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MATHEMATICAL METHODS FOR THE ANALYSIS OF HIERARCHICAL SYSTEMS
II. NUMERICAL METHODS FOR SOLVING GAME-THEORETIC, EQUILIBRIUM AND PARETO OPTIMIZATION PROBLEMS

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PREFACE

This is the second of two papers dealing with mathematical methods that can be used to analyze hierarchical systems.

In this paper, the authors consider the case in which the lower level of a hierarchical system of decision makers is composed of a number of controllable subsystems. If these subsystems are not bounded by common constraints then the analysis is reduced to that of a two-level system consisting of a regulatory center and one lower subsystem. Two types of control are discussed in this case: control of resource use and control through price setting. If, however, there are shared resource-type constraints then it is assumed that the subsystems choose cooperatively from the set of Pareto-optimal alternatives. The problem for the regulatory center is then to maximize its goal function over this set. A number of ways of solving this problem are proposed, and a computational algorithm is given.

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MATHEMATICAL METHODS FOR THE ANALYSIS OF HIERARCHICAL SYSTEMS II. Numerical Methods for Solving Game-Theoretic, Equilibrium, and Pareto Optimization Problems

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1. INTRODUCTION

In this paper (a continuation of [1]), we consider the case in which the lower level of a hierarchical system involves a number of controllable subsystems.

If the lower subsystems are not bounded by common constraints then the analysis reduces to that of a two-level system consisting of a regulatory center and one lower subsystem. This case is discussed in Sections 2 (control of resource use) and 3 (control through prices). In cases where there are shared resource-type constraints we assume that the subsystems make their choices cooperatively from the set of Pareto-optimal alternatives. The problem for the regulatory center then lies in maximizing its goal function over that set. For a linear goal function it is sufficient to consider only extremal points of the Pareto set: an algorithm for doing this is outlined in Section 4.

Another technique is based on the decomposition of the problem by introducing pricing policies for the use of resources. The prices leading to optimal subsystem demands for resources without exceeding total resource availability can be determined by solving a classical competitive equilibrium problem. A computational algorithm for solving this problem is presented in Section 5. All sections have the same structure: they begin with a model which introduces the formal problem under consideration, the computational difficulties are then illustrated by means of an example, and finally a solution algorithm is outlined.

2. AN ALGORITHM FOR ALLOCATION OF SCARCE RESOURCES

2.1. The model

We shall consider a hierarchical model of a production planning problem, which assumes a hierarchical control structure. Let the central control body Π_0 influence the production of branches Π_j , $j = 1,\ldots,n$ through the allocation of primary resources and by setting acceptable levels for environmental pollution, while requiring that a given level of supply is achieved.

We shall also assume that the prices of products in the national economy are based on the consumption of both final products and intermediate products. We shall characterize the activity of every branch by a vector of intensity x^i. Then we have

$$u^i = A_1^i x^i - \sum_{j=1}^{n} y_{ji}$$

$$z^i = A_2^i x^i$$

$$r^i = A_3^i x^i \quad ,$$

where u^i is the final consumption vector, z^i is the level of pollution, and r^i is the consumption of natural resources.

The y_{ji} represent the amounts of intermediate products transferred between branches, where

$$A_4^i x^i \leq \sum_{j=1}^{n} y_{ij} \quad .$$

Let us assume that the center regulates both the level of pollution produced by individual components

$$\sum_{i=1}^{n} z_k^i \leq \alpha_k \ , \quad k = 1,\ldots,K$$

and their consumption of natural resources

$$\sum_{i=1}^{n} r_l^i \leq R_l \ , \quad l = 1,\ldots,L \quad .$$

The goal of the center is to meet the consumption requirements of society u_m^0, i.e.,

$$\min_{m} \frac{\sum_{i=1}^{n} u_m^i}{u_m^0} \to \max_{z,r} \quad ,$$

while the branches attempt to maximize their overall benefit:

$$\sum_{i=1}^{n} \lambda_i[(pA^i x^i - \sum_{j=1}^{n} y_{ji}) + (q \cdot \sum_{j=1}^{n} y_{ji})] \to \max \quad ,$$

where price levels (p,q) are coefficients of commensurability of the benefits of separate branches and are determined by another economic mechanism.

This problem may be viewed as a game G_1. An algorithm for solving this game is given below.

2.2. An algorithm for solving the resource allocation problem

The problem formulated above may be written in the following general form: To determine

$$\max_{u \in D} \left[\min_{x \in T(u)} \sum_{j=1}^{n} k_j x_j \right] = \max_{u \in D} F(u) \quad , \tag{2.1}$$

where

$$T(u) = \{x \in T_0(u) | \sum_{j=1}^{n} c_j x_j = \max_{y \in T_0(u)} \sum_{j=1}^{n} c_j y_j\} \tag{2.2}$$

$$T_0(u) = \{x \in E^n \mid x \geq 0 \, , \sum_{j=1}^{n} a_{ij} x_j = b_i + \sum_{l=1}^{k} b_{il} u_l \, , i = 1,\ldots,m\} \tag{2.3}$$

$$D = \{u \in E^k \mid u \geq 0 \, , \sum_{l=1}^{k} d_{rl} \, u_l \leq d_r \, , r = 1,\ldots,p\} \quad . \tag{2.4}$$

Consider the function $F_0(u) = \sum_{j=1}^{n} k_j x_j$, $x \in T(u)$. This function may not be defined for all values of $u \in E^k$, as the set $T_0(u)$ may be empty. In particular, if the problem (2.1)–(2.4) does not have a solution for any value of u, then $F_0(u)$ is not defined at all. We shall assume further that $u^0 \in D$ exists such that $T_0(u^0) \neq \phi$, where $T(u^0)$ is a bounded set. If $T(u)$ contains more than one element then the function $F_0(u)$ may take several values. We shall use the following notation:

$$F(u) = \min_{x \in T(u)} F_0(u) \quad .$$

Let us now illustrate the problem by means of an example.

Example 1

$$\sum_{j=1}^{6} k_j x_j = x_1 + 5x_2 + 6x_4 + 8x_5 + x_6$$

$$\sum_{j=1}^{6} c_j x_j = x_1 + 5x_2 + 5x_3 + 3x_4 + 2x_5 + x_6$$

$$T_c(u) = \begin{cases} x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 1 \\ 3x_2 + 4x_3 + 7x_4 + 8x_5 + 10x_6 = u \\ x \in E^6 , x \geq 0 \end{cases}$$

$$D = \{u \in E^1 \mid 0 \leq u \leq 10\} \quad .$$

It is clear from Fig. 1 that function $F_0(u)$ may be multivalued; the intermediate function $F(u)$ which should be maximized (the bold line in the figure) appears to be piecewise linear and multiextremal.

