International Institute for Applied Systems Analysis
IIASA
www.iiasa.ac.at

Tradeoff Information in Interactive Multiobjective Linear Programming Methods

Kok, M.

IIASA Working Paper

WP-84-035

May 1984

NOT FOR QUOTATION
WITHOUT PERMISSION
OF THE AUTHOR

TRADEOFF INFORMATION IN INTERACTIVE MULTIOBJECTIVE LINEAR PROGRAMMING METHODS

Matthijs Kok

May 1984
WP-84-35

INTERNATIONAL INSTITUTE FOR APPLIED SYSTEMS ANALYSIS
2361 Laxenburg, Austria

PREFACE

In this paper, Matthijs Kok, a participant in the 1983 Young Scientists' Summer Program, looks at the information given to the decision maker by various interactive methods for multiobjective decision making. He considers a number of common approaches to linear multiobjective decision problems, and shows that in these methods the decision maker usually sees only a part of the available tradeoff information. He then goes on to extend two of these approaches (the reference-point method and the interactive multiple-goal programming method) using duality theory, demonstrating that this yields additional tradeoff information that could be of interest to the decision maker.

This research was carried out as part of the Interactive Decision Analysis Project.

ANDRZEJ WIERZBICKI
Chairman
System and Decision Sciences

ABSTRACT

All of the various methods developed to handle models with multiple objectives require preference information from a decision maker in order to obtain a satisfactory solution. The ability of most decision makers to give *a priori* information about their preference structure is generally weak, but it is assumed that inspection of trial solutions generated during a computer session will help them to formulate their preferences.

In this paper we consider the information that interactive methods can supply to a decision maker. For example, they could provide tradeoff values that could be useful in assessing the interdependence of the objective functions once a trial solution has been obtained. Because there is no unique approach to the multiobjective linear programming (MOLP) problem, several approaches (and scalarization methods) are considered. The relations between the tradeoffs and the dual variables in each of these formulations of the MOLP problem are discussed. These theoretical notions are illustrated by examining the information that is given to a decision maker by some commonly used interactive methods. We show that these methods supply only a part of the available (tradeoff) information. Two existing interactive methods are then extended using the dual variables and duality properties of the problem.

In the next few years we plan to carry out some experiments with decision makers (opinion leaders) in public energy planning to see whether the ideas developed here are actually useful in practice.

TRADEOFF INFORMATION IN INTERACTIVE MULTIOBJECTIVE LINEAR PROGRAMMING METHODS

Matthijs Kok

Delft University of Technology, Delft, The Netherlands

1. INTRODUCTION

Interactive approaches are now used quite widely in multiobjective decision making. The crucial assumption in this type of approach is that a decision maker exists and can provide information on his preferences which makes it possible to obtain a satisfactory compromise solution. (Naturally, this solution depends strongly on the preference structure of the decision maker.)

Many interactive methods have been proposed in the last decade (for reviews see: Cohon, 1978; Hwang and Masud, 1979; Zeleny, 1982; White, 1983b; Chankong and Haimes, 1983. The book by Chankong and Haimes in particular gives an excellent review of concepts and methods in multiobjective programming). In this paper we will investigate the organization of interaction, and particularly the question of what type of information a model should supply to a decision maker to give him/her more insight into the decision problem. The various existing interactive methods differ widely in this regard, each method making different assumptions about the decision maker's behavior. Of course, the question of what information should be given to a decision maker is not easy to answer: the apparent lack of interest in this issue probably has less to do with unawareness of the problem than with the complications involved in addressing it. We believe, however, that investigating the question of what information is available would help us to clarify this issue. Once we know what information is available we can decide which parts of it should be supplied to the decision maker, and on what basis.

There are two main reasons why the information available in multiobjective programming problems is not unique: first, there are different approaches to the problem and, second, different assumptions can be made regarding the decision maker's behavior (these two reasons are not of course independent). Several formulations of the standard multiobjective linear programming

(MOLP) problem are discussed in Section 2, making a distinction between the *optimizing approach* and the *satisficing approach*. In Section 3 we discuss tradeoffs between objective functions. In our opinion, these tradeoffs are very important in assessing possible acceptable solutions of the model. We consider the various types of tradeoffs which have been introduced in the literature.

As with ordinary single-objective linear programming models, every MOLP model has a dual formulation. This will be the main topic of Section 4: the relation between the dual formulation and the tradeoffs described in Section 3 will also be discussed. Section 5 looks at the information offered to the decision maker in some commonly used interactive methods, and shows that only a part of the available information is usually given to the decision maker. This is illustrated in Section 6 by extending two existing interactive methods. Finally, we draw some conclusions from our study in Section 7.

In this paper we consider only *linear* programming problems with multiple objectives. It is, in general, assumed that solutions are non-degenerate and finite. Further, we consider only the values of objective functions, and not the values of decision variables. This does not mean that the latter are not interesting, but, especially in large problems, the question of which decision variables should be shown to the decision maker is problem-dependent. Also, although computer graphics can be very useful in illustrating the values of and changes in the objective functions (see, e.g., Johnson and Louchs, 1980) this will not be discussed here. Finally, we should stress that by "decision maker" we mean any person who is confronted with a decision problem involving multiple objectives (e.g., a manager of an industrial firm, a public policy spokesman, or customers in a shop).

2. PROBLEM FORMULATION

Multiobjective linear programming problems can be formulated in a number of ways. The differences arise both from assumptions made about the behavior of the decision maker and from the mathematical techniques used to calculate (nondominated) solutions.

We may divide MOLP models into two categories: *optimizing models* and *satisficing models*, although this distinction is not as clear as sometimes suggested in the literature. In satisficing models the decision maker has to specify target (or aspiration) levels that he/she wishes to attain, whereas in optimizing models no targets are set. The rationale behind this distinction is behavioral:

the two approaches make different assumptions about the way in which decision makers reach decisions. Historically speaking, the optimization approach was introduced first; the satisficing approach was not developed until the late fifties. These two approaches are now used widely in mathematical programming and, although the satisficing approach has been criticized on a number of grounds, both are generally accepted. We shall look at these approaches in this section.

2.1 Optimizing Models

As mentioned above, we will restrict ourselves to linear models. The problem may be formulated as follows:

$$\max \ Cx \tag{1}$$

subject to

$$Ax = b$$

$$x \geq 0 \ ,$$

where C is a $p \times n$ matrix of objective coefficients, A is an $m \times n$ matrix of constraint coefficients, b is an m vector of right-hand sides, and x is an n vector of decision variables.

Let C^i denote the i-th row of matrix C, and S denote the feasible set: $S = \{x \mid Ax = b, \ x \geq 0\}$. Following Steuer and Choo (1983), $Z \in \mathbb{R}^p$ is the set of all feasible objective vectors, $Z = \{z \mid z = Cx, \ x \in S\}$ (the objective space), and $N \in Z$ denotes the set of all *nondominated* objective vectors (a $\bar{z} \in Z$ is a nondominated objective vector if and only if there does not exist another $z \in Z$ such that $\bar{z}_i \geq z_i$ for all i and $z_i > \bar{z}_i$ for at least one i, $i = 1,2,\ldots,p$). Now $\bar{x}$ is an ***efficient point*** if and only if $\bar{x}$ is feasible and $\bar{x}$ is an inverse image of some $\bar{z} \in N$.

We are interested in nondominated (Pareto-optimal, efficient) solutions of (1). One way of approaching the problem is to use a multiobjective simplex method, which will give all nondominated basic solutions (see, e.g., Zeleny, 1974). However, this approach is not ideal in an interactive environment because the set of all nondominated basic solutions will in general be very large, and there are other, more appropriate, methods for calculating nondominated solutions. We will consider two methods which can be used to transform problem (1) into a scalar optimization problem: the *weighting method* and the *constraint method* (other methods exist, but in our opinion these are less important). It is clear that scalarization is only carried out for technical reasons. However, as will be shown later in this paper, the technical formulation affects the information that can be offered to the decision maker.

