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PREFACE

In this paper, the authors discuss the theory of star-shaped sets and its uses in studying some important classes of nondifferentiable functions. This theory seems to provide tools capable of dealing with many important problems in non-smooth analysis.

This paper is a contribution to research on nondifferentiable optimization currently underway within the System and Decision Sciences Program.

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THE SPACE OF STAR-SHAPED SETS AND ITS APPLICATIONS IN NONSMOOTH OPTIMIZATION

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The study of quasidifferentiable functions is based on the properties of the space of convex sets. One very important concept in convex analysis is that of the *gauge* of a set. However, the definition of a gauge does not require convexity, and therefore the notion of a gauge can be extended beyond convex sets to a much wider class of sets. In this paper the authors develop a theory of gauge functions and study some properties of star-shaped sets. The results are then used to study nonsmooth extremal problems (of which problems involving quasidifferentiable functions represent a special class).

Key words: Gauge, Star-Shaped Sets, Positively Homogeneous Functions, Directional Derivatives, Nonsmooth Optimization, Quasidifferentiable Functions, Necessary Conditions.

1. Introduction

One very important concept in subdifferential calculus is that of *Minkowski duality*, through which every convex compact set is associated with a specific *support function*. The study of quasidifferentiable functions (see [1-3]) is essentially based on the properties of the space of convex sets. Making use of this space, the sum of a convex function and a concave function can be associated with every class of equivalent pairs of convex compact sets.

The concept of a gauge (a guage function of a convex set containing the origin [7]) is very important in convex analysis. However, the definition of a guage does not require the corresponding set to be convex but only to have a "star shape" with respect to its "zero" (origin). For this reason the idea of a gauge is not limited to convex sets, but can be applied to a much wider class of sets altogether (correspondence between gauges and these sets has long been recognized in the geometry of numbers [8]). When doing this it is convenient to consider only those sets which are star-shaped with respect to their zero and which have a continuous gauge. In the present paper, these sets will be called *star-shaped*. It is possible to introduce algebraic operations (called here inverse addition and inverse multiplication by a nonnegative number) within this family of sets in such a way that the natural correspondence between gauges and star-shaped sets becomes an algebraic isomorphism. This allows us to use the standard algebraic technique normally used to construct the space of convex sets to build the space of star-shaped sets. The duality between gauge functions and support functions (which holds in the convex case) allows us to consider the polar operator as a linear mapping from the space of star-shaped sets into the space of convex sets. It is then possible to look at some problems previously studied using the space of convex sets from a different, in some respects more general, standpoint. This is particularly useful in quasidifferential calculus.

In the first part of this paper we study star-shaped sets and their gauges and the family of all star-shaped sets. Algebraic operations and an *order relation* are introduced, and their properties are discussed. The properties of the mapping which associates every star-shaped set with its gauge are also considered. We then define the space of star-shaped sets and study its properties.

The second part of the paper is concerned with applications. Of particular importance is a geometrical interpretation of the directional derivative and its application to quasidifferentiable functions, and a definition of quasidifferentiable mappings. We also discuss the asymptotic behavior of trajectories which are generated by mappings with star-shaped images.

2. Star-shaped sets and gauges

Definition. A closed subset U of the n-dimensional space E_n is called a *star-shaped set* if it contains the origin as an interior point and every ray

$$\mathcal{L}_x = \{\lambda x \mid \lambda \geq 0\} \quad (x \neq 0)$$

does not intersect the boundary of U more than once.

To justify the definition we shall show that a star-shaped set U is star-shaped with respect to its zero, i.e., for all points $x \in U$ the set U contains the interval $[0,x] = \{\lambda x \mid \lambda \in [0,1]\}$. Let us consider the set

$$U_x = U \cap \mathcal{L}_x ,$$

where $x \neq 0$. This set is closed since it is a subset of the ray $\mathcal{L}_x$ and the endpoints of the intervals adjoining it are the boundaries of U. The fact that U is star-shaped implies either that there is no adjoining interval (i.e., $U_x = \mathcal{L}_x$) or that an adjoining interval is unique and of the form

$$\{\nu x \mid \nu \in (\nu',+\infty)\} ,$$

where $\nu' > 0$. In this case $U_x = [0,\nu' x]$.

The star-shape of U with respect to its zero follows immediately from the above, and is equivalent to either of the two relations

$$\lambda U \subset U \quad \forall \lambda \in [0,1] \ ;$$

$$\lambda U \supset U \quad \forall \lambda \geq 1 \ .$$

Recall that a finite function f defined on E_n is called *positively homogeneous* if

$$f(\lambda x) = \lambda f(x) \qquad \forall \lambda \geq 0 \ .$$

Let Ω be a set in E_n, $0 \in \text{int}\ \Omega$. The function

$$|x| \equiv |x|_\Omega = \inf\ \{\lambda > 0 \ |x \in \lambda\Omega\} \tag{1}$$

is called the *gauge* of set Ω (or the Minkowski gauge function). If Ω is convex then the gauge coincides with the gauge function familiar from convex analysis; if Ω is a ball then the gauge is a norm corresponding to this ball.

<u>Theorem 1</u>. *Let* s *be a functional defined on* E_n. *The following propositions are then equivalent:*

(a) the functional s *is positively homogeneous, nonnegative and continuous;*

(b) s *coincides with the gauge of a star-shaped set* Ω*, where* $\Omega = \{x \mid s(x) \leq 1\}$.

<u>Proof</u>. (a) Let s be a positively homogeneous, nonnegative, continuous functional and $\Omega = \{x \mid s(x) \leq 1\}$. Then

$$|x|_\Omega = \inf \{\lambda > 0 \mid s(x) \leq \lambda\} = s(x) .$$

It is easy to check that the set Ω is star-shaped.

(b) Let s coincide with the gauge of a star-shaped set Ω. Since Ω is star-shaped the set

$$\Omega_x = \{\lambda x \mid x \in \lambda\Omega\}$$

is a ray with vertex $|x|x$ (where $|\cdot| = |\cdot|_\Omega$) . This point belongs to Ω and is a boundary point of Ω if $|x| \neq 0$. Since Ω is closed then $\Omega = \{x \mid |x| \leq 1\}$.

It is clear that the gauge is both positively homogeneous and nonnegative. Let us now show that the gauge is continuous. Since the gauge is positively homogeneous it is enough to check that the set $B_1 = \{x \mid |x| \leq 1\}$ is closed and that the set $B_2 = \{x \mid |x| < 1\}$ is open. However, B_1 must be closed since it coincides with Ω. Suppose now that B_2 is not open, that $x \in B_2$ and that there exists a sequence $\{x_k\}$ such that $x_k \to x$, $|x_k| \geq 1$. Without loss of generality we can assume that lim $|x_k| = \nu \geq 1$. Take $y_k = x_k/|x_k|$. Then $|y_k| = 1$ and therefore y_k is a boundary point of Ω . Since $y_k \to x/\nu$ then the point x/ν is also a boundary point of Ω . If $x \neq 0$ it follows that the ray $\mathcal{L}_x$ intersects the boundary of Ω at at least two different points $x/|x|$ and x/ν , which is impossible.

If $|x| = 0$ then the ray $\mathcal{L}_x$ lies entirely in Ω and (from the definition of "star-shaped") does not contain any boundary points of Ω . Thus the gauge of a star-shaped set must also be continuous and the theorem is proved.

Remark. Since the gauge is continuous and int Ω coincides with the set $\{x \mid |x| < 1\}$, Ω must be regular, i.e., it coincides with the closure of its interior.

Let us denote by S the set of all star-shaped subsets of the space E_n, and by K the family of all nonnegative, continuous, positively homogeneous functions defined on E_n .

The following proposition may then be deduced:

Proposition 1. *A mapping $\psi : S \to K$ which associates a guage with every star-shaped set is a bijection.*

The set K is a cone in the space $C_0(E_n)$ of all continuous, positively homogeneous functions defined on E_n . Since every function from $C_0(E_n)$ is completely defined by its trace on the unit sphere $S_1 = \{x \in E_n \mid \|x\| = 1\}$, where $\|x\|$ is the euclidean norm of x , the space $C_0(E_n)$ can be identified with the space $C(S)$ of all functions which are continuous on S and the cone K coincides with the cone of functions which are nonnegative on S . Assume that $C(S)$ (and hence the cone K) are ordered in some natural way: $f_1 \geq f_2 \Leftrightarrow f_1(x) \geq f_2(x) \quad \forall x$.

Let us introduce the following order relation (by anti-inclusion) within the family S of all star-shaped sets:

$$\Omega_1 \geq \Omega_2 \quad \text{if} \quad \Omega_1 \subset \Omega_2 .$$

It follows immediately from the definition of a gauge that the bijection ψ which associates a gauge with every star-shaped set is an isomorphism of ordered sets S and K . In other words, relations $\Omega_1 \subset \Omega_2$ and $|x|_1 \geq |x|_2 \;\; \forall x$ are equivalent (where $|.|_i$ is the gauge of set Ω_i).