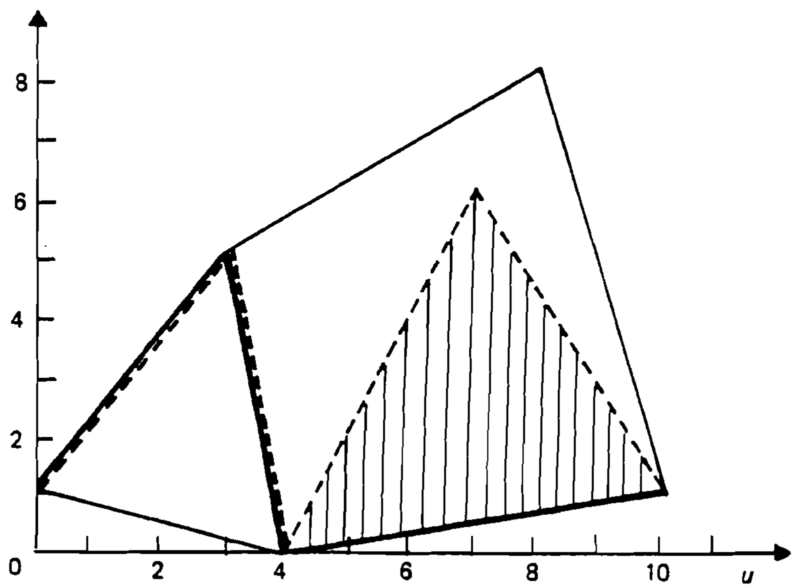

Figure 1. The functions $F_0(u)$, $F(u)$ (bold line) and $F'(u)$ (dashed line) for Example 1.

It is now easy to see that $F'(u)$, defined as $\max_{x \in T(u)} \sum_{j=1}^{n} k_j x_j$ (the dashed line in Fig. 1), is also a piecewise linear function.

We shall use the following notation and definitions in the remainder of this section.

If we take basis $J = \{J,\ldots,J_m\}$ for the problem

$$\max_{x \in T_0} \sum_{j=1}^{n} c_j x_j \tag{2.5}$$

$$T_0 = \{x \in E^n \mid x \geq 0 \, , \sum_{j=1}^{n} a_{ij} x_j = b_i \, , i = 1,\ldots,m\} \quad ,$$

with a matrix $\|a_{ij}\|$ of range m, then the standard form of problem (2.5) is assumed to be

$$\max_{x \in T_0} \left[-\sum_{j=1}^{n} \Delta_j x_j + \Delta_{n+1} \right] \quad ,$$

where

$$\Delta_j = \sum_{i=1}^{m} c_{J_i} \tau_{ij} - c_j \, , \quad j \not\in J$$

$$\Delta_{s_i} = 0 \, , \quad i = 1,\ldots,m$$

$$\Delta_{n+1} = \sum_{i=1}^{m} c_{J_i} \eta_i \quad ,$$

i.e., the matrix $\|i_{ij}\|$ (where $x_{s_i} = -\sum_{j \not\in J} \tau_{ij} x_j + \eta_i$), the vector s_j, $j = 1,\ldots,n$, and the vector of right-hand sides η_i, $i = 1,\ldots,m$, are known. A basis is said to be *permissible* if $\Delta_j \geq 0, i = 1,\ldots,m$, and is called a *pseudobasis* if $\Delta_j \geq 0$, $j = 1,\ldots,n$. We shall say that the basis J is *optimal* if it is a permissible pseudobasis. There is a solution x of problem (2.5) corresponding to each optimal basis.

The following theorem enables us to deal with an optimization problem rather than a maximin problem.

First define the problems

$$\max_{x \in T_0(u)} \sum_{j=1}^{n} (c_j - \delta k_j) x_j \tag{2.6}$$

$$\max_{x \in T_0(u)} \sum_{j=1}^{n} c_j x_j \tag{2.7}$$

and the set

$$T^{\delta}(u) = \{x \in T_0(u) \mid \sum_{j=1}^{n} (c_j - \delta k_j) x_j = \max_{y \in T_0(u)} \sum_{j=1}^{n} (c_j - \delta k_j) y_j\} \quad . \qquad (2.8)$$

We can then state the following theorem:

Theorem 1. *There exists a* $\delta_0 > 0$ *such that for all* $0 < \delta < \delta_0$ *we have* $F_0^{\delta} = \sum_{j=1} k_j x_j$, *where* $x \in T^{\delta}(u)$ *is single-valued and* $F(u) = F_0^{\delta}(u)$.

The proof of this theorem is given in [2].

Theorem 1 provides a basis for a solution algorithm for the problem outlined above. The main idea of the algorithm is to construct the set of all pseudobases of problem (2.6) whose admissible sets include the initial set. The restrictions which define the admissible region are linear and so the optimization problem of the center remains a linear programming problem.

The skeleton of the algorithm is outlined below.

Step 1. Find a pseudobasis $J = \{s_1, \ldots, s_m\}$ of problem (2.7) at $u = u^0$, where $T_0(u^0) \neq \phi$.

Step 2. Extend the set J of indices to $S = \{j \mid \Delta_j(c) = 0\}$.

Solve the problem

$$\sum_{s \in S} k_s x_s \to \min$$

$$\sum_{s \in S} a_{is} x_s = b_i + \sum_{l=1}^{k} b_{il} u_l^0 , \quad i = 1, \ldots, m \quad .$$

(The pseudobasis J of problem (2.6) will be constructed by this method.) This basis will be optimal at all $u \in D$ for which $x_{s_i}(u) \geq 0$. Both these functions and functions $f(u) = \sum_{s_i \in J} k_{s_i} x_{s_i}(u)$ are linear with respect to u.

Step 3. Solve the linear programming problem

$$F_{0J} = \max_{u \in T_y} \sum_{s_i \in J} k_{s_i} x_{s_i}(u)$$

$$T_y = \{u \in D \mid x_{s_i}(u) \geq 0 , i = 1, \ldots, m\} \quad .$$

Step 4. Construct a sequence of pseudobases P of problem (2.6) by finding all the neighboring pseudobases to every current pseudobasis. This may be done by successive exclusion of the s_i and inclusion of the numbers r generated by the double-simplex method:

$$z = \min_{\tau_{ij}<0} \left[-\frac{\Delta_j(c)}{\tau_{ij}} \right]$$

$$\frac{\Delta_r(k)}{\tau_{ir}} = \min_{-\Delta_j(c)/\tau_{ij} = z} \left[\frac{\Delta_j(k)}{\tau_{ij}} \right] .$$

Step 5. Solve the problem given in Step 2 for every pseudobasis P in this sequence.

Step 6. Find $\max_{J\in P} F_{0J}$.

