International Institute for
Applied Systems Analysis
www.iiasa.ac.at

A Generalized Theory of Household Behavior under Rationing

Podkaminer, L.

IIASA Collaborative Paper
October 1984

A GENERALIZED THEORY OF HOUSEHOLD BEHAVIOR UNDER RATIONING

Leon Podkaminer

Institute of Economic Sciences
of the Polish Academy of Sciences,
Warsaw, Poland

PREFACE

Many of today's most significant socioeconomic problems, such as slower economic growth, the decline of some established industries, and shifts in patterns of foreign trade, are international or transnational in nature. But these problems manifest themselves in a variety of ways; both the intensities and the perceptions of the problems differ from one country to another, so that inter-country comparative analyses of recent historical developments are necessary. Through these analyses we attempt to identify the underlying processes of economic structural change and formulate useful hypotheses concerning future developments. The understanding of these processes and future prospects provides the focus for IIASA's project on Comparative Analysis of Economic Structure and Growth.

Our research concentrates primarily on the empirical analysis of interregional and intertemporal economic structural change, on the sources of and constraints on economic growth, on problems of adaptation to sudden changes, and especially on problems arising from changing patterns of international trade, resource availability, and technology. The project relies on IIASA's accumulated expertise in related fields and, in particular, on the data bases and systems of models that have been developed in the recent past.

In this paper, Leon Podkaminer reviews the work of Neary and Roberts on household behavior under rationing and its implications for attempts to "reconstruct Keynesian economics." Podkaminer concludes that the method used by Neary and Roberts to relate rationed to unrationed demand functions, via the concept of "virtual" prices, only holds in certain very restricted circumstances. He then proposes an alternative and more general approach to the evaluation of demand under various rationing schemes when only indirect utility functions are known.

Anatoli Smyshlyaev
Project Leader
Comparative Analysis of
Economic Structure and Growth

A GENERALIZED THEORY OF HOUSEHOLD BEHAVIOR UNDER RATIONING

Leon Podkaminer

Institute of Economic Sciences of the Polish Academy of Sciences,
Warsaw, Poland

ABSTRACT

The results of Neary and Roberts (1980) relating rationed demand functions to the unrationed demand functions via the concept of "virtual" prices are shown to hold only in certain restricted circumstances. An alternative and generalized approach is suggested for the evaluation of effective demand under any arbitrarily-chosen rationing scheme.

1. INTRODUCTION

Many of the recent advances in economic theory and econometrics have been associated with the "disequilibrium revolution," which began with a reawakening of interest in the study of the micro foundations of macroeconomic analysis. A central role in this revolution was played by the implications of household decision making subject to quantity constraints (and not merely to the "orthodox" budget constraint). However, as Neary and Roberts (1980, pp. 25–26) observed, "the basic theory of household behaviour in the presence of rationing remains in an unsatisfactory state. The principal results in this area derived by Tobin and Houthakker (1950–51) and restated by Pollak (1969) apply only for the case where the ration just 'bites', in the sense that the ration levels coincide with the quantities which would have been chosen by an unrationed household facing the same prices and income."

The extension of the theory provided by Neary and Roberts (1980, p. 26) promised "... a complete characterization of rationed demand and supply functions which relates their properties to the properties of the unrationed demand and supply functions without the necessity of explicitly specifying the direct utility function." As this characterization has been utilized for no less

important an endeavor than the reconstruction of Keynesian economics by Neary and Stiglitz (1983), it may be just the right time to subject the Neary and Roberts theory to closer scrutiny — and to offer a proper alternative to it. The structure of this paper is very simple. Section 2 presents the basic features of the Neary—Roberts theory, referred to hereafter simply as N—R. Section 3 discusses a number of difficulties with which the theory is fraught; in particular, its main result is shown to be valid only in the circumstances allowed for by Tobin and Houthakker (1950—51). Sections 4 and 5 then offer an alternative approach to the problem of evaluating demand under various rationing schemes (including schemes not studied before) when only indirect utility functions are known.