2.1.1 The Weighting Method

In this case the problem is formulated as follows:

$$\max \lambda^T Cx \tag{2}$$

subject to

$$Ax = b$$

$$x \geq 0 \quad .$$

Here λ is called the *weighting vector*. Without loss of generality we can assume that $\lambda_i \geq 0$, $i = 1,2,\ldots,p$, $\sum_{i=1}^{p} \lambda_i = 1$.

It is well-known (see, e.g., Zeleny, 1974) that if $\lambda_i > 0$, $i=1,2,\ldots,p$, the solution of (2) is nondominated. On the other hand, if we have an efficient solution $\bar{x}$ then there will exist a $\bar{\lambda}$ (where $\bar{\lambda}_i > 0$, $\sum_{i=1}^{p}\bar{\lambda}_i = 1$) such that $\bar{x}$ is a solution of (2). Furthermore, if $\bar{\lambda}_i = 0$ for some, but not all, $i = 1,2,\ldots,p$, and $\bar{x}$ is the unique solution of (2) with this weighting vector, then $\bar{x}$ is an efficient solution. It is also well-known (see, e.g., Zeleny, 1974) that the weighting vector corresponding to a basic solution is not unique.

Two possibilities can arise when solving (2) with a fixed weighting vector $\bar{\lambda} > 0$: the solution $\bar{x}$ is unique and is called an *efficient basic solution*, or the number of solutions is infinite. In the latter case there is obviously no guideline as to which solution to choose. Perturbation techniques do not give satisfactory results in this situation because they would produce an efficient basic solution, excluding all efficient solutions which are not basic. Because there is no rationale for restricting the set of efficient solutions to efficient basic solutions, problem formulation (2) is unsatisfactory. However, as we shall see in Section 5, this formulation is used in existing interactive methods.

Finally, note that formulation (2) can be used to list all efficient basic solutions by applying parametric optimization methods (see, e.g., Guddat, 1979).

2.1.2 The Constraint Method

In this method the problem is formulated as follows:

$$\max \ (C^i)^T x \tag{3}$$

subject to

$$(C^j)^T x \geq l_j, \quad j = 1,2,\ldots,p; \ j \neq i$$

$$Ax = b$$

$$x \geq 0 \ .$$

Here the elements of the vector $l = (l_1, l_2, \ldots, l_{i-1}, l_{i+1}, l_p)$ are lower bounds to the values of the objective functions.

It can be shown (see, e.g., Chankong and Haimes, 1983) that if $\bar{x}$ is a solution of problem (3) for some i and the solution is unique, then $\bar{x}$ is an efficient solution of problem (1). If this solution is not unique, then $\bar{x}$ may be dominated by another feasible solution, but one of the alternative solutions will be efficient. A sufficient condition for the solution of (3) to be efficient is that the dual variables related to the constraints on the objective functions must be positive. All efficient solutions of problem (1) can be obtained by a suitable choice of vector l, so that no efficient solution is excluded.

2.2 Satisficing Models

In satisficing decision analysis it is assumed that the decision maker can specify target levels for the various objective functions. It could be argued that these target levels cannot be set by the decision maker without some knowledge of the possible solutions, and there is indeed some truth in this. However, we are considering here only *interactive* decision making and in this case the above objection is not valid because the decision maker can change target levels during the interactive process, taking into account information about possible feasible solutions.

The problem formulation is:

$$\min \ y \tag{4}$$

subject to

$$\begin{aligned} Cx + y &\geq \bar{t} \\ Ax &= b \\ x &\geq 0 \\ y &\geq 0 \quad . \end{aligned}$$

Here $\bar{t}$ is the p-vector representing the target (or aspiration) levels specified by the decision maker, and y is the under-achievement vector, $y \in \mathbb{R}^p$. Note that nondominated solutions of (4), if such exist, depend on these target levels, and thus can differ from the nondominated solutions of (1). Note also that in this formulation these levels are attained from below.

It is clear that (4) is a linear problem with multiple objectives, and, as stated before, all nondominated basic solutions can be obtained by the multiobjective simplex method. However, since the number of nondominated basic solutions is generally large, this approach is not very fruitful in interactive programming, so we will take another approach and scalarize problem (4). This can be done in several ways. In this section we shall discuss the two methods which, in our opinion, are the most relevant: the *weighted distance measure approach* and the *achievement function approach*.

2.2.1 The Weighted Distance Measure Approach

Given the target levels $\bar{t} = (\bar{t}_1, \bar{t}_2, \ldots, \bar{t}_p)$ for the objective functions, the weighted distance measure problem may be formulated as follows:*

$$\min \left[\sum_{i=1}^{p} \lambda_i \, | (C^i)^T x - \bar{t}_i |^q \right]^{1/q} \tag{5}$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0 \quad . \end{aligned}$$

*Note the essential difference from problem (4): here over-achievements are also considered.

Here q is an integer parameter, $1 \leq q \leq \infty$, and λ is a weighting vector. The choice of q is not obvious, and only the values $q = 1$ (absolute value norm), $q = 2$ (Euclidean norm) and $q = \infty$ (Tchebycheff norm) represent meaningful geometrical concepts of distance. The value $q = 2$ is often used in economic planning (see, e.g., Hughes-Hallet and Rees, 1983); the value $q = 1$ is also quite popular because then problem (5) remains linear (see, e.g., Hafkamp, 1983). A corresponding linear problem can also be constructed for $q = \infty$. With $q = 1$, problem (5) can be written in linear form as:

$$\min \sum_{i=1}^{p} \lambda_i (d_i^+ + d_i^-) \tag{6}$$

subject to

$$\begin{aligned}
(C^i)^T x - d_i^+ + d_i^- &= \bar{t}_i, \quad i = 1,2,\ldots,p \\
Ax &= b \\
x &\geq 0 \\
d_i^+ &\geq 0, \quad i = 1,2,\ldots,p \\
d_i^- &\geq 0, \quad i = 1,2,\ldots,p \quad .
\end{aligned}$$

The target level $\bar{t}$ may be given by the decision maker; it can also be set equal to the vector $\tilde{t}$ with components:

$$\tilde{t}_i = \max \; (C^i)^T x$$

subject to

$$\begin{aligned}
Ax &= b \\
x &\geq 0 \quad .
\end{aligned}$$

The vector $\tilde{t}$ obtained by solving p ordinary LP problems is often called the ***utopia*** (or ***ideal***) ***point***.

In general, it is only possible to solve (5) when the values of q, $\bar{t}$ and λ have been specified. As we have already seen, the value of q is often selected on purely technical grounds, even though it influences the solution of problem (5). The values of λ (and often $\bar{t}$) have to be assessed by the decision maker.

Earlier comments concerning the weighting vector and the uniqueness of the solutions of problem (2) also hold for problem (6). One final remark should be made about problem (5): it turns out that the solution of this problem depends on the scale of measurement of the objective functions, and therefore a scaling vector should be introduced. (The weighting vector λ can also be interpreted as a scaling vector.)

2.2.2 The Achievement Function Approach

This method, proposed by Wierzbicki (1979), concentrates on the construction of modified utility functions (achievement functions) which express the utility (or disutility) of reaching (or not reaching) given target levels. The function is only used to measure deviations from these levels. The main advantage of this method compared with the previous one is that the achievement function can be constructed in such a way that the solution corresponds to a nondominated solution of problem (1). (It can easily be seen that a solution of problems (4), (5) and (6) is not necessarily nondominated with respect to problem (1). For example, if the target level in these problems is feasible, this (not necessarily nondominated) target level will be the solution.)