It is shown in [2] that an algorithm constructed on the above lines will find $\max_{u\in D} F(u)$. Note that although neither δ nor δ_0 is present in the algorithm, only the pseudobases of problems (2.6) are considered, because if pseudobasis J of problem (2.6) is found then the algorithm uses this basis at Step 4 to obtain neighboring bases of problem (2.6). In actual fact, for every s_i it is necessary to find a number r (by the double-simplex method) which is included in the basis set according to the formula:

$$-\frac{\Delta_r(c-\delta k)}{\tau_{ir}} = \min_{\tau_{ij}<0} \left[-\frac{\Delta_j(c-\delta k)}{\tau_{ij}} \right] .$$

It is easy to see that since $\Delta_j(c - \delta k) = \Delta_j(c) - \delta\Delta_j(k)$ is linear and if δ is sufficiently small, the above procedure is equivalent to the lexicographical procedure carried out in Step 3.

Note that another algorithm for solving this problem, based on a different way of finding the maximin of $F(u)$, is given in [4].

3. A PRICING ALGORITHM

3.1. The model

We shall consider a hierarchical model of price formation in the agricultural sector.

Assume that each agricultural enterprise i functions with an intensity x^i, $i = 1,\dots,n$. Let $u^i = A^i x^i$ represent the production volume of enterprise i and $r^i = B^i x^i$ represent the amount of resources consumed in the production process by the i-th enterprise. Assume that the wholesale prices of the products p and the prices of resources (water, fertilizer, etc.) q are determined by the center in such a way as to get maximum profit from the sale of agricultural products to the consumer at a fixed vector of retail prices v:

$$\sum_{i=1}^{n} (v, u^i) \to \max_{p,q} \quad .$$

The agricultural branch wishes to maximize its benefits

$$\sum_{i=1}^{n} [(p \, , \, A^i x^i) - (q \, , \, r^i)] \to \max_{x}$$

under the condition imposed by the center:

$$\sum_{i=1}^{n} u_j^i \geq \lambda u_j^0 \quad ,$$

where λ is a given level of fulfillment of the state production program u_j , $J = 1,\ldots,m$. When solving this problem we shall assume that the restrictions on resources are not limiting.

3.2. An algorithm for solving the pricing problem

The problem of centralized control of production through price setting may be written in the following general form: To determine

$$\sup_{u \in D} \; \min_{x \in T(u)} \sum_{j=1}^{n} \sum_{l=1}^{k} (k_{lj} u_l + k_{0j}) x_j \quad , \tag{3.1}$$

where

$$T(u) = \{x \in T_0 | \sum_{j=1}^{n} \sum_{l=1}^{k} (c_{lj} u_l + c_{0j}) x_j = \max_{y \in T_0} \sum_{j=1}^{n} \sum_{l=1}^{k} (c_{lj} u_l + c_{0j}) y_j \} \tag{3.2}$$

$$T_0 = \{x \in E^n \mid x \geq 0 \, , \, \sum_{j=1}^{n} a_{ij} x_j = b_i \, , \, i = 1,\ldots,m \} \tag{3.3}$$

$$D = \{u \in E^k \mid u \geq 0 \, , \, \sum_{l=1}^{k} d_{il} u_l \leq d_i \, , \, i = 1,\ldots,p \} \quad . \tag{3.4}$$

We define the functions $F_0(u)$ and $F(u)$ as follows:

$$F_0(u) = \sum_{j=1}^{n} \sum_{l=1}^{k} (k_{lj} u_l + k_{0j}) x_j \, , \quad x \in T(u) \tag{3.5}$$

$$F(u) = \min_{x \in T(u)} \sum_{j=1}^{n} \sum_{l=1}^{k} (k_{lj} + k_{0j})x_j \quad . \tag{3.6}$$

Function $F_0(u)$ may not be single-valued at certain values of u if $T(u)$ has more than one element. Any value of $F_0(u)$ at fixed u may be represented in the following form (here and elsewhere we shall assume that $T_0 \neq \phi$ and T_0 is bounded):

$$F_0(u) = \sum_{i=1}^{t} \lambda_i (\sum_{j=1}^{n} \sum_{l=1}^{k} (k_{lj} u_l + k_{0j})x_j)|_{I_i}$$

where $\lambda_i \geq 0$; $\sum_{i=1}^{t} \lambda_i = 1$; and $I_1, \ldots, I_t$ are the optimal bases of the following linear programming problem:

$$\underset{x \in T_0}{\text{maximize}} \sum_{j=1}^{n} \sum_{l=1}^{k} (c_{lj} u_l + c_{0j})x_j \quad . \tag{3.7}$$

We shall assume that the given problem is nondegenerate. Since

$$F(u) = \min_{x \in T(u)} F_0(u)$$

then

$$F(u) = \min_{I_i} (\sum_{j=1}^{n} \sum_{l=1}^{k} (k_{lj} u_l + k_{0j})x_j)|_{I_i} \quad ,$$

where I_i is a member of the set of optimal bases of problem (3.7) at fixed u.

We shall make use of the definitions and notation given below.

If we are considering a certain basis $J = J_1, \ldots, J_n$ of problem (3.1), then it is assumed that we know its standard form, i.e., matrices τ_{ij}, η_i, $\Delta_j^l(c)$, $\Delta_j^l(k)$. This means that problem (3.1) can be written in the form: To determine

$$\sup_{u \in D} \min_{x \in T(u)} \sum_{j=1}^{n} \sum_{l=1}^{k} -(\Delta_j^l(k)u_l + \Delta_j^0(k))x_j + \sum_{l=1}^{k} \Delta_{n+1}^l(k)u_l + \Delta_{n+1}^0(k) \tag{3.8}$$

$$T(u) = \{x \in T_0 | \sum_{j=1}^{n} \sum_{l=1}^{k} - (\Delta_j^l(c) + \Delta_j^0(c))x_j + \sum_{l=1}^{k} \Delta_{n+1}^l(c)u_l + \Delta_{n+1}^0(c)$$

$$= \max_{y \in T_0} \sum_{j=1}^{n} \sum_{l=1}^{k} - (\Delta_j^l(c)u_l + \Delta_j^0(c))y_j + \sum_{l=1}^{k} \Delta_{n+1}^l(c)u_l + \Delta_{n+1}^0(c)\}$$

$$T_0 = \{x \in E^n | \sum_{j=1}^{n} \tau_{ij} x_j = \eta_i , i = 1, \ldots, m \ , x \geq 0\} \quad . \tag{3.9}$$

It is clear that if $J_i \in J$ then $\Delta^l_{J_i}(c) = \Delta^l_{J_i}(k) = 0,\ i = 1,\ldots,m,\ l = 0,\ldots,k$.

We shall denote the optimal set with basis J for problem (3.7) in set D by

$$T_y = \{u \in D \mid \sum_{l=1}^{k} \Delta^l_j(c)u_l + \Delta^0_j(c) \ge 0 \ , j \notin J\} \quad ,$$

and define the set of indices $S = \{j \mid \Delta^l_j(c) = 0,\ l = 0,\ldots,k\}$. It is clear that $J \subset S$, i.e., S is an extension of J.

