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PREFACE

Many of today's most significant socioeconomic problems, such as slower economic growth, the decline of some established industries, and shifts in patterns of foreign trade, are international or transnational in nature. But these problems manifest themselves in a variety of ways; both the intensities and the perceptions of the problems differ from one country to another, so that intercountry comparative analyses of recent historical developments are necessary. Through these analyses we attempt to identify the underlying processes of economic structural change and formulate useful hypotheses concerning future developments. The understanding of these processes and future prospects provides the focus for IIASA's project on Comparative Analysis of Economic Structure and Growth.

Our research concentrates primarily on the empirical analysis of interregional and intertemporal economic structural change, on the sources of and constraints on economic growth, on problems of adaptation to sudden changes, and especially on problems arising from changing patterns of international trade, resource availability, and technology. The project relies on IIASA's accumulated expertise in related fields and, in particular, on the data bases and systems of models that have been developed in the recent past.

This paper is concerned with the solution algorithm of a nonlinear multisectoral model. The model has been developed at IIASA and falls into the class of so called computable general equilibrium models. The economic theoretical properties of the model, as well as some results of simulations based on it, have been reported elsewhere.

Anatoli Smyshlyaev
Project Leader
Comparative Analysis of
Economic Structure and Growth

CONTENTS

ON THE SOLUTION OF A COMPUTABLE GENERAL EQUILIBRIUM MODEL

József Sivak, Ambrus Tihanyi and Ernö Zalai

1. INTRODUCTION

The 1960s and 1970s were characterized by relatively rapid economic growth and growth itself became a major concern for economic policy makers. Also during this period, national economic planning in the socialist and developing countries became an increasingly sophisticated resource allocation exercise. Planners, interested in various alternatives for allocating resources and in the resulting efficiency gains, were soon able to call upon large-scale linear models of the input-output and programming types.[1]

In these planning exercises, price and relative cost considerations were left outside of the models themselves and a single decision-making unit (and often a single criterion) was generally assumed. These obvious weaknesses and the often exaggerated but close relationship between the principles of mathematical programming and Walrasian general equilibrium theory led to

[1]See, for example, Kornai (1974), Manne (1974), and Taylor (1975) on the use of these models in planning.

the emergence of computable general equilibrium models in the field of economic policy analysis and planning.

The growing number of publications in this field clearly demonstrates that we are witnessing a shift in the methodology concerned with the repercussive processes in a national economy. Apart from Johansen's (1959) pioneering study, it suffices to mention here just a few representative examples, such as Bergman and Pór (1980), Dervis *et al.* (1982), Dixon *et al.* (1982), and Kelly *et al.* (1983). Scarf's (1973) algorithm designed for computing fixed points (equilibria) gave tremendous impetus to the development of later more efficient solution algorithms.

In this paper we are concerned with one specific model developed by Zalai (1980) and, in particular, with its solution technique. This model was designed for planning purposes and particular emphasis was laid on some of the conceptual aspects of adopting such models for central planning. These, however, are not the subject of the present paper.

There is no single prototype model in the field of applied general equibrium analysis and there are no global algorithms among the solution techniques. Although Scarf's algorithm for finding a fixed point has enabled users to solve general equilibrium models, it is known that fixed-point algorithms can only be used for solving rather small models. Thus larger models must be solved using various heuristic methods.

Our experience with a solution technique based on Newton's iteration method is presented later in the paper. Another, rather common technique is to solve the model in question using a series of linear programming problems.[2] It is frequently useful to analyze the mathematical structure of each problem so as to discover the most appropriate and efficient solution techniques. We followed essentially this approach when solving the tentative model developed for the Hungarian economy.

[2]See, for example, Manne *et al.* (1980), Ginsburgh and Waelbroeck (1981), and Pór *et al.* (1982).

2. MATHEMATICAL CHARACTERIZATION OF THE MODEL TO BE SOLVED

2.1. The Equation System

As mentioned in the Introduction, we analyzed the solution possibilities of the model outlined in Zalai (1980). The original description of the model, including the list of variables and parameters, can be found in Appendix I. The model is a nonlinear equation system and throughout our discussion we will refer to the equations by the numbers given in Appendix I. Thus, for example, eqn. (20) refers to $Z_i = Z_{ir} + Z_{id}$. Some of the functional forms were not specific in this version, notably the production capacity functions. Therefore eqns. (8), (11), and (12) had to be made concrete by adopting Cobb-Douglas-type production functions. Instead of the original relations we employed eqns. (F6), (F7), and (F8), whose derivation is documented in Appendix II. Before presenting the solution algorithm adopted we will briefly characterize the mathematical structure of the model.

First of all, we call attention to the fact that the model is linearly homogeneous in a set of variables. More precisely, there is a group of variables x such that if (x_0, y_0) is a feasible solution of the model, then $(\lambda x_0, y_0)$ is also a feasible solution for any positive λ. Therefore, the usual problem of numeraire or normalization appears in this model too. There are several alternative ways to handle this problem. The logic of the solution algorithm we adopted suggested fixing total consumption expenditure (E) at some arbitrary level, which we did as follows:

$$E = \sum_{i=1}^{h} (b_i + \bar{b}_i)$$

This could thus be added as an additional constraint to the model described in Appendix I; alternatively (as we did), E can be treated as a constant parameter rather than a variable.

The resulting equation system specifying the relationships between the variables of the model is presented in Table 1. From now on, this system of equations is considered to be the mathematical basis of the model, and any mention of "the model" refers to this system.

$b \in X$	$E_b \subset E$	X_i	X_{n+1}	M_{iv}	M_{id}	C_i	Z_{iv}	Z_{id}	K_i	I	$\bar M_i$	$\bar C_i$	L_i	$\bar M_{id}$	$\bar M_{iv}$	$\frac{\partial F_i}{\partial L_i}$	W_i	Q_i	$\frac{\partial F_i}{\partial K_i}$	S_i	a_i	V_v	V_d	m_{iv}	P_i	m_{id}	Z_i	P_i^D	EE	P_i^{DI}	C	D	R	P_{n+1}
1.	$X_i + M_{iv} + M_{id} = \sum_{j=1}^{n+1} a_{ij} X_j + C_i + Z_{iv} + Z_{id}$	▦	•	•	•	•	•	•																										
2.	$X_{n+1} = \sum_{j=1}^{n} b_j K_j + I$		•						•	•																								
3.	$\bar M_i = \sum_{j=1}^{n+1} \bar m_{ij} X_j + \bar C_i$	▦	•								•	•																						
4.	$K = \sum_{i=1}^{n} K_i$								•																									
5.	$L = \sum_{i=1}^{n} L_i$												•																					
6.	$\sum_{i=1}^{n} \left(\frac{Z_{id}}{\bar Z_{id}}\right)^{\epsilon_i} P_{id}^{W} Z_{id} - \sum_{i=1}^{n} P_{id}^{W} M_{id} - \sum_{i=1}^{n} P_{id}^{W} \bar M_{id} = D_d$				•			•						•																				
7.	$\sum_{i=1}^{n} P_{iv}^{W} Z_{iv} - \sum_{i=1}^{n} P_{iv}^{W} M_{iv} - \sum_{i=1}^{n} P_{iv}^{W} \bar M_{iv} = D_v$			•			•								•																			
8.	$\frac{\partial F_i}{\partial L_i} = c_i \ell_i \left(\frac{1-\ell_i}{\ell_i}\right)^{1-\ell_i} \left(\frac{W_i}{Q_i}\right)^{1-\ell_i}$															•	•	•																
9.	$\frac{\partial F_i}{\partial K_i} = c_i (1-\ell_i) \left(\frac{\ell_i}{1-\ell_i}\right)^{\ell_i} \left(\frac{Q_i}{W_i}\right)^{\ell_i}$																•	•	•															
10.	$L_i = \ell_i X_i / \frac{\partial F_i}{\partial L_i}$	•											•			•																		
11.	$K_i = (1-\ell_i) X_i / \frac{\partial F_i}{\partial K_i}$	•							•										•															
12.	$S_i = Q_i / \frac{\partial F_i}{\partial K_i}$																	•	•	•														
13.	$a_i = a_i^0 \left(\frac{\theta_{id} V_d P_{id}^{W}}{\theta_{iv} V_v P_{iv}^{W}}\right)^{\sigma_i}$																				•	•	•											
14.	$\bar M_{iv} = a_i \bar M_i$										•				•						•													
15.	$\bar M_{id} = (1-a_i) \bar M_i$										•			•							•													
16.	$m_{iv} = m_{iv}^0 \left(\frac{P_i}{\theta_{iv} V_v P_{iv}^{W}}\right)^{\mu_{iv}}$																					•		•	•									
17.	$m_{id} = m_{id}^0 \left(\frac{P_i}{\theta_{id} V_d P_{id}^{W}}\right)^{\mu_{id}}$																						•		•	•								

Table 1. The equation system.