Choosing one particular scalarizing function, we have the following LP problem:

$$\min \; q \tag{7}$$

subject to

$$Ax = b$$

$$q \geq \gamma_i(\bar{t}_i - (C^i)^{\mathrm{T}}x) + \varepsilon e^{\mathrm{T}}(\bar{t} - Cx), \quad i = 1,\ldots,p$$

$$x \geq 0 \; .$$

Here γ is a scaling (or weighting) vector (e.g., $\gamma_i = 1/\bar{t}_i$, $i = 1,\ldots,p$), ε is a small positive constant and e is the p-dimensional unit vector.

It can easily be shown that the solution $\tilde{x}$ of (7) is efficient for every $\bar{t}$. Suppose that $\tilde{z} = C\tilde{x}$ minimizes (7) but $\tilde{z}$ is not efficient. Then there exists a $\bar{z} \in Z$ such that $\bar{z} \geq \tilde{z}$ and $\bar{z}_i > \tilde{z}_i$ for at least one i, and $e^{\mathrm{T}}(\bar{t} - \bar{z}) < e^{\mathrm{T}}(\bar{t} - \tilde{z})$, so that $\tilde{z}$ cannot be an optimal solution of (7).

Another formulation of this problem is

$$\min \ q \tag{7a}$$

subject to

$$Ax = b$$

$$qe \geq Dy + \gamma^{\mathrm{T}}(\bar{t} - Cx)$$

$$x \geq 0$$

$$y \geq 0 \quad .$$

Here D is a $p \times p$ matrix with elements $d_{ij} = -\varepsilon,\ i \neq j$, and $d_{ii} = 1,\ i,j = 1,\dots,p$, where $0 < \varepsilon << 1$ to ensure that the solutions are nondominated (see Section 4).

In Section 5 we will consider how formulations (1)–(7a) of MOLP problems are used in some existing interactive methods.

3. TRADEOFFS

Suppose that we have two solutions $\bar{x}$ and $\tilde{x}$ of any of the problems (1)–(7). If a decision maker is asked to assess his/her preferences regarding these two solutions (i.e., to decide whether he/she prefers $\bar{x}$ or $\tilde{x}$, is indifferent, or cannot choose between them), he/she must first assess the *tradeoffs*. According to Webster's New World Dictionary, a tradeoff is "an exchange, especially a giving up of benefit, advantage, etc., in order to gain another regarded as more desirable" (Chankong and Haimes, 1983). It is clear that tradeoffs are very important in MOLP problems: any choice between two feasible (nondominated) alternatives, or between several target levels, will necessarily involve the assessment of tradeoffs. Obviously, the tradeoffs between a number of feasible solutions can be presented in different ways: all solutions $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^s$ can be given explicitly, or we can take one solution $\bar{x}^i$, $1 \leq i \leq s$, and give the tradeoffs $T^j = C\bar{x}^j - C\bar{x}^i$, $j = 1,2,\dots,s$, $j \neq i$ with respect to this solution. The second of these methods is particularly appropriate if the decision maker prefers to make pairwise comparisons. In this case we can of course also present all other solutions $\bar{x}^j$, $j = 1,2,\dots,s$, $j \neq i$ for comparison with solution $\bar{x}^i$.

There are no technical reasons for presenting the information in one form rather than another, so the choice of an approach implies that assumptions are being made regarding the behavior of the decision maker. In the case of pairwise comparisons, decision makers seem to prefer to compare alternative solutions $\bar{x}^i$ and $\bar{x}^j$, $j = 1,\ldots,s$, $j \neq i$, rather than to compare the tradeoffs T^j, $j = 1,2,\ldots,s$, $j \neq i$, with respect to solution $\bar{x}^i$ (Zionts and Wallenius, 1983).

Various types of tradeoffs have been introduced in the literature, some of which will be discussed in this section. The first distinction we want to draw is between *indifference tradeoffs* (or *subjective tradeoffs*) and *solution space tradeoffs* (or *objective tradeoffs*). Indifference tradeoffs are assessed by a decision maker without regard for feasibility: he/she has to determine what change in one objective function would compensate for a change in another objective function. These tradeoffs can be used to assess the utility function of the decision maker. Solution space tradeoffs are produced by the computer model from a set of restrictive model constraints. These tradeoffs do not reflect preference information, but can be used by the decision maker to gain insight into the decision problem. We shall now investigate these solution space tradeoffs in more detail.

Solution space tradeoffs are of two types: *partial tradeoffs* and *total tradeoffs* (Haimes and Chankong, 1979; Chankong and Haimes, 1983). This distinction is only useful when there are more than two objective functions. The formal definitions of partial and total tradeoffs are given below.

Consider two feasible alternatives $\bar{x}$ and $\tilde{x}$, for which the values of the objective functions are $C\bar{x} = ((C^1)^T\bar{x},\ldots,(C^p)^T\bar{x})$ and $C\tilde{x} = ((C^1)^T\tilde{x},\ldots,(C^p)^T\tilde{x})$, respectively. Denote the objective function with objective coefficients C^i by number i $(i=1,2,\ldots,p)$. The ratio of the difference between the values of an objective function i for $x = \bar{x}$ and $x = \tilde{x}$ to the difference between the values of the objective function j for $x = \bar{x}$ and $x = \tilde{x}$ will be denoted by $T_{ij}(\bar{x}, \tilde{x})$, where:

$$T_{ij}(\bar{x}, \tilde{x}) = \begin{cases} \dfrac{(C^i)^T\bar{x} - (C^i)^T\tilde{x}}{(C^j)^T\bar{x} - (C^j)^T\tilde{x}}, & (C^j)^T\bar{x} \neq (C^j)^T\tilde{x} \\ \infty, & (C^j)^T\bar{x} = (C^j)^T\tilde{x} \end{cases} .$$

The vector $T_{ij}(\bar{x}, \tilde{x})$ is called a vector of ***partial tradeoffs*** between the objective functions i and j on going from $\bar{x}$ to $\tilde{x}$ if $(C^k)^T\bar{x} = (C^k)^T\tilde{x}$ for all $k = 1,2,\ldots,p$, and $k \neq i,j$. If, on the other hand, $(C^k)^T\bar{x} \neq (C^k)^T\tilde{x}$ for at least one $k = 1,2,\ldots,p$, and $k \neq i,j$, then $T_{ij}(\bar{x}, \tilde{x})$ is called the vector of *total tradeoffs* between objective functions i and j on going from $\bar{x}$ to $\tilde{x}$.

The significance of the partial tradeoff vector is that it enables the decision maker to compare changes in two objectives at a time. It is often claimed that this makes it easier to assign preferences (see, e.g., Chankong and Haimes, 1983).

In continuous problems such as (1) it makes sense to introduce a tradeoff rate. This may be defined as follows (Chankong and Haimes, 1983): given a feasible alternative $\bar{x}$ and a feasible direction $\bar{d}$ emanating from $\bar{x}$ (i.e., there exists an $\bar{\alpha} > 0$ such that $\bar{x} + \alpha\bar{d} \in S$ for $0 \le \alpha \le \bar{\alpha}$), the ***total tradeoff rate*** $t_{ij}(\bar{x}, \bar{d})$ between objective function i and j at $\bar{x}$ along the direction $\bar{d}$ is given by

$$t_{ij}(\bar{x},\bar{d}) = \lim_{\alpha \to 0} T_{ij}(\bar{x} + \alpha\bar{d}, \bar{x}) \quad .$$

The partial tradeoff rate can be introduced in an analogous way: if $\tilde{d}$ is a feasible direction with the property that there exists an $\bar{\alpha} > 0$ such that $(C^k)^{\mathrm{T}}(\bar{x} + \alpha\tilde{d}) = (C^k)^{\mathrm{T}}\bar{x}$ for all $k = 1,\ldots,p$ and $k \neq i,j$, and for all $0 \le \alpha \le \bar{\alpha}$, then the corresponding $t_{ij}(\bar{x}, \tilde{d})$ is called the ***partial tradeoff rate***.

