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Hazard Rates and Probability Distributions: Representation of Random Intensities

Yashin, A.I.

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A.I. Yashin

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INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
2361 Laxenburg, Austria

PREFACE

Recent attempts to apply the results of martingale theory in probability theory have shown that it is first necessary to interpret this abstract mathematical theory in more conventional terms. One example of this is the need to obtain a representation of the dual predictable projections (compensators) used in martingale theory in terms of probability distributions. However, up to now a representation of this type has been derived only for one special case.

In this paper, the author gives probabilistic representations of the dual predictable projection of integer-valued random measures that correspond to jumps in a semimartingale with respect to the σ-algebras generated by this process. The results are of practical importance because such dual predictable projections are usually interpreted as random intensities or hazard rates related to jumps in trajectories: applications are found in such fields as mathematical demography and risk analysis.

ANDRZEJ WIERZBICKI
Chairman
System and Decision Sciences

HAZARD RATES AND PROBABILITY DISTRIBUTIONS: REPRESENTATION OF RANDOM INTENSITIES

A.I. Yashin

International Institute for Applied Systems Analysis, A-2361 Laxenburg, Austria

1. INTRODUCTION

The development of the martingale approach in the theory of random processes has made it possible to formalize and then generalize many of the intuitive notions commonly used in applied fields. One of these is concerned with the concepts of ***hazard*** and ***hazard rate***.

The term ***hazard rate*** is usually associated with the probability of occurrence of some unexpected event or series of such events. This notion, which is popular in risk analysis, corresponds to the idea of a ***compensator*** or ***dual predictable projection*** in martingale theory [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. Many important results from this theory are formulated in terms of compensators: these include convergence of the parameter estimators and conditions for absolute continuity and singularity of the probabilistic measures [5, 6].

Probabilistic representation of the compensators provides a bridge between theory and applications. This paper is concerned with a generalization of Jacod's important result [1] in this area.

2. BASIC NOTATION AND DEFINITIONS

Let $(\Omega, H, \mathbf{H}, \mathbf{P})$ be a probabilistic space, where $\mathbf{H} = (H_t)_{t \geq 0}$ is some non-decreasing right-continuous family of σ-algebras, $H = H_\infty$, and σ-algebra H_0 is completed by $\mathbf{P}$-zero sets from H.

A real-valued random process Y_t, $t \geq 0$, is said to be $\mathbf{H}$-*adapted* if for any $u \geq 0$ random variable Y_u is H_u-measurable.

A non-negative random variable $\mathbf{T}$ is called the $\mathbf{H}$-*stopping time* if the indicator process $Y_t = \mathbf{I}(\mathbf{T} \leq t)$, $t \geq 0$, is $\mathbf{H}$-adapted. We will use the notation $\mathbf{T} \Delta \mathbf{S}$ to describe the stopping time $\tau = \min(\mathbf{T}, \mathbf{S})$.

For any H-stopping time $\mathbf{T}$ there exists a σ-algebra $H_{\mathbf{T}}$ in Ω, generated by events $\mathbf{A}$ from H such that for any $t \geq 0$ we have $\mathbf{A} \cap \{\mathbf{T} \leq t\} \in H_t$.

The H-adapted process m_t is called an H-*martingale* if $\mathbf{E}|m_t| \leq \infty$ for any $t \geq 0$ and $\mathbf{E}(m_t | H_u) = m_u$ for any $t \geq u \geq 0$.

A real-valued H-adapted process is a *local* H-*martingale* if there exists a sequence of H-stopping times $(\mathbf{T}_n)_{n \geq 0}$ such that $\lim_{n \to \infty} \mathbf{T}_n = \infty$ and for any $n \geq 0$ the processes $m_{t \Delta \mathbf{T}_n}$, $t \geq 0$, are uniformly integrable martingales.

A real-valued process Y_t is H-*well-measurable* if the mapping $(\omega, t) \to Y_t$ is measurable with respect to the σ-algebra $\mathbf{W}(\mathrm{H})$ in $\Omega \times (0, \infty)$ generated by all H-adapted, right-continuous processes.

A real-valued process Y_t is H-*predictable* if the mapping $(\omega, t) \to Y_t(\omega)$ is measurable with respect to the σ-algebra $\Pi(\mathrm{H})$ in $\Omega \times (0, \infty)$ generated by all H-adapted, left-continuous processes.

A stopping time $\mathbf{T}$ is said to be H-*predictable* if the process $Y_t = \mathbf{I}(\mathbf{T} \leq t)$, $t \geq 0$, is H-predictable.

The H-adapted process Y_t, $t \geq 0$, is an H-*semimartingale* if it may be represented in the form:

$$Y_t = A_t + M_t, \quad t \geq 0 \ ,$$

where A_t, $t \geq 0$, is a locally integrable variation process and M_t is an H-adapted local martingale.

We shall let $(E_\Delta, \mathbf{B}(E_\Delta))$ denote the measurable space such that $E_\Delta = E \cup \Delta$, where Δ is some auxiliary point, $\mathbf{B}(E_\Delta) = \mathbf{B}(E) \cup \{\Delta\}$, E is Lusin space and $\mathbf{B}(E)$ is the Borelian σ-algebra on E.

We will use the term *random measure* to describe the non-negative transition measure $\eta(\omega; \mathrm{d}t, \mathrm{d}x)$ from (Ω, H) over $(0, \infty) \times E_\Delta$.

Let $\overline{\Pi}(\mathrm{H})$ denote the σ-algebra in $\Omega \times (0, \infty) \times E$ defined by:

$$\overline{\Pi}(\mathrm{H}) = \Pi(\mathrm{H}) \otimes \mathbf{B}(E) \ .$$

A random measure η is called H-*predictable* if for each non-negative $\overline{\Pi}(\mathrm{H})$-measurable function $\mathbf{X}$ the process $(\eta \mathbf{X})_t(\omega)$, $t \geq 0$, defined by

$$(\eta \mathbf{X})_t(\omega) = \int_E \int_0^t \mathbf{X}(\omega, u, x)\, \eta(\omega; \mathrm{d}u, \mathrm{d}x)$$

is H-predictable.

Hereafter we will omit the symbol ω for simplicity.

We will also use the notation $G \vee F$ to describe the σ-algebra in Ω generated by sets from σ-algebras G and F.

3. JACOD'S REPRESENTATION RESULT

Jacod's formula for the random intensity function deals with the case in which environmental factors are random variables and consequently do not change over time. The general process whose intensity is of interest is a sequence of random times and random variables called a multivariate point process.

Some additional formal constructions will be useful in deriving the representation of the random intensity in this particular case.

3.1. Multivariate point processes

According to [1], a multivariate point process is a sequence $(\mathbf{T}_n, \mathbf{Z}_n)_{n\geq 0}$, where the $\mathbf{T}_n$ are H-stopping times and the $\mathbf{Z}_n$ are $H_{\mathbf{T}_n}$-measurable random variables with values in $(E_\Delta, \mathbf{B}(E_\Delta))$. Note that $\mathbf{Z}_n = \Delta$ if and only if $\mathbf{T}_n = \infty$, and that the stopping times $\mathbf{T}_n$ have the following properties :

(i) $\mathbf{T}_1 > 0$,

(ii) $\mathbf{T}_{n+1} > \mathbf{T}_n$, if $\mathbf{T}_n < \infty$,

(iii) $\mathbf{T}_{n+1} = \mathbf{T}_n$, if $\mathbf{T}_n = \infty$.

It follows from these assumptions that sequence $(\mathbf{T}_n)_{n\geq 0}$ has a unique accumulation point $\mathbf{T}_\infty = \lim_{n\to\infty} \mathbf{T}_n < \infty$. We will assume that $\mathbf{T}_\infty = \infty$, $\mathbf{T}_0 = 0$.

A sequence of stopping times $(\mathbf{T}_n)_{n\geq 0}$ satisfying conditions (i)–(iii) is called a univariate point process or simple point process. Any arbitrary discrete-time random process is naturally also a multivariate point process.

A multivariate point process is uniquely characterized by the integer-valued random measure μ on $(0,\infty) \times E$ defined by the equality:

$$\mu((0,t], \Gamma) = \sum_{n\geq 1} \mathbf{I}(\mathbf{T}_n \leq t)\, \mathbf{I}(\mathbf{Z}_n \in \Gamma), \ \ \Gamma \in \mathbf{B}(E), \ \ t \geq 0 \ .$$

In the rest of this paper we shall use μ to denote the integer-valued random measure generated by some multivariate point process $(\mathbf{T}_n, \mathbf{Z}_n)_{n \ge 0}$.

3.2. Dual predictable projections of integer-valued random measures

We shall define H_t^μ as a σ-algebra in Ω generated by the multivariate point process or, equivalently, by the integer-valued random measure μ up to time t:

$$H_t^\mu = \sigma\{\mu((0,u],\Gamma),\quad u \le t,\quad \Gamma \in \mathbf{B}(E)\}$$

and let $\overline{H}_0$ be some fixed σ-algebra in Ω. Denote by $\mathbf{H}_0^\mu$ the non-decreasing family of σ-algebras

$$\mathbf{H}_0^\mu = (H_{0,t}^\mu)_{t \ge 0} \ ,$$

where

$$H_{0,t}^\mu = \overline{H}_0 \vee H_t^\mu \tag{1}$$

are σ-algebras in Ω generated by the union of $\overline{H}_0$ and H_t^μ, $t \ge 0$. The family $\mathbf{H}_0^\mu$ is known to be right-continuous [2].

According to [1], there is one and only one (up to a modification on a $\mathbf{P}$-null set) $\mathbf{H}_0^\mu$-predictable random measure ν_0 on $(0,\infty) \times E$ such that for each non-negative $\Pi(\mathbf{H}_0^\mu)$-measurable function $\mathbf{X}$ we have

$$\mathbf{E} \int_0^t \int_E \mathbf{X}(u,x)\, \mu(\mathrm{d}u,\mathrm{d}x) = \mathbf{E} \int_0^t \int_E \mathbf{X}(u,x)\, \nu_0(\mathrm{d}u,\mathrm{d}x) \ .$$

Measure ν_0 is called the dual $\mathbf{H}_0^\mu$-predictable projection of μ. It turns out that one can choose a version of ν_0 which $\mathbf{P}$-a.s. satisfies the inequality:

$$\nu_0(\{t\},E) \le 1,\quad t \ge 0 \ . \tag{2}$$

We shall use the following equivalent formulation of the above result in this paper:

The random measure ν_0 is characterized by (2) and

(i) the process $\nu_0((0,t],\Gamma)$, $t \ge 0$, is $\mathbf{H}_0^\mu$-predictable for any $\Gamma \in \mathbf{B}(E)$

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$$\mathbf{E}\int_0^t\int_E \mathbf{X}(u,x)\,\mu(\mathrm{d}u,\mathrm{d}x) = \mathbf{E}\int_0^t\int_E \mathbf{X}(u,x)\,\nu_0(\mathrm{d}u,\mathrm{d}x) \ .$$

Measure ν_0 is called the dual $\mathbf{H}_0^\mu$-predictable projection of μ. It turns out that one can choose a version of ν_0 which $\mathbf{P}$-a.s. satisfies the inequality:

$$\nu_0(\{t\},E) \leq 1,\quad t \geq 0 \ . \tag{2}$$

We shall use the following equivalent formulation of the above result in this paper:

The random measure ν_0 is characterized by (2) and

(i) the process $\nu_0((0,t],\Gamma)$, $t \geq 0$, is $\mathbf{H}_0^\mu$-predictable for any $\Gamma \in \mathbf{B}(E)$

(ii) the process $(\mu((0,t],\Gamma) - \nu_0((0,t],\Gamma))$, $t \geq 0$, is an $\mathbf{H}_0^{\mu}$-adapted local martingale.

Dual predictable projections of integer-valued random measures may be interpreted as generalized cumulative random intensity functions.

Remark. Notice here that σ-algebra $\overline{H}_0$ is not necessarily formed by events independent of H_t. For instance, $\overline{H}_0$ could be the σ-algebra in Ω corresponding to the past history of some random process up to time 0. We shall consider the σ-algebra $\overline{H}_0$ generated by a Wiener process up to time ∞.

3.3. Probabilistic representation of random intensity functions

In some senses, Jacod's representation of dual $\mathbf{H}_0^{\mu}$-predictable projections serves as a bridge between the abstract theory of random processes with jumps currently developing in the framework of the martingale approach, and the wide range of applications based largely on the knowledge of probabilistic distribution functions.

In order to express Jacod's result, we have to define regular versions of the $H^{\mu}_{0,\mathbf{T}_n}$-conditional probabilities of events $\{\mathbf{T}_{n+1} \leq u\} \cap \{\mathbf{Z}_{n+1} \in \Gamma\}$, $\{\mathbf{T}_{n+1} \geq u\}$, where $u \geq 0$, $\Gamma \in \mathbf{B}(E)$ and $n = 1,2,\ldots$. This may be done using equalities

$$\mathbf{P}(\mathbf{T}_{n+1} \leq u,\ \mathbf{Z}_{n+1} \in \Gamma \mid H^{\mu}_{0,\mathbf{T}_n}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \leq u)\,\mathbf{I}(\mathbf{Z}_{n+1} \in \Gamma) \mid H^{\mu}_{0,\mathbf{T}_n})$$

$$\mathbf{P}(\mathbf{I}(\mathbf{T}_{n+1} \geq u) \mid H^{\mu}_{0,\mathbf{T}_n}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \geq u) \mid H^{\mu}_{0,\mathbf{T}_n}) .$$

It should be emphasized once again that in this part of the paper we are considering the case in which additional information about events and variables influencing the multivariate point process perceived by the statistician (observer) does not change over time.

The following theorem was proved in [1].