If S contains more than m elements, then the following auxiliary problem may be necessary:

$$\underset{x \in T_{0s}}{\text{minimize}} \sum_{j \in S} \sum_{l=1}^{k} (k_{lj}u_l + k_{0j})x_j \quad , \tag{3.10}$$

where

$$T_{0s} = \{x \in E^n \mid x_j = 0 \ , j \notin S \ , x_s \ge 0 \ \text{if} \ s \in S \ , \sum_{j=1}^{n} a_{ij}x_j = b_i \ , i = 1,\ldots,m\} \quad .$$

A basis composed of elements of set S is a pseudobasis of problem (3.10) if

$$-(\sum_{l=1}^{k} \Delta^l_j(k)u_l + \Delta^0_j(k)) \ge 0 \ , \quad j \in S \setminus I \quad .$$

Finally, we shall introduce the following notation:

$$T_I = \{u \in D \mid -(\sum_{l=1}^{k} \Delta^l_j(k)u_l + \Delta^0_j(k)) \ge 0 \ , j \in S \setminus I\} \quad .$$

Let us now consider the form of functions $F_0(u)$ and $F(u)$.

Example 2. To construct

$$F_0(u) = (u - 2)x_1 + (-5u + 1)x_2 \quad ,$$

where

$$T(u) = \{x \in T_0 \mid -ux_1 + 2ux_2 = \max(-uy_1 + 2uy_2)\}$$

$$T_0 = \left\{ x \in E^4 \mid x \ge 0 \ , \begin{array}{r} x_1 - 2x_2 + x_3 = 2 \\ x_1 + x_2 + x_4 = 5 \end{array} \right\} \quad .$$

The form of $F_0(u)$ is given in Fig. 2: it is easy to see that $F(u)$ is multiextremal and discontinuous.

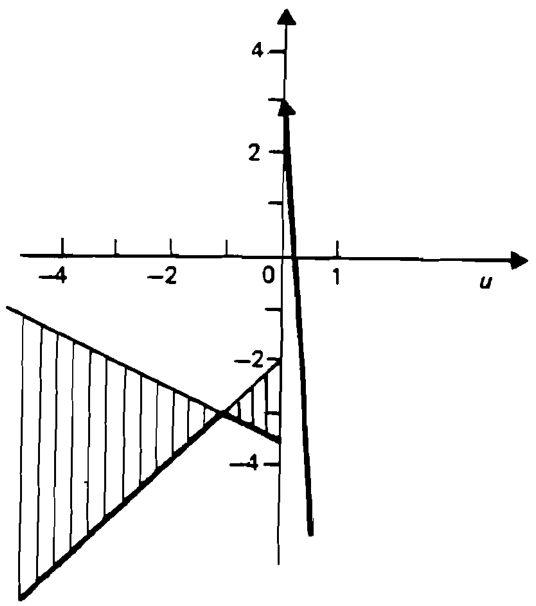

Figure 2. The functions $F_0(u)$ and $F(u)$ (bold line) for Example 2.

Assume that D has an inner point u, i.e., some point $u^* \in D$ exists such that

$$\sum_{l=1}^{k} d_{il} u_l^* < d_i \ , \quad i = 1,\ldots,p \ , \quad u_l^* > 0 \ , \quad l = 1,\ldots,k \quad .$$

Assume also that we know some point $u^0 \in D$. The skeleton of the pricing algorithm is then as outlined below.

Step 1. Find an optimal basis J of problem (3.7) at $u = u^0$.

Step 2. Find the optimal set T_y for basis J in set D, and then find an extension S of basis J. (This extension defines a set of bases which are optimal at the same value of u as basis J. Note that, from the definition of S, the optimal sets of all bases defined by S coincide with T_y.)

Step 3. If T_y has an inner point, find sets T_I for pseudobases I of extension S in problem (3.7). A method for checking the existence of an inner point of S is described in [14] – this can be reduced to the following linear programming problem: To find

$$F_y = \max_{I} \max_{T_y \cap T_I} \left(\sum_{j=1}^{m} \sum_{l=1}^{k} (k_{lj} u_l + k_{0j}) x_j \right) \Big|_I \quad .$$

Step 4. Construct all neighboring bases to basis J using the direct simplex method.

Successive application of Steps 2–4 leads to the construction of a sequence of bases P of problem (3.7) whose optimal sets completely cover set D. It can be shown that $F_1 = \max F_y$ gives the optimal value for the objective of the center in the original problem. To prove this we shall consider a certain sequence u^t, $t = 1,2,\ldots$, which is completely contained in a given T_y, so that $\lim F(u^t) = F_1$. Now take an arbitrary sequence u^t, $t = 1,2,\ldots,u_t \in D$. For the elements of this sequence u^t which belong to arbitrary T_y we deduce, from the definition of function $F(u)$, that $F(u) \le F_1$ as $F(u) \le F_y \le F_1$, and thus the limit of this sequence is not greater than F_1.

4. CONSTRUCTION OF THE EXTREME POINTS OF THE PARETO SET

4.1. Aggregated multiregional model of the world economy (4 × 6)

Within the framework of research carried out by the United Nations on possible strategies for world development and international economic cooperation, a group of American economists headed by W. Leontief has developed a global interbranch model for determining world economic development indices for 1970–2000 [5]. Structurally, the model consists of a set of regional blocks connected by flows of money and goods. Each regional block is composed of two parts: the input–output balance of the branches, and the macroeconomic equations.

The basic model considers 15 regions, of which eight may be regarded as developed and seven as developing, and 45 branches of production. Each regional block is described by 175 constraints and 229 variables. The interregional interactions in the model are fixed by specifying the ratio of imports to gross domestic output on the one hand and the ratio of regional to world export on the other. Different macroeconomic variables are then fixed in the solution procedure, to ensure that the system of linear algebraic equations has a unique solution. We choose as fixed variables those indices which characterize the economic development of the regions (e.g., the rate of increase of the gross national product, the rate of investment, net balance of payments, prices of resources, etc.).

The basic model described above was developed from a number of earlier trial models. The first of these was the two-region, three-branch model suggested by Leontief in his Nobel lecture as an example of world economic ties. The next step was to extend the model to include four regions and six branches.

The main features of the basic model (macroeconomic equations and input–output balances) were reflected in this model, although there were no equations describing financial links.

However, we believe that the approach used in the basic model is too narrow to analyze the possibilities of interregional exchange because (i) trials with the smaller versions of the model do not provide any opportunity to analyze the whole set of conditions and (ii) fixing the proportions of imports and exports restricts the scope of economic interaction.

Research by the Institute of Economics and the Organization of Industry, of the Siberian branch of the USSR Academy of Sciences, has shown that undesirable restrictions can be eliminated if the regional economic development criteria are formulated explicitly, and some constraints on structural exchange are introduced. The problems of global optimization and economic cooperation between regions can then be solved using this model by finding equilibrium exchange prices.

This approach differs from the original model in that it enables one to obtain not only feasible solutions but also efficient (Pareto) solutions which possess equilibrium properties [6,7].