The cone K is a lattice , i.e., if $f_1,\ldots,f_m \in K$ then functions $\underline{f}$ and $\overline{f}$ defined by

$$\underline{f}(x) = \min_i f_i(x), \; \overline{f}(x) = \max_i f_i(x)$$

also belong to K . Let f_i be the gauge of a star-shaped set Ω_i . Then $\underline{f}$ is the gauge of the union $\underline{\Omega} = \bigcup_i \Omega_i$ and $\overline{f}$ is the gauge of the intersection $\overline{\Omega} = \bigcap_i \Omega_i$. This follows from the relations

$$\{\lambda > 0 | x \in \lambda\underline{\Omega}\} = \bigcup_i \{\lambda > 0 | x \in \lambda\Omega_i\} , \qquad (2)$$

$$\{\lambda > 0 | x \in \lambda\overline{\Omega}\} = \bigcap_i \{\lambda > 0 | x \in \lambda\Omega_i\} , \qquad (3)$$

which can be verified quite easily.

Thus, the union and intersection of a finite number of star-shaped sets are themselves star-shaped sets. Furthermore, the union coincides with the infimum and the intersection with the supremum of these sets in lattice S .

Proposition 2. *Let* A *be a set of indices and* U_α *be a star-shaped set with gauge* $|\cdot|_\alpha$. *If the function* $|x| = \inf_{\alpha\in A} |x|_\alpha$ *is continuous, then it is the gauge of the set* $\mathrm{cl} \bigcup_\alpha U_\alpha$. *If the function* $|x| = \sup_{\alpha\in A} |x|_\alpha$ *is finite and continuous, then it is the gauge of the set* $\bigcap_\alpha U_\alpha$.

We shall prove only the first part of the proposition. Since the function $|x| = \inf_{\alpha\in A} |x|_\alpha$ is continuous it follows from Theorem 1 that this function is the gauge of some star-shaped set $\underline{\Omega}$. It is now not difficult to check that

$$\underline{\Omega} = \mathrm{cl} \bigcup_\alpha U_\alpha .$$

Indeed, the continuity of functions $|.|$ and $|.|_\alpha$ implies that

$$\mathrm{int}\, \underline{\Omega} = \{x \mid |x| < 1\} = \{x \mid \inf_\alpha |x|_\alpha < 1\} = \bigcup_\alpha \mathrm{int}\, U_\alpha .$$

Therefore, taking into account the regularity of star-shaped sets we get

$$\underline{\Omega} = \mathrm{cl}\, \mathrm{int}\, \underline{\Omega} = \mathrm{cl} \bigcup_\alpha \mathrm{int}\, U_\alpha = \mathrm{cl} \bigcup_\alpha U_\alpha .$$

This proves the first part of the proposition.

3. Addition and multiplication

The algebraic operations of addition and multiplication by a nonnegative number have been introduced within the family K of gauges of star-shaped sets in a natural way. We shall now introduce corresponding operations within the family S with the help of isomorphism ψ .

Let $\Omega \subset S$, $\lambda \geq 0$. We shall describe the set $\lambda \odot \Omega$ with gauge $|.| = \lambda |.|_\Omega$, where $|.|_\Omega$ is the gauge of Ω , as the *inverse product* of set Ω and number λ .

The set $\Omega_1 \oplus \Omega_2$ with gauge $|.|$ which satisfies the relation

$$|.| = |.|_1 + |.|_2 ,$$

where $|.|_i$ is the gauge of set Ω_i , is called the *inverse sum* of the star-shaped sets Ω_1 and Ω_2 .

It follows from the definition that if $\lambda > 0$ then

$$\lambda \odot \Omega = \frac{1}{\lambda} \Omega .$$

If $\lambda=0$ then the set $\lambda \odot \Omega$ coincides with the entire space E_n .

We shall now describe inverse summation. To do this we require the following elementary proposition.

Proposition 3. *Let* $a_1,\ldots,a_m$ *be nonnegative numbers. Then*

$$a_1 + \ldots + a_m = \min_{\substack{\alpha_i \geq 0 \\ \Sigma\alpha_i = 1}} \max_i \frac{1}{\alpha_i} a_i \tag{4}$$

(where it is assumed that $0/0 = 0$*).*

If $a_i=0$ $\forall i$ then (4) is trivial. Otherwise, for any set $\{\alpha_i\}$ such that $\alpha_i \geq 0$, $\Sigma \alpha_i = 1$ there exists an index j such that

$$\alpha_j \leq \frac{a_j}{\sum_{k=1}^{m} a_k} = \bar{\alpha}_j ,$$

and therefore $\max_i \frac{1}{\alpha_i} a_i \geq \sum_{k=1}^{m} a_k$. At the same time $\max_i \frac{1}{\bar{\alpha}_i} a_i = \sum_{k=1}^{m} a_k$, and this proves the proposition.

Now let us consider star-shaped sets Ω_1 and Ω_2 with gauges $|.|_{\Omega_1}$ and $|.|_{\Omega_2}$ respectively, and let $|.|$ be the gauge of their inverse sum $\Omega_1 \oplus \Omega_2$. Then the following equality holds for every x :

$$|x| = |x|_{\Omega_1} + |x|_{\Omega_2} = \min_{0\leq\alpha\leq 1} \max \left\{ \frac{1}{\alpha} |x|_{\Omega_1} , \frac{1}{1-\alpha} |x|_{\Omega_2} \right\} =$$

$$= \min_{0\leq\alpha\leq 1} \max \left\{ |x|_{\alpha\Omega_1} , |x|_{(1-\alpha)\Omega_2} \right\} = \min_{0\leq\alpha\leq 1} |x|_{\alpha} ,$$

where $|.|_{\alpha}$ is the gauge of set $\alpha\Omega_1 \cap (1-\alpha)\Omega_2$. (It is assumed that $0\cdot\Omega = \bigcap_{\alpha>0} \alpha\Omega$.)

Since the function $|.|$ is continuous it follows from Proposition 2 that

$$\Omega_1 \oplus \Omega_2 = \mathrm{cl} \bigcup_{0\leq\alpha\leq 1} [\alpha\Omega_1 \cap (1-\alpha)\Omega_2] .$$

Note that the role of zero (a neutral element) with respect to summation in a "semilinear space" S is played by the space E_n (since the gauge of E_n coincides with the identity zero). At the same time, E_n is the smallest element of the ordered set S .

We shall now give some computational examples.

<u>Example 1</u>. Consider the following rectangles in E_2^+ :

$$U = [-1,1] \times [-2,2] \ ,$$

$$V_\lambda = [-2\lambda,2\lambda] \times [-\lambda,\lambda] \ .$$

Their inverse sum coincides with an octagon which is symmetric with respect to the coordinate axes. The intersection of this octagon with the first quadrant has the vertices:

$$(0,0) \ , \ \left(\frac{2\lambda}{1+2\lambda} \ , 0\right) , \left(\frac{2\lambda}{1+2\lambda} \ , \ \frac{\lambda}{1+2\lambda}\right) , \left(\frac{\lambda}{2+\lambda} \ , \ \frac{2\lambda}{2+\lambda}\right) , \left(0 \ , \ \frac{2\lambda}{2+\lambda}\right) \ .$$

Rectangles U and V_1 and their inverse sum are shown in Fig. 1. The set $U \oplus V_{10}$ is shown in Fig. 2.

<u>Example 2</u>. Let $U = \{(x,y) \in E_2 | y \leq 1\}$ and $V = \{(x,y) \in E_2 | x \leq 1\}$. The set $U \oplus V$ is depicted in Fig. 3.

<u>Example 3</u>. Sets U and V are presented in Figs. 4(a) and 4(b), respectively; the set $U \oplus V$ coincides with the intersection of U and V (see Fig. 4(c)).

4. The cone of star-shaped sets

We shall now describe the vector space generated by the "cone" of star-shaped sets S for which an order relation (with respect to anti-inclusion) and inverse algebraic operations have been defined.

Let S^2 be the set of pairs (U_1,U_2) , where $U_i \in S$. Let us introduce within S^2 the operations of inverse addition $\oplus$ and inverse multiplication by a number $\odot$, and a preordering

