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EXISTENCE RESULTS AND FINITE HORIZON APPROXIMATES FOR INFINITE HORIZON OPTIMIZATION PROBLEMS

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ABSTRACT

The paper deals with infinite horizon optimization problems. The existence of optimal solutions is obtained as a consequence of an asymptotic growth condition. We also exhibit finite horizon approximates that yield upper and lower bounds for the optimal values and whose optimal solutions converge to the long-term optimal trajectories.

1. INTRODUCTION

Various economic planning problems, in particular in the areas of resource management and capital theory, are inherently infinite time horizon problems of the type P :

$$\begin{aligned} &\text{find } x = (x_t)_{t=1}^{\infty} \text{ with } x_t \in R^n_+ \text{, and such that} \\ &w = \limsup_{T\to\infty} \Sigma_{t=1}^{T} \alpha^{t-1} f_t(x_{t-1}, x_t) \text{ is minimized;} \end{aligned} \tag{1.1}$$

where $x_0 \in R^n_+$ the initial state is given, $\alpha \in (0, 1)$ is a discount factor, and for all t=1,....

$$f_t : R^{2n} \to R \cup \{+\infty\}$$

is a lower semicontinuous function; the effective domain of f_t

$$\text{dom } f_t = \{ (y, z) \mid f_t (y, z) < \infty \}$$

is determined by the constraints imposed on the trajectory $(x_t)_{t=1}^{\infty}$ at time t (in addition to the nonnegativity constraints).

The open-endedness of the future is justified by concerns beyond any finite period, and this feature cannot be conveniently ignored without impairing the validity of the model; this point has been stressed by several economists, see for example [1], [2] and [3]. The conseptual and mathematical elegance of infinite horizon models however is impractical from a computational viewpoint. To actually solve such problems we must usually content ourselves with finite horizon approximates by including some terminal criterion, i.e. we replace P by:

find $(x_t)_{t=1}^{T}$ with $x_t \in R_+^n$, and such that

$$w = \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t (x_{t-1}, x_t) + v_T(x_{T-1}, x_T) \text{ is minimized,} \tag{1.2}$$

for some finite T ; the function $v_T: R^{2n} \to R \cup \{ + \infty\}$ having hopefully almost the same effect on the choice of an optimal trajectory, at least up to time T, as the tail of the series : $\Sigma_{t=1}^{\infty} \alpha^{t-1} f(x_{t-1}, x_t)$.

This paper expands on Grinold's study [4] of the convergence of infimal values and optimal solutions of finite horizon approximates to infinite horizon problems. We extend and strenghten his results in a variety of directions motivated by the following considerations. First we allow for nonlinear dynamics, as well as for nonlinear transition costs ; in Grinold's model [4] nonlinearity appears only in the cost functional in each state-decision variable x_t separately. The results are now applicable to economic models with nonlinear technologies .. in particular, with decreasing returns to scale .. as well as to problems that can be cast in the format of discrete time Bolza type problems, cf. Section 2. Second, we generate both lower and upper bounds that enable us to obtain error bounds for the suggested solution, Grinold [4] is only concerned with lower bounds. Third, we relax the assumptions that the single-period cost function is convex and time stationary.

As part of our development we derive an exsistence result by imposing an asymptotic growth condition, called here Grinold's growth condition, that eliminates from the set of potentially optimal solutions those trajectories whose "average growth" exceeds on equal α^{-1}, the inverse of the discount factor. Ekeland and Scheinkman [5] consider a special version of P and also establish existence but with a growth condition that appears to be much more restrictive than that used here; see also Magill [3] for a related result for a model with linear dynamics and continuous time, and [6, p.93] where Ekeland analyses the one-sector economic growth model of Ramsey.

In Section 2, we give a brief overview of the class of problems that fit the general model (2.1) and in particular bring to the fore a version of P that stresses its dynamical features : the discrete time Bolza type problem; in Section 7 we record our reults in terms of this Bolza model. The basic assumptions that are needed to obtain existence and convergence results are formulated and discussed in Section 3. Section 4 introduces the finite horizon approximates that furnish upper and lower bounds for the optimal value of the infinite horizon optimization problem P.

The main purpose of the remainder of the the paper is to validate the assertion that the finite time horizon problems introduced in Section 4 yield approximate optimal solutions of P. In Section 5 it is shown that with Grinold's growth condition we may naturally limit the decision space to $\ell_1^n(\alpha)$, i.e. those trajectories that have finite "present value". On this $\ell_1^n(\alpha)$-space the (essential) objective function of P is inf-compact which, in turn, guarantees the existence of optimal solutions. The convergence of the optimal solutions of the finite time horizon problems to the optimal solutions of the long-term problem is finally obtained in Section 6 by recasting these problems in the $\ell_1^n(\alpha)$-decision space and then making appeal to the theory of epi-convergence.

2. SOME EXAMPLES

The (abstract) optimization model P, see (1.1), encompasses a wide variety of problems that have been studied in the literature. By way of motivation we begin with a few examples, they should also help us to assess the limitations introduced by the assumptions that we shall impose on P later on.

a) Infinite horizon mathematical programs

Here

$$f_t(x_{t-1}, x_t) = \begin{cases} f_{ot}(x_{t-1}, x_t) \text{ if } f_{it}(x_{t-1}, x_t) \leq 0, \; i = 1, \ldots, m_t \\ \qquad (x_{t-1}, x_t) \in S_t \\ +\infty \text{ otherwise} \end{cases} \tag{2.1}$$

where for $i=0, \ldots, m_t$, the functions f_{it} are finite-valued lower semicontinuous on R^{2n} and $S_t \subset R^{2n}$ is closed.

An important special case arises when the single-period cost function does not depend on t, i.e. for all t

$$m_t \equiv m, \; S_t \equiv S, \text{ and } f_{it} = f_i \text{ for } i=0, \ldots, m.$$

A further specialization is the model studied by Grinold [4] :

$$f_t(x_{t-1}, x_t) = \begin{cases} f_o(x_t) \text{ if } a_i x_{t-1} + b_i x_t \leq \beta_i, \; i=1, \ldots, m, \\ \qquad x_t \in C \subset R^n, \\ +\infty \text{ otherwise} \end{cases} \tag{2.2}$$

Here the dynamics is linearly constrained (a_i and b_i are n-vectors), C is a closed convex set with f_o a convex function; there is no provision for transition costs from x_{t-1} to x_t. Grinold and others [4, Section 8 and References] worked ealier on the infinite horizon linear programming version of this model, i.e. when $f_o(x_t) = cx_t$ and C is a polyhedral set, for example $C = R^n_+$.

Another important special version of (1.1) is when the criterion function only depends on "consumption" such as in the model considered by Ekeland and Scheinkman [5]:

$$f_t(x_{t-1}, x_t) = -\max_{c_t} [u_t(c_t) \mid (x_{t-1}, x_t + c_t) \in S_t] \tag{2.3}$$

where u_t is the utility function at time t and $S_t \subset R^{2n}$, the "production set" is closed.

b) Bolza type problems

Here with $\Delta x_t = x_t - x_{t-1}$, we have

$$f_t(x_{t-1}, x_t) = L_t(x_{t-1}, \Delta x_t) \tag{2.4}$$

where the function $L_t : R^{2n} \to R \cup \{+\infty\}$ is lower semicontinuous.

Finite horizon models of this type were introduced in [7], with extension to infinite horizon studied in [8]. Economic growth models, for example, are most naturally cast in this format [9]. Quite often (2.4) can be restated in the form (2.1) yet Bolza type problems are dealt with more thoroughly ... in Section 7 we transliterate our main results in terms of this model ... and this because of two particular reasons. First, this discrete time version of the classical problem of the Calculus of Variations provides us with the natural bridge to optimal control problems, cf. (2.5) below and more generally the Introduction of [10] where Rockafellar points out the pivotal role played by this class of problems in optimization theory (for dynamical systems). Secondly, we wish to emphasize the fact that L_t may itself be the output of some optimization problem. For example, let

$$D_t = (X_{t-1} \times R^n) \cap \{(x_{t-1}, \Delta x_t) \mid \exists\, u_t \in U_t \text{ s.t. } \Delta x_t = A_t\, x_{t-1} + B_t\, u_t\}$$

where X_{t-1}, U_t are closed sets that correspond to constraints on the state-variables x_{t-1} and controls u_t, A_t and B_t are matrices of approriate dimensions. Further let

$$L_t(x_{t-1}, \Delta x_t) = \begin{cases} \inf_{u_t \in U_t} [c_t(x_{t-1}, u_t) \mid \Delta x_t = A_t\, x_{t-1} + B_t\, u_t\,] \\ \qquad\qquad \text{if} \quad (x_{t-1}, \Delta x_t) \in D_t \\ +\infty \text{ otherwise} \end{cases} \tag{2.5}$$

where $c_t\,(x_{t-1}, u_t)$ is the single-period performance criterion with c_t lower semicontinuous.

Again an important special case of (2.4) is when cost and constraints are time independent, i.e. $L_t \equiv L$ for all t. Further specialization gives us the convex case, the separable case, the linear case, and so on.

3. ASSUMPTIONS

Three basic assumptions enter into play in the derivation of the results :

- problem P is proper : Assumptions 3.1
- Grinold's growth condition : Assumption 3.2
- substainability of tail-stationary trajectories : Assumption 3.3.

The first one can be interpreted as a feasibility condition and will always be viewed as part of the definition of problem P. The second one is the key ingredient in the existence proofs whereas the last assumption is only required to obtain convergence of the finite time approximates (from above).