This study was based on the use of two models: one including 15 regions and 22 branches (15 × 22 model) and the other ten regions and ten branches. Both models were obtained by aggregating branches and regions from the basic model.

However, it is rather more difficult to investigate the structure of the Pareto set than to search for certain Pareto points; it is not possible to use very detailed versions of the model for this purpose and instead variants of the 4 × 6 model have been employed. These variants allow efficient use of complex screening algorithms and provide the opportunity to investigate the general structure of the set of efficient exchanges and equilibrium points.

As mentioned above, the 4 × 6 model was the first step in the construction of the 15 × 45 model, and consequently its macroeconomic part is essentially much simpler in form. In this model the world is divided into two developed regions (North America (I) and all other developed countries (II)) and two developing regions (Latin America (III) and all other developing countries (IV)). The macroeconomic variables of the models include investment I, capital K, employment L and consumption λ. The vector of outputs x consists of traded goods from four branches (agriculture, the extraction industry, light industry and heavy industry), and the "output" of the service and pollution purification branches. Transport is included in the service branch, which consequently has to pay for interregional transportation. Export and import volumes are denoted by $E = (E_1, \ldots, E_4)$ and $M = (M_1, \ldots, M_4)$, respectively.

The input–output equations for region s have the following form:

$$x_i^s = \sum_{j=1}^{6} a_{ij}^s x_j + \gamma_i^s I^s + c_i^s \lambda^s + \sigma_i^s p^s + E_i^s - M_i^s \, , \quad i = 1, \ldots, 4$$

$$x_5^s = \sum_{j=1}^{6} a_{5j}^s x_j + \gamma_5^s I^s + c_5^s \lambda^s + \sigma_5^s p^s + \sum_{j=1}^{4} a_{ij}^s (E_j^s + M_j^s)$$

$$x_6^s = \sum_{j=1}^{6} a_{6j}^s x_j + \gamma_6^s I^s + c_6^s \lambda^s + \sigma_6^s p^s \quad .$$

Here $A^S = \|a_{ij}^s\|$ is a matrix of technology-dependent cost coefficients, the a_{ij}^s is a vector of transport costs, and γ^s is a vector of investment shares. The population of the s-th region is denoted by p^s; this parameter can be varied in different versions of the model. c^s and σ^s are vectors of consumption shares which depend on the consumption level and the size of the population, respectively. Thus, the model uses a linear function to approximate the generally nonlinear dependence of the consumption structure on the consumption level and the population size. Here $\sum_i \sigma_i^s = 0$, i.e., the population dependence affects only the relative demand for various products. In addition, limits can be imposed on outputs from both above and below:

$$x_j \le \bar{x}_j \, , \quad j \in \bar{J}$$

$$x_j \ge \underline{x}_j \, , \quad j \in \underline{J} \quad .$$

Here $\bar{J}$ represents the extraction industry while $\underline{J}$ includes all those branches whose output is a final product.

The macroeconomic constraints consist of a restriction on the availability of labor, a link between output and capital, and a relation between capital and investment:

$$\sum_{j=1}^{6} l_j^s x_j^s + c_L^s \lambda^s + \sigma_L^s p^s \le L^s$$

$$\sum_{j=1}^{6} k_j^s x_j^s + c_K^s \lambda^s + \sigma_K^s p^s = K^s$$

$$c_I^s \lambda^s + r^s K^s + \sigma_I^s p^s = I^s \quad .$$

Here (c_L^s, c_K^s, c_I^s) and $(\sigma_L^s, \sigma_K^s, \sigma_I^s)$ are regional coefficients for the consumption of labor, capital and investment, respectively; these depend on the regional consumption level and population size. Vectors l^s and k^s represent the use of labor and capital by the branches, and L^s is the total amount of labor available, which is assumed to be fixed. The replacement cost of capital is represented by r^s.

The relation between exports and imports is

$$\sum_{i=1}^{4} p_i (E_i^s - M_i^s) \geq 0 \quad ,$$

where $P = (p_1, p_2, p_3, p_4)$ is a price conversion vector. We also have

$$\sum_{s=1}^{4} M_i^s = \sum_{s=1}^{4} E_i^s \, , \quad i = 1,\ldots,4 \quad .$$

We shall take the objective functions of the regions to be maximization of the consumption levels λ^s.

Thus the global economic model reduces to a linear multicriteria problem.

4.2. Definitions and examples

Let a bounded polyhedral set X be defined by the following system of linear constraints:

$$Ax = b \tag{4.1}$$

$$x \geq 0 \quad ,$$

where A is an $m \times n$ matrix, b is a vector, $x \in E^n$ and there are k linear functionals $F_1(x) = c^1 x, \ldots, F_k(x) = c^k x$. The problem is to find all the extreme points of the Pareto set for functionals $F_1(x), \ldots, F_k(x)$ on set X.

An algorithm which does this can be constructed using the following theorem [8–10].

Theorem 2.

(i) *If x^* is an efficient point then a vector $\lambda \in E^k$, $\lambda > 0$, $\sum_{i=1}^{k} \lambda_i = 1$, exists such that x^* is a solution of the linear programming problem*

$$\left[\sum_{l=1}^{k} \lambda_l c^l\right] x \to \max_{x \in X} \quad . \tag{4.2}$$

(ii) *For any $\lambda \in E^k$, $\lambda > 0$, $\sum_{l=1}^{k} \lambda_l = 1$, the solution x^* of problem* (4.2) *is an efficient point.*

The set of parameters λ in the theorem is assumed to be bounded but not closed. Open sets of parameters are not suitable for use with numerical algorithms and thus we derive the following corollary of the theorem, which is the basis for our solution algorithm.

Corollary

(i) *If x^* is an efficient point then a vector $\lambda \in D = \{\lambda \in E^k \mid \lambda_l \geq 1\}$ exists such that x^* is a solution of problem* (4.2).

(ii) *For any $\lambda \in D$, a solution x^* of problem* (4.2) *is an efficient point.*

Assume that X is nondegenerate, i.e., vector b cannot be represented as a linear combination of less than m columns of matrix A. Then any extreme point x is associated with a unique basis $J = \{J_1, \ldots, J_m\}$, and $A_y x_y = b$, $x_y = 0$, $x_y > 0$.

Definition 1. The set

$$T_y = \{\lambda \in E^k \mid \Delta_l^J(\lambda c) \geq 0 \, , \, l = \bar{J}\}$$

is an *optimal set* of basis J.

Here $\bar{J}$ is the complement of J, i.e., $J \cap \bar{J} = \phi$ and $J \cup \bar{J} = \{1, \ldots, n\}$. If $x_J = A_y^{-1} b$ and $T_y \cap D \neq \phi$, then J will be called an optimal basis. The extreme point x which corresponds to this basis will also be an efficient point as it is a solution of problem (4.2) at any $\lambda \in T_y \cap D$.

Thus the problem of constructing the extreme points (if X is nondegenerate) is reduced to that of finding all optimal bases.