$e \in E$	$E_0 \subset E$
18.	$M_{tr} = m_{tr}(X_i - Z_i)$
19.	$M_{td} = m_{td}(X_i - Z_i)$
20.	$Z_i = X_{tr} + X_{td}$
21.	$X_{tr} = X_{tr}^0 \left[\frac{P_i}{r_{tr} v_r P_{tr}^{W}} \right]^{\epsilon_{tr}}$
22.	$X_{td} = X_{td}^0 \left[\frac{P_i}{r_{td} v_d P_{td}^{W}} \right]^{\epsilon_{td}}$
23.	$C_i = c_i + \frac{c_i}{P_i^D} EX$
24.	$\bar C_i = \bar c_i + \frac{\bar c_i}{P_i^W} EX$
25.	$EX = E - \sum_{j=1}^{n} (P_j^D c_j + P_j^W \bar c_j)$
26.	$C = \sum_{i=1}^{n} C_i + \bar C_i$
27.	$I = \frac{\delta}{\theta} C$
28.	$\pi_i = (1 + \bar\omega) w_i$
29.	$Q_i = (\theta_i + R) P_{n+1}$
30.	$P_{n+1} = \sum_{i=1}^{n} P_i^D a_{i,n+1} + \sum_{i=1}^{n} \bar P_i^W \bar a_{i,n+1}$
31.	$\bar P_i^W = \alpha_i(\bar\delta_{tr} v_r \bar P_{tr}^W) + (1 - \alpha_i)(\bar\delta_{td} v_d \bar P_{td}^W)$
32.	$P_j = (\sum_{i=1}^{n} P_i^D a_{ij} + \sum_{i=1}^{n} \bar P_i^W \bar a_{ij} + w_j)(1 + \Pi_j)$
33.	$P_i^D = \frac{P_i}{1 + m_{tr} + m_{td}} + \frac{m_{td}}{1 + m_{tr} + m_{td}} e_{td} v_d P_{td}^W + \frac{m_{tr}}{1 + m_{tr} + m_{td}} e_{tr} v_r P_{tr}^W$
	$\pi \triangleq \sum_{i=1}^{n} (\pi_i + \delta_i)$

Table 1. The equation system (continued).

2.2. The System of Constraints

Let E denote the system of equalities in the model and let $e \in E$ denote an equality of the system. E is the union of its finitely-many disjoint subsets $E_k \subseteq E$, $k \in K \triangleq \{1, 2, ..., 33\}$, and E is specified by specifying E_k, $k \in K$. Note that the "serial number" $k \in K$ in $e \in E_k$ means that e belongs to the class of equations E_k and, in turn, the classification $E = \bigcup_{k \in K} E_k$ shows that we will handle the equalities in E_k $(k \in K)$ in the same way. We will speak, somewhat imprecisely, about the "kth equality" when referring to the $|E_k|$ equalities in E_k, purely for the sake of simplicity.

Using the notation $K_1 \triangleq \{2, 4, 5, 6, 7, 25, 26, 27, 30\}$ and $K_2 \triangleq K \setminus K_1$ yields:

$$k \in K_1 \Rightarrow |E_k| = 1; \; k \in K_2 \Rightarrow |E_k| = n; \tag{2.1}$$

$$n = 19; \; |K_1| = 9; \; |K_2| = 24; \; |E| = 465$$

We now discuss the mathematical relations specified by the equalities $e \in E$ in more detail. Let Q denote the set of mathematical objects in which the mathematical relations between some groups of elements are set up in the model specification. Let $q \in Q$ denote an element in Q. Partition set Q into two disjoint subsets $Q \triangleq U \cup V$; then for every $q \in U$ we say that q is a "parameter" or an "exogenous variable" (these are essentially synonyms), and for every $q \in V$ we say that q is an "endogenous variable."

Using different symbols for each endogenous variable $q \in V$ implies the partition of set V into disjoint classes $V \triangleq \bigcup_{l \in L} V_l$ such that the classes are identified by the following $|L| = 33$ symbols:

$$EE, \hat{C}, X, X_{n+1}, M_r, M_d, C, Z_r, Z_d, K, I, \overline{M}, \overline{C}, L, \overline{M}_d, \hat{W}, R, P_{n+1}, \overline{M}_r, \frac{\partial F}{\partial L}, W,$$
$$Q, \frac{\partial F}{\partial K}, S, \alpha, V_r, V_d, m_r, P, m_d, Z, P^D, P^{DI}.$$

Using the notation $L_1 \triangleq \{X_{n+1}, I, V_r, V_d, EE, \hat{C}, \hat{W}, R, P_{n+1}\}$ and $L_2 \triangleq L \setminus L_1$, we obtain

$$l \in L_1 \Rightarrow |V_l| = 1; \; l \in L_2 \Rightarrow |V_l| = n;$$

$$n = 19, |L_1| = 9, |L_2| = 24, |V| = 465 \tag{2.2}$$

Distinction between any two elements in class V_l, $l \in L_2$ is obtained by a set of indices Ind $\{1,2,...,n\}$ in the mathematical specification such that x_i denotes the ith element of the subset $X \subseteq V, |X| = n$, where $|\cdot|$ denotes the cardinality function.

As for the class U of objects $q \in Q$, the specification implies a classification $U \triangleq \bigcup U_j$ such that the classes are identified by the following $|\Gamma| = 33$ symbols:

α, α_{n+1}, $\overline{m}$, m_{n+1}, s, $\hat{K}$, $\hat{L}$, Z_d, λ, P_d^{WE}, P_d^{WI}, P_r^{WE}, P_r^{WI}, b, $\bar{b}$, $\bar{P}_d^{WI}$, $\bar{P}_r^{WI}$, ζ, ξ, $\bar{\Theta}_d$, Θ_r, Θ_d, $\bar{\Theta}_r$, m_r^o, m_d^o, φ_r, φ_d, c, $\bar{c}$, σ, w, δ, E.

Using the notation $\Gamma_1 \triangleq \{\alpha, \overline{m}\}$, $\Gamma_2 \triangleq \{\bar{K}, \bar{L}, \sigma, E\}$, and $\Gamma_3 \triangleq \Gamma \setminus (\Gamma_1 \cup \Gamma_2)$ we obtain

$$\gamma \in \Gamma_1 \Rightarrow |U_\gamma| = n^2; \tag{2.3}$$

$$\gamma \in \Gamma_2 \Rightarrow |U_\gamma| = 1;$$

$$\gamma \in \Gamma_3 \Rightarrow |U_\gamma| = n;$$

$$n = 19; |\Gamma_1| = 2, |\Gamma_2| = 4, |\Gamma_3| = 27, |U| = 1239.$$

Relations (2.1)-(2.3) mean that ***the mathematical specification is a formalization of the $|E| = 465$ relations*** that hold in the set of the $|U| + |V| = 1704$ mathematical objects associated with economic concepts in the model.

Next, we add the following remarks to the relations (2.1)-(2.3), coupled with the specification in Appendix II:

- Assume that an appropriate set $H_U \subseteq R^U$ of "feasible inputs" is given for the computable values of objects in class U.
- The computable values of the objects in V can be obtained by solving the system of equalities E in the model specification. The economic interpretation of the object in V -- in accordance with the relations formalized in E -- assumes the existence of a set $H_V \subseteq R^{|V|}$, referred to as an "acceptance region."