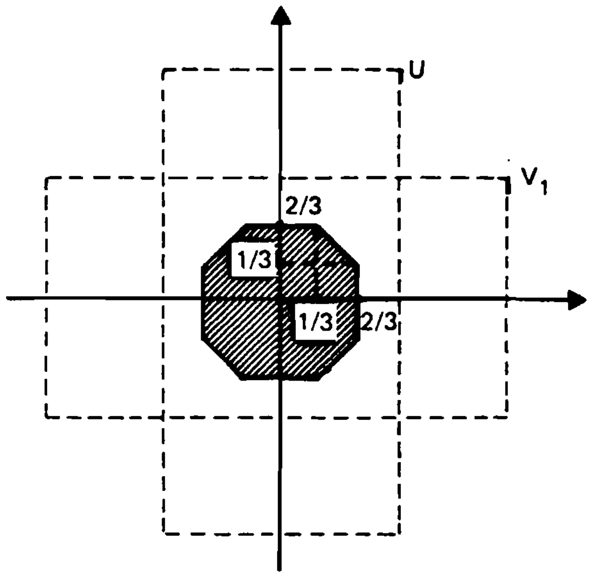

FIGURE 1

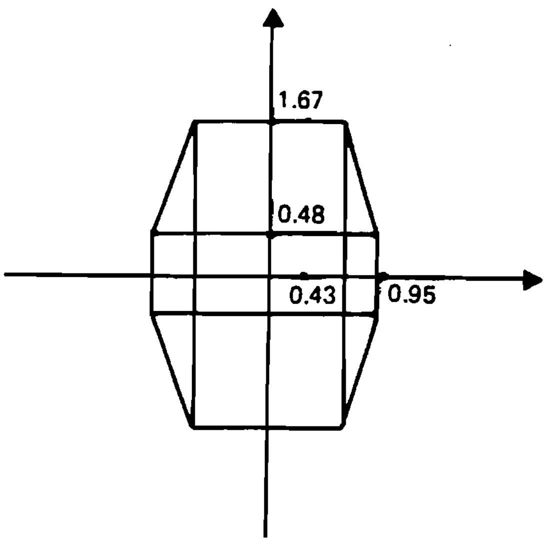

FIGURE 2

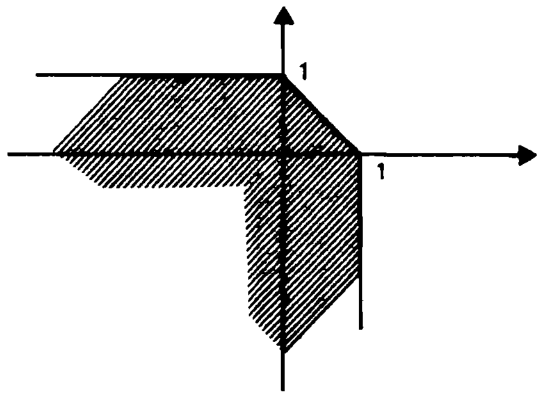

FIGURE 3

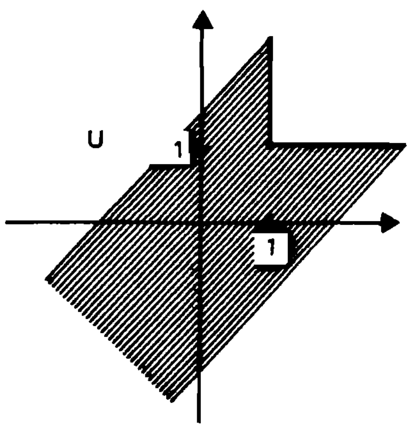

FIGURE 4(a)

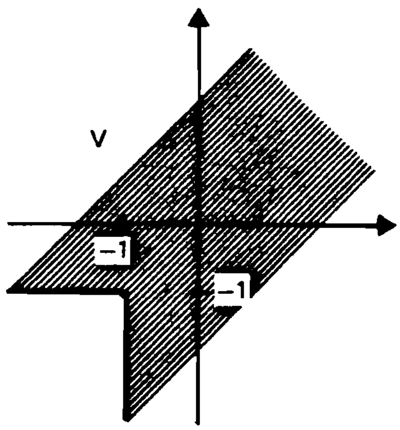

FIGURE. 4(b)

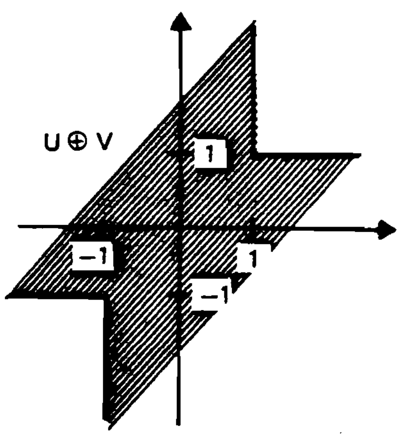

FIGURE 4(c)

relation ≥ and an equivalence relation ~ . These are defined as follows:

$$(U_1,U_2) \oplus (V_1,V_2) = (U_1 \oplus V_1, U_2 \oplus V_2) ,$$

$$\lambda \odot (U_1,U_2) = (\lambda \odot U_1, \lambda \odot U_2) \quad \text{if} \quad \lambda \geq 0 ,$$

$$\lambda \odot (U_1,U_2) = (|\lambda| \odot U_2, |\lambda| \odot U_1) \quad \text{if} \quad \lambda < 0 ,$$

$$(U_1,U_2) \geq (V_1,V_2) \Leftrightarrow U_1 \oplus V_2 \geq U_2 \oplus V_1 ,$$

$$(U_1,U_2) \sim (V_1,V_2) \Leftrightarrow U_1 \oplus V_2 = U_2 \oplus V_1 .$$

We shall now factorize the set S^2 with respect to the equivalence relation ~ . In other words, we shall consider the family T of all classes of equivalent pairs. Since the operators ⊕ and ⊙ produce equivalent pairs when applied to equivalent pairs, the operations for inverse summation and inverse multiplication by a number can be introduced within T in quite a natural way.

The order relation within T is derived naturally from S^2 . An element of T which contains a given pair (U_1,U_2) will be denoted by $[U_1,U_2]$. We shall identify an element U of the set S with the element $[U,E_n]$ of the set T . The equality

$$[U_1,U_2] = [U_1,E_n] \oplus [E_n,U_2] = [U_1,E_n] \ominus [U_2,E_n]$$

(where $\xi \ominus \eta = \xi \oplus (-1) \odot \eta$) then implies that every element of T can be represented as the difference of two elements of S , i.e., T is the smallest vector-ordered space containing S .

For this reason we shall call S the space of star-shaped sets (compare with the space of convex sets).

We shall associate with every pair $(U_1,U_2) \in S^2$ a positively homogeneous function $f = |.|_1 - |.|_2$, where $|.|_i$ is the gauge of U_i . It is clear that two pairs generate the same function if and only if they are equivalent. Hence, the function $f = |.|_1 - |.|_2 \in C_0(E_n)$ is associated with every element $[U_1,U_2]$ of the space T .

Conversely, by representing a continuous positively homogeneous function f in various forms $f = f_1 - f_2$ (where $f_i \in K$), we conclude that every element of the space $C_0(E_n)$ is associated with the class of equivalent pairs $[U_1,U_2]$, where $U_1=\{x|f_i(x)\le 1\}$. Identifying, as above, a star-shaped set U with the element $[U,E_n] \subset T$, we conclude that the mapping

$$[U_1,U_2] \to |.|_1 - |.|_2 \quad (5)$$

is an extension of the bijection $\psi : T \to K$ (which associates a gauge with a star-shaped set) to the bijection $T \to C_0(E_n)$. We shall use the same symbol ψ to denote this bijection and refer to it as a natural isomorphism.

It is clear that ψ preserves both the algebraic operations and the order relation. It is also clear that T, $C_0(E_n)$ and $C(S_1)$ can be viewed as different manifestations of the same ordered vector space.

It is well-known that the space $C(S_1)$ is a vector lattice: its elements $f_1,\dots,f_m$ include a point-wise supremum $\bigvee_{i=1}^{m} f_i$ and a point-wise infimum $\bigwedge_{i=1}^{m} f_i$. In addition, if $f_i=f_{1i}-f_{2i}$ then

$$\bigvee_{i=1}^{m} f_i = \bigvee_{k=1}^{m} (f_{1k} + \sum_{i\neq k} f_{2i}) - \sum_{i=1}^{m} f_{2i} ,$$

$$\bigwedge_{i=1}^{m} f_i = \bigwedge_{k=1}^{m} (f_{1k} + \sum_{i\neq k} f_{2i}) - \sum_{i=1}^{m} f_{2i} .$$

We may now conclude that the space T is also a vector lattice: if $\alpha_1,\ldots,\alpha_m \in T$, $\alpha_i = [U_{1i},U_{2i}]$ then

$$\bigvee_{i=1}^{m} \alpha_i = [\bigcap_{k=1}^{m} (U_{1k} \oplus (\sum_{i\neq k} \oplus)U_{2i}) , (\sum_{i=1}^{m} \oplus)U_{2i}], \quad (6)$$

$$\bigwedge_{i=1}^{m} \alpha_i = [\bigcup_{k=1}^{m} (U_{1k} \oplus (\sum_{i\neq k} \oplus)U_{2i}) , (\sum_{i=1}^{m} \oplus)U_{2i}], \quad (7)$$

where $(\Sigma \oplus)$ denotes the inverse sum of the corresponding terms. From (6) and the relation

$$\bigwedge_{i=1}^{m} \alpha_i = - \bigvee_{i=1}^{m} (-\alpha_i)$$

we conclude that

$$\bigwedge_{i=1}^{m} \alpha_i = [(\sum_{i=1}^{m} \oplus)U_{1i}, \bigcap_{k=1}^{m} (U_{2k} \oplus (\sum_{i\neq k} \oplus)U_{1i})]. \quad (8)$$

Equation (8) is in some respects more convenient than (7).

Let $\alpha = [U_1,U_2]$ be an element of the space of star-shaped sets, and $f = |\cdot|_1 - |\cdot|_2$ be the corresponding positively homogeneous function.

Let $V = \{x \mid f(x) \leq 1\}$. The set V is star-shaped. It is not difficult to check that the element $\alpha^{+} = \alpha \underline{\vee} 0$ coincides with $[V, E_n]$, i.e., that V is the smallest (in the sense of the ordering within S, or the largest with respect to inclusion) star-shaped set with the property $U_1 \supset U_2 \oplus V$.