The concepts introduced in this section will be illustrated in connection with some existing interactive MOLP methods in Section 5.

4. DUALITY

In this section we shall look at the dual formulations of the problems introduced in Section 2. More specifically, we shall investigate the relation between the dual variables and the tradeoffs discussed in Section 3. Duality theorems and their proofs will not be given in this section: we shall simply refer to the relevant literature.

Before introducing the dual problems under consideration, we shall first summarize the properties of the dual formulation for the general MOLP problem (1). The duality properties of ordinary LP problems are:

(i) The primal problem has a finite solution $\Longleftrightarrow$ The dual problem has a finite solution. The optimal values are the same.

(ii) The primal (dual) problem is inconsistent $\Longleftrightarrow$ The dual (primal) problem has no finite optimal value.

(iii) The dual formulation of the dual problem is the primal problem.

The dual formulation of MOLP problem (1) as an MOLP problem with a $p \times m$ matrix of dual variables (Isermann, 1978) has the same properties as the dual formulation of an ordinary LP problem except that the dual formulation of the dual problem is not equivalent to the primal problem. In this case the duality properties are:

(i) For each efficient solution of the primal MOLP problem (1), there exists an efficient solution of the dual problem, with the same value of the objective functions (and vice versa).

(ii) The primal (dual) problem is inconsistent $\Leftrightarrow$ The dual (primal) problem has no finite optimal value.

We shall now introduce the dual formulation. As in Section 2, we first discuss the optimization approach and then the satisficing approach.

4.1 Optimizing Models

Consider problem (1). The dual formulation can be obtained in two ways:

1. Combine the objective functions of problem (1) using a weighting vector $\lambda > 0$ in order to get one right-hand-side vector in the dual problem. The dual problem is now:

$$\min \ b^T u \tag{8}$$

subject to

$$A^T u \geq C^T \lambda \ ,$$

where u is the m-vector of dual variables. Problem (8), which is also the dual of problem (2), can now be seen as a multiparametric LP problem (Hannan, 1978). We can also solve problem (8) for a fixed weighting vector $\bar{\lambda}$ (Kornbluth, 1974).

In this formulation the vector of dual variables u depends on the weighting vector λ. As we have stated before, it is not possible to determine a unique weighting vector λ corresponding to an efficient basic solution, and thus, in this formulation, the dual variables related to an efficient basic solution are not unique. We can conclude that the dual variables in this formulation do not give us much insight into the decision problem.

2. Another dual of problem (1) can be formulated (Isermann, 1977, 1978). In this formulation there is no vector of dual variables, but rather a matrix: each dual variable corresponds not only to a constraint, but also to an objective function. Thus we have a $p \times m$ matrix of dual variables U. Our new dual formulation of (1) is now:

$$\min \; Ub \tag{9}$$

subject to

$$\nexists \; w \geq 0 : A^{\mathrm{T}} Uw \gneq Cw \quad ,$$

where U is the $p \times m$ matrix of dual variables and $\gneq$ is an ordering relation defined by: $x \gneq y$ iff $x > y$ and $x \neq y$. The proof that problem (9) is a dual of problem (1) can be found in Isermann (1978) and Nieuwenhuis (1983).

This dual formulation is based on the characterization of an efficient basic solution $\bar{x}$ of (1) as given in Theorem 1. We must first introduce some notation. Let the coefficient matrix A be partitioned into a square, non-singular $m \times m$ submatrix B corresponding to the basic variables, and a matrix R containing the rest. Similar partitioning can be used for the feasible solution $x = (x_B, x_R)$ and the objective function matrix $C = (C_B, C_R)$.

Theorem 1.
$\bar{x}$ is an efficient basic solution $\Longleftrightarrow \nexists \, w \geq 0 : (C - C_B B^{-1} A)w \gneq 0$.

The proof of this theorem is given in the Appendix. The efficiency of a feasible basic solution can also be characterized in terms of the reduced cost matrix $W = C_R - C_B B^{-1} R$.

Theorem 2.
$\bar{x}$ is an efficient basic solution $\Longleftrightarrow \nexists \; w \geq 0 : Ww \gneq 0$.

The proof of this theorem can also be found in the Appendix.

The dual variables U of problem (9) can be interpreted in the same way as in ordinary single-objective LP problems: the variable U_{lj} indicates how much the value of the l-th objective function changes with a unit change in the right-hand side of the j-th constraint (of course, this change in the solution must remain feasible). These variables give the mutual dependence of the objective functions on the right-hand sides of the constraints, and not on the changes in the objective functions. We can conclude that these dual variables are of little use to the decision maker because they give only the mutual dependence of the objective functions on the right-hand-side vector b.

The reduced-cost matrix W contains information about changes in the objective functions when one unit of a non-basic variable is brought into the basis. Therefore, every column of W can be seen as a vector of tradeoffs between adjacent feasible basic solutions (two basic solutions $\bar{x}$ and $\tilde{x}$ are called adjacent iff $\bar{x}$ and $\tilde{x}$ have $m-1$ basic variables in common)*. Assume that we have obtained a nondominated basic solution $\bar{x}$. It is clear that not all of the adjacent feasible basic solutions (adjacent with respect to $\bar{x}$) are necessarily efficient. It can also happen that an adjacent feasible basic solution is efficient, but that the edge leading to that solution is not efficient (see Section 5).

It is clear that these tradeoffs give the decision maker considerable insight into the mutual dependence of the objective functions. To obtain these tradeoffs it is necessary to calculate a nondominated trial solution. The tradeoffs (the reduced-cost matrix W) can be obtained directly from a multiobjective simplex tableau (Zeleny, 1974; Yu and Zeleny, 1975). Note that the tradeoffs which lead to nondominated solutions of (1) are of special interest to the decision maker. However, it is not necessary to use a multiobjective simplex tableau: the tradeoffs can also be obtained from an ordinary (single-objective) simplex tableau in the following way.

1. Combine the objective functions using a weighting vector $\bar{\lambda} > 0$ to obtain the efficient feasible basic solution $\bar{x}$; this results in one objective function $\bar{\lambda}^T Cx$.

2. Introduce variables z_i, $i=1,2,\ldots,p$, into the model, and add p additional constraints:

$$z_i = \sum_{j=1}^{n} c_{ij} x_j, \quad i=1,2,\ldots,p \quad .$$

* In the terminology of Section 3 each column of W is a total tradeoff vector.

The problem can now be formulated as:

$$\max \ \lambda^{\mathrm{T}} z \tag{2a}$$

subject to

$$Cx - z = 0$$

$$Ax = b$$

$$x \geq 0 \ .$$

The reduced-cost matrix can immediately be obtained from the simplex tableau solving this problem for a fixed $\bar{\lambda}$.

The dual of problem (3) can be written down immediately, since it is a single-objective LP problem. We also have dual variables v_j related to the lower bounds on the objective functions: the variable v_j indicates how much the value of the objective function which is maximized changes with a change of one unit in the lower bound of the objective function j ($j = 1,2,\ldots,i-1,i+1,\ldots,p$).

4.2 Satisficing Models

Consider problem (4). Here again the dual can be obtained in two ways:

1. Combine the objective functions using a weighting vector $\lambda > 0$ in order to get one right-hand-side vector in the dual problem:

$$\max \quad u^{\mathrm{T}} b + v^{\mathrm{T}} \bar{t} \tag{10}$$

subject to

$$uA + vC \leq 0$$

$$v \leq \lambda^{\mathrm{T}} e$$

$$v \geq 0 \ ,$$

where u and v are the m-vector and p-vector, respectively, of dual variables, and e is the p-dimensional unit vector. This has the same drawbacks as the dual formulation (8) of problem (1).