To formulate our conditions we rely on the following construction. For $T=1, \dots,$ let

$$f'_T(y, z) := \inf_{t \geq T} f_t(y, z) \tag{3.1}$$

and, define

$$h_T := \mathrm{cl}\ \mathrm{co}\ f'_T \tag{3.2}$$

to be the lower semicontinuous regularization of the convexification of f'_T. In terms of epigraphs we have

$$\mathrm{epi}\ h_T = \mathrm{cl}\ \mathrm{co}\ (\cup_{t=T}^{\infty} \mathrm{epi}\ f'_t) \tag{3.3}$$

with cl denoting closure and co convex hull, and epi $g = \{ (v, \alpha) \mid \alpha \geq g(v) \}$ is the epigraph of the function g. Of course the functions h_T are convex and lower semicontinuous. Moreover, for all T

$$h_T \leq f_t \qquad \text{for all } t \geq T, \tag{3.4}$$

and with $h := h_1$,

$$h \leq h_2 \leq \ldots\ldots \leq h_T \leq h_{T+1} \leq \ldots\ldots \tag{3.5}$$

3.1 ASSUMPTION

Problem P is proper. This means

i. the function $h > -\infty$

ii. there exists $\tilde{x} = (\tilde{x}_t)_{t=1}^{\infty}$ with $\tilde{x}_t \in R^n_+$ such that

$$\limsup_{T \to \infty} \Sigma_{t=1}^{T} \alpha^{t-1} f_t(\tilde{x}_{t-1}, \tilde{x}_t) < \infty$$

and

$$f_1(\tilde{x}_1, \tilde{x}_1) < \infty$$

Without loss os generality, we assume that $\tilde{x} = 0$. (Otherwise just substitute $f_t(\bullet + \tilde{x}_{t-1}, \bullet + \tilde{x}_t)$ for f_t in the formulation of P).

The essential objective function of P is given by

$$F(x) = \begin{cases} \limsup_{T \to \infty} \Sigma_{t=1}^{T} \alpha^{t-1} f_t(x_{t-1}, x_t) \text{ if for all t, } x_t \in R^n_+ \\ + \infty \text{ otherwise.} \end{cases}$$

Assumption 3.1. ii requires that $F(\tilde{x}) < +\infty$ in addition to $f_1(\tilde{x}_1, \tilde{x}_1) < \infty$, which means that there exists some feasible trajectory $\tilde{x}$ with $(\tilde{x}_1, \tilde{x}_1)$ feasible in time period 1. With $\tilde{x} = 0$, we can think of this condition as "idleness is feasible" if P if concerned with activity analysis, or alternatively as "depletion if acceptable" if P is related to resource management. In terms of model (2.1) this condition becomes

$$f_{o1}(x_o, o) + \Sigma_{t=2}^{\infty} \alpha^{t-1} f_{ot}(0,0) < +\infty$$

and for t = 2, ...

$$(0, 0) \varepsilon S_t, f_{it}(0, 0) \leq 0 \quad \text{for } i=1, \ldots, m_t.$$

$$(x_o, o) \varepsilon S_1, f_{i1}(x_o, o) \leq 0 \text{ for } i=1, \ldots, m_1$$

For example, in the quadratic case, i.e. with $S = R^{2n}$ and

$$f_{it}(y, z) = (y, z)(Q_{it}(y, z)) + p_{it}(y, z) - \beta_{it}$$

with the Q_{it} square matrices, $p_{it} \varepsilon R^{2n}$ and $\beta_{it} \varepsilon R$, this boils down to

$$(x_o, 0)(Q_{i1}(x_o, 0)) + p_{i1}(x_o, 0) \leq \beta_{i1}, i=1, \ldots, m_1$$

$$\beta_{it} \geq 0, \text{ for } i=1, \ldots, m_t \text{ and } t=2, \ldots$$

and

$$\limsup_{T \to \infty} | \beta_{ot} |^{1/t} < \alpha^{-1}.$$

In terms of h, the condition $f_1(\tilde{x}_1, \tilde{x}_1) < \infty$ -- when satisfied at $\tilde{x} = 0$ or after translation of $\tilde{x}$ to 0 -- means that $h(0,0) \leq f_1(0,0) < \infty$. Also $F(0) < \infty$ implies that $h(x_o, 0) \leq f_1(x_o, 0)$ is finite. These conditions , together with the convexity of h, imply that for all $\Theta \varepsilon [0, 1]$, $h((1-\Theta) x_0, 0)$ is finite, and thus in particular that

$$h\,((1-\alpha)\,x_0,\,0) \text{ is finite} \tag{3.6}$$

and it is precisely to obtain this condition that the extra assumption $f_1(\tilde{x}_1, \tilde{x}_1) < \infty$ is needed, see for example the proof of lemma 5.3.

Since for all t, $h \le f_t$, we have that for $x = (x_t)_{t=1}^{\infty}$ with $x_t \in R^n_+$,

$$F(x) \geq \limsup_{T \to \infty} \Sigma_{t=1}^{T} \alpha^{t-1} h(x_{t-1}, x_t),$$

from which follows that

$$F(x) \geq \limsup_{T \to \infty} \frac{1-\alpha^T}{1-\alpha} h(\Sigma_{t=1}^{T} \frac{1-\alpha}{1-\alpha^T} \alpha^{t-1} x_{t-1}, \Sigma_{t=1}^{T} \frac{1-\alpha}{1-\alpha^T} \alpha^{t-1} x_t)$$

by the convexity of h. This, with Assumption 3.1. i does not quite give us $F > -\infty$, but it implies that

$$F(x) > -\infty \text{ for every } x \text{ with } \Sigma_{t=1}^{\infty} \alpha^t x_t < \infty,$$

since the above would imply that

$$F(x) \geq (1-\alpha)^{-1} \inf_{(u,\,v)} [h\,(u,\,v) \mid |u| \leq (1-\alpha)\,\gamma\,,\ |v| \leq (1-\alpha)\alpha^{-1}\,\gamma]$$

with $\gamma > \Sigma_{t=1}^{\infty} \alpha^t |x_t|$ and $|\bullet|$ denoting here the ℓ^1 - norm in R^n.
The last term on the right being finite since h is lower semicontinuous, proper and $h(0, 0) < \infty$, and the variables (u, v) are restricted to a bounded set. We shall see in Section 5 that Assumption 3.2 leads us naturally to restrict the decision space precisely to the trajectories with $\Sigma_{t=1}^{\infty} \alpha^t x_t < \infty$, and thus <u>on that space we have that</u> F <u>is proper</u> whenever P is proper.

The Assumption 3.1.i. is trivially satisfied when the cost structure is monotone nondecreasing with respect to time, i.e. if the sequence $\{f_t(y, z),$ $t=1, \ldots\}$ is monotone nondecreasing for every $(y, z) \varepsilon R^{2n}$, and f_t is convex for every t. Then for all T,

$$h_T = f_T$$

Thus we certainly have not excluded two important special cases that appear to cover nearly all potential applications, namely when the f_t are time independent, i.e. when for all $t=1, \ldots$

$$f_t(y, z) = f(y, z),$$

or, when the goal of the program is to reach certain states at minimum cost, for example

$$f_t(y, z) = f(y, z) + q_t(y, z)$$

where

$$q_t(y, z) = \inf [\text{dist}((u, v), (y, z)) \mid (u, v) \varepsilon D_t],$$

dist is the distance function on $R^{2n} \times R^{2n}$ and $\{D_t, t=1, \ldots\}$ is a decreasing sequence of subsets of R^{2n}, or if q_t is the indicator function of the set D_t:

$$q_t(y,z) = \begin{cases} 0 & \text{if } (y, z) \varepsilon D_t \\ +\infty & \text{otherwise} \end{cases}$$

or still

$$f_t(y,z) = \inf_{u,v} [f(u, v) + \frac{t}{2} \text{dist}^2 ((u, v), (y, z))]$$

which gives us a sequence of functions (known as Moreau - Yosida approximates of f of parameter t^{-1}) converging to f from below .

3.2 ASSUMPTION

Grinold's growth condition. _For every_ $\alpha' \varepsilon [0, \alpha]$ _and_ $z \varepsilon R^n_+$ with $z \neq 0$,

$$(\text{rc } h)(\alpha' z, z) > 0. \tag{3.7}$$

with $h = h_1$ _as defined above, see_ (3.2).

Recall that if $C \subset R^m$ is a nonempty closed convex set, then there exists a largest closed convex cone K such that for all x in C, $x + K \subset C$. This cone is called the _recession cone_ and is usually denoted by rc C. The _recession function_ of a proper lower semicontinuous convex function $g : R^m \to R \cup \{+\infty\}$ is denoted by rc g and defined by the relation

$$\text{epi (rc } g) = \text{rc (epi } g).$$

If g(0) if finite, then

$$(\text{rc } g)(y) = \lim_{\upsilon \to \infty} \lambda_\upsilon h(\lambda_\upsilon^{-1} y^\upsilon) \tag{3.8}$$

where $\lambda_\upsilon \downarrow 0$ and $y^\upsilon \to y$ [11, Section 8].

In the special, but important case when the function $y \mapsto h(y, z)$ is monotonically nonincreasing -- resource management problems would be of that type, for example -- Grinolds's growth condition can be relaxed.

3.3 ASSUMPTION

Strict version of Grinold's growth condition. For every $z \in R^n_+$ with $z \neq 0$

$$(rc\ h)(\alpha z, z) > 0 \tag{3.9}$$

To verify this condition, we could solve the convex program

$$\begin{aligned} &\text{find } z \in R^n_+ \text{ with } z_1 + \cdots + z_n = 1 \text{ such that} \\ &w = (rc\ h)(\alpha z, z) \text{ is minimized} \end{aligned} \tag{3.10}$$

To verify Assumption 3.2 we would need to solve a similar convex program with α replaced by α' and make a parametric analysis as α' varies between 0 and α. For example, when in model (2.2) the cost function is quadratic convex, viz,

$$f_0(y, z) = (y, z)(Q(y, z)) + p(y, z) - \gamma$$

with Q positive semidefinite, $p \in R^{2n}$ and γ a scalar. Then

$$(rc\ h)(\alpha' z, z) = \begin{cases} p(\alpha' z, z) & \text{if } Q(\alpha' z, z) = 0 \\ & \quad a_i x_{t-1} + b_i x_t \leq \beta_i,\ i = 1, \ldots, m, \\ & \quad x_t \in C \subset R^n, \\ +\infty & \text{otherwise,} \end{cases}$$

and (3.10) is a linear programming problem, assuming that C is polyhedral, parameterized by α'.