2. BASIC ASPECTS OF THE NEARY—ROBERTS THEORY

The utility function of the household studied satisfies all the familiar regularity conditions such as quasiconcavity and montonicity. The vector of goods entering the utility function as its argument is partitioned into two distinct subvectors. Retaining the N—R notation, x is the subvector of goods "freely chosen" and y is the subvector of goods "imposed," i.e. those the household is "forced" to consume. If p and q are vectors of "actual" (or "administered") prices associated with x and y, respectively, and b is the lump-sum household income, then the $\tilde{x}$ that is eventually "freely" chosen must satisfy

$$u(\tilde{x},\bar{y}) = \max \{u(x,\bar{y}) : px \leq b - q\bar{y}\} \tag{1}$$

where $u(x,y)$ is the utility function and $\bar{y}$ is the vector of specific quantities the household is "forced" to consume.

The next step is to introduce "virtual" (or support) prices, "which would induce an unrationed household to behave in the same manner as when faced with a given vector of ration constraints." It soon emerges that the actual and the virtual prices for the unrationed goods are identical, so that the term "virtual prices" may be retained exclusively for the vector $\bar{q}$ (as distinct from its actual value q).

The basic N—R result may be summarized in one equation:

$$\tilde{x}(\bar{y},p,q,b) = x(p,\bar{q},b + (\bar{q} - q)\bar{y}) \tag{2}$$

where $\tilde{x}(\bar{y},p,q,b)$ is the vector of rationed demand (and also supply, for goods that the household may sell, such as working time) functions for the freely-

chosen goods, $x(p,q,b)$ is the vector of ordinary unrationed (Marshallian) demand functions for x, and the vector of virtual prices $\bar{q}$ is determined from the equation

$$\bar{y} = y(p,\bar{q},b + (\bar{q} - q)\bar{y}) \tag{3}$$

where $y(p,q,b)$ is the vector of ordinary unrationed (Marshallian) demand (supply) functions for the imposed goods.

According to (2), the function $x(\cdot,\cdot,\cdot)$ is to be evaluated at virtual prices $(p,\bar{q})$ and income inflated (deflated) by the expenditure involuntarily saved (spent) on consumption of the imposed goods, i.e. by $(\bar{q} - q)\bar{y}$. Of course, (2) is well defined provided (3) always has a unique solution $\bar{q}$. As is implied, this is actually the case, at least for $b + (\bar{q} - q)\bar{y} > 0$.

3. THE CRITIQUE

3.1. Determination of the Levels of Forced Activities for (1)

While under the most extreme cases of war economy the household may be forced to supply some goods (e.g. labor or commandeered vehicles), it is usually still free to refuse to purchase or consume (or to resell on the black market), say, the full ration of tobacco to which it is entitled. In contemporary command economies, while rationed in the markets for goods, households are free to consume less than their entitlement. In market economies under a regime of involuntary unemployment, the employees may not be able to sell as much labor as they might wish, yet there is still nothing that forces them to sell some specific amount of it. When the peculiar notion of coercion to consume some quantities of some goods is discarded, a more relevant basic model for an environment with rationing would therefore stipulate the existence of *bounds* on the household's activities (upper for purchases and lower for sales). This can be represented by the following maximization problem:

$$\max \{u(x,y) : px + qy < b, y \lessgtr y^*\} \tag{4}$$

where $y \lessgtr y^*$ translates into $y \geq y^*$ $(y \leq y^*)$ for supplied (demanded) goods subject to *potential* rationing.

The difference between (1) and (4) is not merely semantic. The unique solution to (4), $(\tilde{x},\tilde{y})$, may appear to be actively constrained by some (but not necessarily all) coordinates of the vector y^*. In (1), $\bar{y}$ would therefore consist

of only those coordinates in $\tilde{y}$ that "hit" the respective bounds ($\tilde{y} = y_i{}^*$). There are two conclusions to be drawn at this stage. First, prior to any analysis starting with (1), one must perform the optimization (4) with numerical values for y^* corresponding to the existing (formally administered or actually perceived, though not necessarily eventually constraining) bounds on the household's activities. Formulas (2)–(3) evidently do not apply when $\bar{y}$ is replaced by y^*. (They might apply when $\bar{y}$ is replaced by $\bar{y} = \tilde{y}(p,q,y^*,b)$.) If, however, we are able to solve (4) there is no need for any additional formulas such as (2). Also, all relevant information related to the comparative statics properties of the rationed demand would automatically follow the solution to (4) as regards its sensitivity analysis.