2. Another dual formulation of (4) can be found in Isermann (1977). We first rewrite problem (4) as:

$$\max \ \tilde{C}\tilde{x} \tag{4a}$$

subject to

$$\tilde{A}\tilde{x} = \tilde{b}$$

$$\tilde{x} \geq 0 \ ,$$

where

$$\tilde{C} = [0, -I, 0]$$

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ C & I & -I \end{bmatrix}$$

$$\tilde{b} = \begin{bmatrix} b \\ \bar{t} \end{bmatrix}$$

$$\tilde{x} = \begin{bmatrix} x \\ y \\ s \end{bmatrix} \ .$$

As we have already seen, the dual problem of (4a) is:

$$\min \ \tilde{U}\tilde{b}$$

subject to

$$\not\exists \ w \geq 0 : (\tilde{C} - \tilde{U}\tilde{A})w \gneq 0 \ ,$$

where $\tilde{U} = [U \ V]$. This can be rewritten as:

$$\min \ Ub + V\bar{t} \tag{11}$$

subject to

$$\begin{bmatrix} (UA + VC)w_1 \\ Vw_2 \\ -Vw_3 \end{bmatrix} \gneqq \begin{bmatrix} 0 \\ w_2 \\ 0 \end{bmatrix} \quad \nexists\, w_1, w_2, w_3 \geq 0 \quad ,$$

where U is the $p \times m$ matrix and V the $p \times p$ matrix of dual variables, and $w = (w_1, w_2, w_3)^{\mathrm{T}}$, where w_1 is an n-vector, and w_2 and w_3 are p-vectors. The symbol 0 on the right-hand side denotes a p-dimensional zero vector.

The restrictions in problem (11) stem from the definition of a nondominated solution to (4).

In this formulation the matrix U can be given the same interpretation as in dual problem (8). The dual variables V can be interpreted as follows: variable V_{ij} gives the change in objective function i corresponding to a one-unit change in the lower-bound constraint on objective function j $(i,j = 1,2,\ldots,p)$. It is clear that in this formulation the reduced-cost matrix can be again usefully be given to the decision maker.

The dual formulation of problem (5) will not be considered, since this is generally not a linear problem. Now consider problem (6), the well-known linear goal-programming problem. Because there is only one objective function in this formulation we have an m-vector u and a p-vector v of dual variables.

The dual problem of (6) can be formulated as:

$$\max\ u^{\mathrm{T}}b + v^{\mathrm{T}}\bar{t} \tag{12}$$

subject to

$$uA + vC \leq 0$$

$$v_i \geq -\lambda_i, \quad i = 1,2,\ldots,p$$

$$v_i \leq \lambda_i, \quad i = 1,2,\ldots,p \quad .$$

Again, we can conclude that the dual variables are dependent on the weighting vector λ. However, the dual variables v can give the decision maker insight into the sensitivity of the value of the objective function to changes in the target values.

The dual problem of (7) is:

$$\max\ ub + \sum_{i=1}^{p} v_i(\gamma_i \bar{t}_i + \varepsilon e^{\mathrm{T}}\bar{t}) \tag{13}$$

subject to

$$
\begin{aligned}
uA + v(C + \varepsilon e^{\mathrm{T}} C) &\le 0 \\
e^{\mathrm{T}} v &= 1 \\
v &\ge 0 \\
u &\ge 0 \quad .
\end{aligned}
$$

We cannot prove that the dual variables v are strictly positive; however, this does not mean that solutions of problem (13) are not efficient (see Section 2).

The dual problem of (7a) is:

$$\max \; ub + v\gamma\bar{t} \tag{14}$$

subject to

$$
\begin{aligned}
uA + vC &\le 0 \\
e^{\mathrm{T}} v &= 1 \\
v^{\mathrm{T}} D &\ge 0 \\
v &\ge 0 \quad .
\end{aligned}
$$

It can be proven that in this case $v > 0$ (since $v_i - \varepsilon\sum_{j\neq i} v_j \ge 0$ and $\sum_i v_i = 1$ yields $v_i \ge \varepsilon/(1+\varepsilon) > 0$, $i = 1,\ldots,p$ so the last constraint in problem (7) is redundant). This guarantees that the solution of (7) and (14) is nondominated. We will return to this formulation in Sections 5 and 6.

Concluding this section, we can state that two types of tradeoff information are useful in giving a decision maker more insight into the decision problem:

(i) The dual variables related to the constraints on the values of the objective functions: the sensitivity of the lower bounds can be assessed directly. The dual variables can be in the form of a p-vector (problems (10), (12), (13)) or a $p \times p$ matrix (problem (11)). As we shall see in Section 6, the $p \times p$ matrix of dual variables of problem (11) is also available in a slightly modified formulation of problem (14).

(ii) The reduced-cost matrix W: each column W_j of this matrix is a tradeoff between adjacent basic feasible solutions.

5. EXISTING INTERACTIVE METHODS

Many interactive methods for handling MOLP problems have been proposed in the last decade. It is impossible to discuss all of them here, so we have selected several more or less at random, while still covering a broad class of methods. (For detailed reviews see Chankong and Haimes, 1983; White, 1983b.) We shall look at the following methods:

1. The Zionts and Wallenius method
2. The surrogate-worth-tradeoff method
3. The interactive multiple-goal programming method
4. The reference-point method
5. Steuer's weighted Tchebycheff method

We shall consider applications only to (static) linear problems, although some of these methods can be applied to more general models (e.g., the reference-point method may be used with both nonlinear and dynamic models. In the latter case the discussion of tradeoffs would include the time preferences of the decision maker).

It is not our intention to criticize these methods: we shall simply use them to illustrate the ideas (concerning the tradeoff information given to the decision maker) developed in the preceding sections.

5.1 The Zionts and Wallenius Method

The Zionts and Wallenius method (Zionts and Wallenius, 1976; Zionts and Wallenius, 1983; Zionts, 1983) uses problem (2) (or the equivalent problem (2a)) as the formulation of the MOLP problem. First a trial solution $\bar{x}$ is calculated, using an arbitrary weighting vector in the first iteration. The ***total tradeoff rate*** along one of the edges of the feasible region emanating from the extreme point $\bar{x}$ is then presented to the decision maker. It is clear that only nondominated edges are relevant. The decision maker has to assess these total tradeoff rates, i.e., he/she has to decide if the suggested tradeoff reflects his/her preferences. (The decision maker is also allowed to answer "I don't know".) Using these answers a new weighting vector λ and trial solution are calculated, and the process is repeated. The interaction ends if the decision maker cannot identify preferred tradeoff rates.

The solution of problem (2) generally yields a basic solution. However, if the decision maker has a nonlinear (unknown) utility function, the solution is not necessarily basic: how close the best basic solution found is to the "true" optimal solution depends on the structure of the problem. Zionts (1983) stresses that the objective function is not used as a utility function, but rather "to identify good (and hopefully optimal) alternatives, and present these to the decision maker in helping him to make a decision". This is true, but there is no rationale for excluding all nonbasic nondominated solutions.

Comparing this method with the various approaches to MOLP problems given in Section 2, we conclude that this approach does not make use of lower bounds on the values of the objective functions, nor does it assume "satisficing behavior" on the part of the decision maker. Of course, it is a simple matter to introduce lower or upper bounds on the values of the objective functions.

5.2 The Surrogate-Worth-Tradeoff Method

This method, originally developed by Haimes and Hall (see, e.g., Chankong and Haimes, 1983), uses the *partial tradeoff vector* introduced in Section 3. These partial tradeoffs are calculated using formulation (3). The lower bounds are updated at every iteration, the values for the first iteration being guessed, as in the Zionts and Wallenius method. There is no rule governing which objective should be taken as the objective function in problem (3); however, we recommend that either a dominant objective or one in familiar units should be chosen.