Grinold's growth condition, imposes a restriction on the asymptotic rate of growth of the sequences $(x_t)_{t=1}^{\infty}$ but apparently only in some very restricted directions. We shall see later on that this assumption actually limits the set of feasible solutions to those $(x_t)_{t=1}^{\infty}$ whose rate of growth is eventually less than α^{-1}, i.e. no sustainable growth rate will suffice to compensate for the dampening effect of discounting. In terms of economics, with $\alpha = 1/(1+r)$ where the interest rate $r > 0$ reflects the opportunity cost of capital, Assumption 3.2 quarantees that at very high stock levels the rate of return on additional savings is less that r, i.e. the (endogenous) interest rate of the stock is asymptotically inferior to the best (exogenous) alternative.

To formulate our next assumptions, we need the counterparts of the lower bounding functions h_T. For T=1, ... , let

$$g_T := \sup_{t \geq T} f_t \tag{3.11}$$

where $(\sup_{t \geq T} f_t)(x) = \sup_{t \geq T} f_t(x)$, in terms of epigraphs we have that

$$\text{epi } g_T = \cap_{t=T}^{\infty} \text{epi } f_t, \tag{3.12}$$

The construction here being similar to that of the function h_T, see (3.2). The lower semicontinuity of the functions f_t implies the lower semicontinuity of g_T ; epi g_T is the intersection of closed epigraphs.
Moreover

$$g_1 \geq g_2 \geq .... \geq g_T \geq g_{T+1} \geq .... , \tag{3.13}$$

and for all T,

$$h_T \leq f_T \leq g_T. \tag{3.14}$$

3.4 ASSUMPTION

Sustainability of tail-stationary trajectories. If $F(x) < +\infty$, then

$$\limsup_{T\to\infty} \alpha^{T-1} g_T(x_{T-1}, x_{T-1}) \leq 0 \tag{3.15}$$

with g_T as defined by (3.11).

Observe that (3.15) is satisfied if

$$\limsup_{T\to\infty} \Sigma_{t=T}^{\infty} \alpha^{t-1} g_t(x_{T-1}, x_{T-1}) \leq 0. \tag{3.16}$$

whenever $F(x) < \infty$. If the f_t are time independent, so are the g_t, i.e. $g_t = g$ for all t, and then the two conditions (3.16) and (3.15) are equivalent.

There are really two components to this last assumption which are useful to isolate in order to understands its implications. First, suppose x is feasible, then another feasible solution can be created by following the same trajectory up to time T-1 and staying in state x_{T-1} from then on. And second, for any such modified trajectory $(\bar{x}_t)_{t=1}^{\infty}$ with $\bar{x}_t = x_t$, $t \leq T-1$ and $\bar{x}_t = x_{T-1}$, $t \geq T$, the tail of the series

$$\Sigma_{t=T}^{\infty} \alpha^{t-1} f_t(\bar{x}_{t-1}, \bar{x}_t)$$

becomes less than any positive number, for T sufficiently large.

4. FINITE HORIZON APPROXIMATES

We do not really expect to be able to build finite horizon approximates (1.2) of P whose solutions up to some time T, actually match those of P itself, at least not without first solving P (1.1). At best we may be able to find terminal criteria that yield upper and lower bounds and which would allow us to bracket in this way the optimal value of P.

We begin with <u>approximates from below</u>. We can motivate our construction as follows. Let h_T be as defined in (3.2), i.e. the largest lower semicontinuous function majorized by the f_t for all $t \geq T$. Suppose for the time being that for any feasible trajectory $x = (x_t)_{t=1}^{\infty}$, the convex combination

$$z_T := (1-\alpha) \Sigma_{t=T}^{\infty} \alpha^{t-T} x_t \tag{4.1}$$

of the tail $(x_T, x_{T+1}, \ldots)$ is well defined; in Section 5 we shall see that Grinold's growth condition actually guarantees the existence of z_T. Since

$$(1-\alpha) \Sigma_{t=T}^{\infty} \alpha^{t-T} (x_{t-1}, x_t) = ((1-\alpha) x_{T-1} + \alpha z_T, z_T), \tag{4.2}$$

$$(1-\alpha) \Sigma_{t=T}^{\infty} \alpha^{t-T} = 1,$$

the convexity and the lower semicontinuity of h_T imply that

$$\frac{\alpha^{T-1}}{1-\alpha} h_T ((1-\alpha) x_{T-1} + \alpha z_T, z_T) \leq \limsup_{T' \to \infty} \Sigma_{t=T}^{T} \alpha^{t-1} f_t (x_{t-1}, x_t) \tag{4.3}$$

This suggests choosing the term on the left in (4.3) as terminal criterion in (1.2) to obtain a lower bound for P. We are led to the (finite dimensional) optimization problem P_T:

find $(x_t)_{t=1}^T$ with $x_t \in R_+^n$ such that

$$w = \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t (x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} h_T ((1-\alpha) x_{T-1} + x_T, x_T) \qquad (4.4)$$

and w is minimized.

In view of (4.3) we should not identify the variable x_T that appears in P_T with the T-th state variable but to a discounted version of all future decisions, see (4.1). Roughly speaking we can think of P_T as obtained by averaging constraints and variables from time T on. Of course, we suppose that all quantities that appear here are as in P and that they satisfy the same assumptions. Let

$$V(x_o): = \inf_x F(x) = \inf P \qquad (4.5)$$

$$V_T (x_o) : = \inf P_T \qquad (4.6)$$

denote the infimal values of P and P_T respectively; in the framework of dynamic programming V and V_T are the so-called value functions of P and P_T. Rephrasing the observations that led us to the formulation of the finite horizon problems $\{P_T, T=1, \ldots\}$ in terms of infimal values yields:

4.1 PROPOSITION

Suppose F(x) < ∞, i.e. x is a feasible solution of P, and

$z_T = (1-\alpha) \Sigma_{t=T}^{\infty} \alpha^{t-T} x_t < +\infty$. Then

$$x_1, x_2, \ldots\ldots, x_{T-1}, z_T$$

<u>is feasible for</u> P_T . <u>Moreover</u>

$$\Sigma_{t=1}^{T-1} \alpha^{t-1} f_t (x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} h_T((1-\alpha) x_{T-1} + \alpha z_T, z_T) \leq F(x), \quad (4.7)$$

<u>and hence for all</u> $T=1, \ldots.$

$$V_T (x_o) \leq V(x_o).$$

The construction of the problems P_T is akin to the lower approximates obtained for stochastic optimization problems by substituting for the given measure a discrete probability measure generated by taking conditional expectations and making use of Jensen's inequality, cf [12, Proposition 4.1] for example. Indeed we can view

$$(1-\alpha)\alpha^{t-1} \text{ with } t=1, \ldots.$$

as a probability mass function on the natural numbers. The averaging of the tail corresponds to taking conditional expectation given [1, T-1]. Proposition 4.1 reflects the fact that this gives a lower bound when we substitute h_T for the functions f_t, $t \geq T$. This interpretation also suggests that the lower bound will be tighter if we refine the partitioning with respect to which we take conditional expectations. That is the content of the next proposition whose proof is straightforward.

4.2 PROPOSITION

<u>Suppose the (finite) sequence</u>

$$x_1, x_2, \ldots, x_T, x_{T+1}$$

<u>is a feasible solution of</u> P_{T+1}. <u>Then, with</u>

$$x'_T = (1-\alpha)x_T + \alpha\, x_{T+1},$$

<u>the sequence</u>

$$x_1, x_2, \ldots, x'_T$$

<u>is a feasible solution of</u> P_T, <u>since</u>

$$h_T((1-\alpha)x_{T-1} + \alpha x'_T, x'_T) \leq (1-\alpha)\, f_T(x_{t-1}, x_T) + \alpha h_{T+1}(x_T, x_{T+1}) \quad (4.8)$$

<u>From which it also follows that</u>

$$V_T(x_o) \leq V_{T+1}\,(x_o) \leq V(x_o) \qquad (4.9)$$

Thus, as expected, the sequence $\{V_T(x_o), T=1, \ldots\}$ is monotone nondecreasing and bounded above by $V(x_o)$. That we actually have convergence, when the Assumptions of Section 3 are satisfied, is demonstrated in Section 6. In the process we shall obtain much more, namely the componentwise (i.e. for all t) convergence of the optimal solutions of problems to an optimal solution of P.

Let us also record now that Grinold's growth condition, Assumption 3.3 more exactly is sufficient to guarantee the existence of optimal solutions for P_T.

4.3 PROPOSITION

Suppose P is proper and satisfies the strict version of Grinold's growth condition (Assumptions 3.1 and 3.3). Then for all T=1, ... and all $\beta \in R$, the set

$$\{x_1 \geq 0, \ldots, x_T \geq 0 \mid \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} h_T((1-\alpha)\, x_{T-1} + \alpha\, x_T, x_T) \leq \beta\}. \tag{4.10}$$

is compact, i.e. the essential objective function of P_T is inf-compact. Hence P_T has an optimal solution.

PROOF. Clearly for all β, the set given by relation (4.10) is closed and contained in

$$H_{T,\beta} := \{x_1 \geq 0, \ldots, x_T \geq 0 \mid \Sigma_{t=1}^{T-1} \alpha^{t-1} h(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} h((1-\alpha)x_{T-1} + \alpha\, x_T, x_T) \leq \beta\}$$

since $h = h_1 \leq h_T \leq f_T$ for all T, see (3.3). It is thus sufficient to establish that $H_{T,\beta}$ is bounded to complete the proof, since it would yield the desired compactness from which the existence follows directly; we can then view P_T as minimizing a proper lower semicontinuous function on a compact set.