Thus the mathematical meaning of the relations in E is as follows. A group of equalities indexed by $k \in K_1$ can be considered as the formal definition of a real-valued mapping, and the group of equations indexed by $k \in K_2$ can be considered as a formal definition of a mapping f_k with range $\text{Im } f_k \subseteq R^n$. Therefore the system of equations E can be considered as a definition of a mapping

$$F:H_U \times H_V \to R^{|E|} \quad \text{where Dom } F \subseteq R^{1704}, \quad \text{Im } F \subseteq R^{465} \tag{2.4}$$

Thus, *the mathematical specification of the model has the following concise form:*

$$F(u,v) = \Theta \quad u \in H_U, \quad v \in H_V \tag{2.5}$$

where Θ denotes the "zero element" in the space in question. The details of this specification are shown in Table 1. The left-hand column of the table presents the definitions of the mappings f_k according to the equalities $E_k \subseteq E$ in Appendix I. The order of presentation of the endogenous variables V_l is determined by the order of appearance of the groups of variables in the definitions of mappings f_k, enumerated in the normal way, $k = 1,2,\ldots,33$.

Returning to the actual mathematical specification of (2.5) shown in Table 1, note that it is impossible to obtain a "closed" form of the implicit function $v : H_U \to R^{|E|}$ satisfying equality $F(u,v) = \Theta$, even if an input set H_U that is, in principle, consistent is given. Thus the characterization of the set H_U or the investigation of the consistency of the model must be based on various tentative computations using various inputs of $u \in R^{1239}$. The aim of each such computation is to obtain a solution to the equation $F(u_o,v) = \Theta$ with a fixed value of u_o. Such an investigation needs an effective algorithm for obtaining the solutions to equality $F(u_o,v) = \Theta$ for any input of $u \in H_U$ that is "in principle consistent."

Since our system of equations is too large, common[3] computational techniques cannot be used for its solution. However, the special structure of the system of equations specified enabled us to develop a *special decomposition*

[3]See for example Ortega and Rheinboldt (1970).

method; using this method, we can reduce the solution of the complete system of nonlinear equations to the solution of several smaller systems of nonlinear -- and in some cases even linear – equations.

Before discussing the details of the computational method, we stress that the meaning of the specification in Table 1 is independent of both the order of the indices in the table and the order of the endogenous variables in the head of the table. Thus, one can freely permute both the indices of the equations and the variables. Regarding the special structure of the system of equations, we can obtain the arrangement in Table 2 by permuting both the equations and the endogenous variables appropriately. The well-structured "diagram of variable appearance" shows clearly ***the block diagonal structure of the system of equations studied***. It is this special kind of structure that enables us to solve the system using a decomposition method.

3. THE SOLUTION ALGORITHM

The first goal of the tentative computation was to investigate the consistency of the model. More precisely, we allowed a rather large degree of freedom in the choice of the values of the objects in $V \subseteq Q$ (endogenous variables), i.e., we assumed only those restrictions that follow directly from the mathematical specification (in Table 2) for the set H_V. For example, the object $EE \in V$ in the model can be interpreted as the level of the household excess expenditure, and thus $EE \geqq 0$ should hold.

Observe that the restrictions, such as $0 \leqq \xi \leqq 1$, $\hat{K} \geqq 0$, $Z^0_{id} \geqq 0$, etc., on the values of the parameters and the endogenous variables indexed by $k \in K$ do not ensure that, for any vector $v \in R^{465}$ satisfying equality $F(u_o, v) = \Theta$ with a vector $u_o \in R^{1239}$ fulfilling the mathematical restrictions, component EE of vector v satisfies the inequality $EE \geqq 0$. Therefore we tried to use inputs $u_o \in R^{1239}$ such that the solution $v \in H_V$ to the equation $F(u_o, v) = \Theta$ was appropriate (as regards "rather general" sets H_V). To achieve this, we utilized various consistent data bases obtained by previous model computations.[4]

[4]See Augusztinovics (1981). Boda *et al.* (1982) give a detailed description of the inputs of the model.

$h \in H$	$E_h \subset E$	a_i	P_i^M	V_i	Q_i	$\frac{\partial F_i}{\partial L_i}$	$\frac{\partial F_i}{\partial K_i}$	S_i	m_{iv}	m_{id}	P_i^D	P_i	P_{n+1}	E_{iv}	E_{id}	E_i	ΔX	G_i	C_i	C	I	M_{iv}	M_{id}	K_i	X_i	X_{n+1}	L_i	M_i	M_v	M_d	θ	R	V_v	V_d
13.	$a_i = a_i^0 \left[\frac{\Theta_{id} V_d P_{id}^{W}}{\Theta_{iv} V_v P_{iv}^{W}}\right]^{\gamma_i}$	\																															|	|
31.	$\bar{P}_i^M = a_i(\bar{\Theta}_{iv} V_v \bar{P}_{iv}^{W}) + (1-a_i)(\bar{\Theta}_{id} V_d \bar{P}_{id}^{W})$	\	\																														|	|
28.	$v_i = (1+\bar{\omega})w_i$			\																											|			
29.	$q_i = (\delta_i + R)P_{n+1}$				\								|																			|		
8.	$\frac{\partial F_i}{\partial L_i} = c_i \xi_i \left(\frac{1-\xi_i}{\xi_i}\right)^{1-\xi_i}\left(\frac{V_i}{Q_i}\right)^{1-\xi_i}$			\	\	\																												
9.	$\frac{\partial F_i}{\partial K_i} = c_i(1-\xi_i)\left(\frac{\xi_i}{1-\xi_i}\right)^{\xi_i}\left(\frac{Q_i}{V_i}\right)^{\xi_i}$			\	\		\																											
18.	$S_i = Q_i / \frac{\partial F_i}{\partial K_i}$				\		\	\																										
16.	$m_{iv} = m_{iv}^0 \left[\frac{P_i}{\Theta_{iv} V_v P_{iv}^{W}}\right]^{\mu_{iv}}$								\			\																					|	
17.	$m_{id} = m_{id}^0 \left[\frac{P_i}{\Theta_{id} V_d P_{id}^{W}}\right]^{\mu_{id}}$									\		\																						|
33.	$P_i^D = \frac{P_i}{1+m_{iv}+m_{id}} + \frac{m_{id}}{1+m_{iv}+m_{id}}\Theta_{id} V_d P_{id}^{W} + \frac{m_{iv}}{1+m_{iv}+m_{id}}\Theta_{iv} V_v P_{iv}^{W}$								\	\	\	\																					|	|
32.	$P_j = \left(\sum_{i=1}^{n} P_i^D a_{ij} + \sum_{i=1}^{n} \bar{P}_i^M \bar{m}_{ij} + v_j\right)(1+\pi_j)$		▦					\			▦	\																						
30.	$P_{n+1} = \sum_{i=1}^{n} P_i^D a_{i,n+1} + \sum_{i=1}^{n} \bar{P}_i^M \bar{m}_{i,n+1}$		↔								↔		•																					
21.	$E_{iv} = E_{iv}^0 \left[\frac{P_i}{P_{iv} V_v P_{iv}^{W}}\right]^{\epsilon_{iv}}$											\		\																			|	
22.	$E_{id} = E_{id}^0 \left[\frac{P_i}{P_{id} V_d P_{id}^{W}}\right]^{\epsilon_{id}}$											\			\																			|
20.	$E_i = E_{iv} + E_{id}$													\	\	\																		
26.	$\Delta X = X - \sum_{j=1}^{n} (P_j^D h_j + \bar{P}_j^M \bar{h}_j)$		↔								↔						•																	

Table 2. The structure of the system of equations.