Definition 2. A permissible basis is said to be a *neighboring basis* to another permissible basis if they differ in only one component.

Let J be a permissible basis. Then for any $j \in \bar{J}$, there exists a neighboring basis I such that $j \in I$. This basis may be determined using the simplex rule: if vectors i_j, η are solutions of the equations $A_J i_j = A_j$ and $A_J \eta = b$, then i is such that $\eta_i / \tau_{ij} = \min_{1 \le l \le m} \eta_l / \tau_{lj}$. Such an i must exist because X is bounded and unique (this is a consequence of X being nondegenerate).

Thus $I = \{J \setminus J_i\} \cup j$ and each permissible basis has $n - m$ neighbors.

Definition 3. Any neighboring basis I to an optimal basis J is said to be an *optimal neighboring basis* if the intersection of the optimal sets of J and I is not empty, i.e., $D \cap T_y \cap T_I \neq \phi$. It is clear that if I and J are optimal neighboring bases, then a convex hull of their extreme points or an edge connecting the corresponding extreme points will belong to the Pareto set.

We shall now give some examples which illustrate these definitions.

Example 3. The Pareto set consists of "moustaches". In this case there are bases which are optimal and neighboring but which are not optimal neighboring bases as defined above. We have

$$X = \{x \in E^4 \mid x_1 + x_2 + x_3 + x_4 = 1 \, , \, x \ge 0\}$$

and the following three linear functionals:

$$F_1(x) = 2 + x_1 - 2x_2 - 2x_3$$

$$F_2(x) = 2 - 2x_1 + x_2 - 2x_3$$

$$F_3(x) = 2 - 2x_1 - 2x_2 + x_3 \quad .$$

The reachable set for these functionals is shown in Fig. 3; the bold lines represent the corresponding set of Pareto values. Set X is shown in Fig. 4; the bold lines represent the Pareto set.

Example 4. Here we consider neighboring bases whose optimal sets coincide and bases whose optimal sets are of dimension less than k (where k is the number of functionals). The Pareto set then consists of sides and edges. We have $X = \{x \in E^6 \mid x_1 + x_4 = 1 \, , \, x_2 + x_5 = 1 \, , \, x_3 + x_6 = 1 \, , \, x \ge 0\}$ and the following two linear functionals:

$$F_1(x) = 2 + x_1 + x_2 - x_3$$

$$F_2(x) = 2 - x_1 - x_2 + 2x_3 \quad .$$

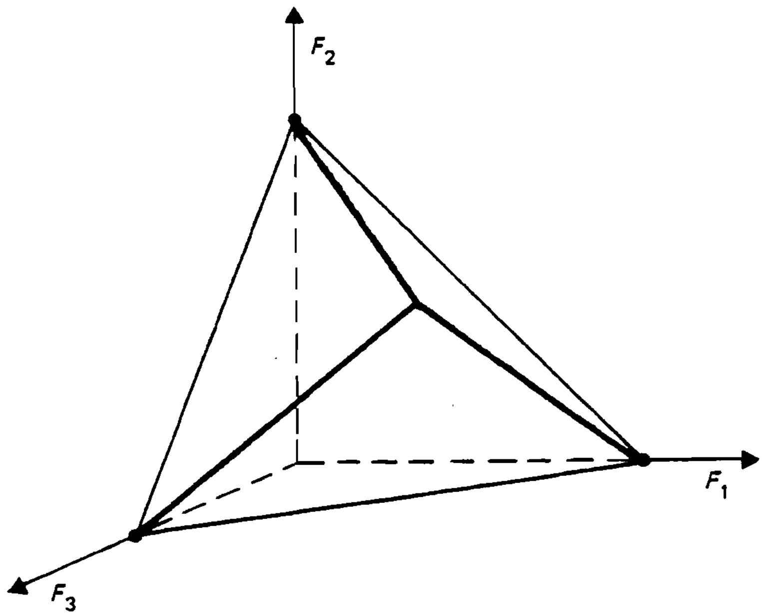

Figure 3. The reachable set of the system of functionals in Example 3 – the bold lines represent the corresponding set of Pareto values.

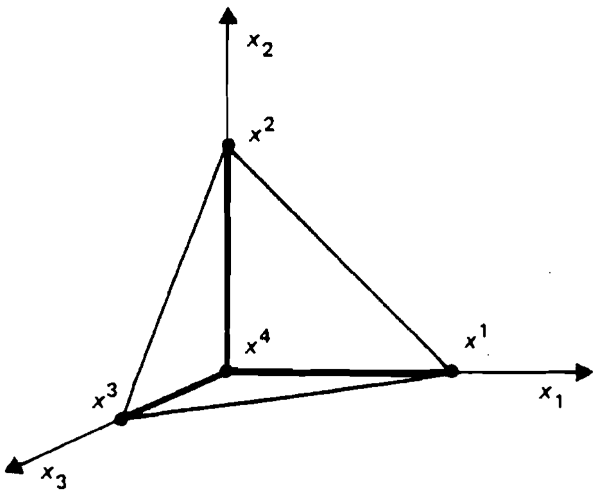

Figure 4. Set X from Example 3 – the bold lines represent the Pareto set.

Set X has eight extreme points of which five are extreme points of the Pareto set. The reachable set of the system of functionals is shown in Fig. 5, while set X is shown in Fig. 6. Bold lines denote the Pareto set in the space of functionals (Fig. 5) and in the space of variables (Fig. 6). The sets

$T_{y_l} \cap D$, $l = 1,\ldots,5$ are illustrated in Fig. 7. We see that sets T_{y_3} and T_{y_4} coincide and are of dimension 1; all the other sets are of dimension 2.

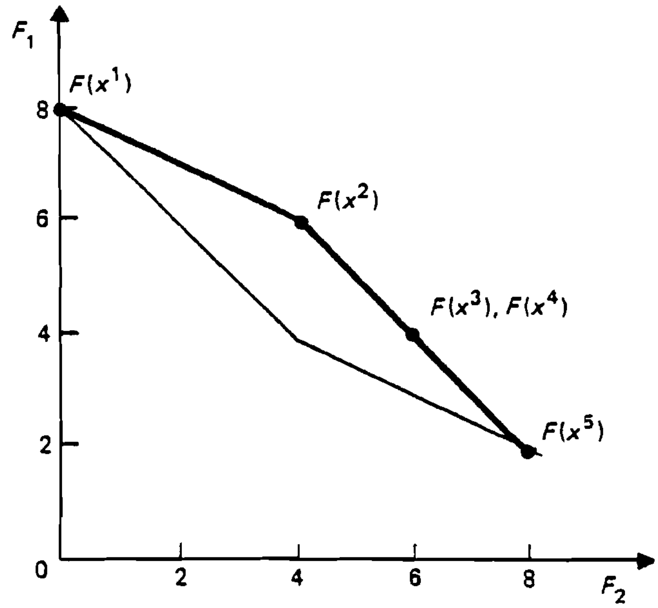

Figure 5. The reachable set of the system of functionals in Example 4 – the bold lines represent the Pareto set in the space of functionals.