Theorem 1. *The following is a representation of a dual* $\mathbf{H}_0^{\mu}$*-predictable projection* ν_0 *of measure* μ*:*

$$\nu_0((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \leq \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1} \Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \leq u,\ \mathbf{Z}_{p+1} \in \Gamma \mid H^{\mu}_{0,\mathbf{T}_p})}{\mathbf{P}(\mathbf{T}_{p+1} \geq u \mid H^{\mu}_{0,\mathbf{T}_p})} . \tag{3}$$

Corollary. Notice that if the σ-algebra $\overline{H}_0$ does not provide the observer with any information about the events, and if all he has at time t is information about events from the history H_t^{μ}, the hazard rate coincides with the dual $\mathbf{H}^{\mu}$-predictable projection ν of the measure μ and the formula for ν is a simple corollary of equation (3):

$$\nu((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \le \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \le u,\ \mathbf{Z}_{p+1} \in \Gamma \mid H_{\mathbf{T}_p}^{\mu})}{\mathbf{P}(\mathbf{T}_{p+1} \ge u \mid H_{\mathbf{T}_p}^{\mu})} . \qquad (3')$$

In the case of a pure point process (a sequence of random times $\mathbf{T}_n$), the formula for the dual $\mathbf{H}^{\mu}$-predictable compensator $A(t)$ becomes:

$$A(t) = \sum_{n=0}^{\infty} \mathbf{I}(\mathbf{T}_n < t \le \mathbf{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \le u \mid H_{\mathbf{T}_p}^{\mathbf{N}})}{\mathbf{P}(\mathbf{T}_{p+1} \ge u \mid H_{\mathbf{T}_p}^{\mathbf{N}})} , \qquad (3'')$$

where the σ-algebras $H_t^{\mathbf{N}}$, $t \ge 0$, are generated by the values of the point process $(\mathbf{T}_n)_{n\ge 0}$ or, equivalently, by the values of the counting process $\mathbf{N}_u$, $u \ge 0$, up to time t.

Equations (3′) and (3′′) produce known results when applied to well-studied processes. Thus, for a Poisson process with deterministic local intensity function $\lambda(t)$, the dual predictable projection $\Lambda(t)$ defined by equation (3′′) coincides with the cumulative hazard rate and is given by the equality

$$\Lambda(t) = \int_0^t \lambda(u)\mathrm{d}u \ .$$

For a finite-state, continuous-time, Markov process ξ_t , $t \ge 0$, with states $\{1,2,\dots,N\}$ and intensities $\lambda_{ij}(t)$, $i,j = (1,2,\dots,N)$, $t \ge 0$, equation (3′) gives the dual predictable projection in the form:

$$\nu_0((0,t],\Gamma) = \int_0^t \sum_{j\in\Gamma} \lambda_{\xi_{u-},\xi_{u-}+j}\ \mathrm{d}u \ .$$

Now assume that the observer has some additional information about the intensity function, for instance, that he knows the value of some random variable $\mathbf{Z}$ which influences the frequency of the jumps. This means that he is dealing with the history $H_{0,t}^{\mu}$ as determined by the equality (1), where σ-algebra $\overline{H}_0$ coincides with the σ-algebra $\sigma(\mathbf{Z})$ generated by random variable $\mathbf{Z}$ in Ω . In this case the observer needs to use the dual $\mathbf{H}_0^{\mu}$-predictable projection of μ as a random intensity function. Thus, in the case of a double stochastic Poisson process $\mathbf{N}_t$, $t \ge 0$, with random intensity function $\mathbf{Z}\lambda(t)$, $t \ge 0$ (where $\mathbf{Z}$ is some positive random variable), it is necessary to use eqn. (3) which gives the following dual $\mathbf{H}_0^{\mu}$-predictable projection of $\mathbf{N}_t$, $t \geq 0$:

$$\Lambda(t,\mathbf{Z}) = \mathbf{Z}\int_0^t \lambda(u)\mathrm{d}u \quad .$$

Note that if two observers have different information about the processes occurring in some real system (for example, if one of them knows the value of the variable $\mathbf{Z}$ and the other does not), they will use different hazard rates to estimate the probability of change. In the case of continuously distributed jump-times in a double stochastic Poisson process, the relation between the two intensities (derived from a comparison between eqns. (3′′) and (3′)) may be represented as follows:

$$\bar{\lambda}(t) = \mathbf{E}(\mathbf{Z}|\, H_t^{\mathbf{N}})\, \lambda(t) \quad , \tag{4}$$

where $\bar{\lambda}(t)$ is the $\mathbf{H}^{\mathbf{N}}$-predictable local hazard rate. This is related to the dual $\mathbf{H}^{\mathbf{N}}$-predictable projection $\bar{\Lambda}(t)$ of $\mathbf{N}_t$ by the equality:

$$\bar{\Lambda}(t) = \int_0^t \bar{\lambda}(u)\,\mathrm{d}u \quad .$$

We shall now prove the relation between $\bar{\lambda}(t)$ and $\lambda(t)$. Consider the integral $G_n(t)$ defined by the equality:

$$G_n(t) = \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathbf{T}_{p+1} \leq u \mid H_{\mathbf{T}_p}^{\mathbf{N}})}{\mathbf{P}(\mathbf{T}_{p+1} \geq u \mid H_{\mathbf{T}_p}^{\mathbf{N}})} \quad ,$$

which is taken from the right-hand side of eqn. (3′′). Let $\varphi_p(u)$ be the density function of conditional distribution $\mathbf{P}(\mathbf{T}_{p+1} \leq u \mid H_{\mathbf{T}_p}^{\mathbf{N}})$. Note that the following equality holds:

$$\varphi_n(u) = \int_0^{\infty} z\,\lambda(u)\, \exp\{-z\int_{\mathbf{T}_n}^{u} \lambda(v)\mathrm{d}v\}\, f(z)\mathrm{d}z \quad ,$$

where $f(z)$ is the density distribution function of random variable $\mathbf{Z}$. Using this equality and noting that for $\mathbf{T}_p \leq u$

$$\exp\{-z\int_{\mathbf{T}_p}^{u} \lambda(v)\mathrm{d}v\} = \mathbf{P}(\mathbf{T}_{p+1} > u \mid H_{\mathbf{T}_p}^{\mathbf{N}}, \mathbf{Z} = z) \quad ,$$

we have

$$G_p(t) = \int_{T_p \Delta t}^{T_{p+1}\Delta t} \mathbf{E}(\mathbf{Z} \mid H^{\mathbf{N}}_{T_p}, T_{p+1} > u)\lambda(u)\,du \quad .$$

Since

$$\mathbf{I}(T_p \leq u < T_{p+1})\, \mathbf{E}(\mathbf{Z} \mid H^{\mathbf{N}}_{T_p}, T_{p+1} > u) = \mathbf{I}(T_p \leq u < T_{p+1})\, \mathbf{E}(\mathbf{Z} \mid H^{\mathbf{N}}_u) \quad ,$$

we can derive the formula for the $\overline{\Lambda}(t)$ from eqn. (3′′).

Remark. Equation (4) shows that to calculate intensity functions $\overline{\Lambda}(t)$ or $\overline{\lambda}(t)$ it is first necessary to calculate the $H^{\mathbf{N}}_t$-conditional mathematical expectation. This problem can be overcome by using an approach based on filtering of the jumping processes (see, for instance, [11, 12]). In the simplest life-cycle models, which are characterized only by stopping time $\mathbf{T}$ (time of death) and are widely used in reliability and demographic analysis, the random intensity describes differences in susceptibility to death or failure [13]. Notice that in the case of life-cycle processes, equation (3′′) gives the following relation between the $\mathbf{H}$-adapted compensator $A(t)$ and the local intensity function $\lambda(t)$:

$$A(t) = \int_0^{t\Delta \mathbf{T}} \lambda(u)\,du \quad .$$

Recall that, from the definition of the compensator, the process

$$M(t) = \mathbf{I}(\mathbf{T} \leq t) - A(t), \quad t \geq 0 \ ,$$

is an H-adapted martingale.

4. GENERAL FORMULA FOR REPRESENTATION OF RANDOM INTENSITY

In spite of the fact that dual predictable projections exist for a wide class of families of σ-algebras, probabilistic representations are known only for σ-algebras with structure (1). However, in practical situations σ-algebras often have a more general structure. In particular, new information may be generated not only by the multivariant point process but also by some additional process η_t. In this case the σ-algebras describing the observation history have the form $H_t = H^{\eta}_t \vee H^{\mu}_t$, where σ-algebras H^{ξ}_t are generated by some process η_t which is observed simultaneously with the multivariate point process $(\mathbf{T}_n, \mathbf{Z}_n)_{n\geq 0}$. Detailed probabilistic characterization of dual predictable projections is often useful when applying the results of the general theory of random processes in practice.

In this section we will give the probabilistic representation of the dual predictable projection of integer-valued random measures corresponding to the jumps of the semimartingale with respect to the family of σ-algebras generated by this process.

4.1. Generation of jumps

Let random process X_t , $t \geq 0$, defined on probability space $(\Omega, H, \mathbf{H}, \mathbf{P})$ be $\mathbf{H}$-adapted, take values in $\mathbf{R}^k$ and have right-continuous, left-limited sampling paths. Denote by $\mathbf{H}^x$ the family of σ-algebras

$$H_t^x = \bigcap_{r>t} \sigma\{X_u, u \leq r\}$$

and assume that H_0^x is completed by the sets from $H^x = H_\infty^x$ with a $\mathbf{P}$-probability of zero. Assume also that process X_t may be represented as follows:

$$X_t = X_0 + \int_0^t A_u \, du + \int_0^t B_u \, dw_u + \int_0^t \int_E x \mu(du, dx) \ , \tag{5}$$

where A_u and B_u are $\mathbf{H}^x$-adapted matrices of appropriate dimensions, matrix B_u is non-singular for any $u \geq 0$, $E = \mathbf{R}^k - \{0\}$, and w_u is an $\mathbf{H}$-adapted, k-dimensional Wiener process independent of X_0.

Note that, in general, the dimension of X_t may be greater than that of the jumping changes. This can mean, for instance, that we are also considering situations with two-component processes of which one is pure jumping and the other is continuous.

It follows from (3) that the measure μ is $\mathbf{H}^x$-adapted and that

$$Z_n = X_{T_n} - X_{T_n -} \ .$$

Let ν^x denote the dual $\mathbf{H}^x$-predictable projection of integer-valued random measure μ. Our main aim is to derive the probabilistic representation of measure ν^x.

4.2. The form of the hazard rate

The main result of this subsection is formulated in terms of auxiliary processes $X_{n,t}$ defined by the equalities:

$$X_{n,t} = X_0 + \int_0^t A_u \mathrm{d}u + \int_0^t B_u \mathrm{d}w_u + \int_0^{\mathrm{T}_n \Delta t} \int_E x \mu(\mathrm{d}u, \mathrm{d}x) \ , \tag{6}$$

where A, B, w, μ have been defined previously. Introducing the measures μ_n defined by the equalities

$$\mu_n((0,t], \Gamma) = \sum_k \mathrm{I}(\mathrm{T}_k \le \mathrm{T}_n \Delta t) \mathrm{I}(\mathrm{Z}_n \in \Gamma), \Gamma \in \mathbf{B}(E), t \ge 0, n \ge 0 \ ,$$

eqn. (6) may be rewritten as follows:

$$X_{n,t} = X_0 + \int_0^t A_u \mathrm{d}u + \int_0^t B_u \mathrm{d}w_u + \int_0^t \int_E x \mu_n(\mathrm{d}u, \mathrm{d}x) \ .$$

Notice that measure μ_n may be considered as a measure of the jumps of the process $X_{n,t}$, $t \ge 0$.

Introduce σ-algebras $H_t^{x_n}$ and $H_t^{\mu_n}$ such that:

$$H_t^{x_n} = \bigcap_{u>t} \sigma\{X_{n,r}, r \le u\} \ ,$$

$$H_t^{\mu_n} = \sigma\{\mu_n((0,t], \Gamma), u \le t, \Gamma \in \mathbf{B}(E)\} \ ,$$

and define the regular versions of the conditional probabilities of events

$$\{\mathrm{T}_{p+1} \le u\} \cap \{\mathrm{Z}_{p+1} \in \Gamma\}, \quad \{\mathrm{T}_{p+1} \ge u\}, \quad u > 0, \Gamma \in \mathbf{B}(E)$$

using the equalities

$$\mathbf{P}(\mathrm{T}_{p+1} \le u, \mathrm{Z}_{p+1} \in \Gamma | H_u^{x_p}) = \mathbf{E}(\mathrm{I}(\mathrm{T}_{p+1} \le u) \mathrm{I}(\mathrm{Z}_{p+1} \in \Gamma) | H_u^{x_p})$$

$$\mathbf{P}(\mathrm{T}_{p+1} \ge u | H_u^{x_p}) = \mathbf{E}(\mathrm{I}(\mathrm{T}_{p+1} \ge u) | H_u^{x_p}) \ .$$

The next assertion is the main result of this paper.

Theorem 2. *Assume that coefficients A and B are such that a strong solution of equation* (3) *exists and is unique. Then we have the following representation of the dual* $\mathbf{H}^x$*-predictable projection* ν^x *of integer-valued random measure* μ*:*

$$\nu^x((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathrm{I}(\mathrm{T}_n < t \le \mathrm{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathrm{T}_p \Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathrm{P}(\mathrm{T}_{p+1} \le u,\, \mathrm{Z}_{p+1} \in \Gamma \mid H_u^{x_p})}{\mathrm{P}(\mathrm{T}_{p+1} \ge u \mid H_u^{x_p})} \quad . \tag{7}$$

This theorem is proved in the Appendix.