Second, let us go so far as to assume that $\bar{y} = \tilde{y}$. Consider a formally extreme (but theoretically important and practically quite plausible) case where the household perceives bounds on all of its activities. Now (4) is equivalent to

$$\max \{u(y) : qy \le b, \, y \lessgtr y^*\} \tag{5}$$

If $\tilde{y} = y^*$ (all bounds are "hit") and $q\tilde{y} = b$ (there is no compulsory saving or dissaving) and yet $\tilde{y}$ is different from the unconstrained household's optimum y^o, given by

$$\max \{u(y) : qy \le b\} \tag{6}$$

then the analysis starting with (1) is not feasible. This is because there are, in this case, exactly n (the dimension of y) alternative $\bar{y}^{(i)}$:

$$\bar{y}^{(1)} = (y_2{}^*, y_3{}^*, \ldots, y_n{}^*), \quad \bar{y}^{(2)} = (y_1{}^*, y_3{}^*, \ldots, y_n{}^*), \quad \ldots, \quad \bar{y}^{(n)} = (y_1{}^*, y_2{}^*, \ldots, y_{n-1}{}^*)$$

each commanding an equally justifiable version of (1), but differing with respect to the definition of the "freely" demanded good. Of course, the optimum solutions to the alternative versions of (1) are the same. Yet their local properties, which are needed for the comparative statics exercises, evidently diverge.

It is also worth noting that the above conclusions hold too when the rationed and unrationed demands (supplies) (i.e. the solutions to (5) and (6)) coincide. This utterly extreme and virtually implausible situation may be interpreted as one of an atomistic household embedded in an economy frozen at a general equilibrium and having perfect knowledge of the market's (and its

own) predicament.

3.2. Need Virtual (Support) Prices for the Rationed Goods Be Unique?

Let us return to (1) and accept $\bar{y}$ as somehow correctly evaluated; presumably this could be done by solving (4) but without paying any attention to the status of the corresponding dual prices (i.e. Lagrange multipliers). Even under these circumstances, it appears that the system of equations (3), which serves N–R as a source of (unique) "support" prices $(\bar{q})$ for the rationed goods, cannot be trusted because in some cases it cannot have a unique solution. Consider, for example, a Cobb–Douglas utility function of the following form

$$u(x,y) = x^{\alpha_1} y_1^{\alpha_2} y_2^{\alpha_3} \quad \alpha_1 + \alpha_2 + \alpha_3 = 1$$

For this case the ordinary unrationed (Marshallian) demand functions $y_i(p_1, q_1, q_2, b)$ are given by

$$y_i(p_1, q_1, q_2, b) = \frac{\alpha_i}{q_i} b \quad i = 1,2 \tag{7}$$

Thus, for (3) we have two equations to determine $\bar{q}_1, \bar{q}_2$:

$$\bar{y}_1 = \frac{\alpha_1}{\bar{q}_1}[b + (\bar{q}_1 - q_1)\bar{y}_1 + (\bar{q}_2 - q_2)\bar{y}_2] \tag{8}$$

$$\bar{y}_2 = \frac{\alpha_2}{\bar{q}_2}[b + (\bar{q}_1 - q_1)\bar{y}_1 + (\bar{q}_2 - q_2)\bar{y}_2] \tag{9}$$

For a number of configurations of parameters $(\alpha_1, \alpha_2, \bar{y}_1, \bar{y}_2, q_1, q_2)$, (8) and (9) are linearly dependent and yield a continuum of solutions for $\bar{q}_1, \bar{q}_2$. (For instance this happens when $\alpha_1 = \alpha_2 = \bar{y}_1 = \bar{y}_2 = q_1 = q_2$.) For numerous other configurations, (8) and (9) possess no meaningful (i.e. positive) solutions. It may therefore be concluded that, even if (8) and (9) have a unique positive solution, its status must remain suspect.