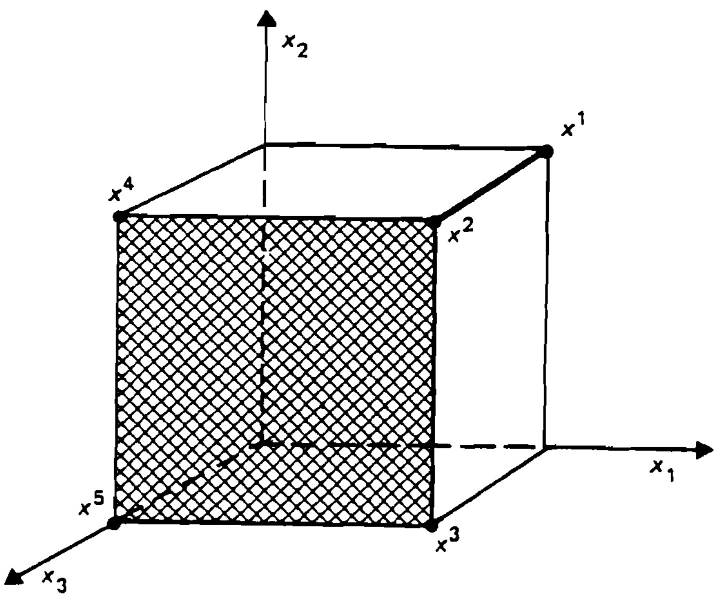

Figure 6. Set X from Example 4 – the bold lines represent the Pareto set in the space of variables.

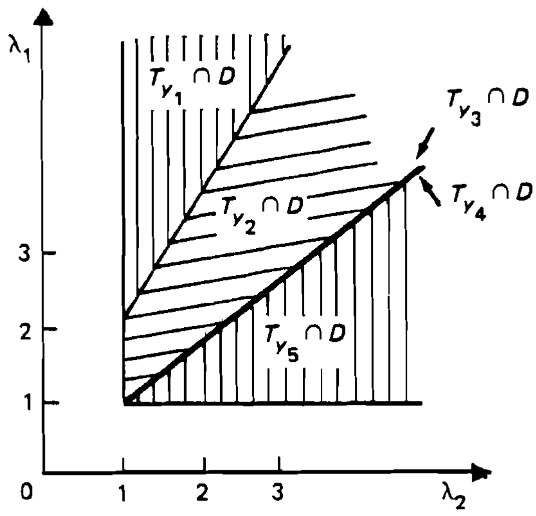

Figure 7. The sets $T_{y_l} \cap D$, $l = 1,\ldots,5$, from Example 4.

The basic idea of the algorithm is as follows: some $\lambda^* \in D$ is chosen and a J is found such that $\lambda^* \in T_y$. All the neighboring optimal bases are then found. Next, all the neighboring optimal bases to these bases are found, and so on until all the neighboring optimal bases to all of the previously identified bases have been found.

The skeleton of the algorithm is then as follows:

Step 1. Choose $\lambda^* \in D$ and find an optimal basis J for the following problem

$$(\lambda^* c)x \to \max_{x \in X} \quad .$$

Step 2. Put basis J in the sequence (the list of bases that have already been found).

Step 3. Take from the sequence any basis J whose neighboring bases have not been found. If there is no such basis then the problem is solved and all the optimal bases have been found.

Step 4. Find all the neighboring bases to basis J and put them in the sequence if they are not already there.

The search for the neighboring optimal bases is carried out as follows:

(a) All the constraints on set T_y which can be turned into strict equalities for points belonging to $T_y \cap D$ are found.

(b) A variable j corresponding to each of the above constraints exists and determines the neighboring basis I. Both bases I and J are optimal on the set

$$D \cap \{\Delta_l^J(\lambda c) \geq 0\ ,\ \ l \in \bar{J} \setminus j\ ,\ \ \Delta_j^J(\lambda c) = 0\}$$

Step 5. Check that the neighboring optimal bases have been found for J, and go to Step 3.

Computations based on the world economic model have been carried out using the algorithm discussed in [14]; the tests are analyzed in [8]. Some inaccuracies have been found in the construction of the Pareto points in test number 3 in [8]; it has been shown that of the 70 extreme points found only 29 are Pareto points (computing time 1 min. 26 sec. on a BESM-6 computer); there are also 121 semi-efficient points (3 min. 10 sec.) and 189 extreme points (2 min. 40 sec.).

5. THE SEARCH FOR EQUILIBRIUM POINTS

5.1. Main definitions and theorems

The world economic models discussed in the previous section can be written in the following form:

$$A^i x^i + G^i e^i + H^i m^i + c^i f_i \leq b^i \tag{5.1}$$

$$x^i \geq 0\ ,\quad e^i \geq 0\ ,\quad m^i \geq 0\ ,\quad f_i \geq 0 \tag{5.2}$$

$$p(e^i - m^i) \geq 0 \quad . \tag{5.3}$$

Here x^i is a vector of regional economic conditions, describing production, consumption, investments, etc.; e^i is an export vector; m^i is an import vector; f_i is an index representing the economic level of the i-th region (e.g., the level of consumption, overall regional product, etc.); p is a vector of prices for traded products; A^i, G^i, H^i are matrices; c^i and b^i are vectors, and $c^i \geq 0$, $i = 1,\ldots,N$.

Assumption 1. The system of constraints comprising (5.1), (5.2), $i = 1,\ldots,N$, and the common balance constraint:

$$\sum_{i=1}^{N} e^i = \sum_{i=1}^{N} m^i \tag{5.4}$$

is consistent, and the reachable set of variables $f_1,\ldots,f_N$ of system (5.1), (5.2), (5.4) is bounded and includes a vector $f > 0$.

Definition 4. The set of vectors $(p^*, x^{*i}, e^{*i}, m^{*i}, f_i^*, \ i = 1,\ldots,N)$ is such that:

(i) for each $i = 1,\ldots,N$, the vector $(x^{*i}, e^{*i}, m^{*i}, f_i^*)$ is a solution of the i-th local problem:

$$f_i \to \max \tag{5.5}$$

subject to (5.1), (5.2), (5.3)

(ii) the point where the general balance restriction

$$\sum_{i=1}^{N} e^{*i} = \sum_{i=1}^{N} m^{*i} \tag{5.6}$$

is satisfied is called an ***equilibrium point*** of the economic interaction model.

Remark 1. Equilibrium points need not necessarily exist; equilibrium points associated with negative prices are also possible.