$b \in X$	$B_b \subset E$	a_i	P_i^{w}	π_i	Q_i	$\frac{\partial F_i}{\partial L_i}$	$\frac{\partial F_i}{\partial K_i}$	S_i	m_{iv}	m_{id}	P_i^{p}	P_i	P_{n+1}	Z_{iv}	Z_{id}	Z_i	ER	C_i	$\bar{C}_i$	C	I	M_{iv}	M_{id}	K_i	X_i	X_{n+1}	L_i	$\bar{M}_i$	$\bar{M}_{iv}$	$\bar{M}_{id}$	Φ	K	Y_r	Y_d
23.	$C_i = c_i + \frac{c_i}{P_i^{p}} ER$										\						|	\																
24.	$\bar{C}_i = \bar{c}_i + \frac{\bar{c}_i}{P_i^{w}} ER$		\														|		\															
26.	$C = \sum_{i=1}^{n} C_i + \bar{C}_i$																	—	—	•															
27.	$I = \frac{1}{s} C$																			•	•														
18.	$M_{iv} = m_{iv}(X_i - Z_i)$								\							\						\			\										
19.	$M_{id} = m_{id}(X_i - Z_i)$									\						\							\		\										
11.	$K_i = (1 - \varepsilon_i) X_i / \frac{\partial F_i}{\partial K_i}$						\																	\	\										
1.	$X_i + M_{iv} + M_{id} = \sum_{j=1}^{n+1} a_{ij} X_j + C_i + Z_{iv} + Z_{id}$													\	\			\				\	\		▦	|									
2.	$X_{n+1} = \sum_{j=1}^{n} a_j K_j + I$																				•			—		•									
10.	$L_i = \varepsilon_i X_i / \frac{\partial F_i}{\partial L_i}$					\																			\		\								
3.	$\bar{M}_i = \sum_{j=1}^{n+1} \bar{m}_{ij} X_j + \bar{C}_i$																		\						▦	|		\							
14.	$\bar{M}_{iv} = \alpha_i \bar{M}_i$																											\	\						
15.	$\bar{M}_{id} = (1 - \alpha_i) \bar{M}_i$																											\		\					
4.	$K = \sum_{i=1}^{n} K_i$																							—											
5.	$L = \sum_{i=1}^{n} L_i$																										—								
6.	$\sum_{i=1}^{n} \left\| \frac{Z_i}{Z_{id}} \right\| P_{id}^{w} Z_{id} - \sum_{i=1}^{n} P_{id}^{w} M_{id} - \sum_{i=1}^{n} P_{id}^{w} \bar{M}_{id} = D_d$														—								—							—					
7.	$\sum_{i=1}^{n} P_{iv}^{w} Z_{iv} - \sum_{i=1}^{n} P_{iv}^{w} M_{iv} - \sum_{i=1}^{n} P_{iv}^{w} \bar{M}_{iv} = D_v$													—								—							—						

Table 2. The structure of the system of equations (continued).

The computation can be performed as follows. Given a vector $u_o \in R^{1239}$ and the mapping

$$\bar{F} : H_V \Rightarrow R^{|E|} \tag{3.1}$$

defined by equality $\bar{F}(v) \triangleq F(u_o, v)$, find a vector $v \in H_V$ such that $\bar{F}(v) = \Theta$.

As mentioned in Section 2, problem (3.1) can be solved by a decomposition technique. This is due to the special structure of the domain $H_V \subseteq R^{|v|}$ of the mapping $\bar{F}$ and its range Im $(\bar{F}) \subseteq R^{|E|}$. Using the notation

$$v^{(1)} \triangleq (\alpha, \bar{P}^{DI}, W, Q, \frac{\partial F}{\partial L}, \frac{\partial F}{\partial K}, S, m_r, m_d, P^D, P, P_{n+1}) \in H_V^{(1)} \triangleq H_V \cap R^{11n+1} ;$$

$$v^{(2)} \triangleq (Z_r, Z_d, Z, EE, C, \bar{C}, \hat{C}, I, M_r, M_d, K, X, X_{n+1}, L, M, \bar{M}_r, \bar{M}_d) \in H_V^{(2)} \triangleq H_V \cap R^{13n+8} ;$$

$$v^{(3)} \triangleq (\bar{W}, R, V_r, V_d) \in H_V^{(3)} \triangleq H_V \cap R^4 ,$$

according to Table 2, $\bar{F} : H_v \to R^{|E|}$ can be defined as follows:

$$F^{(1)} : H_V^{(1)} \times H_V^{(3)} \to R^{|K^{(1)}|};$$

$$F^{(2)} : H_V^{(1)} \times H_V^{(2)} \times H_V^{(3)} \to R^{|K^{(2)}|}; \tag{3.2}$$

$$F^{(3)} : H_V^{(1)} \times H_V^{(2)} \to R^{|K^{(3)}|}$$

where

$K^{(1)} \triangleq \{(13),(31),(28),(29),(8),(9),(12),(16),(17),(33),(32),(30)\};$

$K^{(2)} \triangleq \{(21),(22),(20),(25),(23),(24),(26),(27),(18),(19),(11),(1),(2),(10),(3),(14),(15)\};$

$K^{(3)} \triangleq \{(4),(5),(6),(7)\}.$

If the element $v \in H_V$ is a solution to the system of equations $\bar{F}(v) = \Theta$, then, using the notation $v \triangleq (v^{(1)}, v^{(2)}, v^{(3)}) \in H_V^{(1)} \times H_V^{(2)} \times H_V^{(3)}$, we require

$$F^{(1)}(v^{(1)}, v^{(3)}) = \Theta;$$

$$F^{(2)}(v^{(1)}, v^{(2)}, v^{(3)}) = \Theta; \tag{3.3}$$

$$F^{(3)}(v^{(1)}, v^{(2)}) = \Theta$$

Thus Table 3 can be considered to be the scheme of the decomposition method for solving problem (3.1). Technically, *this method is an iterative process*, which, given any "error parameter" $\varepsilon^{*} > 0$ and an initial value $v^{(3)} \triangleq v_0^{(3)} \in H_V^{(3)}$, successively yields finitely-many ($n = 0,1,\dots,N$) members of the series

$$\rho^{(1)} \triangleq \{v_n^{(1)} \in H_V^{(1)} : n = 0,1,2,\dots\};$$

$$\rho^{(2)} \triangleq \{v_n^{(2)} \in H_V^{(2)} : n = 0,1,2,\dots\};$$

$$\rho^{(3)} \triangleq \{v_n^{(3)} \in H_V^{(3)} : n = 0,1,2,\dots\}.$$

$k \in K$	$\bar{F}: H_V \to R^{\lvert E \rvert}$	$v^{(1)}$	$v^{(2)}$	$v^{(3)}$
$K^{(1)}$	$F^{(1)}(v^{(1)}, v^{(3)}) = \Theta$			
$K^{(2)}$	$F^{(2)}(v^{(1)}, v^{(2)}, v^{(3)}) = \Theta$			
$K^{(3)}$	$F^{(3)}(v^{(1)}, v^{(2)}) = \Theta$			

Table 3.

respect to the jth component of variable $v_n^{(3)} \in V_n^{(3)}$. Approximating it, compute the value of the quotient

$$\frac{\partial \Delta_n^{(i)}}{\partial v_{n,j}^{(3)}} \approx \frac{\Delta_n(v_n^{(3)} + \rho^* e_j) - \Delta_n(v_n^{(3)})}{\rho^*} \tag{3.7}$$

where e_j is the jth unit vector in space R^4. The values of $\Delta_n(v_n^{(3)} + \rho^* e_j)$ for $j = 1,2,3,4$ can be obtained using (3.4)-(3.6).

Step 5: Since we now have a technique for obtaining both the value $\Delta_n(v_n^{(3)})$ of mapping $\Delta_n : V_n^{(3)} \to R^4$ and its Jacobian matrix $J_n(v_n^{(3)}) : R^4 \to R^4$, we can use ***Newton's iteration method***:

$$v_{n+1}^{(3)} \triangleq v_n^{(3)} - J_n^{-1}(v_n^{(3)})\Delta_n(v_n^{(3)}) \tag{3.8}$$

Actually, this completes the description of the method for solving problem (3.1), since element $v^{(3)} \triangleq v_{n+1}^{(3)} \in \rho^{(3)}$ can be an input to Step 1 in the iteration.