At each iteration we solve problem (3) with lower bounds $l_1, l_2, l_{k-1}, l_{k+1}, \ldots, l_p$ and objective function $(C^k)^T x$, and obtain a nondominated solution $\bar{x}$. Let v_j, $j=1,2,k-1, k+1,\ldots,p$ denote the dual variables of these lower bounds, and suppose that $v_j > 0$, $j = 1,2,\ldots,k-1,k+1,\ldots,p$. (If $v_j = 0$ for some j we have to modify this method, see Chankong and Haimes, 1983.) Now each v_j represents the nondominated partial tradeoff rate between $(C^k)^T\bar{x}$ and $(C^j)^T\bar{x}$ when all other objectives are held fixed at their respective values at $\bar{x}$. These tradeoffs are presented to the decision maker, together with the lower bounds. The decision maker is then asked: "Given that $z_i = (C^i)^T\bar{x}$, $i=1,2,\ldots,p$: for all $j = 1,\ldots,p$, how (much) would you like to decrease z_k by v_j units for each one unit increase in z_j with all other z_i remaining unchanged?" (Chankong and Haimes, 1983)

The decision maker also has to determine the "surrogate worth" of the tradeoffs. The method proceeds by changing the lower bounds according to the answers given by the decision makers. Comparing the information that is available and the information that is given to the decision maker, we see that all useful information available in this formulation is actually given to the decision maker. However, the following slightly different formulation makes more information available:

$$\max\ z_k \tag{15}$$

subject to

$$Cx - z = 0$$

$$Ax = b$$

$$z_j \geq l_j, \quad j=1,2,\ldots,k-1,k+1,\ldots,p$$

$$x \geq 0 \quad .$$

This formulation is equivalent to problem (3), but makes the reduced-cost matrix immediately available.

5.3 The Interactive Multiple-Goal Programming Method

This approach (Nijkamp and Spronk, 1980; Spronk, 1981) is also based on formulation (3). In this case p ordinary single-objective LP problems are solved at each iteration, yielding solutions $\bar{z}_i = (C^i)^{\mathrm{T}}\bar{x}^i$, $i=1,\ldots,p$; here $\bar{x}^i$ is the nondominated solution of problem (3) taking $(C^i)^{\mathrm{T}}x$ as the objective function and ignoring all the rest. The vector $\bar{z}$ is the *utopia (ideal) point*. Using the solutions of the p LP problems it is easy to calculate the *nadir point* $\tilde{z}$:

$$\tilde{z}_i = \min_{j=1,\ldots,p} (C^i)^{\mathrm{T}}\bar{x}^j, \quad i=1,\ldots,p \quad .$$

The nadir point is then presented to the decision maker as a trial solution, together with the "potency matrix" containing the utopia point $\bar{z}$ and the nadir point $\tilde{z}$. Next, the decision maker is asked which objective function value should be improved first. The lower bound of this objective function is then updated, possibly using *a priori* preference information. We again calculate a potency matrix, and the decision maker is asked whether the shifts ("sacrifices") counterbalance the proposed improvement in the solution. If so, the decision maker is asked whether the solution should be improved further; if, on the other hand, the sacrifices are judged to be too heavy, the proposed increase in the value of the objective function is obviously too large. In this case a new lower bound is calculated, which in turn has to be evaluated by the decision maker.

One of the charming features of this method is its simplicity. The trial solutions are not efficient, so we cannot speak of tradeoffs between efficient solutions. However, it is again possible to supply the decision maker with more information, as we shall see in Section 6, where this method is extended.

5.4 The Reference-Point Method

The basic idea of the *reference-point method* (Wierzbicki, 1979, 1982; Lewandowski and Grauer, 1982) is to construct an *achievement scalarizing function*. This may be interpreted as the problem of finding the nondominated point "nearest" (in the minimax sense) to any reference point given by the decision maker. The formulation of the MOLP problem is as in problem (7). We shall now discuss this method as it is used in DIDASS, a Dynamic Interactive Decision Analysis and Support System developed at IIASA (Grauer, 1983), setting the coefficient ρ in the achievement scalarizing function equal to the number of objectives. The information given to the decision maker is based on a two-stage model of the decision process: in the first stage the decision support (payoff) matrix is presented to the decision maker (this is a $p \times p$ matrix containing elements $(C^i)^T\bar{x}^j$, where $\bar{x}^j$ is the optimal solution for objective function $(C^j)^T x$); in the second stage the nondominated point "nearest" to the decision maker's reference point is provided. The decision maker can change his reference point at each iteration, leading to a new nondominated solution. In the most recent implementation of DIDASS (Grauer, 1983), the dual variables of problem (14) are also given to the decision maker. These dual variables give the change in the minimum value of the difference between the optimal solution and the reference point corresponding to a change of one unit in the reference point. However, once again more information is potentially available, as we shall see in the extension given in Section 6.

5.5 Steuer's Weighted Tchebycheff Method

This method (Steuer, 1982; Steuer and Choo, 1983) is quite similar to the reference-point method described above. The information given to the decision maker consists of a certain number of nondominated solutions (tradeoffs are implicit), and the decision maker has to assess which he/she prefers. A new selection of nondominated solutions is then calculated (but now from a smaller set) and once again offered to the decision maker.

The number of solutions offered to the decision maker is a matter of judgment; in practice, 5–10 solutions are usually given (Steuer and Harris, 1980). These solutions are calculated in the following way. First, the utopia (ideal) point $\tilde{f}$ is calculated (if there is more than one utopia point, or $\tilde{f}_i = \tilde{f}_j,\ i,j = 1,\ldots,p,\ i \neq j$, then $\tilde{f} = \tilde{f} + \varepsilon,\ \varepsilon > 0$). The distance between any $z \in Z$ and the utopia point $\tilde{f}$ is then measured using the augmented weighted Tchebycheff metric, which is defined as follows:

$$\max_{i=1,\ldots,p} \{\lambda_i(\tilde{f}_i - (C^i)^T x)\} + \varepsilon \sum_{i=1}^{p} (\tilde{f}_i - (C^i)^T x) \quad ,$$

where λ is the weighting vector and ε is a positive scalar sufficiently small to ensure that solutions are nondominated. It can be shown (Steuer and Choo, 1983) that a solution $\bar{z}$ of problem (1) is nondominated if and only if there exists a weighting vector λ such that $\bar{z}$ minimizes the augmented weighted Tchebycheff problem (7). (The formulation (7a) can of course also be used in this method.)

The method then proceeds as follows. A large set of weighting vectors (consistent with preference information obtained from the decision maker in earlier iterations) is generated. Using "filtering" techniques (Steuer and Harris, 1980), several of these vectors are selected and used to solve problem (7), in order to compute maximally dispersed representatives of the set of nondominated objective vectors. This does not, however, mean that there will be a representative set of objective vectors, and therefore these vectors are again "filtered".

The advantage of using problem (7) instead of problem (2) (Steuer, 1977) is that formulation (7) does not exclude nonbasic solutions.

In this method, tradeoff information is provided in the form of maximally dispersed alternatives from a given set: apparently it does not make sense to supply the decision maker with tradeoff information based around a certain solution.

Finally, note that although this method is quite similar to the reference-point method, there are two important differences:

(a) In the reference-point method only one nondominated solution is calculated, while in this Tchebycheff method a selection of nondominated solutions are calculated.

(b) Unlike the reference-point method, this Tchebycheff method does not allow the decision maker to specify the reference point: this is fixed as the utopia point.