The set $H_{T,\beta}$ is closed and convex --- by construction h is lower semi-continuous and convex --- to show that is bounded we prove that its recession cone

$$\text{rc}\, H_{T,\beta} = \{0\}$$

whenever $H_{T,\beta}$ is nonempty. So suppose $\beta \geq F(0)$, by Assumption 3.1 F is finite at 0, and $0 \neq (y_t)_{t=1}^{T} \varepsilon \text{ rc } H_{T,\beta}$. Then for all $\lambda \geq 0$,

$$\beta \geq h(x_0, \lambda y_1) + \Sigma_{t=2}^{T-1} \alpha^{t-1} h(\lambda y_{t-1}, \lambda y_t) + \frac{\alpha^{T-1}}{1-\alpha} h((1-\alpha)\lambda_{T-1} y_{T-1} + \alpha\lambda y_T, \lambda y_T)$$

which implies

$$\beta \geq (1-\alpha)^{-1} h((1-\alpha)x_0 + \lambda\alpha\bar{y}_1, \lambda\bar{y}_1) \tag{4.11}$$

where $\bar{y}_1$ is defined recursively by

$$\bar{y}_T = y_T$$

$$\bar{y}_t = (1-\alpha)\, y_t + \alpha\bar{y}_{t+1} \text{ for } t=T-1, \ldots, 1.$$

The second inequality resulting from the convexity of h. Dividing both sides of (4.11) by λ, letting λ go to $+\infty$ and relying on (3.6), we obtain the following contradiction to Assumption 3.3

$$0 \geq (\text{rc } h)\, (\alpha\bar{y}_1, \bar{y}_1).$$

Hence y must be 0, and this completes the proof. □.

We now turn to approximates from above, here we rely on the upper bounding function $\{\dot{g}_T, T=1, \ldots\}$, cf. (3.11). Suppose $x = (x_t)_{t=1}^{\infty}$ is tail-stationary from time T-1 on. Then

$$F(x) \leq \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} g_T(x_{T-1}, x_{T-1}) \tag{4.12}$$

as follows from (3.11). Motivated by this inequality we introduce the (finite dimensional) optimizaton problem P^T:

find $(x_t)_{t=1}^{T-1}$ with $x_t \in R_+^n$ such that

$$w = \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} g_T(x_{T-1}, x_{T-1}) \text{ is minimized;} \tag{4.13}$$

parameters and functions are as in P. We may think of P_T as the search for the best trajectory which is stationary from time T-1 on. With

$$V^T(x_0) := \inf P^T$$

and straightforward application of (3.14) and (3.13), we obtain:

4.4 PROPOSITION. For all T=1, ...,

$$V(x_0) \leq V^{T+1}(x_0) \leq V^T(x_0) \tag{4.14}$$

The sequence $\{V^T(x_0), T=1, ...\}$ is monotone nonincreasing and bounded below by $V(x_0)$. We prove convergence in Section 6 as part of a general result which also gives us the componentwise convergence of optimal solutions. As one could easily guess, Assumption 3.4 about the sustainability of tail-stationary trajectories plays a key role in that proof.

The existence of optimal solutions for P^T (4.13) is again guaranteed by Grinold's growth condition, the proof is similar to that of Proposition 4.3.

4.5. PROPOSITION

Suppose P is proper and satisfies Grinold's growth condition (Assumptions 3.1 and 3.3). Then for all T=1, ... and $\beta \in R$, the set

$$\{x_1 \geq 0, \ldots, x_{T-1} \geq 0 \mid \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} g_T(x_{T-1}, x_{T-1}) \leq \beta\} \quad (4.15)$$

in compact, i.e. the essential objective function of P^T is inf-compact. Hence P^T has an optimal solution.

PROOF. For every β the set given by relation (4.15) is closed and contained in

$$H_\beta^T = \{x_1, \ldots, x_{T-1} \mid \Sigma_{t=1}^{T-1} \alpha^{t-1} h(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} h(x_{T-1}, x_{T-1})\}$$

as follows from (3.14) and (3.5). The proof will be complete if we show that H_β^T is bounded since it would imply the compactness of the level sets (4.15) of the essential objective of P^T from which the existence of optimal solution follows directly. The function h being lower semicontinuous and convex it follows that set H_β^T is closed and convex. Moreover, it is nonempty if we choose $\beta \geq F(0)$ as follows from assumption 3.1. ii and $h \leq f_t$ for all t. The set H_β^T is then bounded if and only if rc $H_\beta^T = \{0\}$.

Suppose to the contrary that $0 \neq (y)_{t=1}^{T-1} \in$ rc H_β^R. Then for all $\lambda \geq 0$,

$$\beta \geq h(x_0, \lambda y_1) + \Sigma_{t=1}^{T-2} \alpha^{t-1} h(\lambda y_{t-1}, \lambda y_t) + \frac{\alpha^{T-1}}{1-\alpha} h(\lambda_{T-1}, y_{T-1}, \lambda_{T-1}, y_{T-1})$$

and using the convexity, this yields

$$\beta \geq (1-\alpha)^{-1} h((1-\alpha)x_0 + \lambda\alpha\bar{y}_1, \bar{y}_1), \tag{4.16}$$

where $\bar{y}_1$ is defined recursively through

$$\bar{y}_T := \bar{y}_{T-1} := y_{T-1},$$

$$\bar{y}_t := (1-\alpha)y_t + \alpha\bar{y}_{t+1} \text{ for } t = T-2, \ldots, 1.$$

Dividing both sides of the inequality (4.16) by λ, appealing to (3.6) and letting λ go to $+\infty$, we contradict (3.9) since we obtain

$$0 \geq (\mathrm{rc}\; h)(\alpha\bar{y}_1, \bar{y}_1).$$

Hence y must be 0. □.

5. EXISTENCE AND INF-COMPACTNESS

We now study the properties of F, the essential objective of P, and in particular we analyze the implications of Grinold's growth condition, Assumption 3.2 (or 3.3). We first show that all trajectories $x = (x_t)_{t=1}^{\infty}$ of interest for P are bounded in a certain normed space and then show that restricted to that space the function F is weakly inf-compact from which the existence of optimal solutions follows immediately.

Note that if for all t, $f_t = f$ and the constraints implied

$$x_t \in K, \; t = 1, \ldots\ldots,$$

or if we added a constraint of that type, with $K \subset R^n$ compact and f bounded on K X K, then existence and related results could be obtained via the standard method of successive approximations which also gives good error estimates [13, Chapter 6], [14, Chapter 4]. In this paper we do not introduce such artificial (uniform) boundedness conditions on the trajectories $(x_t)_{t=1}^{\infty}$. A fortiori, we shall not require that optimization takes place in the space ℓ_n^{∞} of bounded sequences in R^n. The appropriate space turns out to be

$$\ell_n^1(\alpha) = \{ x = (x_t)_{t=1}^{\infty} \mid || x || : = \Sigma_{t=1}^{\infty} \alpha^t \mid x_t \mid < \infty \} \tag{5.1}$$

as confirmed by the results below; here $| \bullet |$ denotes the ℓ^1-norm in R^n, i.e. $| x_t | : = \Sigma_{j=1}^{n} | x_{tj} |$.

The arguments rely on the asymptotic behavior of "averaged" trajectories. Fix any $\eta \in (0, \alpha]$. Now to each $x = (x_t)_{t=1}^{\infty}$ we associate

$$(u_{T-1}, v_T) := \Sigma_{t=1}^{T} \frac{1-\eta}{1-\eta^T} \eta^{t-1} (x_{t-1}, x_t) \tag{5.2}$$

$$\beta_T := \Sigma_{t=0}^{T} \eta^t \mid x_t \mid = \mid x_0 \mid + \Sigma_{t=1}^{T} \eta^t \mid x_t \mid \tag{5.3}$$

and

$$(y_{T-1}, z_T) := \frac{1-\eta^T}{1-\eta} \beta_T^{-1} (u_{T-1}, v_T) \tag{5.4}$$

Note that u_T and v_T are convex combinations of $(x_0, \ldots\ldots, x_T)$ and $(x_1, \ldots\ldots, x_T)$ respectively, and that y_T and z_T are just scaled versions of these vectors. We have that

$$z_T = \beta_T^{-1} \Sigma_{t=1}^{T} \eta^{t-1} x_t, \tag{5.5}$$

while

$$y_T = \beta_{T+1}^{-1} (\Sigma_{t=1}^{T+1} \eta^{t-1} x_{t-1}) = \beta_{T+1}^{-1} (x_0 + \eta \Sigma_{t=1}^{T} \eta^{t-1} x_t)$$

$$= \beta_{T+1}^{-1} x_0 + \eta \beta_{T+1}^{-1} \beta_T z_T \tag{5.6}$$

Now observe that

$$|z_T| = \beta_T^{-1}\, \eta^{-1}\, \Sigma_{t=1}^{T}\, \eta^t\, |x_t| = \eta^{-1}\, \beta_T^{-1}\, (\beta_T - |x_0|)$$

$$= \eta^{-1} - (\eta\, \beta_T)^{-1}\, |x_0| \tag{5.7}$$

Also

$$|y_T| = \beta_{T+1}^{-1}\, \Sigma_{t=1}^{T+1}\, \eta^{t-1}\, |x_{t-1}| = \beta_{T+1}^{-1}\, \beta_T \tag{5.8}$$

which means by (5.6) that

$$y_T = \beta_{T+1}^{-1}\, x_0 + \eta\, |y_T|\, z_T \tag{5.9}$$

If $|| x || = + \infty$, the case which will be of interest, then the β_T converge monotonically to $+ \infty$. This means that

$$\lim_{T \to \infty} |z_T| = \eta^{-1}$$

the convergence being from below. Also, and this only depends on having $\beta_T > 0$ for T sufficiently large, every cluster point of the sequence $\{ \mid y_T \mid, T=1, \ldots\ldots \}$ belongs to $[0, 1]$. This means that

$$\{ z_T, T=1, \ldots\ldots \} \subset R_+^n \cap \eta^{-1}\, B \tag{5.10}$$

and

$$\{ y_T, T=1, \ldots\ldots \} \subset R_+^n \cap B \tag{5.11}$$

where B is the unit ball in R^n, and hence each one of these sequences admits cluster points.

5.1 LEMMA. Suppose P is proper and satisfies Grinold's growth condition (Assumptions 3.1 and 3.2), and $x = (x_t)_{t=1}^{\infty}$ is such that $||x|| = +\infty$ and either that

$$\limsup_{t\to\infty} |x_t|^{1/t} < \infty \tag{5.12}$$

or there exists $\eta \in (0, \alpha]$ such that

$$\lim_{T\to\infty} z_T = z \tag{5.13}$$

exists with the z_T as defined above (5.5). Then $F(x) = +\infty$.