Sometimes it is more convenient to use another form of representation for ν^x, transforming the conditional probabilities on the right-hand side of (7). For this purpose we introduce the function $F(u,\Gamma)$, making use of the equality

$$F(u,\Gamma) = \mathrm{P}(\mathrm{Z}_{n+1} \in \Gamma \mid H^x_{\mathrm{T}_{n+1}-})\,|_{\mathrm{T}_{n+1}=u},\ \Gamma \in \mathbf{B}(E),\ u \ge 0,\ n \ge 0 \ ,$$

where $\mathrm{P}(\mathrm{Z}_{n+1} \in \Gamma \mid H^x_{\mathrm{T}_{n+1}-})$ is the regular version of the $H^x_{\mathrm{T}_{n+1}-}$-conditional probability of event $\{\mathrm{Z}_{n+1} \in \Gamma\}$. The dual $\mathbf{H}^x$-predictable projection ν^x of measure μ may then be represented in terms of this function as follows:

$$\nu^x((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathrm{I}(\mathrm{T}_n < t \le \mathrm{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathrm{T}_p \Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{F(u,\Gamma)\,\mathrm{d}_u \mathrm{P}(\mathrm{T}_{p+1} \le u \mid H_u^{x_p})}{\mathrm{P}(\mathrm{T}_{p+1} \ge u \mid H_u^{x_p})} \ .$$

5. EXAMPLES

5.1. Conditional Gaussian property

Let process $Y(t)$, $t \ge 0$, satisfy the linear stochastic differential equation

$$\mathrm{d}Y(t) = a_0(t) + a_1(t)\,Y(t)\,\mathrm{d}t + b(t)\,\mathrm{d}w(t),\ Y(0) = Y_0 \ , \tag{8}$$

where Y_0 is a Gaussian random variable with mean m_0 and variance γ_0, $w(t)$ is an $\mathbf{H}$-adapted Wiener process, $\mathbf{H} = (H_t)_{t\ge 0}$ is some non-decreasing, right-continuous family of σ-algebras, and H_0 is completed by $\mathbf{P}$-zero sets from $H = H_\infty$. Denote by $\mathbf{H}^y$ the family of σ-algebras in Ω generated by the values of the random process $Y(u)$, i.e.,

$$\mathbf{H}^y = (H_t^y)_{t\ge 0},\quad H_t^y = \bigcap_{u>t} \sigma\{Y(v),\ v \le u\},\quad t \ge 0 \ .$$

Assume that process $Y(t)$ determines the random rate of occurrence of some unexpected event characterized by the random time $\mathbf{T}$, through the equality:

$$\mathbf{P}(\mathbf{T} > t \mid H_t^y) = \exp\{-\int_0^t Y^2(u)\,\lambda(u)\,\mathrm{d}u\} \ . \tag{9}$$

Notice that process $\mathbf{Z}(u) = Y^2(u)$, $u \geq 0$, may be interpreted as the frailty of an individual changing stochastically over time. Using the terminology of martingale theory one can say that the process

$$A(t) = \int_0^{t \Delta \mathbf{T}} \lambda(u)\, Y^2(u)\, \mathrm{d}u$$

is an $\mathbf{H}^y$-predictable compensator of the life-cycle process $X_t = \mathbf{I}(\mathbf{T} < t)$, $t \geq 0$. This means that the process $M_t = \mathbf{I}(\mathbf{T} < t) - A(t)$, $t \geq 0$ is an $\mathbf{H}^y$-adapted martingale. Associating the stopping time $\mathbf{T}$ with the time of death, we may describe the process $Y^2(t)$, $t \geq 0$, as the age-specific mortality rate for an individual with history $Y_0^t = \{Y(u)\}$, $0 \leq u \leq t$.

Letting $\bar{\lambda}(t)$, $t \geq 0$, denote the observed age-specific mortality rate we have $\bar{\lambda}(t) = \lambda(t)\, \bar{\mathbf{Z}}(t)$, $t \geq 0$, where $\bar{\mathbf{Z}}(t) = \mathbf{E}\{ Y^2(t) \mid \mathbf{T} > t \}$ [13].

In order to calculate the observed mortality rate $\bar{\lambda}(t)$, $t \geq 0$, it is necessary first to calculate the second moment of the conditional distribution of the $Y(u)$ given the event $\{\mathbf{T} \geq 0\}$. It turns out that this moment may be calculated quite easily using the result of the following theorem.

Theorem 3. *Assume that process $Y(t)$ and stopping time $\mathbf{T}$ are related through eqns.* (8) *and* (9). *Then the conditional distribution of $Y(t)$ given $\{\mathbf{T} \geq t\}$ is Gaussian. The parameters of this distribution, i.e., the mean m_t and the variance γ_t, are given by the following equations:*

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} = a_0(t) + a_1(t)\, m_t - 2 m_t \gamma_t \lambda(t), \quad m_0 \ . \tag{10}$$

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = 2 a_1(t)\, \gamma_t + b^2(t) - 2\lambda(t)\, \gamma_t^2, \quad \gamma_0 \ . \tag{11}$$

The formula for $\bar{\lambda}(t)$ is then $\bar{\lambda}(t) = \lambda(t)\,(m_t^2 + \gamma_t)$.

This theorem may be proved in a similar way as the conditional Gaussian property for processes governed by stochastic differential equations of the diffusion type (see [14]).

6. APPENDIX: PROOF OF THEOREM 2

The proof uses representation (3) for ν_0. It turns out that if the σ-algebra $\overline{H}_0$ in eqn. (1) is of a particular form (which will be specified later) then the $H^{\mu}_{0,\mathbf{T}_n}$-conditional probabilities of events $\{\mathbf{T}_{p+1} \le u\} \cap \{\mathbf{Z}_{p+1} \in \Gamma\}$ and $\{\mathbf{T}_{p+1} \ge u\}$ will $\mathbf{P}$-a.s. coincide with the $H^{x_n}_u$-conditional probabilities of these events on the integration intervals in (7). Representation of measure ν_0 through $H^{x_n}_u$-conditional probabilities makes it easier to prove its $\mathbf{H}^x$-predictability property. It is then easy to check that the process

$$(\mu((0,t],\Gamma) - \nu_0((0,t],\Gamma)),\ \ t \ge 0$$

is an $\mathbf{H}^x$-adapted martingale for any $\Gamma \in \mathbf{B}(E)$. The fact that ν^x is unique shows that ν^x and ν_0 coincide $\mathbf{P}$-a.s. Representation (7) is derived from (3) through substitution of the conditional probabilities. Several auxiliary results will be useful in the proof of Theorem 2: these are derived in the following subsections.

6.1. Auxiliary σ-algebras

Introduce the auxiliary right-continuous families of σ-algebras $\mathbf{H}^w$, $\mathbf{H}^{\mu}$, $\mathbf{H}^{w,\mu}$, $\mathbf{H}^w_n$, $\mathbf{H}^{w,\mu}_0$ and $\overline{\mathbf{H}}^w$, where

$$\mathbf{H}^w = (H^w_t)_{t\ge0},\quad H^w_t = \sigma\{w_u\,,\, u \le t\}\vee\sigma(X_0),\ \ H^w = H^w_\infty\ ,$$

$$\mathbf{H}^{\mu} = (H^{\mu}_t)_{t\ge0},\quad H^{\mu}_t = \sigma\{\mu((0,u],\Gamma)\,,\, u \le t\,,\, \Gamma \in \mathbf{B}(E)\},\ \ H^{\mu} = H^{\mu}_\infty\ ,$$

$$\mathbf{H}^{w,\mu} = (H^{w,\mu}_t)_{t\ge0},\quad H^{w,\mu}_t = H^w_t \vee H^{\mu}_t\ ,$$

$$\mathbf{H}^w_n = (H^w_{n,t})_{t\ge0},\quad H^w_{n,t} = H^w_t \vee H^{\mu}_{\mathbf{T}_n} = H^w_t \vee H^{\mu_n}_\infty\ ,$$

$$\mathbf{H}^{w,\mu}_0 = (H^{w,\mu}_{0,t})_{t\ge0},\quad H^{w,\mu}_{0,t} = H^w \vee H^{\mu}_t\ ,$$

$$\overline{\mathbf{H}}^w = (\overline{H}^w_t)_{t\ge0},\quad \overline{H}^w_t = \overline{H} \vee H^w_t\ ,$$

where $\overline{H}$ is some σ-algebra in Ω and σ-algebras H^w_0 and H^{μ}_0 are completed by $\mathbf{P}$-zero sets from σ-algebras H^w and H^{μ}, respectively.

Recall also that family $\mathbf{H}^x$ is defined as follows:

$$\mathbf{H}^x = (H^x_t)_{t\ge0},\quad H^x_t = \bigcap_{r>t} \sigma\{X_u, u \le r\}\ .$$

6.2. Existence of H^x-predictable projections

We shall now establish that dual H^x-predictable projections exist and are unique.

Lemma 1. *The dual H^x-predictable projections ν^x of the integer-valued random measure μ exist and are unique.*

Proof. Note that the sets $[0]\ E\ ]\mathrm{T}_n, \mathrm{T}_{n+1}]\ E$ belong to $\overline{\Pi}$ and have measure

$$M_\mu(\mathrm{d}\omega, \mathrm{d}u, \mathrm{d}x) = \mathrm{P}(\mathrm{d}\omega)\,\mu(\mathrm{d}u, \mathrm{d}x)$$

less than or equal to 1. This means that measure M_μ is σ-finite on $((\Omega\ (0,t]\ E), \overline{\Pi})$. From [1], this implies that the lemma is true.

6.3. Characterization of $\mathrm{H}^{w,\mu}$-stopping times

For any $t \geq 0$ let

$$H^w_{t,\infty} = \sigma\{w_r - w_u,\ r \geq u \geq t\} \ .$$

The next assertion is a generalization of Lemma 3.2 in Jacod's paper [1].

Lemma 2. *Let* T *be the* $\mathrm{H}^{w,\mu}$*-stopping time. For any* $n \geq 0$ *there exists a random variable* S^n *such that indicator* $\mathrm{I}(\mathrm{S}^n \geq u)$ *is* $H^w_{n,u}$*-measurable for any* $u \geq 0$ *and the following equality holds:*

$$\mathrm{I}\{\mathrm{T} \geq u\}\,\mathrm{I}\{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\} = \mathrm{I}\{\mathrm{S}^n \geq u\}\,\mathrm{I}\{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\} \ .$$

Proof. It follows from the definition of the σ-algebra H^μ_u that the following families of sets coincide:

$$H^\mu_{u-} \cap \{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\} = H^\mu_{\mathrm{T}_n} \cap \{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\}$$

and consequently the families of sets

$$(H^w_u \vee H^\mu_{u-}) \cap \{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\} = H^w_u \vee H^\mu_{\mathrm{T}_n} \cap \{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\}$$

also coincide. Take the set $\{\mathrm{T} < u\}$ from $H^w_u \vee H^\mu_{u-}$, and find the set D_u from $H^w_u \vee H^\mu_{\mathrm{T}_n}$ such that

$$\{\mathrm{T} < u\} \cap \{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\} = \mathrm{D}_u \cap \{\mathrm{T}_n < u \leq \mathrm{T}_{n+1}\} \ .$$

Note that for $r < u$ we now have

$$[(\mathbf{D}_r \cap \{\mathbf{T}_n < r\}) \cup (\mathbf{D}_u \cap \{\mathbf{T}_n < u\})] \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} = \mathbf{D}_u \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} .$$

Define $\mathbf{S}^n$ by the equalities

$$\{\mathbf{S}^n < u\} = \bigcup_{r \le u} (\mathbf{D}_r \cap \{\mathbf{T}_n < r\}) ,$$

where the r are rational numbers. We then obtain

$$\{\mathbf{T} < u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} = \{\mathbf{S}^n < u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+l}\}$$

or

$$\{\mathbf{T} \ge u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} = \{\mathbf{S}^n \ge u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} .$$

thus completing the proof of Lemma 2.

6.4. Representation of martingales

The following result plays a fundamental role in the analysis of the predictability property.

Lemma 3. *Let $\mathbf{Z}_t$ be a right-continuous, left-limited, square-integrable, $\overline{\mathbf{H}}^w$-martingale process. Then an $\overline{\mathbf{H}}^w$-adapted process $f(u,\omega)$, $u > 0$, exists such that*

$$\mathbf{E} \int_0^t f^2(u,\omega) \mathrm{d}u < \infty, \quad t \ge 0$$

and

$$\mathbf{Z}_t = \mathbf{Z}_0 + \int_0^t f(u,\omega) \mathrm{d}w_u .$$

The proof of this lemma is similar to that of Theorem 5.5 in [14]. The following well-known result is important in the proof of some auxiliary assertions.

Lemma 4. *Let $\mathbf{L}$ be some vector space of bounded real functions defined on Ω. Assume that it contains the constant 1, is closed with respect to uniform convergence, and is such that for any uniformly bounded increasing sequence of non-negative functions f_n, $n \ge 0$, $f_n \in \mathbf{L}$, the function $f = \lim_{n \to \infty}$ also belongs to $\mathbf{L}$. Let Q be a subset of $\mathbf{L}$ which is closed with respect to multiplication. Then* *the space* L *contains all bounded functions, measured with respect to the* σ*-algebra* H *generated by the elements of* Q.

Remark. This result is known as the monotonic class theorem, and is proved in [15]. The theorem is also true if

(a) L is closed with respect to monotonic and uniform convergence and

(b) Q is the algebra and $1 \in Q$

or

(a′) L is a set of functions closed with respect to monotonic convergence to the bounded function and

(b′) Q is a vector space closed with respect to operation Λ (maximum of two functions) and $1 \in Q$.