Step 6: The fact that the process terminates after performing finitely-many steps is guaranteed by checking whether inequality $||\Delta_n|| < \varepsilon^*$ holds (see Step 3), i.e., whether the inequalities

$$||v_{n+1}^{(3)}|| < ||v_n^{(3)}|| \text{ and } ||v_{n+1}^{(3)}|| / ||v_n^{(3)}|| > (1 - \varepsilon^{**}) \tag{3.9}$$

hold, where ε^{**} is the so-called "parameter of convergence" of the process.

It is obvious that the method described above is only one possible technique for solving problem (3.1) (or more precisely, for solving the system of equations). And of course, as mentioned earlier, we have no method for deciding whether the system of equations (3.1) has any solution or not. This question can be answered by performing the procedure above. The ideas outlined can also be utilized in the solution of other systems of equations that have a decomposable structure.

Having obtained an element $v_n^{(3)} \in \rho^{(3)}$, the method works as follows:

Step 1: Obtain the element $v_n^{(1)} \in \rho^{(1)}$, which is the value of the implicit function $v^{(1)}: H_V^{(3)} \Rightarrow H_V^{(1)}$ defined by

$$F^{(1)}(v^{(1)}, v^{(3)}) = \Theta \tag{3.4}$$

at argument $v^{(3)} \triangleq v_n^{(3)}$; and denote $v_n^{(1)} \triangleq v^{(1)}(v_n^{(3)})$.

Step 2: On obtaining elements $v_n^{(3)}$ and $v_n^{(1)}$, obtain the element $v_n^{(2)} \in \rho^{(2)}$, which is the value of the implicit function $v^{(2)}: H_V^{(3)} \times H_V^{(1)} \Rightarrow H_V^{(2)}$ defined by

$$F^{(2)}(v^{(1)}, v^{(2)}, v^{(3)}) = \Theta \tag{3.5}$$

at argument $(v^{(3)}, v^{(1)}) \triangleq (v_n^{(3)}, v_n^{(1)})$; and denote:

$v_n^2 \triangleq v^{(2)}(v_n^{(3)}, v_n^{(1)})$.

Step 3: Equalities (3.6) hold by definition for elements $v_n^{(3)} \in H_V^{(3)}, v_n^{(1)} \in H_V^{(1)}, v_n^{(2)} \in H_V^{(2)}$, where

$$F^{(1)}(v_n^{(1)}, v_n^{(3)}) = \Theta;$$

$$F^{(2)}(v_n^{(1)}, v_n^{(2)}, v_n^{(3)}) = \Theta; \tag{3.6}$$

$$F^{(3)}(v_n^{(1)}, v_n^{(2)}) \triangleq \Delta_n(v_n^{(3)}).$$

The process terminates if $\|\Delta_n\| < \varepsilon^*$ holds. The $N(\varepsilon^*) \triangleq n$ iterations done yield a so-called ε^*-approximating solution $v \triangleq (v_n^{(1)}, v_n^{(2)}, v_n^{(3)})$ satisfying inequality $\|\bar{F}(v)\| < \varepsilon^*$ for the system of equations $\bar{F}(v) = \Theta$ (cf. (3.3) above).

Step 4: Calculate the computational approximation of the Jacobian matrix of the mapping $\Delta_n : V_n^{(3)} \Rightarrow R^4$ defined in an appropriately small (e.g., with radius $2\rho^*$) vicinity $V_n^{(3)} \subset H_V^{(3)} \subset R^4$ of point $v_n^{(3)} \in H_V^{(3)}$ at $v_n^{(3)}$. The (i,j)th entry of the 4x4 Jacobian matrix J_n is the first partial derivative of the ith component of mapping Δ_n with

When actually solving the problem, we first studied the actual structure of the model and then modified and rationalized the steps of the general algorithm so that the computation became easier.[5] Now we present details of these modifications.

The first significantly special feature of the model is that the subproblem (3.4), which is to be solved in Step 1, is in principle analogous to the original problem (3.1). Observe that (3.1) requires that we find an element $v \in R^{465}$, which can be obtained by solving the system of equations $F(u_0, v) = \Theta$ with a *given* element $u \triangleq u_0 \in R^{1239}$; and (3.4) requires that we find an element $v_n^{(1)} \in R^{20}$, which can be obtained by solving the system of nonlinear equations $F^{(1)}(v^{(1)}, v_n^{(3)}) = \Theta$ with a *given* element $v^{(3)} \triangleq v_n^{(3)} \in R^4$. From this it follows directly that, bearing in mind the structure of the system of equations, we need an appropriate decomposition method for solving subproblem (3.4). Using the notation $\bar{F}^{(1)}(v^{(1)}) \triangleq F^{(1)}(v^{(1)}, v_n^{(3)})$ and $v^{(1)} \triangleq (v^{(1,1)}, v^{(1,2)}, v^{(1,3)}, v^{(1,4)})$, we need to solve the equation $\bar{F}^{(1)}(v^{(1)}) = \Theta$ (this problem is shown in Table 4). To solve it we must, for example, obtain the components of solution $v^{(1)}$:

$$v^{(1,1)} \triangleq (\alpha, \bar{P}^{DI}) \in R^{2n};$$

$$v^{(1,2)} \triangleq (Q, \frac{\partial F}{\partial L}, \frac{\partial F}{\partial K}, S) \in R^{4n};$$

$$v^{(1,3)} \triangleq (m_T, m_d, P^D) \in R^{3n};$$

$$v^{(1,4)} \triangleq (P, P_{n+1}) \in R^{n+1}.$$

This method is an iterative algorithm whose "scheme" is shown in Table 5.

[5]The method was implemented on the computer of OTSzK (the Computing Center of the Hungarian National Central Planning Board) by several colleagues, including Lajos Laszlo, Sigitas Povilaitis, and Laszlo Zeöld.

$F^{(1)}: H_V^{(1)} \times H_V^{(s)} \to \Theta \in R^{11n+1}$	α	$\overline{P}^{DI}$	W	Q	$\frac{\partial F}{\partial L}$	$\frac{\partial F}{\partial K}$	S	m_r	m_d	P^D	P	P_{n+1}
$f_1^{(1)}(\alpha) = \Theta$	╲											
$f_2^{(1)}(\alpha, \overline{P}^{ID}) = \Theta$	╲	╲				∅			∅		∅	
$f_3^{(1)}(W) = \Theta$			╲									
$f_4^{(1)}(Q, P_{n+1}) = \Theta$				╲								│
$f_5^{(1)}(W, Q, \frac{\partial F}{\partial L}) = \Theta$			╲	╲	╲				∅			
$f_6^{(1)}(W, Q, \frac{\partial F}{\partial K}) = \Theta$			╲	╲		╲						
$f_7^{(1)}(Q, \frac{\partial F}{\partial K}, S) = \Theta$				╲		╲	╲					
$f_8^{(1)}(m_r, P) = \Theta$								╲			╲	
$f_9^{(1)}(m_d, P) = \Theta$		∅			∅				╲		╲	
$f_{10}^{(1)}(P^D, P, m_r, m_d) = \Theta$								╲	╲	╲	╲	
$f_{11}^{(1)}(P^D, P, \overline{P}^{DI}, S) = \Theta$		╲					╲			╲	╲	
$f_{12}^{(1)}(P_{n+1}, P^D, P^{DI}) = \Theta$		—								—		•

Table 4.

$k \in K^{(1)}$	$F^{(1)}: H_V^{(1)} \to R^{11n+1}$	$v^{(1,1)}$	$v^{(1,2)}$	$v^{(1,3)}$	$v^{(1,4)}$
$K^{(1,1)}$	$F^{(1,1)}(v^{(1,1)}) = \Theta$				
$K^{(1,2)}$	$F^{(1,2)}(v^{(1,1)}, v^{(1,2)}, v^{(1,4)}) = \Theta$				
$K^{(1,3)}$	$F^{(1,3)}(v^{(1,3)}, v^{(1,4)}) = \Theta$				
$K^{(1,4)}$	$F^{(1,4)}(v^{(1,1)}, v^{(1,2)}, v^{(1,3)}, v^{(1,4)}) = \Theta$				

Table 5.