PROOF. The argument follows the same pattern as the proofs of [3, Theorem 4.1], [15, Theorem 1]. We begin by showing that with (5.12) and (5.13), the sequence $\{ (y_{T-1}, z_T), T=1, \ldots\ldots \}$ admits a cluster point (y, z) with $y = \eta' z$, $z \neq 0$ and $\eta' \in [0, \alpha]$. Suppose first that (5.12) holds. Using (5.5), (5.9) and (5.8) we see that

$$y_{T-1} = \beta_T^{-1} x_0 + \eta z_T - \eta^T x_T / \beta_T$$

From (5.10) we know that some subsequence of the sequence $\{ z_T, T=1, \ldots\ldots \}$ converges to some z with $|z| = \eta^{-1}$. Since β_T^{-1} goes to 0, it would follow that

$$\lim_{T\to\infty} y_{T-1} = \eta z$$

provided that

$$\lim_{T\to\infty} \eta^T |x_T| / (\eta^T |x_T| + \ldots + \eta |x_1| + |x_0|) = 0,$$

and to guarantee this we choose $\eta = \min[\rho^{-1}, \alpha]$ where $\rho < \limsup_{t\to\infty} |x_t|^{1/t}$, see [15, Lemma 1] for the details. Now suppose that (5.13) is satisfied. Then some subsequence of $\{ |y_{T-1}| \; T=1, \ldots \}$ will converge to a $\Theta \in [0, 1]$. Restricting ourselves to this subsequence of $\{|y_{T-1}|, T=1, \ldots\}$ it follows by (5.9) that it converges to : $\eta (\lim_{T\to\infty} |y_{T-1}|) \lim_{T\to\infty} z_{T-1} = \eta \Theta z = \eta' z$, where $\eta' \in [0, \alpha]$ and $z = \lim_{T\to\infty} z_T$ with $|z| = \eta^{-1}$.

For the rest of the proof we assume that actually

$$\lim_{T\to\infty} (y_{T-1}, z_T) = (\eta' z, z)$$

with $\eta' \in [0, \alpha]$; there is no loss of generality in doing so since all assertions remain valid if we work only with a converging subsequence. For the sake of the argument, let us assume that

$$\limsup_{T\to\infty} \Sigma_{t=1}^{T} \eta^{t-1} f_t(x_{t-1}, x_t) < \gamma < +\infty$$

Since for all t, $f_t \geq h$ and $h = h_1$, as defined by (3.2), is convex, from (5.2) we obtain

$$h(u_{T-1}, v_T) \leq \frac{1-\eta}{1-\eta^T} \Sigma_{t=1}^{T} \eta^{t-1} f_t(x_{t-1}, x_t) \leq \frac{1-\eta}{1-\eta^T} \gamma$$

for T sufficiently large. Reexpressing this in terms of (y_{T-1}, x_T) and dividing both sides by $\lambda_T = (1-\eta)(1-\eta^T)^{-1} \beta_T$ yields

$$\lambda_T^{-1} h(\lambda_T y_{T-1}, \lambda_T z_T) \leq \beta_T^{-1} \gamma .$$

Since β_T and λ_T tend to $+\infty$ with T, from (3.8) and the limiting properties of the sequence $(y_{T-1}, z_T)_{T=1}^{\infty}$ we obtain

$$(\mathrm{rc}\ h)(\eta' z, z) \leq 0 ,$$

which contradicts Grinold's growth condition (3.7). Hence

$$\limsup_{T\to\infty} \Sigma_{t=1}^{T} \eta^{t-1} f_t(x_{t-1}, x_t) = +\infty .$$

But now recall that $\eta \leq \alpha$, therefore we also have

$$\limsup_{T\to\infty} \Sigma_{t=1}^{T} \alpha^{t-1} f_t(x_{t-1}, x_t) = F(x) = +\infty ,$$

(the positive part of the sum already dominates the negative part with the parameter η less than α), and this completes the proof. □

The conditions (5.12) and (5.13) cover all cases that seem to be of interest. However, it is possible to generate trajectories, with more than exponential growth and for which the "averaged" trajectories $\{ z_T, T=1, \ldots\ldots \}$ do not converge. Such trajectories must have very wild tails! And for these, Grinold's growth condition would not be sufficient to guarantee that they correspond to $F(x) = +\infty$, we would need to impose much more constringent growth conditions to handle such exotic cases.

By Lemma 5.1 we may safely restrict optimization to those sequences $(x_t)_{t=1}^{\infty}$ in $\ell_n^1(\alpha)$ whose natural pairing is with ℓ_n^{∞}. This plays a significant role in the convergence results of the next section but it also has some bearing on the question of the existence of optimal solutions. Weak convergence in $\ell_n^1(\alpha)$ is characterized by having "componentwise" convergence, thus a (filtered) collection of points $\{ x^{\nu} = (x_t^{\nu})_{t=1}^{\infty}, \nu \in N \}$ weakly converges to x if and only if

$$\lim_{\nu \in N} x_t^{\nu} = x_t \qquad \text{for all } t, \tag{5.14}$$

which would also be sufficient for strong convergence if $\{ x^{\nu}, \nu \in N \}$ is a sequence. This set-up will provide us with the topological framework for the study of the properties of P. Henceforth, we will think of P as being defined on $\ell_n^1(\alpha)$ with the essential objective function now given by:

$$F(x) = \begin{cases} \limsup_{t \to \infty} \Sigma_{t=1}^{T} \alpha^{t-1} f_t(x_{t-1}, x_t) \text{ if } x \in \ell_n^1(\alpha)_+ \\ \qquad \text{with } \ell_n^1(\alpha)_+ = \{x \in \ell_n^1(\alpha) \mid x_t \geq 0, t=1, \dots \} \\ +\infty \text{ otherwise} \end{cases} \tag{5.15}$$

5.2 PROPOSITION. Suppose P is proper (Assumption 3.1). Then F is a proper, weakly lower semicontinuous function.

PROOF. Properness of F on $\ell_n^1(\alpha)$ has been argued in Section 3 in connection with Assumption 3.1. Lower semicontinuity is obtained as a consequence of a version of Fatou's Lemma. By Assumption 3.1. i, the function h ($\leq f_t$ for all t) is proper and convex, so let a be an affine function majorized by h. Then for all t,

$$q_t = f_t - a \geq 0.$$

Now consider a collection $\{x^\nu \in \ell_n^1(\alpha),\ \nu \in N\}$ converging weakly to x. For all $\nu \in N$ and T, we set

$$\kappa_{\nu,T} := \Sigma_{t=1}^{T} \alpha^{t-1} q_t(x_{t-1}^\nu, x_t^\nu)$$

Since the quantities involved are nonnegative, we have that for all ν, the $\kappa_{\nu T}$ are monotonically nondecreasing with T and thus

$$\kappa_\nu := \lim_{T\to\infty} \kappa_{\nu,T} \tag{5.16}$$

is well defined, possibly with value $+\infty$. Hence for all T

$$\liminf_{\nu \in N} \kappa_{\nu,T} \leq \liminf_{\kappa \in N} \kappa_\nu ,$$

then taking lim sup with respect to T on both sides (which of course does not affect the right-hand side) and using (5.16), we obtain

$$\limsup_{t\to\infty} \liminf_{\nu\to N} \kappa_{\nu,T} \leq \liminf_{\nu\in N} \limsup_{T\to\infty} \kappa_{\nu,T} \tag{5.17}$$

Now, note that

$$\limsup_{T\to\infty} \kappa_{\nu,T} = F(x^\nu) - A(x^\nu) \tag{5.18}$$

where A is the affine functional on $\ell_n^1(\alpha)_+$ generated by replacing f_t by a in (5.15). Also, since f_t is lower semicontinuous so is q_t and thus

$$\liminf_{\nu\in N} \kappa_{\nu,T} = \liminf \Sigma_{t=1}^{T} \alpha^{t-1} g_t(x_{t-1}^\nu, x_t^\nu) \geq \Sigma_{t=1}^{T} \alpha^{t-1} g_t(x_{t-1}, x_t)$$

Taking lim sup with respect to T, and combining this with (5.17) and (5.18), yields

$$\liminf_{\nu\in N} F(x^\nu) - A(x^\nu) \geq F(x) - A(x).$$

Since A is an affine functional and the x^ν converge weakly, this inequality gives us the weak lower semicontinuity of F since it implies that

$$\liminf_{\nu\in N} F(x^\nu) \geq F(x). \quad \square$$

The existence of optimal solutions could now very well be settled by requiring that dom F, the effective domain is contained in a weakly compact set; the level sets of F being weakly closed it would give us the weak inf-compactness of F. The following growth condition would suffice: <u>for some sequence of nonnegative numbers</u> λ_t, t=1, ... <u>with</u>

$$\Sigma_{t=1}^{\infty} \alpha^t \lambda_t < \infty$$

<u>we have for all</u> $x \in \text{dom } F$ that $|x_t| < \lambda_t$ <u>for all</u> t. Then every trajectory in dom F is uniformly summable and weak compactness then follows, see [16, p. 20] for example. In particular this argument shows that we may allow feasible trajectories to grow exponentially at a rate less than α^{-1}. When this is translated in the language of capital theory, the condition

$$|x_t| \leq \lambda_t \text{ with } \limsup_{t\to\infty} (\lambda_t)^{1/t} < \alpha^{-1}$$

is nothing else than the requirement that the rate of impatience, as defined by Fisher [17] exceeds the rate of capital growth. Otherwise we would embark on a path of capital accumulation [3]. We shall see however, that we do not need to introduce weak compactness artificially, in fact it is already there in some way, as a consequence of the assumptions we have been working with so far. We begin with a lemma involving the function

$$H(x) := \begin{cases} \limsup_{T\to\infty} \Sigma_{t=1}^{T} \alpha^{t-1} \; h(x_{t-1}, x_t) \text{ if } x \in \ell^1_n(\alpha)_+ \\ \\ +\infty \text{ otherwise} \end{cases} \tag{5.19}$$

Since $h \leq f_t$ for all t, cf (3,4) and (3.5), we have that

$$H \leq F \text{ on } \ell^1_n(\alpha).$$

If P is proper, we have that H is <u>proper and weakly lower semicontinuous</u>, as follows from Proposition 5.2 (replacing in the definition of F every f_t with h). Moreover, H is <u>convex</u> (convexity of h) and H(0) is finite.