We shall now introduce a norm $\|\cdot\|$ within the space $C_0(E_n)$. If $f \in C_0(E_n)$ then

$$\|f\| = \max_{x \in E_n} \frac{|f(x)|}{\|x\|} ,$$

where $\|\cdot\|$ is the euclidean norm in E_n. The corresponding norm in $C(S_1)$ is $\|f\| = \max_{z \in S_1} |f(z)|$.

In what follows we shall use the equality

$$\|f\| = \inf \{\lambda \geq 0 \mid -\lambda \|x\| \leq f(x) \leq \lambda \|x\| , \ \forall x \in E_n\} .$$

Let B be the unit ball in E_n. The element $e = (B, E_n)$ of the space T corresponds to the function $\|\cdot\|$, and the element $-e = (E_n, B)$ to the function $-\|\cdot\|$.

Let us define the following norm in T:

$$\|\alpha\| = \inf \{\lambda > 0 \mid -\lambda \odot e \leq \alpha \leq \lambda \odot e\} ,$$

where $\alpha \in E_n$.

If $\alpha = [U_1, U_2]$ then

$$\|\alpha\| = \inf \{\lambda > 0 \mid U_1 \supset U_2 \oplus \frac{1}{\lambda} B \ ; \ U_2 \supset U_1 \oplus \frac{1}{\lambda} B\} .$$

For a star-shaped set U we set $\|U\| = \|[U,E_n]\|$ and therefore

$$\|U\| = \inf\{\lambda > 0 | \lambda U \supset B\} \equiv \inf\{\lambda > 0 | U \supset \lambda \odot B\} .$$

Let X be a star-shaped compact set in E_n , and Σ be some subset of the family $S(X)$ of all star-shaped subsets of X . Let $U \in \Sigma$, and $|.|_U$ be the gauge of U . We shall consider the sets

$$\partial_U = \{x \mid |x|_U = 1\}$$

(the boundary of U) and

$$\partial_U^\varepsilon = \{x | 1 - \varepsilon \leq |x|_U \leq 1 + \varepsilon\} .$$

Proposition 4. *Let a subset Σ of the space $S(X)$ be closed in the topology of the space of star-shaped sets T. The set Σ is compact in this topology if and only if*

(i) *there exists a neighborhood $\tilde{B}$ of zero such that $\tilde{B} \subset U \ \forall U \in \Sigma$;*

(ii) *for every $\varepsilon>0$ there exists a $\delta>0$ such that $\partial_U + B_\delta \subset \partial_U^\varepsilon$, where $B_\delta = \{x | \|x\| < \delta\}$.*

Proof. Let us consider the set $\Sigma_{|.|}$ of all functions from $C(S_1)$-- this represents a contraction (on S_1) of the gauges of sets from Σ . The fact that the set Σ is compact is equivalent to the set $\Sigma_{|.|}$ being compact. By the Arzelá-Ascoli theorem this property of $\Sigma_{|.|}$ is equivalent to this set being bounded and equicontinuous. It is clear that condition (i) is satisfied if and only if $\Sigma_{|.|}$ is bounded.

We shall now show that condition (ii) is equivalent to $\Sigma_{|.|}$ being equicontinuous, assuming that (i) holds. Let condition (ii) be satisfied and $U \in \Sigma$.

If $|x|_U = 1$ and $\|x-y\| < \delta$, then $y \in \delta_U^{\varepsilon}$, i.e.,

$$\big|\, |y|_U - 1 \,\big| < \varepsilon ,$$

or, equivalently,

$$\big|\, |y|_U - |x|_U \big| < \varepsilon .$$

Set

$$x' = \frac{x}{\|x\|} , \quad y' = \frac{y}{\|y\|} .$$

Since X is compact there exists a number R such that $\|x\| < R$ $\forall\, x \in X$. Condition (i) implies that there exists an r such that $\|x\| > r$ (from the relation $|x|_U = 1$ for $U \in \Sigma$).

We shall now evaluate the difference $x'-y'$:

$$|x'-y'| = \left| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right| = \left| \frac{x\|y\|-y\|x\|}{\|x\|\cdot\|y\|} \right| =$$

$$= \left| \frac{x\|y\|-y\|y\|+y\|y\|-y\|x\|}{\|x\|\cdot\|y\|} \right| = \left| \frac{\|y\|(x-y) + y(\|y\|-\|x\|)}{\|x\|\cdot\|y\|} \right| \leq$$

$$\leq \frac{\|y\|\cdot\|x-y\| + \|y\| \, \big| \|x\| - \|y\| \big|}{\|x\|\cdot\|y\|} \leq \frac{2\|x-y\|}{\|x\|} < \frac{2\delta}{r} = \delta' .$$

This means that functions from $\Sigma_{|\cdot|}$ are equicontinuous.

Assume that the functions from $\Sigma_{|\cdot|}$ are equicontinuous as proposed above, i.e., for every $\varepsilon'>0$ there exists a $\delta'>0$ such that

$$\left| |x'|_U - |y'|_U \right| < \varepsilon' \qquad \forall\, U \in \Sigma ,$$

where $\|x'\|=1$, $\|y'\|=1$, and $\|x'-y'\|<\delta'$.

Let R and r be the numbers defined above. Fix $\varepsilon > 0$ and let $\varepsilon' < \varepsilon/R$. For ε' let us find δ' , the existence of which has already been established. Choose $\delta > 0$ such that

$$\delta < (\varepsilon - R\varepsilon')r \text{ and } \frac{\delta}{\sqrt{r(r-\delta)}} < \delta' .$$

From this inequality and from the identity

$$\|x\|\cdot\|y\|\cdot\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|^2 = \|x-y\|^2 - (\|x\|-\|y\|)^2$$

it follows that the relation $\|x-y\| < \delta$ implies the inequality

$$\left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\| < \delta' .$$

Let $U \in \Sigma$ and $x \in \partial_U$, i.e., $|x|_U=1$ and $y \in x + B_\delta$. Take

$$x' = \frac{x}{\|x\|} , \; y' = \frac{y}{\|y\|} .$$

Now

$$\Big| |x|_U - |y|_U \Big| = \Big| |x'|_U \cdot \|x\| - |y'|_U \cdot \|y\| \Big| =$$

$$= \Big| |x'|_U \cdot \|x\| - \|x\| \cdot |y'|_U + \|x\| \cdot |y'|_U - |y'_U| \cdot \|y\| \Big| \leq$$

$$\leq \|x\| \cdot \Big| |x'|_U - |y'|_U \Big| + |y'|_U \Big| \|x\| - \|y\| \Big| \leq$$

$$\leq R\varepsilon' + \frac{1}{r} \|x-y\| < \varepsilon .$$

Here we have used the inequality $|y'|_U \leq 1/r$, i.e.,

$$\Big| \|y\|_U - 1 \Big| \leq \varepsilon ,$$

or, equivalently, $y \in \partial_U^\varepsilon$. This completes the proof.

Remark. Let Σ be the family of all convex compact sets belonging to X for which condition (i) of Proposition 4 is satisfied. Then it is not difficult to show that set $\Sigma_{|\cdot|}$ is equicontinuous and therefore Σ is compact.

5. The space of convex sets

In conjunction with the space of star-shaped sets T , we shall consider the space of convex sets M (see [5, chapter I]. Recall that this space consists of classes of equivalent pairs [U,V], where U and V are convex compact sets in E_n and the equivalence relation is defined by

$$(U_1,V_1) \sim (U_2,V_2) \Leftrightarrow U_1 - V_2 = U_2 - V_1 .$$

The algebraic operations in M are defined as follows:

$$[U_1,V_1] + [U_2,V_2] = [U_1+U_2,V_1+V_2] ,$$

$$\lambda[A,B] = \begin{cases} [\lambda A,\lambda B] & \text{if } \lambda \geq 0 , \\ [\lambda B,\lambda A] & \text{if } \lambda < 0 . \end{cases}$$

The order relation $\geq$ is given by

$$[U_1,V_1] \geq [U_2,V_2] \text{ if } U_1-V_2 \supset U_2-V_1 .$$

Let L be the subspace of the space $C_0(E_n)$ which consists of functions which can be represented by the sum of a convex function and a concave function. The mapping $\Phi : M \rightarrow L$ defined by

$$\Phi([U,V])(x) = \max_{u\in U} (u,x) + \min_{v\in V} (v,x) \tag{9}$$

is an algebraic and ordering isomorphism (it is, of course, assumed that L is provided with natural algebraic operations and an order relation).

The inverse mapping Φ^{-1} associates an element $[\underline{\partial}p,\bar{\partial}q]$ from M with a function $p + q \in L$ (here $\underline{\partial}p$ is the subdifferential of the sublinear functional p and $\bar{\partial}q$ is the superdifferential of the superlinear functional q).

Let us consider a subset U of the space E_n . Let U^o denote its polar:

$$U^o = \{x | (u,x) \le 1 \quad \forall u \in U\} .$$

Here (and in (9)) (y,x) is the scalar product of y and x . Let us recall the main properties of the polar:

(i) The set U^o is convex and closed; $0 \in U^o$.

(ii) If U is convex and closed and $0 \in U$, then $U^{oo}=U$.

(iii) U is compact if and only if $0 \in \text{int } U^o$.

(iv) Let U be a convex closed set, with $0 \in U$. Then the gauge function of U coincides with the support function of the polar U^o and the support function of U coincides with the gauge function of the polar.