Given an initial value $v^{(1,4)} \triangleq v_0^{(1,4)}$ and an error parameter ε_1^*, the method involves the following steps:

Step 1.1: Obtain a solution $v^{(1,1)} = v_m^{(1,1)}$ to the equation $F^{(1,1)}(v^{(1,1)}) = \Theta$

Step 1.2: On obtaining $v_m^{(1,4)}$ and $v_m^{(1,1)}$, obtain a solution $v^{(1,2)} \triangleq v_m^{(1,2)}$ to the equation $F^{(1,3)}(v_m^{(1,1)}, v^{(1,2)}, v_m^{(1,4)}) = \Theta$.

Step 1.3: Obtain a solution $v^{(1,3)} \triangleq v_m^{(1,3)}$ to the equation $F^{(1,3)}(v^{(1,3)}, v_m^{(1,4)}) = \Theta$.

Step 1.4: Obtain a value $\Delta_m^{(1)} \triangleq F^{(1,4)}(v_m^{(1,1)}, v_m^{(1,2)}, v_m^{(1,3)}, v_m^{(1,4)}) \in R^{11n+1}$. If $||\Delta_m^{(1)}|| \leqq \varepsilon_{(1)}^*$, then the process terminates; and, having performed $M(\varepsilon^*) \triangleq m$ iterations, it yields an $\varepsilon_{(1)}^*$-approximating solution $v^{(1)} \triangleq (v_m^{(1,1)}, v_m^{(1,2)}, v_m^{(1,3)}, v_m^{(1,4)})$ to the equation $F^{(1)}(v^{(1)}) = \Theta$. If

$||\Delta_m^{(1)}|| > \varepsilon_{(1)}^*$, then obtain a value $v_{m+1}^{(1,4)} \in R^{20}$ and continue to the steps below.

Step 1.5: Obtain the Jacobian matrix J_m of the mapping $\Delta^{(1)}: R^{20} \to R^{20}$ at the point $v_m^{(1,4)}$. (Use Steps 1.1-1.4.)

Step 1.6: Obtain a value $v_{m+1}^{(1,4)}$ by the formula $v_{m+1}^{(1,4)} \triangleq v_m^{(1,4)} - J_m^{-1}\Delta_m^{(1)}$ according to Newton's iteration process. In order that the process should terminate in finitely-many steps, besides checking the inequality $||\Delta_m^{(1)}|| < \varepsilon_{(1)}^*$ in Step 1.4, check whether or not the following inequalities hold:

$$||\Delta_{m+1}^{(1)}|| \leqq ||\Delta_m^{(1)}||$$

and

$$||\Delta_{m+1}^{(1)}|| / ||\Delta_m^{(1)}|| \geqq (1 - \varepsilon_{(1)}^{**}).$$

However, as is frequently the case, there is still "a fly in the ointment":

-- First, *at each iteration* ($m \leq M(\varepsilon^*)$) *the system of nonlinear equalities* $F^{(1,3)}(v^{(1,3)}, v_m^{(1,4)}) = \Theta$ *must be solved.* Since this system is fairly large, consisting of $3n = 57$ nonlinear equalities in unknown $v^{(1,3)} \in R^{57}$, it might seem doubtful whether it is possible to find an effective solution technique.

-- Second, though this is not so important, to perform Step 1.5, i.e., to obtain a matrix approximating the Jacobian matrix J_m, we must perform the computation in Steps 1.1-1.4 for $n + 1 = 20$ initial values at each ($m = 0,1,\ldots,M$) iteration. And this means that, to solve subproblem $F^{(1)}(v^{(1)}) = \Theta$, we must solve the system of equations in 57 unknowns at least $(n + 2)M(\varepsilon_{(1)}^*)$ times. Considering that this algorithm is a "subroutine" of the original problem (3.1), we must therefore solve the system of equations in 57 unknowns at least $5 \cdot N(\varepsilon^*)(n + 2)M(\varepsilon_{(1)}^*)$ times during the tentative computation, causing a considerable increase in the run time required for the solution.

These two observations forced us to examine the question of how to find an "efficient" computational algorithm for performing Step 1. (We do not propose to discuss the details of this mathematical investigation here.)

After due consideration, we chose a method *different* from the decomposition technique represented by Steps 1.1-1.6 and developed for solving subproblem (3.4). This alternative approach is due to Andras Pór, who, having done some computation with models similar to the one we were actually studying,[6] called our attention to some remarkable methods of reduction. Namely, *using Banach-Tychonow's fixed-point theorem*, one can see that *the solution of the system of equations* $F^{(1)}(v^{(1)}) = \Theta$ *can be based on the contractive property of the mapping* $\Delta^{(1)}: R^{n+1} \to R^{n+1}$ defined by implicit functions $v^{(1,2)}: R^{n+1} \to R^{4n}$, $v^{(1,3)}: R^{n+1} \to R^{3n}$ and the mapping $F^{(1,4)}: R^{11n+1} \to R^{n+1}$. Thus we solve the problem in *Step 1* using an *iterative process* with an initial value $v^{(1,4)} = v_0^{(1,4)}$ chosen appropriately. Using the values $v_m^{(1,1)}, v_m^{(1,2)}, v_m^{(1,3)}$ obtained in Steps 1.1-1.3 and using the element $v^{(1,4)} \triangleq v_m^{(1,4)} \in R^{n+1}$, the mth iteration obtains the element $v_{m+1}^{(1,4)} \triangleq F^{(1,4)}(v_m^{(1,1)}, v_m^{(1,2)}, v_m^{(1,3)}, v_m^{(1,4)})$.

The solution of the problem in Step 2 is not difficult since, using the notation $\bar{F}^{(2)}(v^{(2)}) \triangleq F^{(2)}(v_n^{(1)}, v^{(2)}, v^{(3)})$, the system of equations has the form $\bar{F}^{(2)}(v^{(2)}) = \Theta$; and, being a system of linear equations, this system can be solved by inverting its coefficient matrix.

When implementing the tentative computation, we utilized the fact that the matrix representation of the linear mapping $\bar{F}^{(2)}: R \to R$, which is presented in Table 2, contains a lot of empty cells with a special structure (a quasi-triangular matrix). Using the Gaussian elimination method, we reduced the solution of the system of linear equations $\bar{F}^{(2)}(v^{(2)}) = \Theta$ to inverting an $(n + 1)\times(n + 1)$ matrix.

We do not wish to describe here the more trivial steps of the computation. Note, however, that some reduction can be achieved in the method used for the tentative computation by modifying it further.[7] We have not ourselves

[6] See Bergman and Pór (1980).

[7] The modifications that either were applied or could be applied in the future were pointed out and

implemented these modifications to date, since we do not think that the computation time needed to perform Step 2 is unduly long in comparison with the total computation time for the algorithm. But if others wish to develop or extend the model, or to develop a dynamic version of it, they should carefully consider the efficiency of the solution techniques used and then take into account the further reduction possibilities mentioned above.

The solution algorithm outlined above, as we have explained, exploits to a great extent the special mathematical features of the model. It is in general a rather difficult problem to check whether a large computable general equilibrium model has any solution, and if so, whether it is unique or not. There are no efficient global algorithms yet available, unlike the situation for linear programming models. The development of special solution algorithms for a particular class of models therefore seemed the most suitable approach.

Despite the model-specific feature of the solution algorithm discussed, it still allows for several modifications of the model specification. Some of these necessitate minor revisions of the algorithm. We will not discuss here the various alternative spcifications that can be solved by the same algorithm, but we will illustrate the possible extension of the algorithm with just one example.