Let us now write down the dual problem to linear programming problem (5.5):

$$\begin{gathered} y^i A^i \geq 0\,, \quad y^i G^i - z_i p \geq 0\,, \quad y^i H^i + z_i p \geq 0 \\ \\ y^i c^i \geq 1\,, \quad y^i \geq 0\,, \quad z_i \geq 0\,, \quad y^i b^i \to \min \quad . \end{gathered} \tag{5.7}$$

Here y^i is a vector of estimates of constraints (5.1) and z_i is an estimate of constraints (5.3).

Remark 2. Problems (5.7), $i = 1,\ldots,N$, depend on the value of parameter p: it is possible that the problem is consistent at some values of p and unbounded at others.

Remark 3. If problem (5.7) is consistent and its functional is limited, then the solution $(y^i\ , z_i)$ is not necessarily unique.

Remark 4. Consider the restriction $y^i c^i \geq 1$. If the functional of problem (5.7) is greater than 0, then this will be an equality; if the functional equals 0, then (y^{*i}, z_i^*) will be among the solutions, where $y^{*i} c^i = 1$.

Definition 5. An equilibrium point $(p, x, l, m, f_i, \ i = 1,\ldots,N)$ is said to be an *equilibrium point of class* $Z > 0$ if for each $i = 1,\ldots,N$ a solution (y, z) of problem (5.7) at $p = p^i$ exists such that $z_i^* > 0$. All other equilibrium points are called equilibrium points of class $Z = 0$. The regional linear programming problem depending on the following parameters plays an important role in the search for equilibrium points:

$$v \in V = \{v \in E^N \mid v \geq 0 \, , \sum_{i=1}^{N} v_i = 1\}$$

$$\sum_{i=1}^{N} e^i = \sum_{i=1}^{N} m^i \, , \quad A^i x^i + G^i e^i + H^i m^i + c^i f_i \leq b^i \, , \quad i = 1,\ldots,N \quad ,$$

$$\rho v_i = f_i \, , \quad i = 1,\ldots,N \, , \quad x^i \geq 0 \, , \quad e^i \geq 0 \, , \quad m^i \geq 0 \, , \quad i = 1,\ldots,N \tag{5.8}$$

$$\rho \to \max \quad .$$

The problem dual to this is

$$\eta^i A^i \geq 0 \, , \quad \eta^i G^i - q \geq 0 \, , \quad \eta^i H^i + q \geq 0 \, , \quad i = 1,\ldots,N$$

$$\eta^i c^i - \xi_i = 0 \, , \quad i = 1,\ldots,N$$

$$\sum_{i=1}^{N} \xi_i v_i = 1 \tag{5.9}$$

$$\eta^i \geq 0 \, , \quad i = 1,\ldots,N$$

$$\sum_{i=1}^{N} \eta^i b^i \to \min \quad .$$

Note that it follows from Assumption 1 that these problems are consistent at any $v \in V$ and that the value of the functional is positive. The following algorithm is based on theorems given in [11]:

Theorem 3. *Let* $(p, x, e, m, f, i = 1,...,N)$ *be an equilibrium point of class* $Z > 0$ *and* $v^* = (f_1^*,...,f_N^*)/(\sum_{i=1}^N f_i^*)$. *Then*

(i) *the vector* $(\sum_{l=1}^N f_l^*, x^{*i}, e^{*i}, m^{*i}, f_i^*, \ i = 1,...,N)$ *is a solution of problem* (5.8) *at* $v = v^*$

(ii) *among the solutions of problem* (5.9) *at* $v = v^*$ *is a vector* $(q^*, \eta^{*i}, \xi_i^*, \ i = 1,...,N)$ *such that*

(a) $\xi_i^* > 0, \ i = 1,...,N$

(b) $\eta^{*i} b^i - \xi_i^* v_i^* \rho^* = 0, \ i = 1,...,N,$

where $\rho^* = \sum_{i=1}^N \eta^{*i} b^i$ *is the optimal value of the functional in problem* (5.9).

Theorem 4. *Let vector* $(q^*, \eta^{*i}, \xi_i^*, \ i = 1,...,N)$ *be a solution of problem* (5.8) *at* $v \in V$, *where*

(i) $\xi_i^* > 0, \ i = 1,...,N$

(ii) $\eta^{*i} b^* - \xi^{*i} v_i^* \rho^* = 0, \ i = 1,...,N.$

If $(\rho^*, x^{*i}, e^{*i}, m^{*i}, f_i^*, \ i = 1,...,N)$ *is a solution of problem* (5.8) *at* $v = v^*$ *then* $(q^*, x^{*i}, e^{*i}, m^{*i}, f_i^*, \ i = 1,...,N)$ *is an equilibrium point of class* $Z > 0$.

Definition 6. A parameter $v \in V$ is called an *equilibrium parameter* if the conditions of Theorem 4 are satisfied.

It follows from Theorems 3 and 4 that for each equilibrium point of class $Z > 0$ there is a corresponding equilibrium parameter. The converse is also true: for each equilibrium parameter there is at least one corresponding equilibrium point of class $Z > 0$.

To check whether a parameter $v \in V$ in problem (5.8) is an equilibrium parameter it is first necessary to obtain the dependence of the solution of the dual problem on the parameters. We shall divide the set of parameters V into optimal polyhedra of separate bases. The set of solutions to the dual problem for parameters from the interior of the polyhedra either contains one point or has the form of a convex hull of a number of extreme points. It is easy to see that each extreme point depends inversely on the parameters. The solution of the dual problem for parameters from the edges of the polyhedra is a convex hull of two extreme points; each extreme point is related to the parameters by a fractional-linear law.

5.2. The skeleton of the algorithm

Step 1. $v^* \in V$ is chosen, and an optimal basis of problem (5.8) is found at $v = v^*$.

Step 2. The dependence of the basis variables on v is determined:

$$x_{J_i}^J(v) = \left[\sum_{l=1}^{N} t_{il}^J v_l\right] / \left[\sum_{l=1}^{N} d_l^J v_l\right] .$$

The same is done for the dual variables:

$$y_i(v) = p^J / \left[\sum_{l=1}^{N} d_l^J v_l\right] .$$

Step 3. A polyhedron T_y is constructed in the V-optimal space of basis J. The essential conditions of set $T_y = \{x_{J_i}^J(v) \geq 0\}$ are determined, i.e., those conditions whose violation changes the set.

Step 4. The system of neighboring pseudobases is constructed as in the resource allocation algorithm (see Section 2). (We consider the essential conditions and the set in space V where these bases are optimal.)

Step 5. Repeat Steps 2–4 until the set of parameters V is completely covered by sets T_y.

Step 6. The equilibrium parameters v in space $T_y \cap V$ should be checked using conditions (i) and (ii) from Theorem 2.

To implement an algorithm based on this skeleton we have to find a solution of a system of algebraic equations, the order of which depends on the structure of set T_y. Various methods of solving such a system for sets T_y of dimension $(k - 1)$, $(k - 2)$, 0 have been suggested: however, there are as yet no methods available for other cases.

Experiments with the 4×6 world economic model have led to some interesting conclusions about the equilibrium price structure [12,13].

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