2)(iv) is verified. Hence (x,k) is the solution to (w,K,Γ) .

Step 5. We consider here an arbitrary function w satisfying the assumption (★). Set

$$\alpha(t) := \liminf_{t' \to t+} \frac{\|w(t') - w(t)\|}{t' - t} + 1$$

and we consider the function $\bar{w} : \left[0 \text{ , } \int_0^T \alpha(s)ds\right] \to \mathbb{R}^n$ defined by

$$\bar{w}\left(\int_0^t \alpha(s)ds\right) := w(t)$$

Then

$$\liminf_{t' \to t+} \frac{\bar{w}\left(\int_0^{t'} \alpha(s)ds\right) - \bar{w}\left(\int_0^t \alpha(s)ds\right)}{\int_0^{t'} \alpha(s)ds - \int_0^t \alpha(s)ds} \leqslant \liminf_{t' \to t+} \frac{|w(t')-w(t)|}{t' - t} \cdot \frac{1}{\alpha(t)}$$

$$\leqslant 1$$

By the previous part there exist continuous functions $(\bar{x},\bar{k})$ satisfying (3.2)(i)-(iv) for all $\tau \in \left[0 \text{ , } \int_0^T \alpha(s)ds\right]$ and $\|\dot{\bar{k}}(\tau)\| \leqslant \frac{M}{\nu}$. For all $t \in [0,T]$ set

$$x(t) = \bar{x}\left(\int_0^t \alpha(s)ds\right) \quad ; \quad k(t) = \bar{k}\left(\int_0^t \alpha(s)ds\right)$$

Then $d|k|_t \leqslant \frac{M}{\nu}\,\alpha(t)dt$. Clearly (3.2)(i),(ii) are satisfied. Moreover $\dot{k}(t) \in \Gamma(x(t))$ implies (3.2)(iii). Exactly as in step 4 we verify that (3.2)(iv) holds.

Step 6. (Uniqueness). Suppose that A do not depend on x and that there exists $c > 0$ such that the map $x \to A\Gamma(x) \cap B + cx$ is monotone. Then if (x_1,k_1) and (x_2,k_2) are solutions to (w,K,Γ) we obtain

$$\frac{d}{dt}\ \frac{1}{2}\ \|\sqrt{A}k_1 - \sqrt{A}k_2\|^2(t) = \langle A\dot{k}_1(t)-A\dot{k}_2(t)\ ,\ k_1(t)-k_2(t)\rangle =$$

$$= \langle A\dot{k}_1(t)-A\dot{k}_2(t)\ ,\ x_2(t)-x_1(t)\rangle\ .$$

By monotonicity, using that $\dot{k}_i(t) \in \Gamma(x_i(t))$ $i=1,2$, we have

$$\frac{d}{dt}\ \frac{1}{2}\ \|\sqrt{A}k_1 - \sqrt{A}k_2\|^2(t) \leqslant c(\|\dot{k}_1(t) + \dot{k}_2(t)\|)\|k_2(t)-k_1(t)\|^2$$

Integrating on $[0,t]$ the above inequality we get

$$\frac{\nu}{2}\ \|k_1(t)-k_2(t)\|^2 \leqslant \frac{1}{2} \langle A(k_1(t)-k_2(t))\ ,\ k_1(t)-k_2(t)\rangle \leqslant$$

$$\leqslant c \int_0^t (\|\dot{k}_1(s)\|+\|\dot{k}_2(s)\|)\ \|k_1(s)-k_2(s)\|^2\, ds$$

The Gronwall inequality implies then that

$$\|k_1(t) - k_2(t)\|^2 \leqslant 0$$

Hence $\quad k_1 = k_2 \quad$ and $\quad x_1 = w-k_1 = w-k_2 = x_2$.

6 . Proof of Theorem 3.4.

The last statement (the uniqueness) follows from Theorem 3.3. We shall proceed with a proof of existence using results from [24]. Note first that if $w \in C^1(\mathbb{R}_+,\mathbb{R}^n)$ then by Theorem 3.3 the problem

(w,K,Γ) has a solution (x,k) . Since $\Gamma(x)$ has a compact sole which does not contain zero for all $x \in \partial K$ we can find $s(x) \in S^{n-1}$ and $\rho(x) > 0$ such that

$$\langle \gamma, s(x) \rangle \geqslant \rho(x) \qquad \text{for all} \quad \gamma \in \Gamma(x) \cap S^{n-1}$$

Because Γ has a closed graph for all $x \in \partial K$ there exists $R(x) > 0$ such that

$$(x' \in \partial K \cap (x+R(x)B)) \Rightarrow \langle \gamma, s(x) \rangle \geqslant \rho(x)/2 \qquad \text{for all} \quad \gamma \in \Gamma(x) \cap S^{n-1}$$

As in the proof of Theorem 3.3 it is not restrictive to assume that K is compact. Then the boundary ∂K can be covered by a finite number of open balls $B(x_i, R(x_i))$.

On the other hand the monotonicity of the map $x \to A(x)\Gamma(x) \cap B + cx$ implies that $A(x)\Gamma(x) \subset N_K^{\infty}$ and hence by assumptions

$$A(x)\ \Gamma(x) = N_K^{\infty}(x)$$

Let $w_i \in C^{\infty}(\mathbb{R}_+, \mathbb{R}^n)$ be a sequence converging to w uniformly on compacts. By theorem 3.3 there exists a solution (x_i, k_i) to (w_i, K, Γ) . By the results from [24] we know that a subsequence $\{(x_{ij}, k_{ij})\}_{j \geqslant i}$ converges to a solution (x,k) of problem (w,K,Γ) . (To prove it one has to use the monotonicity to show the precompactness of set $\{(x_i, k_i)\}_{i \geqslant 1}$ and verify that cluster points of $\{(x_i, k_i)\}_{i \geqslant 1}$ are solutions to (x,K,Γ) , see [24]).

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