4. BEYOND THE QUANTITY BOUNDS

The basic approach to the evaluation of demand under rationing prescribed by (4) needs three distinct additions. First, the quantity constraints $y \le y^*$ $(y \ge y^*)$ correspond to a regime of rationing whereby various potentially lacking (unsold) goods are treated separately. In practice, however,

whenever the rationing takes on some institutionalized shape, there are rationing schemes that allow some degree of substitution among the various goods subject to rationing. Thus, the households (and firms) are "given" some total "ration points" $(b^{(1)}, b^{(2)}, \ldots, b^{(m)})$ and vectors of the "ration point prices" $(\pi^{(1)}, \pi^{(2)}, \ldots, \pi^{(m)})$ are prescribed so that the budget constraint and the upper (lower) bounds are complemented by a system of linear inequalities:

$$y\pi^{(j)} \leq b^{(j)} \quad j = 1,2,\ldots,m \tag{10}$$

Secondly, a not uncommon device under rationing stipulates that "hard" quantity constraints $(y \leq y^*)$ be replaced by presumably "softer" variable pricing. Therefore the "actual" price the household (firm) pays depends on the quantity purchased: $q = q(y)$. (This trick is often used in the pricing of electric power or in rationing water consumption in arid regions. Also, rents in publicly-owned housing tend to vary with the amount of living space per family member.) Quite often the "hard" quantity constraint for a good is replaced by discontinuous pricing:

$$q_j(y_j) = \begin{cases} q_j^- & \text{for } y_j \leq y_j^* \\ q_j^+ & \text{for } y_j > y_j^* \end{cases} \quad (q_j^- < q_j^+)$$

A counterpart to (4) allowing for the existence of systems of "ration point prices" and soft rather than hard quantity constraints is quite straightforward to develop. It is given by the following problem:

$$\text{maximize } u(x,y), \tag{11}$$

$$\text{subject to } px + q(y)y \leq b \text{ and } y\pi^{(j)} \leq b^{(j)} \quad j = 1,2,\ldots,m$$

It may be worth noting that (11), still a concave programming problem, continues to possess a unique solution. However, its analytical (explicit) derivation upon the formulation of the first-order (Lagrange) conditions is inappropriate, since Kuhn–Tucker conditions are relevant here. The appropriate operational approach for the determination of the optimum is, of course, any version of the gradient method. (Nonsmoothness in pricing $q(y)$ may of course dictate the application of nonsmooth gradient methods.) Also, the fact that the optimum solution is expected to be accompanied by sets of subderivatives with respect to the parameters (prices, rations, income) — and not merely unique derivatives — must not be overlooked either.

5. EVALUATION OF DEMAND UNDER RATIONING WHEN ONLY INDIRECT UTILITY FUNCTIONS ARE KNOWN: A DUAL TO ROY'S IDENTITY

The arguments presented in this section are based upon Podkaminer (1983).

Problem (11) presupposes knowledge of the utility function in its ordinary form. However, both economic theory and even empirical economic research increasingly rely on utility functions in indirect forms, or alternatively, on expenditure or cost-of-utility functions. While the underlying household preferences (or the production correspondences of firms) are — in theory — equally well described by direct or indirect utility functions (or expenditure functions), theoretical economic analysis using indirect utility forms is much more convenient and powerful, especially when the utility is homothetic. Also, the econometric estimation of the indirect utility function is much easier to perform without postulating that it has oversimplified properties (such as constant elasticity of substitution, absence of complementarity, etc.). However, the estimated indirect utility functions cannot always be easily transformed into the corresponding direct ones. We do not know the direct utility functions for such widely cited indirect utility functions as Houthakker's indirect addilog, Diewert's generalized Leontief cost functions, or various translog forms (including the AIDS of Deaton and Muellbauer).