6.5. Predictability of $\mathbf{H}_n^w$-well-measurable processes

It turns out that $\mathbf{H}_n^w$-well-measurable processes have the following remarkable property:

Lemma 5. *Let* Y_t^n *be an arbitrary* $\mathbf{H}_n^w$*-well-measurable process. Then process* $Y_t^n \mathbf{I}\{\mathbf{T}_n < t\}$ *is* $\mathbf{H}_n^w$*-predictable.*

Proof. Let $\mathbf{T}$ be an arbitrary $\mathbf{H}_n^w$-stopping time, and denote by $\lambda(t)$ the dual $\mathbf{H}_n^w$-predictable projection of non-decreasing process $\mathbf{I}(\mathbf{T} \le t)$. From the definition of $\lambda(t)$ the process $\mathbf{Z}_t = \mathbf{I}(\mathbf{T} \le t) - \lambda(t)$ is an $\mathbf{H}_n^w$-martingale.

Now consider the process $v_u = w_{\mathbf{T}_n+u} - w_{\mathbf{T}_n}$, and define $\mathbf{H}_n^v = (H_{n,t}^v)_{t\ge 0}$, where $H_{n,t}^v = \sigma\{v_u, u \le t\} \vee H_{n,T_n}^w$.

Observe that $H_{n,t}^v = H_{n,T_n+t}^w$ and consequently that family $\mathbf{H}_n^v$ coincides with family $\mathbf{H}_n^{n,w} = (H_{n,T_n+t}^w)_{t\ge 0}$. It is not difficult to check that v_u is a Wiener process with respect to $\mathbf{H}_n^v$ and that $\mathbf{Z}_t^n = \mathbf{Z}_{\mathbf{T}_n+t}$ is an $\mathbf{H}_n^{n,w}$-martingale process.

From Lemma 3 we have the following representation of $\mathbf{Z}_t^n$:

$$\mathbf{Z}_t^n = \mathbf{Z}_0^n + \int_0^t f_n(u,\omega)\mathrm{d}v_u \quad ,$$

or, in terms of $\mathbf{Z}_t$,

$$\mathbf{Z}_{\mathbf{T}_n+u} = \mathbf{Z}_{\mathbf{T}_n} + \int_{\mathbf{T}_n}^{\mathbf{T}_n+u} f_n(\mathbf{T}_n + r, \omega)\mathrm{d}w_{\mathbf{T}_n+r} \quad .$$

Taking $T_n + u = t$ we get

$$Z_t \, I(T_n < t) = \left[Z_{T_n} + \int_{T_n}^{t} f_n(u,\omega) dw_u \right] I(T_n < t) \ .$$

The right-hand side of this equality is an H_n^w-predictable process. Remembering the definition of Z_t , we deduce that the process $I(T \le t) I(T_n < t)$ is H_n^w-predictable.

The result of the lemma may then be derived from the monotonic class theorem [15].

6.6. Characterization of $H^{w,\mu}$-predictable processes

The following assertion describes the structure of $H^{w,\mu}$-predictable processes.

Lemma 6. *An $H^{w,\mu}$-adapted process Z_t is $H^{w,\mu}$-predictable if and only if, for any $n \ge 0$, there exists an H_n^w-well-measurable process Y_t^n such that*

$$Y_t^n I(T_n < t \le T_{n+1}) = Z_t \, I(T_n < t \le T_{n+1}) \ . \tag{A.1}$$

Proof

Necessity. Consider the process $Z_t = I(t \le T)$, where T is an arbitrary $H^{w,\mu}$-stopping time. It follows from Lemma 1 that

$$I(t \le T) I(T_n < t \le T_{n+1}) = I(t \le S^n) I(T_n < t \le T_{n+1}) \ ,$$

which leads to equality (B) with $Y_t^n = I(t \le S^n)$. That these conditions are necessary may be proved from the monotonic class theorem.

Sufficiency. Observe that for an arbitrary H_n^w-adapted process Y_t, the process $I\{T_n \le t\} \, Y_t$ is $H^{w,\mu}$-adapted. This is because

$$(H_t^w \vee H_{T_n}^\mu) \cap \{T_n \le t\} = (H_t^w \vee H_{t-}^\mu) \cap \{T_n \le t\}, \, n \ge 0 \ ,$$

and any arbitrary set from $(H_t^w \vee H_{t-}^\mu) \cap \{T_n \le t\}$ is $H_t^w \vee H_t^\mu$-measurable. Left-continuous H_n^w-adapted processes Y_t generate left-continuous $H^{w,\mu}$-adapted processes $I\{T_n \le t\} \, Y_t$. This means that the following inclusion is true:

$$\Pi(H_n^w) \cap]| T_n , T_{n+1}]| \subseteq \Pi(H^{w,\mu}) \ , \tag{A.2}$$

where $\Pi(\mathbf{H}_n^w)$ and $\Pi(\mathbf{H}^{w,\mu})$ are σ-algebras for $\mathbf{H}_n^w$- and $\mathbf{H}^{w,\mu}$-predictable sets respectively, and $]|\mathbf{T}_n,\mathbf{T}_{n+1}]|$ is the stochastic interval corresponding to the stopping times $\mathbf{T}_n$ and $\mathbf{T}_{n+1}$. The inclusion (A.2) yields:

$$\bigcup_n]|\mathbf{T}_n,\mathbf{T}_{n+1}]| \cap \Pi(\mathbf{H}_n^w) \subseteq \Pi(\mathbf{H}^{w,\mu}) \ . \tag{A.3}$$

From Lemma 5, the process $Y_t^n \mathbf{I}(\mathbf{T}_n \le t)$ is $\mathbf{H}_n^w$-predictable. Inclusion (A.3) shows that the process $\sum_n Y_t^n \mathbf{I}_{]|\mathbf{T}_n,\mathbf{T}_{n+1}]|}$ (which according to equality (A.1) coincides with process $\mathbf{Z}_t$) is $\mathbf{H}^{w,\mu}$-predictable. This completes the proof.

6.7. A property of conditional distributions

Let H, G, F be σ-algebras in Ω. Assume that they are complete with respect to measure $\mathbf{P}$ and such that $G \subseteq H$, $F \subseteq H$. The next statement will then be useful in analyzing the form of the dual predictable projection.

Lemma 7. *Let* $\mathbf{B} \in H$, $\mathbf{P}(\mathbf{B}) > 0$ *be such that the families of sets* $F \cap \mathbf{B}$ *and* $G \cap \mathbf{B}$ *coincide* $\mathbf{P}$-*a.s. Then for any* H-*measurable integrable random variable* η *the following equality holds:*

$$\mathbf{I}(\mathbf{B})\,\mathbf{E}(\eta \mid G \vee \mathbf{B}) = \mathbf{I}(\mathbf{B})\,\mathbf{E}(\eta \mid F \vee \mathbf{B}) \ .$$

Proof. For any $A \in H$ define the measure $\mathbf{P}^{\mathbf{B}}(A)$ as follows:

$$\mathbf{P}^{\mathbf{B}}(\mathbf{A}) = \frac{\mathbf{P}(A \cap \mathbf{B})}{\mathbf{P}(\mathbf{B})} \ .$$

Let $\mathbf{E}^{\mathbf{B}}$ denote the mathematical expectation with respect to $\mathbf{P}^{\mathbf{B}}$. The families of sets $G \cap \mathbf{B}$ and $F \cap \mathbf{B}$ form σ-algebras of the subsets of set $\mathbf{B}$ that, generally speaking, are not σ-algebras in Ω. Since these families are complete with respect to measure $\mathbf{P}^{\mathbf{B}}$, they coincide $\mathbf{P}^{\mathbf{B}}$-a.s. with the σ-algebras $G \vee \mathbf{B}$ and $F \vee \mathbf{B}$, respectively.

It follows from the conditions of the lemma that for any $\mathbf{A} \in H$ we have

$$\mathbf{P}^{\mathbf{B}}(\mathbf{A} \mid G \cap \mathbf{B}) = \mathbf{P}^{\mathbf{B}}(\mathbf{A} \mid F \cap \mathbf{B}), \quad \mathbf{P}^{\mathbf{B}}\text{-a.s.} \ ,$$

or, equivalently,

$$\mathbf{P}^{\mathbf{B}}(A \mid G \vee \mathbf{B}) = \mathbf{P}^{\mathbf{B}}(\mathbf{A} \mid F \vee \mathbf{B}), \quad \mathbf{P}^{\mathbf{B}}\text{-a.s.}$$

and thus the following equality holds $\mathbf{P}^{\mathbf{B}}$-a.s. for any bounded random variable η:

$$\mathbf{E}^{\mathbf{B}}(\eta \mid G \vee \mathbf{B}) = \mathbf{E}(\eta \mid F \vee \mathbf{B}) \ .$$

This may be rewritten in the form

$$\mathbf{I}(\mathbf{B})\ \mathbf{E}^{\mathbf{B}}(\eta \mid G \vee \mathbf{B}) = \mathbf{I}(\mathbf{B})\ \mathbf{E}^{\mathbf{B}}(\eta \mid F \vee \mathbf{B}), \quad \mathbf{P}\text{-a.s.}$$

or

$$\mathbf{I}(\mathbf{B})\ \mathbf{E}(\eta \mid G \vee \mathbf{B}) = \mathbf{I}(\mathbf{B})\, \mathbf{E}(\eta \mid F \vee \mathbf{B}), \quad \mathbf{P}\text{-a.s.} \ ,$$

thus completing the proof.

6.8. Some properties of conditional mathematical expectations

The next assertion will be useful in proving the predictable characterization of some random measures.

Lemma 8. *Let* $\mathbf{A} \in H^{\mu}_{\mathbf{T}_{n+1}}$. *Then the following equalities are true for any* $t > 0$ *:*

$$\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\, \mathbf{I}(\mathbf{A}) \mid H^{w} \vee H^{\mu}_{\mathbf{T}_n}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\, \mathbf{I}(\mathbf{A}) \mid H^{w}_{t} \vee H^{\mu}_{\mathbf{T}_n}) \qquad (\mathrm{A}.4)$$

$$\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \ge t) \mid H^{w} \vee H^{\mu}_{\mathbf{T}_n}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \ge t) \mid H^{w}_{t} \vee H^{\mu}_{\mathbf{T}_n}) \ .$$

Proof. Since B_u is non-singular for any $u \ge 0$, the process w_t may be represented as follows:

$$w_t = \int_0^t B_u^{-1}(\mathrm{d}X^c_u - A_u\, \mathrm{d}u) \ ,$$

where

$$X^c_t = X_t - \int_0^t \int_E x\, \mu(\mathrm{d}u, \mathrm{d}x) \ .$$

This shows that the process w_t is $\mathbf{H}^x$-adapted and leads to the inclusion:

$$H^{w}_{t} \vee H^{\mu}_{t-} \subseteq H^{x}_{t} \ . \qquad (\mathrm{A}.5)$$

Consider now the bounded random variables X_1, X_2, X_3 which are measurable with respect to σ-algebras H^{w}_{t}, $H^{\mu}_{\mathbf{T}_n}$ and $H^{w}_{t,\infty}$, respectively. Note that X_3 does not depend on events from H_t and consequently H^{x}_{t} since $H^{x}_{t} \subseteq H_t$.

Define $d = \mathbf{E}(X_1X_2X_3\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}))$. Using the $H^w \vee H^\mu_{\mathrm{T}_n}$-measurability of the product $X_1X_2X_3$ this can be rewritten as

$$d = \mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w \vee H^\mu_{\mathrm{T}_n}) \ .$$

Observe now that the product $X_1X_2\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A})$ is H^w_t-measurable and consequently H^x_t-measurable. Using the fact that X_3 is independent of the events of σ-algebra H^x_t we obtain

$$d = \mathbf{E}(X_1X_2\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}))\mathbf{E}X_3 \ . \tag{A.6}$$

Since $\mathbf{I}(\mathrm{T}_{n+1} \le t) = \mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathrm{T}_n \le t)$ equation (A.6) may be rewritten as follows:

$$d = \mathbf{E}(X_1X_2\mathbf{I}(\mathrm{T}_n \le t)\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^\mu_{\mathrm{T}_n}, \{\mathrm{T}_n < t\}))\mathbf{E}X_3 \ .$$

Noting that events from $(H^w_t \vee H^\mu_{\mathrm{T}_n}) \cap \{\mathrm{T}_n < t\}$ also belong to H^x_t and since X_3 is independent of H^x_t we get

$$\begin{aligned} d &= \mathbf{E}(X_1X_2X_3\mathbf{I}(\mathrm{T}_n \le t)\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^\mu_{\mathrm{T}_n}, \{\mathrm{T}_n < t\})) \\ &= \mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^\mu_{\mathrm{T}_n})) \ . \end{aligned}$$

Thus

$$\mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w \vee H^\mu_{\mathrm{T}_n})) = \mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^\mu_{\mathrm{T}_n})) \ .$$

Using the monotonic class theorem we prove the first part of the lemma.

In a similar way it is possible to prove the equalities:

$$\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} < t)\mathbf{I}(\mathbf{A}) \mid H^w \vee H^\mu_{\mathrm{T}_n}) = \mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} < t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^\mu_{\mathrm{T}_n}) \ ,$$

which yield

$$\mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \ge t) \mid H^w \vee H^\mu_{\mathrm{T}_n}) = \mathbf{E}(\mathbf{I}(\mathrm{T}_{n+1} \ge t) \mid H^w_t \vee H^\mu_{\mathrm{T}_n}) \ .$$

6.9. Predictability analysis of the ν_0

The following assertion is an important step towards the proof of the main result.