5.3 LEMMA. Suppose P is proper and satisfies the strict version of Grinold's growth condition (Assumptions 3.1 and 3.3). Then H is weakly if-compact, i.e. for all $\beta \varepsilon R$

$$\mathrm{lev}_\beta H = \{x \mid H(x) \leq \beta\}$$

is weakly compact.

PROOF. Of course it suffices to consider the case when $\mathrm{lev}_\beta F$ is nonempty, since H(0) is finite (Assumption 3.1. ii) let us assume that $\beta \geq H(0)$. This means that $0 \varepsilon \mathrm{lev}_\beta H$. Since $\mathrm{lev}_\beta H$ is weakly closed (Proposition 5.2), to prove weak compactness we only need to show that $\mathrm{lev}_\beta H$ is weak sequentially compact (Eberlein-Smulian Theorem) and in $\ell^1_n(\alpha)$ this actually turns out to be the same as strong precompactness.

We first intend to show that $\mathrm{lev}_\beta H$ is locally weak sequentially compact at 0. To see this, consider the continuous linear functional

$$x \to \langle e, x\rangle = \Sigma_{t=1}^{\infty} \alpha^t e_t x_t$$

where for all t, $e_t = (1,1, \ldots, 1)$. Note that $\langle e, x\rangle = \|x\|$ whenever $x \varepsilon$ dom H. Now let

$$V: = \mathrm{lev}_\beta H \cap \{x \mid \langle e, x\rangle \leq 1\}$$

This is a closed weak neighborhood of 0 relative to $\mathrm{lev}_\beta H$. Pick any sequence $\{x^\nu, \nu=1, \ldots\} \subset V$; we must exhibit a convergent subsequence characterized by (5.14). If some subsequence converges in norm to 0, there is nothing to prove, so we suppose that for some $\bar{\gamma} > 0$

$$\bar{\gamma} \leq \|x^\nu\| = \langle e, x^\nu \rangle \leq 1$$

for all ν. Passing to a subsequence, if necessary, we may assume that

$$\lim_{\nu \to \infty} \|x^\nu\| = \gamma \in [\bar{\gamma}, 1].$$

Observe that for all ν and all t:

$$\frac{\alpha^t |x_t^\nu|}{\|x^\nu\|} \in [0, 1]$$

Therefore by a standard diagonal procedure we can extract a subsequence $\{x^\nu, \nu \in N'\}$ such that

$$\lim_{\nu \in N'} \frac{\alpha^t x_t^\nu}{\|x^\nu\|} =: \frac{\alpha^+ x_t}{\gamma} \text{ for } t=1, \ldots .$$

implying the (weak) convergence of the $\{x^\nu, \nu \in N'\}$ to $x: = (x_t)_{t=1}^\infty$.

Next we prove that the set $lev_\beta H$ is norm-bounded. Suppose to the contrary that there exists a sequence $\{x^\nu \in lev_\beta H, \nu=1, \ldots\}$ such that $\lim_{\nu \to \infty} \|x^\nu\| = +\infty$. Define

$$y^\nu: = x^\nu / \|x^\nu\|$$

assuming that $\|x^\nu\| > 0$ for all ν. Since $\|y^\nu\| = 1$,

$$y^\nu \in V \text{ for all } \nu$$

and, passing to a subsequence if necessary, there exists $y = \lim_{\nu\to\infty} y^\nu$ as follows from the weak sequential compactness of V. Since the y^ν converge weakly to y, in particular we have

$$1 = \lim_{\nu\to\infty} \|x^\nu\| = \lim_{\nu\to\infty} \langle e, y^\nu\rangle = \langle e, y\rangle = \|y\|.$$

Since $\mathrm{lev}_\beta H$ is convex our construction would imply that $0 \neq y \in \mathrm{rc}\ \mathrm{lev}_\beta H$ (with $y_t \geq 0$ for all t). Then

$$\beta \geq \limsup_{T\to\infty} [h(x_0, \lambda y_1) + \Sigma_{t=2}^{T} \alpha^{t-1} h(\lambda_{t-1}, y_{t-1}, \lambda y_t)]$$

for all $\lambda \geq 0$, using the fact that $0 \in \mathrm{lev}_\beta H$. The convexity of h now yields

$$\beta \geq \limsup_{T\to\infty} \frac{1-\alpha^T}{1-\alpha} h(\frac{1-\alpha}{1-\alpha^T}(x_0 + \lambda\alpha\, \Sigma_{t=1}^{T-1} \alpha^{t-1} y_t, \lambda\, \Sigma_{t=1}^{T}\alpha^{t-1} y_t)) \quad (5.22)$$

As T goes to $+\infty$, α^T goes to 0 and

$$z: = \Sigma_{t=1}^{\infty} (1-\alpha)\alpha^{t-1} y_t,$$

is well defined since $\|y\| < \infty$, with $z \neq 0$ since $y \neq 0$. Dividing both sides of (5.22) by λ and letting λ go to ∞, we obtain

$$0 = \lim_{\lambda\to\infty} \lambda^{-1}\beta \geq \lim_{\lambda\to\infty} \lambda^{-1} (1-\alpha)^{-1} h((1-\alpha)x_0 + \lambda\alpha z, \lambda z),$$

which with formula (3.8) and condition (3.6), a consequence of Assumption 3.1, implies

$$0 \geq \mathrm{rc}\ h(\alpha z, z)$$

contradicting (3.9). Hence y must be 0, and thus $\mathrm{lev}_\beta H$ is bounded.

To complete the proof it suffices to observe that local weak sequentially compactness and boundedness yield weak sequential compactness. □

5.4 THEOREM. Existence. <u>Suppose</u> P <u>is proper and satisfies the strict version of Grinold's growth condition (Assumptions</u> 3.1 <u>and</u> 3.3). <u>Then the essential objective function F of P is weakly inf-compact, and hence there exists optimal solutions of</u> P.

PROOF. Since $H \leq F$ (5.20), F is weakly lower semicontinuous, it follows that for all $\beta \in R$, $\mathrm{lev}_\beta F$ is a weakly closed subset of the weakly compact set $\mathrm{lev}_\beta H$ (Lemma 5.3) and thus $\mathrm{lev}_\beta F$ is also weakly compact.

Since F is proper, the inf-compactness implies the existence of optimal solutions. □

The preceeding theorem suggests that we could restrict ourselves to trajectories that satisfy some uniform summability condition, but at the outset we do not know the pertinent parameters and it would be inappropriate to introduce them artificially. Let us stress here the fact that these last results very much depend on having $\alpha < 1$, with $\alpha = 1$ we are in another ballpark and the rules of the game are then quite different.

6. CONVERGENCE OF FINITE HORIZON APPROXIMATES

We embed each finite horizon into an equivalent infinite dimensional problem (in $\ell^1_n(\alpha)$, and then rely on the convergence results for the infima of epi-convergent functions, cf. [18, Section 2] for a review of its highlights. This technique was used by Back [9] in a related context, in his work on infinite horizon economies under uncertainty.

To P_T (4.4), the finite horizion problem giving lower estimate, we associate

$$F_T \colon \ell^1_n(\alpha) \to R \cup \{+\infty\}$$

that will play the role of the essential objective function of the corresponding problem in $\ell^1_n(\alpha)$. Let

$$F_T(x) := \begin{cases} \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_{t-1}, x_t) + \frac{\alpha^{T-1}}{1-\alpha} h_T((1-\alpha)x_{T-1} + \alpha z_T, z_T) \\ \qquad \text{if } x \in \ell^1_n(\alpha)_+ \text{ and } z_T = (1-\alpha)\,\Sigma_{t=T}^{\infty} \alpha^{t-T} x_t \\ +\infty \text{ otherwise,} \end{cases} \tag{6.1}$$

h_T being as usual the function defined in Section 3, see (3.2). The definition of F_T is motivated by the construction that led us to P_T. The optimization problem

$$\text{find } x \in \ell^1_n(\alpha) \text{ such that } F_T(x) \text{ is minimized} \tag{6.2}$$

can be viewed as an $\ell_n^1(\alpha)$-version of P_T. Indeed, if $F_T(x) < +\infty$, then $(x_1, \ldots, x_{T-1}, z_T)$ is a feasible solution of P_T, and on the other hand if $(x_t)_{t=1}^T$ is a feasible solution of P_T, the sequence $x = (x_1, \ldots, x_{T-1}, x_T, x_T, \ldots)$ is feasible for (6.2) since then $z_T = x_T$. In particular, we have that

$$V_T(x_0) = \inf_{x \in \ell_n^1(\alpha)} F_T(x)$$

with $V_T(x_0)$ given by (4.6). Thus from (4.9) it follows that the infima of the F_T are monotonically nondecreasing with T and bounded above by

$$V(x_0) = \inf_{x \in \ell_n^1(\alpha)} F(x)$$

with F as defined by (5.15). This is not too surprising since as an immediate consequence of (3.5), we have that

$$F_1 \leq \ldots \leq F_T \leq F_{T+1} \leq \ldots \leq F. \tag{6.5}$$

Thus $\{F_T, T=1, \ldots\}$ is a monotone nondecreasing sequence of functions bounded above by Fand since F is weakly lower semicontinuous we can establish epi-convergence by showing that the F_T pointwise converge to F. Note that here, <u>epi-convergence is always to be understood in terms of the weak topology</u>.