In some simulations based on the discussed model, the ruble trade flows (Z_{ir}, M_{ir}, and $\overline{M}_{ir}$) were held constant. This meant that these variables became constant parameters, and consequently some equations had to be dropped, while others assumed an altered meaning. The real problem, which made it impossible to use directly the algorithm described above, was caused by the change in the determination of the ruble import share variables (m_{ir} and α_i). These were no longer relative price dependent variables; and therefore it was no longer possible to determine their values, simultaneously with those of the relative prices, in Step 1. But, fortunately, minor revision of the algorithm and the use of simple iteration techniques were enough to overcome this problem. Starting with some initial values for these share variables, we constantly updated them after each step of a full iteration and terminated the process with additional constraints that assured their convergence.

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built into the computer program of the model by Lajos Laszlo and Sigitas Povilaitis.

Thus, the above example shows that the simple algorithm developed can be modified even for some cases that basically alter the mathematical structure of the model. These possibilities are, however, limited. Therefore, there is still a great need for the development of more general, global, and at the same time efficient techniques.

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APPENDIX I: FORMAL STATEMENT OF THE MODEL

Endogenous Variables

X_j	gross output in sector $j = 1,2,\ldots,n$
M_{ir}, M_{id}	competitive ruble and dollar import of commodity $i = 1,2,\ldots,n$
X_{ij}	use of domestic-import composite commodity $i = 1,2,\ldots,n$ in sector $j = 1,2,\ldots n, n+1$
Z_i, Z_{ir}, Z_{id}	total, ruble, and dollar export of commodity i
X_{n+1}	total gross investments
I	total net investments at base price level
$\overline{M}_i, \overline{M}_{ir}, \overline{M}_{id}$	total, ruble, and dollar noncompetitive import of commodity $i = 1,2,\ldots,n$
$\overline{M}_{ij}$	use of noncompetitive import commodity $i = 1,2,\ldots,n$ in sector $j = 1,2,\ldots,n,n+1$
$\overline{C}_i$	total private and public consumption of noncompetitive import commodity $i = 1,2,\ldots,n$
K_j	capital used in sector $j = 1,2,\ldots,n$
L_j	labor employed in sector $j = 1,2,\ldots,n$
S_j	(optimal) user cost of labor and capital per unit of output in sector $j = 1,2,\ldots,n$

W_j	user cost of labor in sector $j = 1,2,\ldots,n$
W	net rate of return requirement (tax) on labor
Q_j	user cost of capital in sector $j = 1,2,\ldots,n$
R	net rate of return requirement (tax) on capital
$\overline{m}_i$	share of ruble import in total noncompetitive import of commodity $i = 1,2,\ldots,n$
m_{ir}, m_{id}	proportions of competitive ruble and dollar imports of commodity $i = 1,2,\ldots,n$
P_j	domestic seller price of commodity $j = 1,2,\ldots,n$ produced
P_{jd}^E	dollar export price of commodity $j = 1,2,\ldots,n$
V_r, V_d	exchange rate of rubles and dollars
$\overline{P}_i^{DI}$	average domestic price of noncompetitive import of commodity $i = 1,2,\ldots,n$
P_i^D	average price of domestic-import composite commodity $i = 1,2,\ldots,n$
E	total consumption expenditure
EE	excess expenditure level
C	total consumption at base price level

Exogenous Variables and Parameters

s_j	capital replacement rate in sector $j = 1,2,\ldots,n$
δ_j	depreciation rate in sector $j = 1,2,\ldots,n$
K	total capital stock
L	total labor
Z_{id}^o, Z_{ir}^o	
	parameters in the export functions
$\varepsilon_{ir}, \varepsilon_{id}$	
λ_i	negative reciprocal of dollar export demand elasticities in sector $i = 1,2,\ldots,n$
$P_{id}^{WE}, P_{ir}^{WE},$	
$P_{id}^{WI}, P_{ir}^{WI},$	world market export and import prices of commodity i (ruble-dollar, competitive-noncompetitive import)
$\overline{P}_{id}^{WI}, \overline{P}_{ir}^{WI}$	
D_d, D_r	target surplus or deficit on dollar and ruble foreign trade balance
a_{ij}	input coefficient of domestic-import composite commodity $i = 1,2,\ldots,n$ in sector $j = 1,2,\ldots,n,n+1$

$\overline{m}_i^o, \rho_i$	parameters in the determination of the area composition of the noncompetitive import of commodity $i = 1,2,\ldots,n$
m_{ir}^o, m_{id}^o	
	parameters in the import functions, $i = 1,2,\ldots,n$
μ_{ir}, μ_{id}	
$b_i, \overline{b}_i$	fixed (base) amount of total consumption of commodity $i = 1,2,\ldots,n$
$c_i, \overline{c}_i$	fixed structure of excess consumption of commodity $i = 1,2,\ldots,n$
σ	real consumption net investment ratio
w_j	wage coefficient in sector $j = 1,2,\ldots,n$

Balancing Equations

Intermediate Commodities

$$X_i + M_{ir} + M_{id} = \sum_{j=1}^{n+1} X_{ij} + C_i + Z_{ir} + Z_{id} \qquad i = 1,2,\ldots,n \tag{1}$$

$$X_{n+1} = \sum_{j=1}^{n} s_j K_j + I \tag{2}$$

Noncompetitive Imports

$$\overline{M}_i = \sum_{j=1}^{n+1} \overline{M}_{ij} + \overline{C}_i \qquad i = 1,2,\ldots,n \tag{3}$$

Primary Factors

$$K = \sum_{j=1}^{n} K_j \tag{4}$$

$$L = \sum_{j=1}^{n} L_j \tag{5}$$

Trade Balances

$$\sum_{i=1}^{n} \left(\frac{Z_{id}^o}{Z_{id}} \right)^{\lambda_i} P_{id}^M Z_{id} - \sum_{i=1}^{n} P_{id}^{WI} M_{id} - \sum_{i=1}^{n} \overline{P}_{id}^{WI} \overline{M}_{id} = D_d \tag{6}$$

$$\sum_{i=1}^{n} P_{ir}^{WE} Z_{ir} - \sum_{i=1}^{n} P_{ir}^{WI} M_{ir} - \sum_{i=1}^{n} \overline{P}_{ir}^{WI} \overline{M}_{ir} = D_r \tag{7}$$

Technological Choice

$$X_j = F_j(L_j, K_j) = \zeta_j L_j^{\xi_j} K_j^{1-\xi_j} \qquad j = 1,2,\ldots,n \tag{8}$$

$$X_{ij} = a_{ij} X_j \quad i = 1,2,\ldots,n \quad j = 1,2,\ldots,n,n+1 \tag{9}$$

$$\overline{M}_{ij} = \overline{m}_{ij} X_j \quad i = 1,2,\ldots,n \quad j = 1,2,\ldots,n,n+1 \tag{10}$$

$$S_j \frac{\partial F_j}{\partial L_j} = W_j \quad j = 1,2,\ldots,n \tag{11}$$

$$S_j \frac{\partial F_j}{\partial K_j} = Q_j \quad j = 1,2,\ldots,n \tag{12}$$

Import and Export Functions
Noncompetitive Imports

$$\overline{m}_i = \overline{m}_i^o \left(\frac{V_d \overline{P}_{id}^{WI}}{V_r \overline{P}_{ir}^{WI}} \right)^{\rho_i} \quad i = 1,2,\ldots,n \tag{13}$$

$$\overline{M}_{ir} = \overline{m}_i \overline{M}_i \quad i = 1,2,\ldots,n \tag{14}$$

$$\overline{M}_{id} = (1 - \overline{m}_i)\overline{M}_i \quad i = 1,2,\ldots,n \tag{15}$$