(v) Let U_1 and U_2 be convex and closed and let $0 \in U_1$, $0 \in U_2$. Then the relations $U_1 \supset U_2$, $U_1^o \subset U_2^o$ are equivalent and

$$(U_1+U_2)^o = U_1^o \oplus U_2^o ,$$

$$(\lambda U)^o = \frac{1}{\lambda}U^o \quad \text{if} \quad \lambda > 0 .$$

Now let us consider star-shaped convex sets U_1 and U_2 . Since $0 \in \text{int } U_i$, the polar U_i^o is compact. Since the gauge $|.|_i$ of the set U_i coincides with the support function of the polar U_i^o , the following relation holds:

$$|x|_1 - |x|_2 = \max_{l\in U_1^o} (l,x) - \max_{l\in U_2^o} (l,x) =$$

$$= \max_{l\in U_1^o} (l,x) + \min_{l\in[-U_2^o]} (l,x) . \qquad (10)$$

Let ψ and Φ be mappings defined by formulas (5) and (9) respectively, α be an element of the space T containing the pair (U_1, U_2) , and β be an element of the space M containing the pair $(U_1^o, -U_2^o)$. From (10) it follows that

$$\psi\alpha = \Phi\beta$$

and hence

$$\beta = (\Phi^{-1}\psi)(\alpha) .$$

The operator $\pi = \Phi^{-1}\psi$ defines the operation of taking the polar (π is the polar operator). It is defined on the subspace T_c of the space T which consists of elements α such that there exists a pair $(U,V) \in \alpha$, where U and V are convex sets.

It is clear that T_c is a linear space (this follows from the equivalence of the convexity of a star-shaped set and that of its gauge).

The set of values of the operator π coincides with the space of convex sets. Indeed, for $\beta \in M$ it is always possible to find a pair $(U,V) \in \beta$ such that $0 \in U$, $0 \in V$.

Then $U = (U^{\circ})^{\circ}$, $V = (V^{\circ})^{\circ}$ so that $\beta=\pi\alpha$, where $\alpha=[U^{\circ},V^{\circ}] \in T_c$.

From the properties of the polar it follows that the operator π is linear and order-preserving.

6. Quasidifferentiability and a geometrical interpretation of directional derivatives

The space of star-shaped sets can be used to provide a geometrical interpretation of directional derivatives. Let f be a function defined on an open set $\Omega \subset E_n$ and suppose that at a point $x \in E_n$ we can construct the directional derivative of f:

$$\frac{\partial f(x)}{\partial g} = f'_x(g) \equiv \lim_{\alpha\to+0} \frac{1}{\alpha}[f(x+\alpha g) - f(x)] ,$$

where the function $f'_x(g)$ is continuous in g.

Since the functional f'_x is positively homogeneous, an element of the space T of star-shaped sets is associated with f'_x. In other words, a pair of star-shaped sets (U,V) exists such that

$$f'_x(g) = \min \{\lambda > 0 | g \in \lambda U\} - \min \{\lambda > 0 | g \in \lambda V\}$$

or, equivalently,

$$f'_x(g) = \min \{\lambda > 0 | g \in \lambda U\} + \max \{\lambda < 0 | g \in (-\lambda) V\}. \qquad (11)$$

Note (from equation (11)) that the pairs (U,V) and (U_1,V_1) represent the derivative of f if and only if they are equivalent. Let us denote the set U in (11) by $\underline{d}f(x)$ and the set V by $\bar{d}f(x)$. Invoking the properties of the space T of star-shaped sets, it is possible to state rules for algebraic operations over functions and the corresponding pairs:

$$\underline{d}(f_1+f_2)(x) = \underline{d}f_1(x) \oplus \underline{d}f_2(x) ,$$

$$\bar{d}(f_1+f_2)(x) = \bar{d}f_1(x) \oplus \bar{d}f_2(x) ,$$

$$\underline{d}(f_1 \cdot f_2)(x) = f_1(x) \odot \underline{d}f_2(x) \oplus f_2(x) \odot \underline{d}f_1(x) ,$$

$$\bar{d}(f_1 \cdot f_2)(x) = f_1(x) \odot \bar{d}f_2(x) \oplus f_2(x) \odot \bar{d}f_1(x) .$$

Using formulas (6) and (7) and the rules for differentiability of the maximum function it is easy to find

$$\underline{d}(\max_i f_i(x)) , \bar{d}(\max_i f_i(x)) , \underline{d}(\min_i f_i(x)) , \bar{d}(\min_i f_i(x)) .$$

It is clear that a function f is quasidifferentiable at x if and only if there exist convex sets $\underline{d}f(x)$ and $\bar{d}f(x)$. In this case

$$\underline{d}f(x) = [\underline{\partial}f(x)]^{\circ} , \bar{d}f(x) = [-\bar{\partial}f(x)]^{\circ} ,$$

where $\underline{\partial}f(x)$ and $\bar{\partial}f(x)$ are a subdifferential and a superdifferential , respectively, of f at x .

We shall now present a geometrical interpretation of necessary conditions for a minimum. It is based on the following lemma.

Lemma 1. *Let a functional* f *be directionally differentiable at* $x \in E_n$, *the derivative* $f'_x(g)$ *be continuous in* g *and* K *be a cone in* E_n . *Then*

(i) *The relation*

$$\min_{g \in K} f'_x(g) = 0$$

is satisfied if and only if $\underline{d}f(x) \cap K \subset \bar{d}f(x)$.

(ii) *The relation*

$$\max_{g \in K} f'_x(g) = 0$$

is satisfied if and only if $\bar{d}f(x) \cap K \subset \underline{d}f(x)$.

Proof. Let us write $f'_x(g)$ in the form

$$f'_x(g) = |g|_1 - |g|_2 ,$$

where $|.|_1$ is the gauge of the set $\underline{d}f(x)$ and $|.|_2$ is the gauge of the set $\bar{d}f(x)$. Assume that

$$\min_{g \in K} f'_x(g) = 0 \quad \text{and } g \in \underline{d}f(x) \cap K .$$

Then $|g|_1 \le 1$ and $|g|_1 - |g_2| \ge 0$, so that $|g|_2 \le 1$, which is equivalent to the inclusion $g \in \bar{d}f(x)$. Thus, we have $\underline{d}f(x) \cap K \subset \bar{d}f(x)$.

Arguing from the other direction, suppose that this last inclusion holds. For a $g \in K$ such that $|g|_1 > 0$ let us find a $\lambda > 0$ such that $|\lambda g|_1 = 1$. Then $\lambda g \in \underline{d}f(x)$. But since $\lambda g \in \bar{d}f(x)$ we have the inequality $|\lambda g|_2 \le 1$. This means that $|g|_1 - |g|_2 = f'_x(g) \ge 0$. Thus, if $|g|_1 = 0$ then $|g|_2 = 0$ (since $|g|_2 \le |g|_1$) .

Part (ii) of the lemma can be proved in the same way.

Let $x \in \Omega \subset E_n$. By γ_x we shall denote the cone of feasible directions of the set Ω at the point x , i.e., $g \in \gamma_x$ if $x + \alpha g \in \Omega \ \ \forall \alpha \in (0,\alpha_0]$, where α_0 is some positive number (which depends on x and g).

Let Γ_x denote the cone of feasible (in a broad sense) directions of Ω at x : $g \in \Gamma_x$ if for any $\varepsilon>0$ there exists an element $g_\varepsilon \in B_\varepsilon(g) \equiv \{q \mid \|q-g\| < \varepsilon\}$ and a number $\alpha_\varepsilon \in (0,\varepsilon)$ such that $x + \alpha_\varepsilon g_\varepsilon \in \Omega$.

A functional f defined on an open set $\Omega \subset E_n$ is said to be uniformly directionally differentiable at $x \in \Omega$ if for any $g \in E_n$ and $\varepsilon>0$ there exist numbers $\delta>0$ and $\alpha_0>0$ such that

$$|f(x+\alpha q) - f(x) - \alpha f'_x(q)| < \alpha\varepsilon \quad \forall q \in B_\delta(g) \ , \ \forall \alpha \in (0,\alpha_0] \ .$$

It is shown in [5, chapter I] that a directionally differentiable, locally Lipschitzian function is also uniformly directionally differentiable.

<u>Theorem 2</u>. *Let* $x^* \in \Omega$ *be a minimum point of* f *on* Ω . *If* f *is directionally differentiable at* x^* *and* $f'_{x^*}(g)$ *is continuous in* g *then*

$$\underline{d}f(x^*) \cap \gamma_{x^*} \subset \bar{d}f(x^*) \ . \tag{12}$$

If f *is uniformly differentiable at* x^* *then*

$$\underline{d}f(x^*) \cap \Gamma_{x^*} \subset \bar{d}f(x^*) \ . \tag{13}$$

Corollary. *If* f *attains its minimal value at an interior point of the set* Ω , *then* $\underline{d}f(x^*) \subset \bar{d}f(x^*)$.

Remark. If f is quasidifferentiable and the sets $\underline{d}f(x^*)$ and $\bar{d}f(x^*)$ are convex then the relation $\underline{d}f(x^*) \subset \bar{d}f(x^*)$ is equivalent to the inclusion

$$-\bar{\partial}f(x^*) \subset \underline{\partial}f(x^*) \ ,$$

which is familiar from quasidifferential calculus.