Lemma 9. *For any* $\Gamma \in \mathbf{B}(E)$ *the process* $\nu_0((0,t],\Gamma)$, $t \geq 0$, *is* $\mathrm{H}^{w,\mu}$*-predictable.*

Proof. It follows from Lemmas 5 and 6 that the dual $\mathrm{H}_0^{w,\mu}$-predictable projection of integer-valued random measure $\mu(du,dx)$ may be represented as follows:

$$\nu_0((0,t],\Gamma) = \sum_{n=}^{\infty} \mathrm{I}(\mathrm{T}_n < t \leq \mathrm{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathrm{T}_p\Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{d_u \mathrm{P}(\mathrm{T}_{p+1} \leq u,\ \mathrm{Z}_{p+1} \in \Gamma \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})}{\mathrm{P}(\mathrm{T}_{p+1} \geq u \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})} .$$

Observe that the function on the right-hand side of this equality immediately following the indicator $\mathrm{I}(\mathrm{T}_n < t \leq \mathrm{T}_{n+1})$ is $H_t^w \vee H_{\mathrm{T}_n}^{\mu}$-measurable, with right-continuous, left-limited sampling paths. This means that the function is H_n^w-well-measurable. From Lemma 3 the process

$$\mathrm{I}(\mathrm{T}_n < t) \sum_{p=0}^{n} \int_{\mathrm{T}_p\Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{d_u \mathrm{P}(\mathrm{T}_{p+1} \leq u,\ \mathrm{Z}_{p+1} \in \Gamma \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})}{\mathrm{P}(\mathrm{T}_{p+1} \geq u \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})}$$

is H_n^w-predictable, and consequently (from Lemma 4) the process $\nu_0((0,t],\Gamma)_{t\geq 0}$ is $\mathrm{H}^{w,\mu}$-predictable for any $\Gamma \in \mathbf{B}(E)$. This completes the proof.

6.10. Measure ν as the dual $\mathrm{H}^{w,\mu}$-predictable projection of μ

The next two lemmas give the probabilistic form of the dual $\mathrm{H}^{w,\mu}$-predictable projection of μ.

Lemma 10. *For any* $\Gamma \in \mathbf{B}(E)$ *the process*

$$Y_t^{\Gamma} = \mu((0,t],\Gamma) - \nu_0((0,t],\Gamma), \quad t \geq 0 ,$$

is an $\mathrm{H}^{w,\mu}$*-adapted local martingale.*

Proof. From the definition of the ν_0, Y_t^{Γ} is an $\mathrm{H}_0^{w,\mu}$-adapted local martingale for any $\Gamma \in \mathbf{B}(E)$. Introduce the process

$$X_t^{\Gamma} = \mathbf{E}(Y_t^{\Gamma} \mid H_t^{w,\mu}), \quad t \geq 0 .$$

It is easy to see that X_t is an $\mathrm{H}^{w,\mu}$-adapted local martingale. However, it follows from Lemma 6 that the process Y_t^{Γ} is $\mathrm{H}^{w,\mu}$-adapted and consequently coincides with X_t^{Γ}, thus proving the lemma.

The following assertion provides a probabilistic characterization of the dual $\mathrm{H}^{w,\mu}$-predictable projection of measure μ.

Lemma 11. *The dual* $\mathrm{H}^{w,\mu}$*-predictable projection of integer-valued random measure* μ *coincides with the process* ν_0.

This may be proved using Lemmas 6 and 7 and the uniqueness of the dual $\mathrm{H}^{w,\mu}$-predictable projection of μ.

6.11. Probabilistic form of the dual H^x-predictable projection of μ

The fact that eqn. (4) has a strong solution for X_t yields the inclusion

$$H_t^x \subseteq H_t^w \vee H_t^\mu \ ,$$

which, together with (11), shows that σ-algebras $H_t^w \vee H_t^\mu$ and H_t^x coincide. This in turn means that the classes of $\mathrm{H}^{w,\mu}$- and H^x-predictable processes coincide, and consequently that ν_0 is H^x-predictable.

The introduction of a non-singularity condition for B_u, $u \geq 0$, means that for any $n \geq 0$ we have:

$$w_t = \int_0^t B_u^{-1}(\mathrm{d}X_{n,u}^c - A_u \mathrm{d}u) \ ,$$

where

$$X_{n,t}^c = X_{n,t} - \int_0^t \int_E x \mu_n(\mathrm{d}u, \mathrm{d}x), \ n > 0, \ t \geq 0 \ .$$

It follows from these equalities that process w_t is H^{x_n}-adapted and consequently that

$$H_t^w \vee H_t^{\mu_n} \subseteq H_t^{x_n} \ .$$

Note also that equations (6) have a strong, unique solution for $X_{n,t}$, $n \geq 0$. This fact yields the inverse inclusion:

$$H_t^{x_n} \subseteq H_t^w \vee H_t^{\mu_n}$$

and consequently

$$H_t^{x_n} = H_t^w \vee H_t^{\mu_n} \ .$$

From the definition of $H_t^{\mu_n}$ we have

$$H_t^{\mu_n} \cap \{\mathrm{T}_n < t\} = H_{\mathrm{T}_n}^{\mu_n} \cap \{\mathrm{T}_n < t\} = H_{\mathrm{T}_n}^{\mu} \quad \{\mathrm{T}_n < t\}$$

and thus

$$(H_t^{w} \vee H_{\mathrm{T}_n}^{\mu}) \cap \{\mathrm{T}_n < t\} = (H_t^{w} \vee H_t^{\mu_n}) \cap \{\mathrm{T}_n < t\} = H_t^{x_n} \cap \{\mathrm{T}_n < t\} \ .$$

Substituting the $H_t^{w} \vee H_{\mathrm{T}_n}^{\mu}$-conditional probabilities in eqn. (3) by $H_t^{x_n}$-conditional probabilities we obtain:

$$\nu_0((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathrm{I}(\mathrm{T}_n < t \le \mathrm{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathrm{T}_p \Delta t}^{\mathrm{T}_{p+1} \Delta t} \frac{d_u \mathrm{P}(\mathrm{T}_{p+1} \le u,\ \mathrm{Z}_{p+1} \in \Gamma \mid H_u^{x_p})}{\mathrm{P}(\mathrm{T}_{p+1} \ge u \mid H_u^{x_p})} \ .$$

From Lemma 7 and the coincidence of the σ-algebras $H_t^{w} \vee H_t^{\mu}$ and H_t^{x} for any $t \ge 0$, we deduce that process

$$Y_t^{\Gamma} = \mu((0,t],\Gamma) - \nu_0((0,t],\Gamma),\ \ t \ge 0$$

is an $\mathbf{H}^x$-adapted local martingale. The uniqueness of the dual $\mathbf{H}^x$-predictable projection means that measures ν_0 and ν^x coincide, thus completing the proof of Theorem 2.

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(ii) the process $(\mu((0,t],\Gamma) - \nu_0((0,t],\Gamma))$, $t \geq 0$, is an H_0^μ-adapted local martingale.

Dual predictable projections of integer-valued random measures may be interpreted as generalized cumulative random intensity functions.

Remark. Notice here that σ-algebra $\overline{H}_0$ is not necessarily formed by events independent of H_t. For instance, $\overline{H}_0$ could be the σ-algebra in Ω corresponding to the past history of some random process up to time 0. We shall consider the σ-algebra $\overline{H}_0$ generated by a Wiener process up to time ∞.

3.3. Probabilistic representation of random intensity functions

In some senses, Jacod's representation of dual H_0^μ-predictable projections serves as a bridge between the abstract theory of random processes with jumps currently developing in the framework of the martingale approach, and the wide range of applications based largely on the knowledge of probabilistic distribution functions.

In order to express Jacod's result, we have to define regular versions of the H_{0,T_n}^μ-conditional probabilities of events $\{\mathrm{T}_{n+1} \leq u\} \cap \{\mathrm{Z}_{n+1} \in \Gamma\}$, $\{\mathrm{T}_{n+1} \geq u\}$, where $u \geq 0$, $\Gamma \in \mathbf{B}(E)$ and $n = 1,2,\ldots$. This may be done using equalities

$$\mathbf{P}(\mathrm{T}_{n+1} \leq u,\ \mathrm{Z}_{n+1} \in \Gamma \,|\, H_{0,\mathrm{T}_n}^\mu) = \mathbf{E}(\mathrm{I}(\mathrm{T}_{n+1} \leq u)\mathrm{I}(\mathrm{Z}_{n+1} \in \Gamma) \,|\, H_{0,\mathrm{T}_n}^\mu)$$

$$\mathbf{P}(\mathrm{I}(\mathrm{T}_{n+1} \geq u) \,|\, H_{0,\mathrm{T}_n}^\mu) = \mathbf{E}(\mathrm{I}(\mathrm{T}_{n+1} \geq u) \,|\, H_{0,\mathrm{T}_n}^\mu) \ .$$

It should be emphasized once again that in this part of the paper we are considering the case in which additional information about events and variables influencing the multivariate point process perceived by the statistician (observer) does not change over time.

The following theorem was proved in [1].

Theorem 1. *The following is a representation of a dual H_0^μ-predictable projection ν_0 of measure μ:*

$$\nu_0((0,t],\Gamma) = \sum_{n=0}^{\infty} \mathrm{I}(\mathrm{T}_n < t \leq \mathrm{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathrm{T}_p\Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{\mathrm{d}_u \mathbf{P}(\mathrm{T}_{p+1} \leq u,\ \mathrm{Z}_{p+1} \in \Gamma \,|\, H_{0,\mathrm{T}_p}^\mu)}{\mathbf{P}(\mathrm{T}_{p+1} \geq u \,|\, H_{0,\mathrm{T}_p}^\mu)} \ . \tag{3}$$

Corollary. Notice that if the σ-algebra $\overline{H}_0$ does not provide the observer with any information about the events, and if all he has at time t is information about events from the history H_t^{μ}, the hazard rate coincides with the dual $\mathbb{H}^{\mu}$-predictable projection ν of the measure μ and the formula for ν is a simple corollary of equation (3):

$$\nu((0,t],\Gamma) = \sum_{n=0}^{\infty} I(T_n < t \le T_{n+1}) \sum_{p=0}^{n} \int_{T_p \Delta t}^{T_{p+1}\Delta t} \frac{d_u P(T_{p+1} \le u,\ Z_{p+1} \in \Gamma \mid H^{\mu}_{T_p})}{P(T_{p+1} \ge u \mid H^{\mu}_{T_p})} . \qquad (3')$$

In the case of a pure point process (a sequence of random times T_n), the formula for the dual $\mathbb{H}^{\mu}$-predictable compensator $A(t)$ becomes:

$$A(t) = \sum_{n=0}^{\infty} I(T_n < t \le T_{n+1}) \sum_{p=0}^{n} \int_{T_p \Delta t}^{T_{p+1}\Delta t} \frac{d_u P(T_{p+1} \le u \mid H^{\mathbf{N}}_{T_p})}{P(T_{p+1} \ge u \mid H^{\mathbf{N}}_{T_p})} . \qquad (3'')$$

where the σ-algebras $H_t^{\mathbf{N}}$, $t \ge 0$, are generated by the values of the point process $(T_n)_{n\ge 0}$ or, equivalently, by the values of the counting process $\mathbf{N}_u$, $u \ge 0$, up to time t.

Equations (3') and (3'') produce known results when applied to well-studied processes. Thus, for a Poisson process with deterministic local intensity function $\lambda(t)$, the dual predictable projection $\Lambda(t)$ defined by equation (3'') coincides with the cumulative hazard rate and is given by the equality

$$\Lambda(t) = \int_0^t \lambda(u)du .$$

For a finite-state, continuous-time, Markov process ξ_t , $t \ge 0$, with states $\{1,2,\ldots,N\}$ and intensities $\lambda_{ij}(t)$, $i,j = (1,2,\ldots,N)$, $t \ge 0$, equation (3') gives the dual predictable projection in the form:

$$\nu_0((0,t],\Gamma) = \int_0^t \sum_{j\in\Gamma} \lambda_{\xi_{u-},\xi_{u-}+j}\, du .$$

Now assume that the observer has some additional information about the intensity function, for instance, that he knows the value of some random variable $\mathbf{Z}$ which influences the frequency of the jumps. This means that he is dealing with the history $H^{\mu}_{0,t}$ as determined by the equality (1), where σ-algebra $\overline{H}_0$ coincides with the σ-algebra $\sigma(\mathbf{Z})$ generated by random variable $\mathbf{Z}$ in Ω . In this case the observer needs to use the dual $\mathbb{H}_0^{\mu}$-predictable projection of μ as a random intensity function. Thus, in the case of a double stochastic Poisson process $\mathbf{N}_t$, $t \ge 0$, with random intensity function $\mathbf{Z}\lambda(t)$, $t \ge 0$ (where $\mathbf{Z}$ is some positive random variable), it is necessary to use eqn. (3) which gives the following dual $\mathbf{H}_0^{\mu}$-predictable projection of $\mathbf{N}_t$, $t \geq 0$:

$$\Lambda(t, \mathbf{Z}) = \mathbf{Z} \int_0^t \lambda(u) du \ .$$

Note that if two observers have different information about the processes occurring in some real system (for example, if one of them knows the value of the variable $\mathbf{Z}$ and the other does not), they will use different hazard rates to estimate the probability of change. In the case of continuously distributed jump-times in a double stochastic Poisson process, the relation between the two intensities (derived from a comparison between eqns. (3″) and (3′)) may be represented as follows:

$$\bar{\lambda}(t) = \mathbf{E}(\mathbf{Z} | H_t^{\mathbf{N}}) \, \lambda(t) \ , \tag{4}$$

where $\bar{\lambda}(t)$ is the $\mathbf{H}^{\mathbf{N}}$-predictable local hazard rate. This is related to the dual $\mathbf{H}^{\mathbf{N}}$-predictable projection $\bar{\Lambda}(t)$ of $\mathbf{N}_t$ by the equality:

$$\bar{\Lambda}(t) = \int_0^t \bar{\lambda}(u) \, du \ .$$

We shall now prove the relation between $\bar{\lambda}(t)$ and $\lambda(t)$. Consider the integral $G_n(t)$ defined by the equality:

$$G_n(t) = \int_{\mathbf{T}_p \Delta t}^{\mathbf{T}_{p+1} \Delta t} \frac{d_u \mathbf{P}(\mathbf{T}_{p+1} \leq u \mid H_{\mathbf{T}_p}^{\mathbf{N}})}{\mathbf{P}(\mathbf{T}_{p+1} \geq u \mid H_{\mathbf{T}_p}^{\mathbf{N}})} \ ,$$

which is taken from the right-hand side of eqn. (3″). Let $\varphi_p(u)$ be the density function of conditional distribution $\mathbf{P}(\mathbf{T}_{p+1} \leq u \mid H_{\mathbf{T}_p}^{\mathbf{N}})$. Note that the following equality holds:

$$\varphi_n(u) = \int_0^{\infty} z \, \lambda(u) \exp\{-z \int_{\mathbf{T}_n}^{u} \lambda(v) dv\} \, f(z) dz \ ,$$

where $f(z)$ is the density distribution function of random variable $\mathbf{Z}$. Using this equality and noting that for $\mathbf{T}_p \leq u$

$$\exp\{-z \int_{\mathbf{T}_p}^{u} \lambda(v) dv\} = \mathbf{P}(\mathbf{T}_{p+1} > u \mid H_{\mathbf{T}_p}^{\mathbf{N}}, \mathbf{Z} = z) \ ,$$

we have

$$G_p(t) = \int_{T_p \Delta t}^{T_{p+1}\Delta t} \mathbf{E}(\mathbf{Z} \mid H^{\mathbf{N}}_{T_p}, \mathrm{T}_{p+1} > u)\, \lambda(u)\, du \quad .$$