6.1 PROPOSITION. <u>Suppose P is proper (Assumption</u> 3.1). <u>Then</u>

$$\{F_T(x), T=1, \ldots\} \uparrow F(x) \tag{6.6}$$

<u>for all</u> $x \in \ell_n^1(\alpha)$, <u>which implies that</u>

$$F = \operatorname*{epi-lim}_{T\to\infty} F_T \tag{6.7}$$

PROOF. It suffices to show that if $x \in \ell_n^1(\alpha)_+$ then $F(x) = \lim_{T\to\infty} F_T(x)$, and this convergence wil follow from the definitions of F_T and F if we show that

$$\liminf_{T\to\infty} \frac{\alpha^{T-1}}{1-\alpha}\ h_T((1-\alpha)\, x_{T-1} + \alpha z_T, z_T) \geq 0$$

or that

$$\liminf_{T\to\infty} \alpha^{T-1}\ h(z_{T-1}, z_T) \geq 0$$

since $h \leq h_T$ for all T, and

$$z_{T-1} = (1-\alpha)\, x_{T-1} + \alpha z_T.$$

But observe that $(\alpha^t z_t)_{t=0}^{\infty}$ is a monotone nonincreasing sequence in R_+^n converging to 0 since $\|x\| < \infty$. Hence

$$\liminf_{T\to\infty} \alpha^{T-1}\ h(z_{T-1}, z_T) = \liminf_{T\to\infty} \alpha^{T-1}\ h(\alpha^{-T+1}(\alpha^{T-1} z_{T-1}, \alpha^{T-1} z_T))$$

$$= (\mathrm{rc}\ h)(0,0) = 0$$

This gives us (6.6). Now, since F is weakly lower semicontinuous, epi-convergence can be verified directly, such as in [18. Proposition 4.23, or more immediately by observing that monotonicity implies (weak) equi-lower semicontinuity [19, Definition 2.17] which yields epi-convergence as a consequence of pointwise convergence [19 Corollary 2.19]. □

Assuming P is proper (Assumption 3.1), the functions F_T are <u>weakly lower semicontinuous</u>, the proof of Proposition 5.2 applies equally well, F and F_T satisfying the same conditions. Moreover, since for all t, $h \leq h_t \leq f_t$,

$$H \leq F_T \text{ for all } T \tag{6.8}$$

with H as in Section 5, see (5.19). Hence, for all T, the F_T are <u>proper</u>, <u>weakly inf-compact</u> functions, whenever the strict version of Grinold growth condition is satisfied (Assumption 3.3); we rely here on Lemma 5.3. This guarantees the existence of points $x \in \ell_n^1(\alpha)$ that minimize F_T. All of this should not come as much of a surpise since Assumptions 3.1 and 3.3 are exactly those we used to obtain the existence of optimal solutions for the finite horizon problems P_T, consult Proposition 4.3. In fact, it is easy to verify that if

$$\bar{x} = (\bar{x}_t)_{t=1}^{\infty} \in \text{argmin } F_T$$

where argmin G: = $\{x \mid G(x) \leq \inf G\}$, then

$$(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_{T-1}, z_T = (1-\alpha) \Sigma_{t=1}^{T} \alpha^{t-1} \bar{x}_t),$$

is an optimal solution of P_T. Similarly, if

$$(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_T)$$

solves P_T, then

$$\bar{x} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_T, \bar{x}_{T+1} = \bar{x}_T, \bar{x}_{T+2} = \bar{x}_T, \ldots) \in \operatorname{argmin} F_T.$$

Thus, we can identify the optimal solution of P_T with those of the optimization problem (6.2) in $\ell_n^1(\alpha)$.

6.2 THEOREM. Consider problem P (1.1) and the finite horizon approximates $\{P_T, T=1, \ldots\}$ (4.4). Suppose that P is proper, satisfies Grinold's growth condition, and that this implies that the feasible solutions of P are in $\ell_n^1(\alpha)$. Then, the sequence

$$\{V_T(x_0), T=1\} \text{ converges from below to } V(x_0). \tag{6.9}$$

Moreover, P and all the problems P_T admit optimal solutions, and given any sequence $\{x_T, T=1, \ldots\}$ of optimal solutions of P_T, it admits at least one cluster point $x = (x_t)_{t=1}^{\infty}$ such that

$$x_t = \lim_{T\to\infty} x_t^T \text{ for all } t, \tag{6.10}$$

and any such cluster point solves the long term problem P. Finally, if x solves P, then there exists a sequence of real numbers $\varepsilon_T \downarrow 0$ and $\bar{x}^T = (\bar{x}_t^T)_{t=1}^T$ such that $\bar{x}^T$ is an ε_T-optimal solution of P_T -- i.e. up to ε_T, $\bar{x}^T$ solves P_T -- and for all t, $x_t = \lim_{T\to\infty} \bar{x}_t^T$

PROOF. The assumptions allow us to identify P with minimizing F on $\ell_n^1(\alpha)$ and the P_T minimizing F_T on $\ell_n^1(\alpha)$. Now, let us choose β such that $\beta \geq \inf F$ and define

$$K := \mathrm{lev}_\beta H$$

where H is as defined (5.19) in Section 5. We have

$$\min F = \min_K F = V(x_0)$$

and for all T, see (6.8),

$$\min F_T = \min_K F_T = V_T(x_0)$$

We write min instead of inf since we know that the infima are actually attained. Since K is compact (Lemma 5.3), and $F = \underset{T\to\infty}{\text{epi-lim}}\, F_T$ (Proposition 6.1), it follows

$$\liminf\, (\min_K F_T) \geq \min_K F.$$

cf. [20, Proposition 2.1], [21]. Combining this with what precedes and (4.9) of Proposition 4.2, we obtain (6.9).

Since epi-convergence implies

$$\limsup_{T\to\infty}\, (\operatorname{argmin} F_T) \subset \operatorname{argmin} F$$

and whenever $\inf F = \lim\limits_{T\to\infty} (\inf F_T)$

$$\operatorname{argmin} F = \bigcap_{\varepsilon>0} \liminf_{T\to\infty} (\varepsilon\text{-argmin } F_T),$$

see [21, Theorems 2 and 3] or the epi-convergent version of [17, Proposition 3.12], we now obtain all the remaining assertions using (i) the fact that for all T, (argmin F_T) $\subset K$ and (ii) that (6.10) characterizes weak convergence on $\ell_n^1(\alpha)$. □.

We now turn to $\{P^T, T=1, \ldots\}$, the finite horizon approximates that yield upper bounds. We essentially proceed in the same manner as above, however, we shall now need to introduce Assumption 3.4 on the sustainability of tail-stationary trajectories to obtain convergence. To each problem P^T, defined by (4.13), we associate

$$F^T(x) := \begin{cases} \Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_t, x_{t-1}) + \frac{\alpha^{T-1}}{1-\alpha} g_T(x_{T-1}, x_{T-1}) \\ \qquad \text{if } x \in \ell_n^1(\alpha)_+ \text{ and } x_t = x_{T-1} \text{ for } t=T, \ldots, \\ \\ +\infty \text{ otherwise} \end{cases} \tag{6.11}$$

where g_T is as in (3.11), the pointwise supremum of the f_t with $t \geq T$. The optimization problem

$$\text{find } x \in \ell_n^1(\alpha) \text{ such that } F^T(x) \text{ is minimized} \tag{6.12}$$

can thus be viewed as an $\ell_n^1(\alpha)$-version of P^T. As for P_T and F_T, we can identify feasible solutions of P^T and F^T. In fact the correspondence here is one-to-one, so that in particular we can identify optimal solutions of P^T with elements $x \in \ell_n^1(\alpha)$ that minimize F^T, and vice-versa. We also have that

$$V^T(x_0) = \inf_{x \,\varepsilon\, \ell^1_n(\alpha)} F^T(x),$$

where $V^T(x_0)$ is the infimal value of P^T and thus, as a consequence of Proposition 4.4, we know that the infima $\{\inf F^T,\ T=1, \ldots\}$ form a nonincreasing sequence bounded below by $V(x_0) = \inf F$. To obtain convergence we again rely on the following fact:

6.3 PROPOSITION. Suppose P is proper and tail-stationary trajectories are sustainable (Assumptions 3.1 and 3.4). Then

$$F = \operatorname*{epi-lim}_{T \to \infty} F^T \tag{6.13}$$

PROOF. This time we verify directly the definition of epi-convergence [18, Section 2]. We have to show that for any $x \,\varepsilon\, \ell^1_n(\alpha)$

(i) for all $\{x^{\nu} \varepsilon\, \ell^1_n(\alpha),\ \nu=1, \ldots\}$ converging weakly to x

$$\liminf_{\nu \to \infty} F^{\nu}(x^{\nu}) \geq F(x) \tag{6.14}$$

and

(ii) for some sequence $\{x^{\nu} \,\varepsilon\, \ell^1_n(\alpha),\ \nu=1, \ldots\}$ converging weakly to x,

$$\limsup_{\nu \to \infty} F^{\nu}(x)^{\nu} \leq F(x). \tag{6.15}$$

The first condition (6.14) follows from the weak lower semicontinuity of F (Proposition 5.2) which implies

$$\liminf_{\nu \to \infty} F(x^\nu) \geq F(x),$$

and the fact that for any $x \in \ell_n^1(\alpha)$

$$F^\nu(x) \geq F(x);$$

to see this observe that $F^\nu(x) = +\infty$ if x is not tail-stationary for $t \geq \nu-1$ and if it is tail-stationary then by definition of g_ν, in particular (3.13) and (3.14),

$$\frac{\alpha^{T-1}}{1-\alpha} g_\nu (x_{\nu-1}, x_{\nu-1}) \geq \limsup_{\nu' \to \infty} \Sigma_{t=\nu}^{\nu'} \alpha^{t-1} f_t(x_{\nu-1}, x_{\nu-1})$$

To obtain (6.15) for some sequence $\{x^\nu, \nu=1, \ldots\}$ converging weakly to x we construct it as follows: set

$$\begin{aligned} x_t^\nu &= x_t && \text{for } t = 0, \ldots, \nu-1 \\ x_t^\nu &= x_{\nu-1} && \text{for } t = \nu, \ldots . \end{aligned}$$

Then

$$F^\nu(x^\nu) = \Sigma_{t=1}^{\nu-1} \alpha^{t-1} f_t(x_{t-1}, x_t) + \frac{\alpha^{\nu-1}}{1-\alpha} g_\nu(x_{\nu-1}, x_{\nu-1})$$

and taking $\limsup_{\nu\to\infty}$ on both sides yields (6.15) since the second term in the sum is non-positive by the sustainablity of tail-stationary trajectories assumption (3.15). □.