Analogous necessary conditions for a constrained extremum of a quasidifferentiable function can be obtained from (12) and (13).

The values $a = \left| \min_{\|g\|=1} f'_x(g) \right|$, $b = \max_{\|g\|=1} f'_x(g)$ are called the rates of steepest descent and steepest ascent, respectively, of f on E_n .

Proposition 5. *The following relations hold:*

$$a = \inf \{\lambda > 0 \mid \bar{d}f(x) \supset \underline{d}f(x) \oplus \lambda \odot B\} \ , \tag{14}$$

$$b = \inf \{\lambda > 0 \mid \underline{d}f(x) \supset \bar{d}f(x) \oplus \lambda \odot B\} \ . \tag{15}$$

Proof. Note that

$$a = - \min_{\|g\|=1} f'_x(g) = \max_{\|g\|=1} (-f'_x(g)) = \inf \{\lambda>0 \mid -f'_x(g) \le \lambda \|g\| \} \ ,$$

$$b = \max_{\|g\|=1} f'_x(g) = \inf \{\lambda > 0 \mid f'_x(g) \le \lambda \|g\|\} .$$

Since $f'_x(g) = |g|_1 - |g|_2$, where $|.|_1$ is the gauge of the set $\underline{d}f(x)$ and $|.|_2$ is the gauge of the set $\bar{d}f(x)$, we immediately arrive at (14) and (15).

Note that

$$\max \{a,b\} = \|f'_x(g)\| = \|[\underline{d}f(x) , \bar{d}f(x)]\| .$$

7. Differentiability of star-shaped-set-valued mappings

We shall now use the space of star-shaped sets to derive a definition of differentiability for star-shaped-set-valued mappings. Let $a : \Omega \to S$ be a mapping, where Ω is an open set in E_n and S is the family of all star-shaped subsets of the space E_m . Identifying S with the cone of elements of space T with the form $[U,E_n]$, we can assume that a operates into the Banach space T . The mapping a is said to be strongly star-shaped directionally differentiable at $x \in \Omega$ if there exists a mapping $a'_x : E_n \to T$ such that for every $g \in E_n$ and sufficiently small $\alpha > 0$ the following relation holds:

$$[a(x + \alpha g) , a(x)] = \alpha \odot a'_x(g) \oplus o(\alpha) , \qquad (16)$$

where $\frac{o(\alpha)}{\alpha} \xrightarrow[\alpha\to+0]{} 0$. Here the convergence is in the metric of space T .

Let

$$a'_x(g) = [a^+_x(g) , a^-_x(g)] , \quad o(\alpha) = [o^+(\alpha) , o^-(\alpha)] .$$

Then (16) can be reformulated as follows:

$$[a(x+\alpha g)\ ,\ a(x)] = [\alpha \odot a_x^+(g) \oplus o^+(\alpha)\ ,\ \alpha \odot a_x^-(g) \oplus o^-(\alpha)]\ .$$

Since the pairs of sets on both sides of this equality define the same element of the space T , they are equivalent, i.e.,

$$a(x+\alpha g) \oplus \alpha \odot a_x^-(g) \oplus o^-(\alpha) = a(x) \oplus \alpha \odot a_x^+(g) \oplus o^+(\alpha)\ . \qquad (17)$$

Thus, a mapping a is strongly star-shaped directionally differentiable if and only if there exist mappings $a_x^- : E_n \to S$, $a_x^+ : E_n \to S$ which satisfy (17).

Remark. Several other definitions of the derivative of a mapping have been proposed. These are based on the use of the space of convex sets and the derivative of the support function of a mapping (see, for example, [5, chapter II]).

Let us associate a gauge $|.|_x$ with each set a(x) . This means that we define a mapping (an abstract function) $x \to |.|_x$ with values in $C_0(E_n)$. It follows from the definition that a mapping a is strongly star-shaped differentiable if and only if this abstract function is directionally differentiable (in the topology of space $C_0(E_n)$).

We shall now consider an example.

Let f(x,y) be a function defined on $\Omega \times E_m$ (where Ω is an open set in E_n). Assume that it is nonnegative, continuous and continuously differentiable with respect to x in its domain. Suppose also that f is positively homogeneous in y :

$$f(x,\lambda y) = \lambda f(x,y) \quad \forall\, \lambda \geq 0 \ .$$

Set

$$a(x) = \{y \,|\, f(x,y) \leq 1\} \ .$$

It is easy to check that the gauge $|\cdot|_x$ of the set $a(x)$ coincides with the function $f(x,\cdot)$. From the properties of f it now follows that the mapping a is directionally differentiable and that the function

$$y \longrightarrow \left(\frac{\partial f(x,y)}{\partial x} , g \right)$$

corresponds to $a_x'(g)$ (through a natural isomorphism).

Note the following relations between strong differentiability and algebraic operations:

1. Let $a_1 : \Omega \to S$ and $a_2 : \Omega \to S$ be strongly directionally differentiable mappings, and let $a_1 \oplus a_2$ be their inverse sum:

$$(a_1 \oplus a_2)(x) = a_1(x) \oplus a_2(x) \quad \forall\, x \in \Omega \ .$$

Then the mapping $a_1 \oplus a_2$ is directionally differentiable and

$$(a_1 \oplus a_2)_x'(g) = (a_1)_x'(g) \oplus (a_2)_x'(g) \ .$$

2. Let a mapping $a : \Omega \to S$ and a function $f : \Omega \to E_1$ be directionally differentiable. Then the mapping $b: x \to f(x) \odot a(x)$ is directionally differentiable and

$$b'_x(g) = f'_x(g) \odot a(g) \oplus f(x) \odot a'_x(g) .$$

To prove these two assertions it is necessary to view the mappings $\Omega \to S$ as single-valued mappings $\Omega \to T$ and to make use of the properties of directional derivatives of single-valued operators. The following property can be proved in the same way:

3. Let mappings $F : \Sigma \to \Omega$ and $a : \Omega \to S$ be directionally differentiable and a be Lipschitzian. Then the mapping $b(x)=a(Fx$ is also directionally differentiable and $b'_x(g)=a'_{Fx}(F'_x(g))$.

We say that a strongly directionally differentiable mapping a is strictly quasidifferentiable if its derivative $a'_x(g)$ belongs to the subspace T_c of space T or, equivalently, if there exists a representation $a'_x(g) = [a^-_x(g) , a^+_x(g)]$, where sets $a^-_x(g)$ and $a^+_x(g)$ are convex.

The function $\mu(x,y) = |y|_x$, where $|.|_x$ is the gauge of set $a(x)$, is called the gauge function of the mapping a . If a is strongly quasidifferentiable (in g), then the function μ is directionally differentiable and the following equality holds:

$$\mu'_x(x,y,g) = |y|_- - |y|_+ = \max_{l \in A_g} (l,y) + \min_{l \in B_g} (l,y) ,$$

where $|.|_-$ and $|.|_+$ are the gauges of the sets $a^-_x(g)$ and $a^+_x(g)$, respectively, and $A_g = [a^-_x(g)]^o$, $B_g = - [a^+_x(g)]^o$. The element $[A_g,B_g] = \pi(a'_x(g))$ of the space of convex sets (where π is the polar operator) is called a *quasidifferential* of the mapping a in direction g .

Let a mapping a have convex images and the polar mapping a^o be defined by

$$a^o(x) = [a(x)]^o .$$

Applying the polar operator π to the equality

$$[a(x+\alpha g), E_n] = [a(x), E_n] \oplus \alpha \odot a'_x(g) \oplus o(\alpha) ,$$

we obtain

$$[a^o(x+\alpha g), 0] = [a^o(x), 0] + \alpha\pi(a'_x(g)) + \pi \cdot o(\alpha) .$$

This provides a proof of the following theorem.

Theorem 3. *If a mapping* a *possesses the property of strong (star-shaped) quasidifferentiability, this is equivalent to saying that a strong (convex) derivative of mapping* a^o *exists.*

8. Weakly star-shaped directional differentiability

Let a mapping $a : \Omega \to S$ have gauge function μ . We say that a is weakly (star-shaped) differentiable in a direction g if for any $y \in E_n$ the partial derivative $\mu'_x(x,y,g)$ exists. Note that the function $y \to \mu'_x(x,y,g)$ is not even required to be continuous.

We shall now discuss in detail the conditions necessary for the partial derivative $\mu'_x(x,y,g)$ to exist. Let $a : E_n \to 2^{E_m}$ be a mapping. Fix $x \in E_n$, $y \in a(x)$, and $g \in E_n$. Let

$$\gamma(x,y,g) = \{v \in E_m \mid \exists \alpha_0 > 0 : y + \alpha v \in a(x+\alpha g) \quad \forall \alpha \in (0,\alpha_0]\}. \quad (18)$$

$$\Gamma(x,y,g) = \text{cl}\ \gamma\ (x,y,g) .$$

We say (see [4]) that the mapping $a : E_n \to 2^{E_m}$ allows first-order approximation at $x \in E_n$ in the direction $g \in E_n$ if for any numerical sequence $\{\alpha_k\}$ such that $\alpha_k \to +0$ and any convergent sequence $\{y_k\}$ such that $y_k \in a(x+\alpha_k g)$, $y_k \to y$, the representation $y_k = y + \alpha_k v_k + o(\alpha_k)$ holds, where

$$v_k \in \Gamma(x,y,g) \ , \ \alpha_k v_k \to 0 \ , \quad y \in a(x) \ .$$

Assume also that a is a continuous mapping and that the topology of S is induced from the Banach space T . This is equivalent to saying that the mapping $x \to |.|_{a(x)} = \mu(x,\cdot)$ is continuous.