But our inability to state the direct utility function for (11) in an explicit form does not remove the possibility of solving (11) whenever the indirect utility function corresponding to $u(x,y)$ is known. First, let us introduce some notation. Let $u(z) = u(x,y)$ be the (not explicitly known) direct utility function and $g(v)$ its indirect form. Under familiar conditions relating to quasiconcavity and monotonicity of $u(z)$, we know that $g(v)$ satisfying

$$g(v) = \max_{z} \{u(z) : vz \leq 1\}$$

is quasiconvex and nonincreasing. Conversely (see Diewert 1974, p. 124), under quasiconvexity and nonincreasing monotonicity of $g(v)$, it is clear that $u(z)$ satisfies

$$u(z) = \min_{v} \{g(v) : vz \leq 1\} \tag{12}$$

The ability to compute the value of the direct utility function for any $z = (x,y)$, even when only the indirect utility function is known (12), suffices for the

efficient computation of the optimum solution to (11). This is because modern gradient methods of mathematical (concave) programming theory require just that and no more. The quasigradient methods do without exact derivatives of the maximand. Instead, they run on approximations given — in our context — by finite differences $h^{-1}[u(z + he_i) - u(z)]$, where e_i is the ith unit vector and h is a positive scalar (see Bräuninger 1981).

Also, it may be worth noting that the exact gradient of the totally unconstrained (even by the budget inequality) ordinary utility function may be relatively easily computed for any commodity bundle when only the indirect utility function is known. The following "dual" to the classical Roy's Identity holds:

$$\nabla_z u(z) = -\hat{\lambda}\hat{v} \tag{13}$$

where $\hat{v}$ is the solution to the optimization problem:

$$\min_v \{g(v) : vz \leq 1\}$$

and $\hat{\lambda}$ is the corresponding optimal Lagrange multiplier for the constraint $vz \leq 1$. Expression (13) can be immediately demonstrated. Roy's Identity states that

$$\nabla_v g(v) = -\lambda^* z^*$$

where z^* is the solution to the optimization problem:

$$\max_z \{u(z) : zv \leq 1\}$$

and λ^* is the corresponding optimal Lagrange multiplier for the constraint $vz \leq 1$. Noting that Roy's Identity also applies to (12), we arrive at (13). It is also worth noticing that the first-order conditions applicable to (13) are

$$\nabla_z u(z) = \lambda^* v$$

Together with (13), these imply that $\lambda^* = -\hat{\lambda}$.

6. CONCLUSIONS

While the theory proposed by Neary and Roberts appears to be incorrect, there is still the possibility of a computational determination of effective demand under rationing, irrespective of whether direct or indirect utility functions (or expenditure functions) for households (firms) are known. The method

suggested here works quite well, even under rationing regimes not often visualized in contemporary economic theory but very familiar from economic practice ("softening" devices for quantity constraints through variable pricing and partial substitutability of rations). The price paid for this approach is the need to rely on computational iterative rather than analytical methods. Comparative statics exercises, which are the gist of much abstract economic work, can still be easily performed with respect to the solutions arrived at, though they would more accurately be termed "sensitivity analyses." It appears, however, that the indeterminate conclusions as to the impacts of infinitesimal variations in the parameters on the endogenous economic variables must be expected with much greater frequency than one might normally be used to. This, however, may just be part of a lesson of broader significance concerning the status of economic theories.

REFERENCES

Bräuninger, J. (1981) A globally convergent version of Robinson's algorithm for general nonlinear programming problems without using derivatives. *Journal of Optimization Theory and Applications*, 35:195–216.

Diewert, W.E. (1974) Applications of duality theory. In M.D. Intriligator and D.A. Kendrick (Eds.), *Frontiers of Quantitative Economics*, vol. II. Amsterdam, North-Holland.

Neary, J.P. and K.W.S. Roberts (1980) Household behaviour under rationing. *European Economic Review*, 13:25–42.

Neary, J.P. and J.E. Stiglitz (1983) Towards a reconstruction of Keynesian economics: expectations and constrained equilibria. *Quarterly Journal of Economics*, 98:199–228.

Podkaminer, L. (1983) *Evaluation of Demand Under Rationing When Only Indirect Utility Functions are Known: A Note*. Collaborative Paper CP-83-16. Laxenburg, Austria: International Institute for Applied Systems Analysis.

Pollak, R.A. (1969) Conditional demand functions and consumption theory. *Quarterly Journal of Economics*, 83:60–78.

Tobin, J. and H.S. Houthakker (1950–51) The effects of rationing on demand elasticities. *Review of Economic Studies*, 18:140–153.