Competitive Imports

$$m_{ir} = m_{ir}^o \left(\frac{P_i}{V_r P_{ir}^{WI}} \right)^{\mu_{ir}} \quad i = 1,2,\ldots,n \tag{16}$$

$$m_{id} = m_{id}^o \left(\frac{P_i}{V_d P_{id}^{WI}} \right)^{\mu_{id}} \quad i = 1,2,\ldots,n \tag{17}$$

$$M_{ir} = m_{ir}(X_i - Z_i) \quad i = 1,2,\ldots,n \tag{18}$$

$$M_{id} = m_{id}(X_i - Z_i) \quad i = 1,2,\ldots,n \tag{19}$$

Exports

$$Z_i = Z_{ir} + Z_{id} \quad i = 1,2,\ldots,n \tag{20}$$

$$Z_{ir} = Z_{ir}^o \left(\frac{P_i}{V_r P_{ir}^{WE}} \right)^{\varepsilon_{ir}} \quad i = 1,2,\ldots,n \tag{21}$$

$$Z_{id} = Z_{id}^o \left(\frac{P_i}{V_d P_{id}^{WE}} \right)^{\varepsilon_{id}} \quad i = 1,2,\ldots,n \tag{22}$$

Final Demand Equations

$$C_i = b_i + \frac{c_i}{\sum_{i=1}^{n} P_i^D c_i} EE \quad i = 1,2,\ldots,n \tag{23}$$

$$\overline{C}_i = \overline{b}_i + \frac{\overline{c}_i}{\sum_{i=1}^{n} \overline{P}_i^{DI} \overline{c}_i} EE \quad i = 1,2,\ldots,n \tag{24}$$

$$EE = E - \sum_{j=1}^{n} \left[P_j^D b_j + \overline{P}_j^{DI} \overline{b}_j \right] \tag{25}$$

$$C = \sum_{i=1}^{n} C_i + \sum_{i=1}^{n} \overline{C}_i \tag{26}$$

$$C - \sigma \cdot I = 0 \quad \text{or } I = \overline{I} \tag{27}$$

Prices and Costs

$$W_j = (1 + W)w_j \quad j = 1,2,\ldots,n \tag{28}$$

$$Q_j = (\delta_j + R)P_{n+1} \quad j = 1,2,\ldots,n \tag{29}$$

$$P_{n+1} = \sum_{i=1}^{n} P_i^D a_{i,n+1} + \sum_{i=1}^{n} \overline{P}_i^{DI} \, \overline{m}_{i,n+1} \tag{30}$$

$$\overline{P}_i^{DI} = \overline{m}_i V_r \overline{P}_{ir}^{WI} + (1 - \overline{m}_i) V_d \overline{P}_{id}^{WI} \quad i = 1,2,\ldots,n \tag{31}$$

$$P_j = \sum_{i=1}^{n} P_i^D a_{ij} + \sum_{i=1}^{n} \overline{P}_i^{DI} \overline{m}_{ij} + s_j \quad j = 1,2,\ldots,n \tag{32}$$

$$P_i^D = \frac{1}{1 + m_{id} + m_{ir}} P_i + \frac{m_{id}}{1 + m_{id} + m_{ir}} V_d P_{id}^{WI} + \frac{m_{ir}}{1 + m_{id} + m_{ir}} V_r P_{ir}^{WI} \quad i = 1,2,\ldots,n \tag{33}$$

APPENDIX II: MATHEMATICAL TRANSFORMATION OF THE PRODUCTION RELATIONS

Consider the economic interpretation of the equalities $X_j = F_j(L_j, K_j)$ $(j = 1,2,\ldots,n)$ (these equalities are denoted by (8) in the description of the model). The jth equality, which is called the production function, represents the relations between the output (X_j) of the jth "producer"--in this case, of the "jth coordination sector"--the labor (L_j) employed to obtain this output, and the capital (K_j) used in sector j. To study the consequences of the theoretical assumptions behind the production function, we need to discuss the mathematical specification of the mapping $F_j: R_+^2 \to R_+$ in (8).

To make the implementation of the model easier, we restrict our investigation to "production relations" that can be represented by first-order homogeneous functions, F_j, and we use Cobb-Douglas-type functions F_j in the first tentative computations. Thus we assume that (8) has the form

$$X_j = \zeta_j L_j^{\xi_j} K_j^{(1-\xi_j)} \qquad (j = 1,2,\ldots,n) \tag{F.1}$$

where ζ_j and ξ_j $(j = 1,2,\ldots,n)$ are real parameters such that $\zeta_j > 0$ and $0 \leq \xi_j \leq 1$.

We now examine the "behavioral" rules of the jth producer. Symbols W_j and Q_j $(j = 1,2,\ldots,n)$ denote the jth producer's costs per unit of output when it employs labor

$$l_j = \frac{L_j}{X_j} \quad (j = 1,2,\ldots,n)$$

and uses capital (F.2)

$$k_j = \frac{K_j}{X_j} \quad (j = 1,2,\ldots,n)$$

respectively. (Here we utilize the fact that the function specified by (F.1) is a first-order homogeneous function.) Therefore the jth producer's cost is $l_j W_j + k_j Q_j$ per unit of output. Producer j's wish to minimize its cost, bearing in mind equalities (F.1) and (F.2), can thus be represented by the following problem:

$$l_j W_j + k_j Q_j \to \min$$

subject to (F.3)

$$1 = \zeta_j l_j^{\xi_j} k_j^{(1-\xi_j)}$$

The behavior of producer j is said to be rational if it chooses the minimum expenditures L_j, K_j for producing its gross production X_j; i.e., if $\underline{l}_j$ and $\underline{k}_j$ denote the solution to problem (F.2), the equalities

$$L_j = \underline{l}_j X_j \quad (j = 1,2,\ldots,n)$$

(F.4)

$$K_j = \underline{k}_j X_j \quad (j = 1,2,\ldots,n)$$

hold. The consequence of such behavior can be seen in the solution of problem (F.3).

Problem (F.3) can be solved by the Lagrange multiplier method. After simple computation, we obtain

$$\underline{l}_j = \left[\frac{W_j(1-\xi_j)}{Q_j \xi_j}\right]^{\xi_j} \qquad (j = 1,2,\ldots,n)$$

(F.5)

$$\underline{k}_j = \left[\frac{W_j(1-\xi_j)}{Q_j \xi_j}\right]^{(1-\xi_j)} \qquad (j = 1,2,\ldots,n)$$

The partial differentiation of equalities (8) with respect to L_j and K_j yields

$$\frac{\partial F_j}{\partial L_j} = \zeta_j \xi_j \left[\frac{1-\xi_j}{\xi_j}\right]^{(1-\xi_j)} \left[\frac{W_j}{Q_j}\right]^{(1-\xi_j)}$$

and (F.6)

$$\frac{\partial F_j}{\partial K_j} = \zeta_j (1-\xi_j) \left[\frac{\xi_j}{1-\xi_j}\right]^{\xi_j} \left[\frac{Q_j}{W_j}\right]^{\xi_j}$$

respectively; moreover it yields the relations

$$L_j \frac{\partial F_j}{\partial L_j} = \xi_j X_j$$

and

$$K_j \frac{\partial F_j}{\partial K_j} = (1-\xi_j) X_j$$

From (F.6) and the two latter relations, we obtain

$$L_j = \xi_j X_j \Big/ \frac{\partial F_j}{\partial L_j} \quad (j = 1,2,\dots,n)$$

(F.7)

$$K_j = (1-\xi_j) X_j \Big/ \frac{\partial F_j}{\partial K_j} \quad (j = 1,2,\dots,n)$$

It is known from the specification of the model that variable $S_j (j = 1,2,\dots,n)$, the user cost of labor and capital per unit of output in sector j, can be defined by equalities $S_j = W_j L_j + Q_j K_j \ (j = 1,2,\dots,n)$; and thus, assuming that the production functions $F_j \ (j = 1,2,\dots,n)$ are first-order homogeneous functions and therefore that equalities (11) and (12) hold, we need only the equalities

$$S_j = Q_j \Big/ \frac{\partial F_j}{\partial K_j} \quad (j = 1,2,\dots,n) \tag{F.8}$$

derived from (12). Thus, substituting the equalities (8), (11), and (12) in the original specification by the equalities (F.6), (F.7), and (F.8) obtained above, respectively, we obtain the system of equations on which our computation was based.