Fix an element $y_0 \in E_m$, and for $x \in \Omega$ take $V(x) = [\mu(x,y_0),+\infty)$. We shall now describe the set $\Gamma_V(x,\cdot,g)$ (the closure of set $\gamma_V(x,\cdot,g)$ constructed from formula (18). Let $\lambda \in V(x)$. The relation $v \in \gamma_V(x,\lambda,g)$ means that, for α sufficiently small, we have

$$\mu(x+\alpha g,y_0) \le \lambda + \alpha\ v\ . \quad (19)$$

If $\lambda > \mu(x,y_0)$ then (19) is valid for every v (with α sufficiently small). If $\lambda = \mu(x,y_0)$ then (19) can be rewritten in the form

$$v \ge \frac{1}{\alpha}[\mu(x+\alpha g,y_0) - \mu(x,y_0)] .$$

Now we have

$$\Gamma_V(x,\lambda,g) = \begin{cases} (-\infty, +\infty) , & \lambda > \mu(x,y_0) \\ [\bar{\mu}'_x(x,y_0,g) , +\infty) , & \lambda = \mu(x,y_0) \end{cases}$$

where

$$\underline{\mu}'_x(x,y_0,g) = \overline{\lim_{\alpha\to+0}} \frac{1}{\alpha}[\mu(x+\alpha g,y_0) - \mu(x,y_0)] .$$

<u>Proposition 6</u>. *A mapping* a *is weakly star-shaped directionally differentiable at* x *if and only if the mapping* V *allows first-order approximation in every direction for all* $y_0 \in E_n$.

<u>Proof</u>. 1. Let V be such that first-order approximation is allowed in a direction g , and $\alpha_k \to +0$. Then

$$\mu(x+\alpha_k g,y_0) \to \mu(x,y_0)$$

and therefore

$$\mu(x+\alpha_k g,y_0) = \mu(x,y_0) + \alpha_k v_k + o(\alpha_k) ,$$

where $v_k \geq \bar{\mu}'_x(x,y_0,g)$. This leads to

$$\mu'_x(x,y_0,g) = \underline{\lim} \frac{1}{\alpha_k}[\mu(x+\alpha_k g,y_0) - \mu(x,y_0)] \geq \bar{\mu}'_x(x,y_0,g) .$$

2. Let a be directionally differentiable. Then the derivative $\mu'_x(x,y_0,g)$ exists for every $y_0 \in a(x)$, $g \in E_n$. Let $\lambda_k \to \lambda$, $\lambda_k \in V(x+\lambda_k g)$. Then

$$\lambda_k \geq \mu(x+\alpha_k g, y_0) = \mu(x,y_0) + \alpha_k \mu_x'(x,y_0,g) + o(\alpha_k) .$$

If $\lambda=\mu(x,y_0)$, set $v_k=\alpha_k\mu_x'(x,y_0,g)$ and we have a representation which is used in the definition of the first-order approximation.

If $\lambda > \mu(x,y_0)$ then this representation is obvious, and the proposition is proved.

Remark. The gauge function can be viewed as a minimum function with dependent constraints

$$\mu(x,y_0) = \min_{\lambda \in V(x)} \lambda ,$$

and therefore its differentiability can be studied with the help of a theorem by Demyanov [4]. However, this theorem is proved under the assumption that V allows first-order approximation. Proposition 6 shows that this assumption is absolutely essential in the case under consideration.

It is clear that the inverse sum of weakly differentiable mappings is also weakly differentiable. If a is weakly differentiable, $f(x) \geq 0$ and f is a directionally differentiable function, then the mapping $b(x) = f(x) \odot a(x)$ is also weakly differentiable.

Let $a_i : \Omega \to S$ $(i \in 1:N)$ be a weakly directionally differentiable mapping. Then the union of these mappings $\underline{a}(x) = \bigcup_{i \in 1:N} a_i(x)$ and their intersection $\bar{a}(x) = \bigcap_{i \in 1:N} a_i(x)$ are also weakly directionally differentiable. If μ_i is the gauge of the mapping a_i then the derivatives of the gauge functions $\bar{\mu}$ and $\underline{\mu}$ of the mappings $\bar{a}$ and $\underline{a}$ are described by the following equations:

$$\bar{\mu}'_x(x,y,g) = \max_{i \in R(x,y)} \mu'_i(x,y,g) ,$$

$$\underline{\mu}'_x(x,y,g) = \min_{i \in Q(x,y)} \mu'_i(x,y,g) ,$$

where $R(x,y) = \{i \in 1:N \mid \bar{\mu}(x,y) = \mu_i(x,y)\}, Q(x,y) = \{i \in 1:N \mid \underline{\mu}(x,y) = \mu_i(x,y)\}$.

We shall now consider some examples of weakly differentiable mappings.

Example 4. Let $l : E_n \to E_m$ be directionally differentiable and set

$$a(x) = \{y \mid (l(x),y) \le 1\} .$$

It is clear that $a(x)$ is a star-shaped set, with gauge

$$|y|_x = \mu(x,y) = \max \{(l(x),y) , 0\} .$$

The derivative of $\mu(x,y)$ at x in direction g (where y is fixed) exists and is given by

$$\mu'_x(x,y,g) = \begin{cases} (l'_x(g),y) & \text{if} \quad (l(x),y) > 0 , \\ 0 & \text{if} \quad (l(x),y) < 0 , \\ \max \{(l'_x(g),y) , 0\} & \text{if} \quad (l(x),y) = 0 . \end{cases}$$

Thus the mapping a is at least weakly differentiable. The function $y \to \mu'_x(x,y,g)$ may be discontinuous, and in this case the mapping a is not strongly differentiable.

Example 5. Let $a(x) = \{y \mid (l_i(x), y) \le 1 \ , \ i \in 1:k\}$, where the $l_i : E_n \to E_m$ are directionally differentiable mappings. Take $a_i(x) = \{y \mid (l_i(x), y) \le 1\}$. Since $a(x) = \bigcap_i a_i(x)$ we deduce that the gauge μ of mapping a is of the form

$$\mu(x,y) = \max_{i \in 0:k} (l_i(x), y) \ ,$$

where $l_0(x) = 0 \quad \forall x \in E_n$.

The function μ is directionally differentiable for any fixed y and

$$\mu_x'(x,y,g) = \max_{i \in R(x,y)} ((l_i)_x' g, y) \ ,$$

where $R(x,y) = \{i \mid \mu(x,y) = (l_i(x), y)\}$.

Example 6. Let $l_{ij} : E_n \to E_m (i \in 1:k(j) \ ; \ j \in 1:p)$ be directionally differentiable mappings

$$a_j(x) = \{y \mid (l_{ij}(x), y) \le 1 \quad \forall i \in 1:k(j)\} \ ,$$

$$a(x) = \bigcup_{j=1}^{p} a_j(x) \ .$$

The gauge function μ of mapping a is given by

$$\mu(x,y) = \min_{j \in 1:p} \ \max_{i \in 0:k(j)} (l_{ij}(x), y) \ ,$$

where $l_{0j}(x) = 0 \quad \forall j \in 1:p \ ; \quad \forall x \in E_n$.

The function μ is directionally differentiable and hence the mapping a is weakly differentiable.

Example 7. Let

$$a(x) = \bigcup_{i=1}^{k} g_i(x)U_i ,$$

where U_i , $i \in 1:k$, are star-shaped sets in E_m , the g_i are functions defined in E_n , and the set

$$\Omega = \{x \mid g_i(x) > 0 \quad \forall\ i \in 1:k\}$$

is not empty. For $x \in \Omega$ the set $a(x)$ is star-shaped with gauge

$$|y|_x = \mu(x,y) = \min_i \frac{|y|_i}{g_i(x)} ,$$

where $|.|_i$ is the gauge of set U_i .

It is clear that the mapping a is weakly differentiable. Analogously, the mapping

$$a(x) = \bigcap_{i=1:k} g_i(x)U_i$$

is also weakly differentiable with gauge

$$\mu(x,y) = \max_i \frac{|y|_i}{g_i(x)} .$$

Example 8. Let $F : E_n \to E_m$ be a directionally differentiable mapping with coordinate functions f_i , $i \in 1:m$. Take

$$\Omega = \{x \mid f_i(x) > 0 \quad \forall\ i \in 1:m\}$$

and assume that Ω is not empty. Consider the mapping

$$a(x) = \{y \in E_m | y \le F(x)\} = F(x) - E_m^+$$

defined on Ω .

Since a(x) can be rewritten in the form

$$a(x) = \{y \Big| \frac{y_i}{f_i(x)} \le 1\} ,$$

it is clear that the gauge of the mapping a is

$$\mu(x,y) = \max_i \frac{y_i}{f_i(x)} .$$

It is possible to introduce the notion of quasidifferentiability for weak derivatives as well as strong derivatives. We say that a mapping $a : \Omega \to S$ is *weakly quasidifferentiable* if for every $y \in E_n$ there exist convex compact sets A_y and B_y such that

$$\mu_x'(x,y,g) = \max_{l \in A_y} (l,g) + \min_{l \in B_y} (l,g) ,$$

where μ is the gauge of mapping a .