Since

$$\mathrm{I}(\mathrm{T}_p \le u < \mathrm{T}_{p+1})\, \mathbf{E}(\mathbf{Z} \mid H^{\mathbf{N}}_{T_p}, \mathrm{T}_{p+1} > u) = \mathrm{I}(\mathrm{T}_p \le u < \mathrm{T}_{p+1})\, \mathbf{E}(\mathbf{Z} \mid H^{\mathbf{N}}_u) \ ,$$

we can derive the formula for the $\overline{\Lambda}(t)$ from eqn. (3″).

Remark. Equation (4) shows that to calculate intensity functions $\overline{\Lambda}(t)$ or $\overline{\lambda}(t)$ it is first necessary to calculate the $H^{\mathbf{N}}_t$-conditional mathematical expectation. This problem can be overcome by using an approach based on filtering of the jumping processes (see, for instance, [11, 12]). In the simplest life-cycle models, which are characterized only by stopping time $\mathbf{T}$ (time of death) and are widely used in reliability and demographic analysis, the random intensity describes differences in susceptibility to death or failure [13]. Notice that in the case of life-cycle processes, equation (3″) gives the following relation between the H-adapted compensator $A(t)$ and the local intensity function $\lambda(t)$:

$$A(t) = \int_0^{t\Delta \mathrm{T}} \lambda(u)\, du \quad .$$

Recall that, from the definition of the compensator, the process

$$M(t) = \mathrm{I}(\mathrm{T} \le t) - A(t), \quad t \ge 0 \ ,$$

is an H-adapted martingale.

4. GENERAL FORMULA FOR REPRESENTATION OF RANDOM INTENSITY

In spite of the fact that dual predictable projections exist for a wide class of families of σ-algebras, probabilistic representations are known only for σ-algebras with structure (1). However, in practical situations σ-algebras often have a more general structure. In particular, new information may be generated not only by the multivariant point process but also by some additional process η_t. In this case the σ-algebras describing the observation history have the form $H_t = H^{\eta}_t \vee H^{\mu}_t$, where σ-algebras H^{ξ}_t are generated by some process η_t which is observed simultaneously with the multivariate point process $(\mathbf{T}_n, \mathbf{Z}_n)_{n \ge 0}$. Detailed probabilistic characterization of dual predictable projections is often useful when applying the results of the general theory of random processes in practice.

In this section we will give the probabilistic representation of the dual predictable projection of integer-valued random measures corresponding to the jumps of the semimartingale with respect to the family of σ-algebras generated by this process.

4.1. Generation of jumps

Let random process X_t, $t \geq 0$, defined on probability space $(\Omega, H, \mathrm{H}, \mathrm{P})$ be H-adapted, take values in $\mathbf{R}^k$ and have right-continuous, left-limited sampling paths. Denote by H^x the family of σ-algebras

$$H_t^x = \bigcap_{r>t} \sigma\{X_u, u \leq r\}$$

and assume that H_0^x is completed by the sets from $H^x = H_\infty^x$ with a P-probability of zero. Assume also that process X_t may be represented as follows:

$$X_t = X_0 + \int_0^t A_u \mathrm{d}u + \int_0^t B_u \mathrm{d}w_u + \int_0^t \int_E x \mu(\mathrm{d}u, \mathrm{d}x) \ , \tag{5}$$

where A_u and B_u are H^x-adapted matrices of appropriate dimensions, matrix B_u is non-singular for any $u \geq 0$, $E = \mathbf{R}^k - \{0\}$, and w_u is an H-adapted, k-dimensional Wiener process independent of X_0.

Note that, in general, the dimension of X_t may be greater than that of the jumping changes. This can mean, for instance, that we are also considering situations with two-component processes of which one is pure jumping and the other is continuous.

It follows from (3) that the measure μ is H^x-adapted and that

$$Z_n = X_{T_n} - X_{T_n -} \ .$$

Let ν^x denote the dual H^x-predictable projection of integer-valued random measure μ. Our main aim is to derive the probabilistic representation of measure ν^x.

4.2. The form of the hazard rate

The main result of this subsection is formulated in terms of auxiliary processes $X_{n,t}$ defined by the equalities:

$$X_{n,t} = X_0 + \int_0^t A_u \mathrm{d}u + \int_0^t B_u \mathrm{d}w_u + \int_0^{\mathbf{T}_n \Delta t} \int_E x\mu(\mathrm{d}u,\mathrm{d}x) \ , \tag{6}$$

where A, B, w, μ have been defined previously. Introducing the measures μ_n defined by the equalities

$$\mu_n((0,t], \Gamma) = \sum_k \mathbf{I}(\mathbf{T}_k \le \mathbf{T}_n \Delta t)\, \mathbf{I}(\mathbf{Z}_n \in \Gamma),\ \Gamma \in \mathbf{B}(E),\ t \ge 0,\ n \ge 0 \ ,$$

eqn. (6) may be rewritten as follows:

$$X_{n,t} = X_0 + \int_0^t A_u \mathrm{d}u + \int_0^t B_u \mathrm{d}w_u + \int_0^t \int_E x\mu_n(\mathrm{d}u,\mathrm{d}x) \ .$$

Notice that measure μ_n may be considered as a measure of the jumps of the process $X_{n,t}$, $t \ge 0$.

Introduce σ-algebras $H_t^{x_n}$ and $H_t^{\mu_n}$ such that:

$$H_t^{x_n} = \bigcap_{u>t} \sigma\{X_{n,r}, r \le u\} \ ,$$

$$H_t^{\mu_n} = \sigma\{\mu_n((0,t], \Gamma), u \le t, \Gamma \in \mathbf{B}(E)\} \ ,$$

and define the regular versions of the conditional probabilities of events

$$\{\mathbf{T}_{p+1} \le u\} \cap \{\mathbf{Z}_{p+1} \in \Gamma\}, \quad \{\mathbf{T}_{p+1} \ge u\}, \quad u > 0, \Gamma \in \mathbf{B}(E)$$

using the equalities

$$\mathbf{P}(\mathbf{T}_{p+1} \le u,\ \mathbf{Z}_{p+1} \in \Gamma | H_u^{x_p}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{p+1} \le u)\, \mathbf{I}(\mathbf{Z}_{p+1} \in \Gamma) \,|\, H_u^{x_p})$$

$$\mathbf{P}(\mathbf{T}_{p+1} \ge u \,|\, H_u^{x_p}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{p+1} \ge u) | H_u^{x_p}) \ .$$

The next assertion is the main result of this paper.

Theorem 2. *Assume that coefficients A and B are such that a strong solution of equation (3) exists and is unique. Then we have the following representation of the dual $\mathbf{H}^x$-predictable projection ν^x of integer-valued random measure μ:*

$$\nu^x((0,t],\Gamma) = \sum_{n=0}^{\infty} I(T_n < t \le T_{n+1}) \sum_{p=0}^{n} \int_{T_p \Delta t}^{T_{p+1} \Delta t} \frac{d_u P(T_{p+1} \le u,\ Z_{p+1} \in \Gamma | H_u^{x_p})}{P(T_{p+1} \ge u \mid H_u^{x_p})} . \quad (7)$$

This theorem is proved in the Appendix.

Sometimes it is more convenient to use another form of representation for ν^x, transforming the conditional probabilities on the right-hand side of (7). For this purpose we introduce the function $F(u,\Gamma)$, making use of the equality

$$F(u,\Gamma) = P(Z_{n+1} \in \Gamma | H^x_{T_{n+1}-})|_{T_{n+1}=u},\ \Gamma \in B(E),\ u \ge 0,\ n \ge 0 ,$$

where $P(Z_{n+1} \in \Gamma | H^x_{T_{n+1}-})$ is the regular version of the $H^x_{T_{n+1}-}$-conditional probability of event $\{Z_{n+1} \in \Gamma\}$. The dual $\mathbf{H}^x$-predictable projection ν^x of measure μ may then be represented in terms of this function as follows:

$$\nu^x((0,t],\Gamma) = \sum_{n=0}^{\infty} I(T_n < t \le T_{n+1}) \sum_{p=0}^{n} \int_{T_p \Delta t}^{T_{p+1} \Delta t} \frac{F(u,\Gamma)\, d_u P(T_{p+1} \le u \mid H_u^{x_p})}{P(T_{p+1} \ge u \mid H_u^{x_p})} .$$

5. EXAMPLES

5.1. Conditional Gaussian property

Let process $Y(t)$, $t \ge 0$, satisfy the linear stochastic differential equation

$$dY(t) = a_0(t) + a_1(t)\, Y(t)\, dt + b(t)\, dw(t),\ Y(0) = Y_0 , \quad (8)$$

where Y_0 is a Gaussian random variable with mean m_0 and variance γ_0, $w(t)$ is an H-adapted Wiener process, $\mathbf{H} = (H_t)_{t \ge 0}$ is some non-decreasing, right-continuous family of σ-algebras, and H_0 is completed by $\mathbf{P}$-zero sets from $H = H_\infty$. Denote by $\mathbf{H}^y$ the family of σ-algebras in Ω generated by the values of the random process $Y(u)$, i.e.,

$$\mathbf{H}^y = (H_t^y)_{t \ge 0},\quad H_t^y = \bigcap_{u > t} \sigma\{Y(v),\ v \le u\},\quad t \ge 0 .$$

Assume that process $Y(t)$ determines the random rate of occurrence of some unexpected event characterized by the random time $\mathbf{T}$, through the equality:

$$P(\mathbf{T} > t \mid H_t^y) = \exp\{-\int_0^t Y^2(u)\, \lambda(u)\, du\} . \quad (9)$$

Notice that process $\mathbf{Z}(u) = Y^2(u)$, $u \geq 0$, may be interpreted as the frailty of an individual changing stochastically over time. Using the terminology of martingale theory one can say that the process

$$A(t) = \int_0^{t \Delta \mathbf{T}} \lambda(u)\, Y^2(u)\, \mathrm{d}u$$

is an $\mathbf{H}^y$-predictable compensator of the life-cycle process $X_t = \mathbf{I}(\mathbf{T} < t)$, $t \geq 0$. This means that the process $M_t = \mathbf{I}(\mathbf{T} < t) - A(t)$, $t \geq 0$ is an $\mathbf{H}^y$-adapted martingale. Associating the stopping time $\mathbf{T}$ with the time of death, we may describe the process $Y^2(t)$, $t \geq 0$, as the age-specific mortality rate for an individual with history $Y_0^t = \{Y(u)\}$, $0 \leq u \leq t$.

Letting $\bar{\lambda}(t)$, $t \geq 0$, denote the observed age-specific mortality rate we have $\bar{\lambda}(t) = \lambda(t)\,\bar{\mathbf{Z}}(t)$, $t \geq 0$, where $\bar{\mathbf{Z}}(t) = \mathbf{E}\{Y^2(t) \mid \mathbf{T} > t\}$ [13].

In order to calculate the observed mortality rate $\bar{\lambda}(t)$, $t \geq 0$, it is necessary first to calculate the second moment of the conditional distribution of the $Y(u)$ given the event $\{\mathbf{T} \geq 0\}$. It turns out that this moment may be calculated quite easily using the result of the following theorem.

Theorem 3. *Assume that process $Y(t)$ and stopping time $\mathbf{T}$ are related through eqns.* (8) *and* (9). *Then the conditional distribution of $Y(t)$ given $\{\mathbf{T} \geq t\}$ is Gaussian. The parameters of this distribution, i.e., the mean m_t and the variance γ_t, are given by the following equations:*

$$\frac{\mathrm{d}m_t}{\mathrm{d}t} = a_0(t) + a_1(t)\, m_t - 2m_t \gamma_t \lambda(t), \quad m_0 \ . \tag{10}$$

$$\frac{\mathrm{d}\gamma_t}{\mathrm{d}t} = 2a_1(t)\,\gamma_t + b^2(t) - 2\lambda(t)\,\gamma_t^2, \quad \gamma_0 \ . \tag{11}$$

The formula for $\bar{\lambda}(t)$ is then $\bar{\lambda}(t) = \lambda(t)\,(m_t^2 + \gamma_t)$.

This theorem may be proved in a similar way as the conditional Gaussian property for processes governed by stochastic differential equations of the diffusion type (see [14]).