6. EXTENSIONS OF TWO METHODS

In this section we propose extensions of two of the methods discussed in the preceding section: the interactive multiple-goal programming method and the reference-point method. These extensions are concerned only with the information about possible alternatives and tradeoffs in objective space. The purpose of this section is to show that the decision maker can be given more information than the methods suggest when he/she is assessing trial solutions generated during a computer session. This information can be made available on request: e.g., instead of giving the system a new reference point, the decision maker may ask the system for tradeoff information around a certain trial solution.

6.1 Extension of the Interactive Multiple-Goal Programming Method

In every major iteration of this method (see Section 5.3) we optimize each objective function separately, with lower limits imposed on all other objective functions. If the solutions are unique, this results in at most p efficient solutions. If the solutions are not unique, one of them will be efficient. In the original method these solutions are not shown to the decision maker, but this could be done at his/her request. Moreover, $(p-1)$ dual variables are associated with each solution; these are related to the minimum value constraints of the objective functions (it obviously makes no sense to formulate a minimum value constraint for the objective function which is being optimized). Thus we have a total of $p \times (p-1)$ dual variables available. It seems rather excessive to present a matrix containing all these dual variables to the decision maker (although the decision maker could of course request to see them if he/she wanted) – it would be more reasonable to proceed as follows. If the decision maker wants to increase the minimum value of a particular objective function j, then the system should give him/her the option of seeing the $(p-1)$ dual variables of this constraint in the $(p-1)$ LP problems. These dual variables give the corresponding "losses" in the values of the objective functions in the neighborhood of the optimal solution. All that is required is to ask the decision maker:

> "Do you want to see the changes in the objective functions caused by increasing the minimum value of one objective function by one unit? If yes, indicate which objective function."

Another possibility is that the decision maker is interested in the changes in the minimum values of the objective functions caused by decreasing the maximum value of objective function j. In this case we can give the inverse of the dual variables to the decision maker.

The total tradeoff vectors (i.e., the reduced-cost matrix) are also available, but we will not discuss their use in this method.

Finally, applications of this method show that the dual variables are often used to obtain information about the mutual dependence of the objective functions (see, e.g., van Driel et al., 1983). However, this is done *ad hoc* and not by changing the options available in the method.

6.2 Extension of the Reference-Point Method

The reference-point method as described in Section 5.4 does not provide the decision maker with tradeoff information. This means that the decision maker has to specify a new reference point without any knowledge of nondominated solutions in the neighborhood of the calculated nondominated solution. To see what tradeoff information could be given we rewrite problem (7) as

$$\min\ q \tag{7b}$$

subject to

$$\begin{aligned} z - Dy + qe + s &= \bar{t} \\ Ax &= b \\ Cx - z &= 0 \\ x, y, z &\geq 0 \ . \end{aligned}$$

We shall set

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 & 0 & 0 \\ C & -I & 0 & 0 & 0 \\ 0 & I & -D & e & I \end{bmatrix}$$

and let $\tilde{B}$ be the basic matrix corresponding to an optimal basic solution of (7b).

Suppose that we have calculated an optimal basic solution $\bar{x}$ of problem (7b) with $\bar{z} = C\bar{x}$. What tradeoff information can be given to the decision maker?

First we look at the dual variables of the constraints with the reference point on their right-hand side. These dual variables give only the sensitivity of the optimal value of the objective function q to changes in the reference point. It would be more useful to know the sensitivities of the values of all objective functions $\bar{z} = C\bar{x}$ to changes in the reference point. This information is available: we can obtain the sensitivity to (small) changes μ in the reference point $\bar{t}$ from the simplex tableau. A tradeoff vector is available for each component of the reference point, so we have a $p \times p$ matrix of tradeoff vectors (some or all of which may be zero). This matrix is a part of the $\tilde{B}^{-1}$ matrix, which itself is part of the simplex tableau (for more details see Despontin and Vincke, 1977). Note that this tradeoff matrix has exactly the same interpretation as the matrix of dual variables V in problem (10).

The decision maker may then use this tradeoff information to choose a new reference point. However, we can also proceed in another way (Despontin and Vincke, 1977; Isermann, 1977). The decision maker chooses an objective which he/she wants to improve. The corresponding tradeoff vector v is then offered to the decision maker and he/she has to specify a stepsize τ in the direction of the tradeoff vector v, such that a new nondominated solution $\bar{\bar{z}}$ is reached:

$$\bar{\bar{z}} = \bar{z} + \tau v; \quad 0 \le \tau \le \bar{\tau} \quad ,$$

where $\bar{\tau}$ is the upper bound on the stepsize; above $\bar{\tau}$ the $\bar{\bar{z}}$ becomes infeasible.

There are of course other ways to extend and modify the reference-point method as it is now used in DIDASS (see, e.g., Kallio et al., 1981). Whether this will turn out to be fruitful or not depends on the capabilities of the decision maker (or, more precisely, on whether our assumptions concerning these capabilities are correct).

7. CONCLUSIONS

The question of which tradeoff information should be given to a decision maker is a central issue in interactive MOLP methods. We have shown that, in general, more information is available than is actually given to the decision maker. Of course, it can be argued that we cannot give the decision maker all the available information because otherwise he/she would not be able to see the wood for the trees. However, if the information is made available to the decision maker only on request this argument is not valid. Furthermore, decision makers might require different types of information at the beginning and end of a computer session – decision makers familiar with computerized decision support systems may not want the same information as inexperienced users. The drawback of making more information available is of course that the interaction between the decision maker and the computer becomes more complicated. The problem is to find a compromise between the quality of information available to the decision maker and the complexity of the interaction. This paper only provides a framework for investigating this question; much research still remains to be done.

APPENDIX: PROOFS OF THEOREMS 1 AND 2

Theorem 1

$\bar{x}$ is an efficient basic solution $\Leftrightarrow$ $\nexists\ w \geq 0$: $(C - C_B B^{-1} A)\, w \gneqq 0$.

Proof (Isermann, 1978).

(i) Let $\bar{x}$ be an efficient basic solution. Then:

$$\exists\ \bar{\lambda} > 0, \quad \forall x \in S:\ \bar{\lambda}^{\mathrm{T}} C \bar{x} > \bar{\lambda}^{\mathrm{T}} C x \quad .$$

We can write:

$$C\bar{x} = C_B B^{-1} b = C_B B^{-1} A x$$

so that

$$\bar{\lambda}^{\mathrm{T}} C\bar{x} = \bar{\lambda}^{\mathrm{T}} C_B B^{-1} b = \bar{\lambda}^{\mathrm{T}} C_B B^{-1} A x \quad .$$

We now have a single-objective LP problem with the optimality condition

$$\bar{\lambda}^{\mathrm{T}} C - \bar{\lambda}^{\mathrm{T}} C_B B^{-1} A \leq 0 \quad ,$$

or, equivalently:

$$\bar{\lambda}^{\mathrm{T}} (C - C_B B^{-1} A) \leq 0 \quad .$$

We now invoke Motzkin's theorem of the alternative (see, e.g., Mangasarian, 1969):

$$\{\exists\ \bar{\lambda} : \bar{\lambda} > 0,\ \ (C_B B^{-1} A - C)^{\mathrm{T}} \bar{\lambda} \geq 0\}$$

$$\Leftrightarrow$$

$$\{\nexists\ \tilde{\mu} \gneqq 0,\ \ \nexists\ \bar{\mu} \geq 0 : \tilde{\mu} + (C^B B^{-1} A - C)\bar{\mu} = 0\} \quad .$$

The last part is equivalent to

$$\nexists\ w \geq 0 : (C - C_B B^{-1} A) w \gneqq 0 \quad .$$

(ii) The condition

$$\nexists\ w \geq 0 : (C - C_B B^{-1} A) w \gneqq 0$$

implies

$$\nexists\, x \in S : Cx \gneq C_B B^{-1} Ax \quad ,$$

where $C_B B^{-1} Ax = C_B B^{-1} b = C\tilde{x}$, and $\tilde{x}$ is a feasible basic solution. We then have $\nexists\, x \in S: \;\; Cx \gneq C\tilde{x}$, so that $\tilde{x}$ is efficient.