We can now produce the parallel version of Theorem 6.2 for finite time approximates from above. Before we do so let us observe that the functions $\{F^T, T=1, \ldots\}$ are also weakly inf-compact provided that P is proper and satisfies the strict version of Grinold's growth condition. Indeed since $F^T \geq H$ -- with H as defined by (5.19) in connection with Lemma 5.3 -- and H is weakly inf-compact, it suffices to see that F^T is the restriction to a closed linear space (tail-stationarity for x_t with $t \geq T-1$) of the function

$$\Sigma_{t=1}^{T-1} \alpha^{t-1} f_t(x_t, x_{t-1}) + \limsup_{T'\to\infty} \Sigma_{t=T}^{T'} \alpha^{t-1} g_T(x_t, x_{t-1})$$

which is weakly lower semicontinuous by Proposition 5.2. Thus, for all T the infimum is then attained, which we can also express by writing

$$\operatorname{argmin} F^T \neq \emptyset.$$

All of this being derived with exactly the same assumptions that we used to assert the existence of optimal solutions of P^T, see Proposition 4.5.

6.4 THEOREM. <u>Consider Problem</u> P (1.1) <u>and the finite horizon approximates</u> $\{P^T,$ $T=1, \ldots\}$ (4.13). <u>Suppose that</u> P <u>is proper, satisfies Grinold's growth condition, that this implies that feasible solutions of</u> P <u>are in</u> $\ell_n^1(\alpha)$, <u>and that tail-stationary trajectories are sustainable. Then the sequence</u>

$$\{V^T(x_0);\ T=1, \ldots\} \text{ converges from above to } V(x_0). \tag{6.16}$$

<u>Moreover,</u> P <u>and all problems</u> P^T <u>admit optimal solutions, and given any sequence</u> $\{x^T, T=1, \ldots\}$ <u>of optimal solutions of</u> P^T, <u>it admits at least one cluster point</u>

$$x = (x_t)_{t=1}^{\infty} \text{ such that}$$

$$x_t = \lim_{T\to\infty} x_t^T \qquad \text{for all } t, \tag{6.17}$$

<u>and any such cluster point solves the long term problem</u> P. <u>Finally, if</u> x <u>solves</u> P, <u>then there exist a sequence of real numbers</u> $\varepsilon_T \downarrow 0$ <u>and</u> $\bar{x}^T = (\bar{x}_t^T)_{t=1}^{T-1}$ such <u>that</u> x^T <u>is an</u> ε_T<u>-optimal solution of</u> P^T -- <u>i.e. up to</u> ε_T, $\bar{x}^T$ <u>solves</u> P^T -- <u>and for all</u> t, $x_t = \lim_{T\to\infty} \bar{x}_t^T$.

PROOF. The assumptions allow us to identify P with minimizing F on $\ell_n^1(\alpha)$ and the P^T with minimizing F^T on $\ell_n^1(\alpha)$. Since by Proposition (6.3) $F = \text{epi-}\lim_{T\to\infty} F^T$, it follows, see [20] or [21] for example, that

$$\limsup_{T\to\infty} (\inf F^T = V^T(x_0)) \leq V(x_0) = \inf F.$$

which gives us (6.16), since we already know that $\{V^T(x_0), T=1, \ldots\}$ is a nonincreasing sequence (Proposition 4.4).

The remainder of the proof is identical to that of Theorem 6.2, except that in order to claim that for all T, argmin F^T is contained in a weakly compact set we choose this time

$$K := \mathrm{lev}_{\beta} H$$

with $\beta \geq \min F^1$. □

Let us conclude by observing that if the optimal solution of P was unique, for example if the f_t were strictly convex, then Theorems 6.2 and 6.4 would assert that this optimal solution is the unique cluster point (componentwise) of the optimal solutions of the $\{P_T, T=1, \ldots\}$ and $\{P^T, T=1\}$ provided naturally that P satisfies the assumptions of Section 3.

7. BOLZA TYPE PROBLEMS

The purpose here is to record the assumptions and the structure of the approximating finite horizon problems when P is a problem of the Bolza type, to which we already referred in Section 2. The infinite horizon problem, that we designate by B, then reads

$$\text{find } x = (x_t)_{t=1}^{\infty} \text{ such that } x_t \in R_+^n \text{ and}$$

$$w = \lim_{T\to\infty} \Sigma_{t=1}^{T} L_t(x_{t-1}, \Delta x_t) \text{ is minimized} \tag{7.1}$$

with x_0 the initial state fixed, $\alpha \in (0,1)$ a discount factor,

$$\Delta x_t = x_t - x_{t-1},$$

and for all t, $L_t : R^{2n} \to R \cup \{+\infty\}$ is a lower semicontinuous function. Setting

$$f_t(x_{t-1}, x_t) := L_t(x_{t-1}, \Delta x_t)$$

gives the connection with the formulation (1.1) of P.

Again, for T=1, ..., let

$$h_T: = \text{cl co} (\inf_{t\geq 1} L_t), \tag{7.2}$$

with h: = h_1 and

$$g_T = \sup_{t \geq T} L_t. \tag{7.3}$$

ASSUMPTION 7.1 Problem B is proper. <u>This means</u>

(i) <u>the function</u> $h > -\infty$

(ii) <u>there exists</u> $\tilde{x} = (\tilde{x}_t)_{t=1}^{\infty}$ <u>with</u> $x_t \in R^n_+$ such that
$\limsup_{T\to\infty} \Sigma_{t=1}^{T} \alpha^{t-1} L_t(\tilde{x}_{t-1}, \Delta\tilde{x}_t) < \infty$

and

$L_1(\tilde{x}_1, 0) < \infty.$

ASSUMPTION 7.2 Grinold's growth condition. <u>For every</u>

$\alpha' \in [0,\alpha]$ <u>and</u> $z \in R^n_+$ <u>with</u> $z \neq 0$,

$$\mathrm{rc}\; h(\frac{\alpha'}{1-\alpha'} z, z) > 0 \tag{7.4}$$

ASSUMPTION 7.3. Sustainability of tail-stationary trajectories. <u>If</u> $x = (x_t)_{t=1}^{\infty}$ <u>is feasible, i.e.</u> $x_t \in R^n_+$ <u>for all</u> t, <u>and</u>

$$\limsup_{T\to\infty} \Sigma_{t=1}^{T} \alpha^{t-1} L_t(x_{t-1}, \Delta x_t) < \infty,$$

then

$$\limsup_{T\to\infty} \alpha^{T-1} g_T(x_{T-1}, 0) \leq 0 \tag{7.5}$$

Approximates from below B_T:

find $(x_t)_{t=1}^{T}$ with $x_t \in R_+^n$,

$$w = \Sigma_{t=1}^{T-1} \alpha^{t-1} L_t(x_{t-1}, \Delta x_t) + \frac{\alpha^{T-1}}{1-\alpha} h_T((1-\alpha)x_{T-1} + \alpha x_T, (1-\alpha)\Delta x_T) \quad (7.6)$$

and w is minimized.

Approximate from above B^T:

find $(x_t)_{t=1}^{T-1}$ with $x_t \in R_+^n$,

$$w = \Sigma_{t=1}^{T-1} \alpha^{t-1} L_t(x_{t-1}, \Delta x_t) + \frac{\alpha^{T-1}}{1-\alpha} g_T(x_{T-1}, 0), \quad (7.7)$$

and w is minimized

All results of sections 4.5 and 6 can now be rephrased in a straightforward manner.

REFERENCES

[1] Gale, D. "On optimal development in a multi-sector economy", Review of Economic Studies, 34 (1967), 1-18.

[2] Weitzman, M. "Duality theory for infinite horizon convex models", Management Science, 19 (1973), 783-789.

[3] Magill, M. "Infinite horizon programs", Econometrica, 49 (1981), 679-711.

[4] Grinold, R.C. "Convex infinite horizon programs", Mathematical Programming, 25 (1983), 64-82.

[5] Ekeland, I. Scheinkman, J. "Tranversality conditions for some infinite horizon discrete time optimization problems", Technical Report 411, Inst. Mathem. Studies in the Social Sciences, Stanford Univ., 1983.

[6] Ekeland, I, Thurnbull, T. "Infinite dimensional optimization and convexity", The University of Chicago Press, Chicago, 1983.

[7] Rockafellar, R.T., Wets, R. "Deterministic and stochastic optimization problems of Bolza type in discrete time", Stochastics, 10 (1983).

[8] Flåm, S.D. "Duality in stochastic discrete time models of resource management", CMI-report 832050-4, Chr. Michelsen Institute, Bergen, Norway, 1983; forthcoming in Mathematics of Operations Research.

[9] Back, K. "Optimality and equilibrium in infinite horizon economies under uncertaity", Ph.D. Theses, Univ. of Kentucky, Lexington, 1982.

[10] Rockafellar, R.T. "Conjugate convex functions in optimal control and the calculus of variations", J. Math. of anal. and its applic., 32 (1970), 174-222.

[11] Rockafellar, R.T. "Convex Analysis", Princeton University Press, Princeton, 1970.

[12] Birge, J., Wets, R. "Designing approximation schemes for stochastic optimization problems, in particular stochastic programs with recourse", IIASA Working paper, 83-111; forthcoming in Mathematical Programming Study.

[13] Ross, S. "Applied probability models with optimization applications", Holden-Day, San Francisco, 1970.

[14] Huston, V, Pym, J. "Applications of functional analysis and operator theory", Academic Press, New York, 1980.

[15] Grinold, r.C. "Finite horizon approximates of infinite horizon linear programs" Mathemathical Programming, 12 (1977), 1-17.

[16] Meyer, P. "Probability and potentials", Blaisdell, Waltham, 1966.

[17] Fisher, I. "Theory of interest", New York, Augustus M. Kelley, (1965).

[18] Attouch, H., Wets, R. "A convergence theory for saddle functions", Trans. Amer. Math. Soc., 280 (1983), 1-41.

[19] Dolecki, S., Salinetti, G, Wets, R. "Convergence of functions: equi-semicontinuity", Trans. Amer. Math. Soc., 276 (1983), 409-429.

[20] Wets, R. "On a compactness theorem for epi-convergent sequences of functions", in Proceed. Rio de Janerio congress on Mathematical Programming, eds. R. Cottle, M. Kelmanson and B. Korte, North-Holland, 1984.

[21] Attouch, H. "Variational convergence for functions and operators", Pitman Research Notes in Mathematics, London, 1984.

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