6. APPENDIX: PROOF OF THEOREM 2

The proof uses representation (3) for ν_0. It turns out that if the σ-algebra $\overline{H}_0$ in eqn. (1) is of a particular form (which will be specified later) then the H^{μ}_{0,T_n}-conditional probabilities of events $\{\mathrm{T}_{p+1} \le u\} \cap \{\mathrm{Z}_{p+1} \in \Gamma\}$ and $\{\mathrm{T}_{p+1} \ge u\}$ will $\mathbf{P}$-a.s. coincide with the $H^{x_n}_u$-conditional probabilities of these events on the integration intervals in (7). Representation of measure ν_0 through $H^{x_n}_u$-conditional probabilities makes it easier to prove its $\mathbf{H}^x$-predictability property. It is then easy to check that the process

$$(\mu((0,t],\Gamma) - \nu_0((0,t],\Gamma)),\ \ t \ge 0$$

is an $\mathbf{H}^x$-adapted martingale for any $\Gamma \in \mathbf{B}(E)$. The fact that ν^x is unique shows that ν^x and ν_0 coincide $\mathbf{P}$-a.s. Representation (7) is derived from (3) through substitution of the conditional probabilities. Several auxiliary results will be useful in the proof of Theorem 2: these are derived in the following subsections.

6.1. Auxiliary σ-algebras

Introduce the auxiliary right-continuous families of σ-algebras $\mathbf{H}^w$, $\mathbf{H}^{\mu}$, $\mathbf{H}^{w,\mu}$, $\mathbf{H}^w_n$, $\mathbf{H}^{w,\mu}_0$ and $\overline{\mathbf{H}}^w$, where

$$\mathbf{H}^w = (H^w_t)_{t\ge 0}\,,\quad H^w_t = \sigma\{w_u\,,\ u \le t\} \vee \sigma(X_0),\ \ H^w = H^w_\infty\ ,$$

$$\mathbf{H}^{\mu} = (H^{\mu}_t)_{t\ge 0}\,,\quad H^{\mu}_t = \sigma\{\mu((0,u],\Gamma)\,,\ u \le t,\ \Gamma \in \mathbf{B}(E)\},\ \ H^{\mu} = H^{\mu}_\infty\ ,$$

$$\mathbf{H}^{w,\mu} = (H^{w,\mu}_t)_{t\ge 0}\,,\quad H^{w,\mu}_t = H^w_t \vee H^{\mu}_t\ ,$$

$$\mathbf{H}^w_n = (H^w_{n,t})_{t\ge 0}\,,\quad H^w_{n,t} = H^w_t \vee H^{\mu}_{\mathrm{T}_n} = H^w_t \vee H^{\mu_n}_\infty\ ,$$

$$\mathbf{H}^{w,\mu}_0 = (H^{w,\mu}_{0,t})_{t\ge 0}\,,\quad H^{w,\mu}_{0,t} = H^w \vee H^{\mu}_t\ ,$$

$$\overline{\mathbf{H}}^w = (\overline{H}^w_t)_{t\ge 0},\quad \overline{H}^w_t = \overline{H} \vee H^w_t\ ,$$

where $\overline{H}$ is some σ-algebra in Ω and σ-algebras H^w_0 and H^{μ}_0 are completed by $\mathbf{P}$-zero sets from σ-algebras H^w and H^{μ}, respectively.

Recall also that family $\mathbf{H}^x$ is defined as follows:

$$\mathbf{H}^x = (H^x_t)_{t\ge 0},\quad H^x_t = \bigcap_{r>t} \sigma\{X_u, u \le r\}\ .$$

6.2. Existence of H^x-predictable projections

We shall now establish that dual H^x-predictable projections exist and are unique.

Lemma 1. *The dual H^x-predictable projections ν^x of the integer-valued random measure μ exist and are unique.*

Proof. Note that the sets $[0]\ E\ ]T_n, T_{n+1}]\ E$ belong to $\overline{\Pi}$ and have measure

$$M_\mu(d\omega, du, dx) = P(d\omega)\,\mu(du, dx)$$

less than or equal to 1. This means that measure M_μ is σ-finite on $((\Omega\ (0,t]\ E), \overline{\Pi})$. From [1], this implies that the lemma is true.

6.3. Characterization of $H^{w,\mu}$-stopping times

For any $t \ge 0$ let

$$H^w_{t,\infty} = \sigma\{w_r - w_u,\ r \ge u \ge t\}\ .$$

The next assertion is a generalization of Lemma 3.2 in Jacod's paper [1].

Lemma 2. *Let $\mathbf{T}$ be the $H^{w,\mu}$-stopping time. For any $n \ge 0$ there exists a random variable S^n such that indicator $I(S^n \ge u)$ is $H^w_{n,u}$-measurable for any $u \ge 0$ and the following equality holds:*

$$I\{T \ge u\}\, I\{T_n < u \le T_{n+1}\} = I\{S^n \ge u\}\, I\{T_n < u \le T_{n+1}\}\ .$$

Proof. It follows from the definition of the σ-algebra H^μ_u that the following families of sets coincide:

$$H^\mu_{u-} \cap \{T_n < u \le T_{n+1}\} = H^\mu_{T_n} \cap \{T_n < u \le T_{n+1}\}$$

and consequently the families of sets

$$(H^w_u \vee H^\mu_{u-}) \cap \{T_n < u \le T_{n+1}\} = H^w_u \vee H^\mu_{T_n} \cap \{T_n < u \le T_{n+1}\}$$

also coincide. Take the set $\{T < u\}$ from $H^w_u \vee H^\mu_{u-}$, and find the set D_u from $H^w_u \vee H^\mu_{T_n}$ such that

$$\{T < u\} \cap \{T_n < u \le T_{n+1}\} = D_u \cap \{T_n < u \le T_{n+1}\}\ .$$

Note that for $r < u$ we now have

$$[(\mathbf{D}_r \cap \{\mathbf{T}_n < r\}) \cup (\mathbf{D}_u \cap \{\mathbf{T}_n < u\})] \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} = \mathbf{D}_u \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} .$$

Define $\mathbf{S}^n$ by the equalities

$$\{\mathbf{S}^n < u\} = \bigcup_{r \le u} (\mathbf{D}_r \cap \{\mathbf{T}_n < r\}) ,$$

where the r are rational numbers. We then obtain

$$\{\mathbf{T} < u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} = \{\mathbf{S}^n < u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\}$$

or

$$\{\mathbf{T} \ge u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} = \{\mathbf{S}^n \ge u\} \cap \{\mathbf{T}_n < u \le \mathbf{T}_{n+1}\} ,$$

thus completing the proof of Lemma 2.

6.4. Representation of martingales

The following result plays a fundamental role in the analysis of the predictability property.

Lemma 3. *Let Z_t be a right-continuous, left-limited, square-integrable, $\overline{\mathbf{H}}^w$-martingale process. Then an $\overline{\mathbf{H}}^w$-adapted process $f(u,\omega)$, $u > 0$, exists such that*

$$\mathbf{E} \int_0^t f^2(u,\omega) du < \infty, \quad t \ge 0$$

and

$$Z_t = Z_0 + \int_0^t f(u,\omega) dw_u .$$

The proof of this lemma is similar to that of Theorem 5.5 in [14]. The following well-known result is important in the proof of some auxiliary assertions.

Lemma 4. *Let $\mathbf{L}$ be some vector space of bounded real functions defined on Ω. Assume that it contains the constant 1, is closed with respect to uniform convergence, and is such that for any uniformly bounded increasing sequence of non-negative functions f_n, $n \ge 0$, $f_n \in \mathbf{L}$, the function $f = \lim_{n \to \infty}$ also belongs to $\mathbf{L}$. Let Q be a subset of $\mathbf{L}$ which is closed with respect to multiplication. Then* *the space* L *contains all bounded functions, measured with respect to the σ-algebra H generated by the elements of Q.*

Remark. This result is known as the monotonic class theorem, and is proved in [15] . The theorem is also true if

(a) **L** is closed with respect to monotonic and uniform convergence and

(b) Q is the algebra and $1 \in Q$

or

(a′) **L** is a set of functions closed with respect to monotonic convergence to the bounded function and

(b′) Q is a vector space closed with respect to operation Λ (maximum of two functions) and $1 \in Q$.

6.5. Predictability of H_n^w-well-measurable processes

It turns out that H_n^w-well-measurable processes have the following remarkable property:

Lemma 5. *Let Y_t^n be an arbitrary H_n^w-well-measurable process. Then process $Y_t^n \mathrm{I}\{\mathrm{T}_n < t\}$ is H_n^w-predictable.*

Proof. Let **T** be an arbitrary H_n^w-stopping time, and denote by $\lambda(t)$ the dual H_n^w-predictable projection of non-decreasing process $\mathrm{I}(\mathrm{T} \le t)$. From the definition of $\lambda(t)$ the process $\mathrm{Z}_t = \mathrm{I}(\mathrm{T} \le t) - \lambda(t)$ is an H_n^w-martingale.

Now consider the process $v_u = w_{\mathrm{T}_n+u} - w_{\mathrm{T}_n}$, and define $\mathrm{H}_n^v = (H_{n,t}^v)_{t \ge 0}$, where $H_{n,t}^v = \sigma\{v_u, u \le t\} \vee H_{n,T_n}^w$.

Observe that $H_{n,t}^v = H_{n,T_n+t}^w$ and consequently that family H_n^v coincides with family $\mathrm{H}_n^{n,w} = (H_{n,T_n+t}^w)_{t \ge 0}$. It is not difficult to check that v_u is a Wiener process with respect to H_n^v and that $\mathrm{Z}_t^n = \mathrm{Z}_{\mathrm{T}_n+t}$ is an $\mathrm{H}_n^{n,w}$-martingale process.

From Lemma 3 we have the following representation of Z_t^n:

$$\mathrm{Z}_t^n = \mathrm{Z}_0^n + \int_0^t f_n(u,\omega)\mathrm{d}v_u \quad ,$$

or, in terms of Z_t,

$$\mathrm{Z}_{\mathrm{T}_n+u} = \mathrm{Z}_{\mathrm{T}_n} + \int_{\mathrm{T}_n}^{\mathrm{T}_n+u} f_n(\mathrm{T}_n + r, \omega)\mathrm{d}w_{\mathrm{T}_n+r} \quad .$$

Taking $T_n + u = t$ we get

$$Z_t \, I(T_n < t) = \left[Z_{T_n} + \int_{T_n}^{t} f_n(u,\omega) dw_u \right] I(T_n < t) \ .$$

The right-hand side of this equality is an H_n^w-predictable process. Remembering the definition of Z_t , we deduce that the process $I(T \le t) I(T_n < t)$ is H_n^w-predictable.

The result of the lemma may then be derived from the monotonic class theorem [15].

6.6. Characterization of $H^{w,\mu}$-predictable processes

The following assertion describes the structure of $H^{w,\mu}$-predictable processes.

Lemma 6. *An $H^{w,\mu}$-adapted process Z_t is $H^{w,\mu}$-predictable if and only if, for any $n \ge 0$, there exists an H_n^w-well-measurable process Y_t^n such that*

$$Y_t^n I(T_n < t \le T_{n+1}) = Z_t \, I(T_n < t \le T_{n+1}) \ . \tag{A.1}$$

Proof

Necessity. Consider the process $Z_t = I(t \le T)$, where T is an arbitrary $H^{w,\mu}$-stopping time. It follows from Lemma 1 that

$$I(t \le T) I(T_n < t \le T_{n+1}) = I(t \le S^n) I(T_n < t \le T_{n+1}) \ ,$$

which leads to equality (8) with $Y_t^n = I(t \le S^n)$. That these conditions are necessary may be proved from the monotonic class theorem.

Sufficiency. Observe that for an arbitrary H_n^w-adapted process Y_t, the process $I\{T_n \le t\} Y_t$ is $H^{w,\mu}$-adapted. This is because

$$(H_t^w \vee H_{T_n}^\mu) \cap \{T_n \le t\} = (H_t^w \vee H_{t-}^\mu) \cap \{T_n \le t\}, \ n \ge 0 \ ,$$

and any arbitrary set from $(H_t^w \vee H_{t-}^\mu) \cap \{T_n \le t\}$ is $H_t^w \vee H_t^\mu$-measurable. Left-continuous H_n^w-adapted processes Y_t generate left-continuous $H^{w,\mu}$-adapted processes $I\{T_n \le t\} Y_t$. This means that the following inclusion is true:

$$\Pi(H_n^w) \cap]] T_n, T_{n+1}]] \subseteq \Pi(H^{w,\mu}) \ , \tag{A.2}$$

where $\Pi(\mathbf{H}_n^w)$ and $\Pi(\mathbf{H}^{w,\mu})$ are σ-algebras for $\mathbf{H}_n^w$- and $\mathbf{H}^{w,\mu}$-predictable sets respectively, and $]|\mathbf{T}_n,\mathbf{T}_{n+1}]|$ is the stochastic interval corresponding to the stopping times $\mathbf{T}_n$ and $\mathbf{T}_{n+1}$. The inclusion (A.2) yields:

$$\bigcup_n]|\mathbf{T}_n,\mathbf{T}_{n+1}]| \cap \Pi(\mathbf{H}_n^w) \subseteq \Pi(\mathbf{H}^{w,\mu}) \ . \tag{A.3}$$

From Lemma 5, the process $Y_t^n \mathbf{I}(\mathbf{T}_n \le t)$ is $\mathbf{H}_n^w$-predictable. Inclusion (A.3) shows that the process $\sum_n Y_t^n \mathbf{I}_{]|\mathbf{T}_n,\mathbf{T}_{n+1}]|}$ (which according to equality (A.1) coincides with process $\mathbf{Z}_t$) is $\mathbf{H}^{w,\mu}$-predictable. This completes the proof.