Theorem 2

$\bar{x}$ is an efficient basic solution $\Leftrightarrow$ $\nexists\, w \geq 0 : \;\; Ww \gneq 0$

Proof

$\nexists\, w \geq 0 \;\; : \; W w \gneq 0$

$\nexists\, w \geq 0 \;\; : \; (C_R - C_B B^{-1} R) w \gneq 0$

$\forall x_B \quad : Rx_R + Bx_B = b \; ; \; \nexists\, x_R \geq 0 : C_B x_B + C_R x_R - C_B B^{-1} b \gneq 0 \;\; \Leftrightarrow$

$\nexists\, x \in S \;\; : \;\; Cx - C_B B^{-1} b \gneq 0 \qquad \Leftrightarrow \qquad \nexists\, x \in S \;\; : \;\; Cx \gneq C\bar{x} \qquad \Leftrightarrow$

$\bar{x}$ is an efficient basic solution.

REFERENCES

Chankong, V. and Y.Y. Haimes (1983) *Multiobjective Decision Making, Theory and Methodology*. North-Holland Series in Systems Science and Engineering, North-Holland, Amsterdam.

Cohon, J.L. (1978) *Multiobjective Programming and Planning*. Academic Press, New York.

Despontin, M. and P. Vincke (1977) Multiple criteria economic policy, in M. Roubens (Ed.), *Advances in Operations Research*. North-Holland, Amsterdam.

Driel, G.J. van, C. van Ravenzwaaij, J. Spronk and F.R. Veeneklaas (1983) *Grenzen en mogelijkheden van het economisch stelsel in Nederland*, WRR, V. 40, Den Haag (in Dutch).

Grauer, M. (1983) A dynamic interactive decision analysis and support system (DIDASS). User's guide (May 1983). Working Paper WP-83-60, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Guddat, J. (1979) Parametrische Optimierung und Vektor Optimierung, in K. Lommatzsch (Ed.), *Anwendungen der Linearen Parametrischen Optimierung*, pp. 54–75. Akademie-Verlag, Berlin (East) (in German).

Hafkamp, W.A. (1983) *Triple Layer model, a national–regional economic–environmental model for the Netherlands*. Dissertation, Amsterdam.

Haimes, Y.Y. and C. Chankong (1979) Kuhn–Tucker multipliers as tradeoffs in multiobjective decision-making analysis, *Automatica* 15, pp. 59–72.

Hannan, E.L. (1978) Using duality theory for identification of primal efficient points and for sensitivity analysis in multiple objective linear programming, *Journal of the Operational Research Society* 29, pp. 643–649.

Hughes-Hallett, A. and H. Rees (1983) *Quantitative Economic Policies and Interactive Planning: A Reconstruction of the Theory of Economic Policy*. Cambridge University Press, Cambridge.

Hwang, C. and A. Masud (1979) *Multiple Objective Decision Making: Methods and Applications*. Springer-Verlag, Berlin.

Isermann, H. (1977) The relevance of duality in multiple objective linear programming, *TIMS Studies in the Management Sciences* 6, pp. 241–262. North-Holland, Amsterdam.

Isermann, H. (1978) On some relations between a dual pair of multiple objective linear programs, *Zeitschrift für Operations Research* 22, pp. 33–41.

Johnson, L.E. and D.P. Louchs (1980) Interactive multiobjective planning using computer graphics, *Computers and Operations Research* 7, pp. 89–97.

Kallio, M., A. Lewandowski and W. Orchard Hays (1981) An implementation of the reference point approach for multiobjective optimization, in G.B. Dantzig, M.A.H. Dempster and M. J. Kallio (Eds.), *Large-Scale Linear Programming*, pp. 1025–1054. Collaborative Proceedings CP-81-S1, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Kornbluth, J.S.H. (1974) Duality, indifference and sensitivity analysis in multiple objective linear programming, *Operational Research Quarterly* 25, pp. 599–614.

Lewandowski, A. and M. Grauer (1982) The reference point optimization approach – methods of efficient implementation, in M. Grauer, A. Lewandowski and A.P. Wierzbicki (Eds.), *Multiobjective and Stochastic Optimization*, pp. 353–376. Collaborative Proceedings CP-82-S12, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Mangasarian, O.L. (1969) *Nonlinear Programming*. McGraw-Hill Book Company, New York.

Nieuwenhuis, J.W., (1983) Technical note: About Isermann duality, *Journal of Optimization Theory and Applications* 41, pp. 481–490.

Nijkamp, P. and J. Spronk (1980) Interactive multiple goal programming: an evaluation and some results, in G. Fandel and T. Gal (Eds.), *Multiple Criteria Decision Making: Theory and Applications*, pp. 278–293. Springer-Verlag, Berlin.

Spronk, J. (1981) *Interactive Multiple Goal Programming: Applications to Financial Planning*. Martinus Nyhoff, Boston.

Steuer, R.E. (1977) An interactive multiple objective linear programming procedure, *TIMS Studies in the Management Sciences* 6, pp. 225–239. North-Holland, Amsterdam.

Steuer, R.E. (1982) On sampling the efficient set using weighted Tchebycheff metrics, in M. Grauer, A. Lewandowski and A.P. Wierzbicki (Eds.), *Multiobjective and Stochastic Optimization*, pp. 335–352. Collaborative Proceedings CP-82-S12, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Steuer, R.E. and F.W. Harris (1980) Intra-set point generation and filtering in decision and criterion space, *Computers and Operations Research* 7, pp. 41–53.

Steuer, R.E. and E.U. Choo (1983) An interactive weighted Tchebycheff procedure for multiple objective programming, *Mathematical Programming* 26, pp. 326–344.

White, D.J. (1983a) The foundations of multi-objective interactive programming – some questions, in P. Hansen (Ed.), *Essays and Surveys on Multiple Criteria Decision Making*, pp. 406–415. Springer-Verlag, Berlin.

White, D.J. (1983b) A selection of multi-objective interactive programming methods, in S. French, R. Hartley, L.C. Thomas and D.J. White (Eds.), *Multi-Objective Decision Making*, pp. 99–126. Academic Press.

Wierzbicki, A.P. (1979) The use of reference objectives in multiobjective optimization – theoretical implications and practical experience. Working Paper WP-79-66, International Institute for Applied Systems Analysis, Laxenburg, Austria.

Wierzbicki, A.P. (1982) A mathematical basis for satisficing decision making, *Mathematical Modeling* 3, pp. 391–405.

Yu, P.L. and M. Zeleny (1975) The set of all nondominated solutions in linear cases and a multicriteria simplex method, *Journal of Mathematical Analysis and Applications* 49, pp. 430–468.

Zeleny, M. (1974) *Linear Multiobjective Programming*. Lecture Notes in

Economics and Mathematical Systems, Springer-Verlag, Berlin.

Zeleny, M. (1982) *Multiple Criteria Decision Making*. McGraw-Hill.

Zionts, S. (1983) A report on a project on multiple criteria decision making, 1982, in P. Hansen (Ed.), *Essays and Surveys on Multiple Criteria Decision Making*, pp. 416–430. Springer-Verlag, Berlin.

Zionts, S. and J. Wallenius (1976) An interactive programming method for solving the multiple criteria problem, *Management Science* 22, pp. 652–663.

Zionts, S. and J. Wallenius (1983) An interactive multiple objective linear programming method for a class of underlying nonlinear utility functions, *Management Science* 29, pp. 519–529.