We shall now consider one application of weak quasidifferentiability to extremal problems.

Let Z be a set described by $Z = \{x \in \Omega | y \in a(x)\}$, where a is a mapping defined on an open set $\Omega \subset E_n$ and operating into the set S of star-shaped subsets of E_n ; y is a fixed vector from E_m . In other words, $Z = a^{-1}(y)$. (A more general case is discussed in [9].) It is necessary to construct the cone of feasible directions of Z at $x \in Z$.

If μ is the gauge function of mapping a then

$$Z = \{x' \in \Omega \mid \mu(x',y) \le 1\} .$$

If a is weakly quasidifferentiable we can consider the cones:

$$\gamma_1 = \{g \mid \mu'_x(x,y,g) < 0\} , \quad \gamma_2 = \{g \mid \mu'_x(x,y,g) \le 0\} .$$

Let γ_x denote the cone of feasible directions of Z at z . Then $\gamma_1 \subset \gamma_x \subset \gamma_2$. From [5, chapter 1, proposition 2 § 10] , it follows that if

$$G_x(-\bar{\partial}_x \mu(x,y)) \not\subset G_x(\underline{\partial}_x \mu(x,y)) ,$$

where

$$G_x(V) = \{\mu \in V \mid \mu(x) = \max_{\nu \in V} \nu(x)\}$$

and $\bar{\partial}_x \mu(x,y)$ and $\underline{\partial}_x \mu(x,y)$ are respectively a superdifferential and a subdifferential of function μ with respect to x , then

$$\text{cl } \gamma_1 = \text{cl } \gamma_2 = \gamma_2 .$$

Consider the following example. Let

$$a(x) = \{v \in E_m \mid v \le Fx\} ,$$

where F is a quasidifferentiable mapping with coordinate functions f_i, $i \in 1:m$; $y = \mathbb{1} = (1,\dots,1)$. Then (see Example 8 above)

$$\mu(x,y) = \max_i \frac{1}{f_i(x)} = \frac{1}{\min_i f_i(x)} .$$

The inequality $\mu(x,y) \le 1$ is equivalent to both

$$\min_i f_i(x) \ge 1$$

and

$$\max_i g_i(x) \le 0 ,$$

where $g_i(x) = 1 - f_i(x)$.

9. Trajectories of star-shaped mappings

Let us now discuss the asymptotic behavior of trajectories generated by a star-shaped mapping. Problems of this type commonly arise in mathematical economics, where they are studied under additional convexity assumptions. The same problems without the convexity assumptions have been discussed in [6].

Let X be a star-shaped compact set in E_n . A Hausdorff continuous mapping $a : X \to \Pi_{st}(X)$ defined on X is called a *discrete dispersible dynamic system* (D^3-system). Here $\Pi_{st}(X)$ is the family of all star-shaped subsets of set X .
A sequence $\{x_i \mid i \in 0,1,\dots\}$ of elements of X such that

$$x_{i+1} \in a(x_i) \quad i \in 0,1,\dots$$

is called a *trajectory* of the D^3-system a .

A nonempty subset Ω of set X is called a *semiinvariant set* of the D^3-system a if $a(\Omega) \subset \Omega$. Take

$$P_a(\xi) = \mathrm{cl} \bigcup_{t=1}^{\infty} a^t(\xi)$$

for $\xi \in \Pi_{st}(X)$, where $a^{t+1}(\xi) = a(a^t(\xi))$.

A point $x \in X$ is called a *Poisson stable point* of D^3-system a if $x \in P_a(x) = P_a(\{x\})$.

A set $\tilde{M} \in \Pi_{st}(X)$ is called a *turn-pike set* of D^3-system a if $\rho(x_t,\tilde{M}) \to 0$ for any trajectory $\{x_t\}$ of this system. Let M denote the intersection of all turn-pike sets.

A functional h defined on X is said to be *in equilibrium* if h is continuous, $h(x) \geq 0 \;\; \forall x \in X$, and

$$h(y) \leq h(x) \qquad \forall x \in X , \; y \in a(x) .$$

Let the functional h be in equilibrium. Take

$$(h \circ a)(x) = h(a(x)) = \max \{h(y) \mid y \in a(x)\}$$

for $x \in X$ and set

$$W_h = \{x \in X \mid h(x) = (h \circ a)(x)\} ,$$

$$W = \bigcap_h W_h ,$$

where the intersection is taken over all functionals in equilibrium. It is shown in [6] that $W \supset M$.

Let Σ be a compact subset of the space $\Pi_{st}(X)$ in the topology induced from the space of star-shaped sets T .

A mapping $a : X \to \Sigma$ is called *quasihomogeneous* if

$$a(\lambda x) \supset \lambda a(x) \quad \forall \lambda \in [0,1) .$$

In addition, $a(\mu x) \subset \mu a(x) \quad \forall \mu > 1$ for quasihomogeneous mappings. Some examples of quasihomogeneous mappings are given below.

1. *Concave mappings* (under the additional assumption $0 \in a(0)$) . A mapping a defined on a convex compact set X is concave if

$$a(\alpha x+\beta y) \supset \alpha a(x) + \beta a(y) \quad \forall \alpha,\beta \geq 0 , \quad \alpha+\beta = 1 .$$

2. *Homogeneous mappings of degree* δ . A mapping a is homogeneous of degree δ if it follows from $x, \lambda x \in X$ that

$$a(\lambda x) = \lambda^{\delta} a(x) .$$

<u>Proposition 7</u>. *If a mapping* $a : X \to \Sigma$ *is quasihomogeneous then the function*

$$h(x) = \begin{cases} |x|_{\xi} & \text{if } x \notin \xi , \\ 1 & \text{if } x \in \xi , \end{cases}$$

is in equilibrium, where ξ *is a star-shaped semi-invariant set.*

<u>Proof</u>. If $x \in \xi$ then $h(x) = 1$. If $y \in a(x)$ then $y \in a(x) \subset \xi$ since ξ is semi-invariant, and hence $h(y) = 1$.

Let $x \notin \xi$ and $y \in a(x)$. Then

$$h(x) = |x|_\xi = \inf \{\lambda > 0 | x \in \lambda\xi\} > 1 .$$

If $y \in \xi$ then $h(y) = 1 < h(x)$. Let $y \notin \xi$. Then, using the inequality $h(x) = \bar{\lambda} > 1$, the quasihomogeneity of the mapping a and the semi-invariance of ξ , we obtain

$$y \in a(x) \subset a(\bar{\lambda}\xi) \subset \bar{\lambda}a(\xi) \subset \bar{\lambda}\xi .$$

Therefore $h(y) = |y| \leq \bar{\lambda} = h(x)$. This implies that h is in equilibrium and proves the proposition.

Lemma 2. *If C is a compact star-shaped set, then for any* η *there exists an* $\varepsilon > 0$ *such that*

$$C + \varepsilon B \subset (1+\eta)C .$$

Proof. Assume the converse to be true. Suppose that there exist sets $\{g_k\}$, $\{v_k\}$, $g_k \in B$, $v_k \in C$, $v_k \to v$ and a number $\eta' > 0$ such that $v_k+g_k/k \notin (1+\eta')C$. Taking the limit as $k \to \infty$ we obtain $v \notin (1+\eta')C$, which contradicts the inclusion $v \in C$ and thus proves the lemma.

Theorem 4. *If* $a : X \to \Pi_{st}(X)$ *is a quasihomogeneous mapping and* $a(x) \in \Sigma$ *for every* x , *then* $W = M = H$ *where* H *is the family of all Poisson stable points.*

Proof. It is necessary to check the inclusions $H \supset W$, $M \supset H$.

1. We shall first verify $H \supset W$. If $x \notin H$, then $x \notin P_a(x)$. The set $P_a(x)$ is star-shaped (since $a(x) \in \Sigma$ and Σ is compact) and semi-invariant. Let h be the function defined in Proposition 7 with respect to set $\xi = P_a(x)$. Then h is in equilibrium and since $x \notin P_a(x)$ we have $h(x) > 1$.

However, we also have $a(x) \in P_a(x)$ and therefore $(h \circ a)(x)=1$. Thus $x \notin W_h$ and hence $x \notin W$.

2. To verify $M \supset H$, we first let $x \in H$, i.e., $x \in P_a(x)$. From Lemma 2 it is clear that for every $\varepsilon \in (0,1)$ there exists a number t such that

$$(1-\varepsilon)x \in a^t(x) . \tag{20}$$

Consider a sequence of positive numbers $\{\varepsilon_k\}$ such that $\prod_{k=1}^{\infty}(1-\varepsilon_k)$ converges to some number $\nu \in (0,1)$. Using (20) and the quasihomogeneity of a we can construct a trajectory $\chi = \{x_t\}$ starting from x and containing the subsequence $\{x_{t_j} = \prod_{k=1}^{j}(1-\varepsilon_k)x\}$.

This means that νx is a limit point of the trajectory x and therefore $\nu x \in M$. Since ν is an arbitrary number we conclude that $x \in M$. This completes the proof of the theorem.

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