6.7. A property of conditional distributions

Let H, G, F be σ-algebras in Ω. Assume that they are complete with respect to measure $\mathbf{P}$ and such that $G \subseteq H$, $F \subseteq H$. The next statement will then be useful in analyzing the form of the dual predictable projection.

Lemma 7. *Let* $\mathbf{B} \in H$, $\mathbf{P}(\mathbf{B}) > 0$ *be such that the families of sets* $F \cap \mathbf{B}$ *and* $G \cap \mathbf{B}$ *coincide* $\mathbf{P}$*-a.s. Then for any* H*-measurable integrable random variable* η *the following equality holds:*

$$\mathbf{I}(\mathbf{B})\,\mathbf{E}(\eta \mid G \vee \mathbf{B}) = \mathbf{I}(\mathbf{B})\,\mathbf{E}(\eta \mid F \vee \mathbf{B}) \ .$$

Proof. For any $A \in H$ define the measure $\mathbf{P}^{\mathbf{B}}(A)$ as follows:

$$\mathbf{P}^{\mathbf{B}}(\mathbf{A}) = \frac{\mathbf{P}(A \cap \mathbf{B})}{\mathbf{P}(\mathbf{B})} \ .$$

Let $\mathbf{E}^{\mathbf{B}}$ denote the mathematical expectation with respect to $\mathbf{P}^{\mathbf{B}}$. The families of sets $G \cap \mathbf{B}$ and $F \cap \mathbf{B}$ form σ-algebras of the subsets of set $\mathbf{B}$ that, generally speaking, are not σ-algebras in Ω. Since these families are complete with respect to measure $\mathbf{P}^{\mathbf{B}}$, they coincide $\mathbf{P}^{\mathbf{B}}$-a.s. with the σ-algebras $G \vee \mathbf{B}$ and $F \vee \mathbf{B}$, respectively.

It follows from the conditions of the lemma that for any $\mathbf{A} \in H$ we have

$$\mathbf{P}^{\mathbf{B}}(\mathbf{A} \mid G \cap \mathbf{B}) = \mathbf{P}^{\mathbf{B}}(\mathbf{A} \mid F \cap \mathbf{B}), \quad \mathbf{P}^{\mathbf{B}}\text{-a.s.} \ ,$$

or, equivalently,

$$\mathbf{P}^{\mathbf{B}}(A \mid G \vee \mathbf{B}) = \mathbf{P}^{\mathbf{B}}(\mathbf{A} \mid F \vee \mathbf{B}), \quad \mathbf{P}^{\mathbf{B}}\text{-a.s.}$$

and thus the following equality holds P^B-a.s. for any bounded random variable η:

$$E^B(\eta \mid G \vee B) = E(\eta \mid F \vee B) .$$

This may be rewritten in the form

$$I(B)\, E^B(\eta \mid G \vee B) = I(B)\, E^B(\eta \mid F \vee B), \quad P\text{-a.s.}$$

or

$$I(B)\, E(\eta \mid G \vee B) = I(B)\, E(\eta \mid F \vee B), \quad P\text{-a.s.} \; ,$$

thus completing the proof.

6.8. Some properties of conditional mathematical expectations

The next assertion will be useful in proving the predictable characterization of some random measures.

Lemma 8. *Let* $A \in H^{\mu}_{T_{n+1}}$. *Then the following equalities are true for any* $t > 0$ *:*

$$E(I(T_{n+1} \le t)\, I(A) \mid H^w \vee H^{\mu}_{T_n}) = E(I(T_{n+1} \le t)\, I(A) \mid H^w_t \vee H^{\mu}_{T_n}) \qquad (A.4)$$

$$E(I(T_{n+1} \ge t) \mid H^w \vee H^{\mu}_{T_n}) = E(I(T_{n+1} \ge t) \mid H^w_t \vee H^{\mu}_{T_n}) \; .$$

Proof. Since B_u is non-singular for any $u \ge 0$, the process w_t may be represented as follows:

$$w_t = \int_0^t B_u^{-1}(\mathrm{d}X_u^c - A_u\, \mathrm{d}u) \; ,$$

where

$$X_t^c = X_t - \int_0^t \int_E x\, \mu(\mathrm{d}u, \mathrm{d}x) \; .$$

This shows that the process w_t is H^x-adapted and leads to the inclusion:

$$H^w_t \vee H^{\mu}_{t-} \subseteq H^x_t \; . \qquad (A.5)$$

Consider now the bounded random variables X_1, X_2, X_3 which are measurable with respect to σ-algebras H^w_t, $H^{\mu}_{T_n}$ and $H^w_{t,\infty}$, respectively. Note that X_3 does not depend on events from H_t and consequently H^x_t since $H^x_t \subseteq H_t$.

Define $d = \mathbf{E}(X_1X_2X_3\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}))$. Using the $H^w \vee H^{\mu}_{\mathbf{T}_n}$-measurability of the product $X_1X_2X_3$ this can be rewritten as

$$d = \mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w \vee H^{\mu}_{\mathbf{T}_n}) .$$

Observe now that the product $X_1X_2\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A})$ is H^w_t-measurable and consequently H^x_t-measurable. Using the fact that X_3 is independent of the events of σ-algebra H^x_t we obtain

$$d = \mathbf{E}(X_1X_2\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}))\mathbf{E}X_3 . \tag{A.6}$$

Since $\mathbf{I}(\mathbf{T}_{n+1} \le t) = \mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{T}_n \le t)$ equation (A.6) may be rewritten as follows:

$$d = \mathbf{E}(X_1X_2\mathbf{I}(\mathbf{T}_n \le t)\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^{\mu}_{\mathbf{T}_n}, \{\mathbf{T}_n < t\}))\mathbf{E}X_3 .$$

Noting that events from $(H^w_t \vee H^{\mu}_{\mathbf{T}_n}) \cap \{\mathbf{T}_n < t\}$ also belong to H^x_t and since X_3 is independent of H^x_t we get

$$\begin{aligned} d &= \mathbf{E}(X_1X_2X_3\mathbf{I}(\mathbf{T}_n \le t)\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^{\mu}_{\mathbf{T}_n}, \{\mathbf{T}_n < t\})) \\ &= \mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^{\mu}_{\mathbf{T}_n})) . \end{aligned}$$

Thus

$$\mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w \vee H^{\mu}_{\mathbf{T}_n})) = \mathbf{E}(X_1X_2X_3\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \le t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^{\mu}_{\mathbf{T}_n})) .$$

Using the monotonic class theorem we prove the first part of the lemma.

In a similar way it is possible to prove the equalities:

$$\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} < t)\mathbf{I}(\mathbf{A}) \mid H^w \vee H^{\mu}_{\mathbf{T}_n}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} < t)\mathbf{I}(\mathbf{A}) \mid H^w_t \vee H^{\mu}_{\mathbf{T}_n}) ,$$

which yield

$$\mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \ge t) \mid H^w \vee H^{\mu}_{\mathbf{T}_n}) = \mathbf{E}(\mathbf{I}(\mathbf{T}_{n+1} \ge t) \mid H^w_t \vee H^{\mu}_{\mathbf{T}_n}) .$$

6.9. Predictability analysis of the ν_0

The following assertion is an important step towards the proof of the main result.

Lemma 9. *For any* $\Gamma \in \mathrm{B}(E)$ *the process* $\nu_0((0,t],\Gamma)$, $t \geq 0$, *is* $\mathrm{H}^{w,\mu}$*-predictable.*

Proof. It follows from Lemmas 5 and 6 that the dual $\mathrm{H}_0^{w,\mu}$-predictable projection of integer-valued random measure $\mu(du,dx)$ may be represented as follows:

$$\nu_0((0,t],\Gamma) = \sum_{n=}^{\infty} \mathrm{I}(\mathrm{T}_n < t \leq \mathrm{T}_{n+1}) \sum_{p=0}^{n} \int_{\mathrm{T}_p\Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{d_u \mathrm{P}(\mathrm{T}_{p+1} \leq u,\ \mathrm{Z}_{p+1} \in \Gamma \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})}{\mathrm{P}(\mathrm{T}_{p+1} \geq u \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})} .$$

Observe that the function on the right-hand side of this equality immediately following the indicator $\mathrm{I}(\mathrm{T}_n < t \leq \mathrm{T}_{n+1})$ is $H_t^w \vee H_{\mathrm{T}_n}^{\mu}$-measurable, with right-continuous, left-limited sampling paths. This means that the function is H_n^w-well-measurable. From Lemma 3 the process

$$\mathrm{I}(\mathrm{T}_n < t) \sum_{p=0}^{n} \int_{\mathrm{T}_p\Delta t}^{\mathrm{T}_{p+1}\Delta t} \frac{d_u \mathrm{P}(\mathrm{T}_{p+1} \leq u,\ \mathrm{Z}_{p+1} \in \Gamma \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})}{\mathrm{P}(\mathrm{T}_{p+1} \geq u \mid H_u^w \vee H_{\mathrm{T}_p}^{\mu})}$$

is H_n^w-predictable, and consequently (from Lemma 4) the process $\nu_0((0,t],\Gamma)_{t\geq 0}$ is $\mathrm{H}^{w,\mu}$-predictable for any $\Gamma \in \mathrm{B}(E)$. This completes the proof.

6.10. Measure ν as the dual $\mathrm{H}^{w,\mu}$-predictable projection of μ

The next two lemmas give the probabilistic form of the dual $\mathrm{H}^{w,\mu}$-predictable projection of μ.

Lemma 10. *For any* $\Gamma \in \mathrm{B}(E)$ *the process*

$$Y_t^{\Gamma} = \mu((0,t],\Gamma) - \nu_0((0,t],\Gamma),\ \ t \geq 0 ,$$

is an $\mathrm{H}^{w,\mu}$*-adapted local martingale.*

Proof. From the definition of the ν_0, Y_t^{Γ} is an $\mathrm{H}_0^{w,\mu}$-adapted local martingale for any $\Gamma \in \mathrm{B}(E)$. Introduce the process

$$X_t^{\Gamma} = \mathbf{E}(Y_t^{\Gamma} \mid H_t^{w,\mu}),\ \ t \geq 0 .$$

It is easy to see that X_t is an $\mathrm{H}^{w,\mu}$-adapted local martingale. However, it follows from Lemma 6 that the process Y_t^{Γ} is $\mathrm{H}^{w,\mu}$-adapted and consequently coincides with X_t^{Γ}, thus proving the lemma.

The following assertion provides a probabilistic characterization of the dual $\mathrm{H}^{w,\mu}$-predictable projection of measure μ.

Lemma 11. *The dual* $\mathrm{H}^{w,\mu}$*-predictable projection of integer-valued random measure* μ *coincides with the process* ν_0.

This may be proved using Lemmas 6 and 7 and the uniqueness of the dual $\mathrm{H}^{w,\mu}$-predictable projection of μ.

6.11. Probabilistic form of the dual H^x-predictable projection of μ

The fact that eqn. (4) has a strong solution for X_t yields the inclusion

$$H_t^x \subseteq H_t^w \vee H_t^\mu \ ,$$

which, together with (11), shows that σ-algebras $H_t^w \vee H_t^\mu$ and H_t^x coincide. This in turn means that the classes of $\mathrm{H}^{w,\mu}$- and H^x-predictable processes coincide, and consequently that ν_0 is H^x-predictable.

The introduction of a non-singularity condition for B_u, $u \geq 0$, means that for any $n \geq 0$ we have:

$$w_t = \int_0^t B_u^{-1}(\mathrm{d}X_{n,u}^c - A_u \mathrm{d}u) \ ,$$

where

$$X_{n,t}^c = X_{n,t} - \int_0^t \int_E x \mu_n(\mathrm{d}u, \mathrm{d}x), \ n > 0, \ t \geq 0 \ .$$

It follows from these equalities that process w_t is H^{x_n}-adapted and consequently that

$$H_t^w \vee H_t^{\mu_n} \subseteq H_t^{x_n} \ .$$

Note also that equations (6) have a strong, unique solution for $X_{n,t}$, $n \geq 0$. This fact yields the inverse inclusion:

$$H_t^{x_n} \subseteq H_t^w \vee H_t^{\mu_n}$$

and consequently

$$H_t^{x_n} = H_t^w \vee H_t^{\mu_n} \ .$$

From the definition of $H_t^{\mu_n}$ we have

$$H_t^{\mu_n} \cap \{T_n < t\} = H_{T_n}^{\mu_n} \cap \{T_n < t\} = H_{T_n}^{\mu} \quad \{T_n < t\}$$

and thus

$$(H_t^{w} \vee H_{T_n}^{\mu}) \cap \{T_n < t\} = (H_t^{w} \vee H_t^{\mu_n}) \cap \{T_n < t\} = H_t^{z_n} \cap \{T_n < t\} \ .$$

Substituting the $H_t^{w} \vee H_{T_n}^{\mu}$-conditional probabilities in eqn. (3) by $H_t^{z_n}$-conditional probabilities we obtain:

$$\nu_0((0,t],\Gamma) = \sum_{n=0}^{\infty} I(T_n < t \le T_{n+1}) \sum_{p=0}^{n} \int_{T_p \Delta t}^{T_{p+1} \Delta t} \frac{d_u P(T_{p+1} \le u,\ Z_{p+1} \in \Gamma \mid H_u^{z_p})}{P(T_{p+1} \ge u \mid H_u^{z_p})} \ .$$

From Lemma 7 and the coincidence of the σ-algebras $H_t^{w} \vee H_t^{\mu}$ and H_t^{z} for any $t \ge 0$, we deduce that process

$$Y_t^{\Gamma} = \mu((0,t],\Gamma) - \nu_0((0,t],\Gamma), \quad t \ge 0$$

is an $\mathbf{H}^z$-adapted local martingale. The uniqueness of the dual $\mathbf{H}^z$-predictable projection means that measures ν_0 and ν^z coincide, thus completing the proof of Theorem 2.

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