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Study of Solutions to Differential Inclusions by the "Pipe Method"

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STUDY OF SOLUTIONS TO DIFFERENTIAL
INCLUSIONS BY THE "PIPE METHOD"

Jean-Pierre Aubin

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PREFACE

A pipe of a differential inclusion is a set-valued map associating with each time t a subset P(t) of states which contains a trajectory of the differential inclusion for any initial state x_0 belonging to P(0). As in the Liapunov method, knowledge of a pipe provides information on the behavior of the trajectory. In this paper, the characterization of pipes and non-smooth analysis of set-valued maps are used to describe several classes of pipes.

This research was conducted within the framework of the Dynamics of Macrosystems study in the System and Decision Sciences Program.

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STUDY OF SOLUTIONS TO DIFFERENTIAL INCLUSIONS BY THE "PIPE METHOD"

Jean-Pierre Aubin

INTRODUCTION

We propose a method analogous to Liapunov's second method for studying the asymptotic properties of solutions to differential equations or inclusions.

Let F be a set-valued map from a subset $K \subset \mathbb{R}^n$ to $\mathbb{R}^n$ describing the dynamics of the system:

$$(0.1) \qquad x'(t) \in F(x(t)), \quad x(0) = x_o \text{ given in } K$$

Let V be a function from a neighborhood of K to $\mathbb{R}_+$.

All methods falling under the name of Liapunov's second method deal with the fact that inequalities of the type

$$(0.2) \qquad \forall x \in K, \quad \inf_{v \in F(x)} \langle v'(x), v \rangle \leq - W(x,v)$$

(where W is a non-negative function on Graph(F)) imply, under reasonable assumptions, that

$$(0.3) \qquad \begin{cases} \forall x_o \in K, \text{ there exists a solution } x(\cdot) \text{ to } (0.1) \text{ such} \\ \text{that } t \to V(x(t) \text{ is decreasing} \end{cases}$$

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According to the choice of the function V (and the function W),
we then can deduce information upon the behavior of some solutions
to (0.1); in particular, that

$$(0.4) \qquad \forall t \geq 0, \ V(x(t)) \leq V(x(t_o)).$$

(We refer to Aubin-Cellina [1984], Chapter 6, for a presentation
of the Liapunov method for non-differentiable functions V and for
differential inclusions.) The "pipe method" we propose aims for
the same objectives. A "pipe" P is a set-valued map

$$(0.5) \qquad t \in \mathbb{R}_+ \to P(t) \subset K$$

which is related to the dynamics described by F by

$$(0.6) \qquad \forall t \geq 0, \ \forall x \in P(t), \ F(t,x) \cap DP(t,x) \neq \emptyset$$

where

$$(0.7) \qquad DP(t,x) := \{v \in \mathbb{R}^n \mid \liminf_{h \to 0+} d(v, \frac{P(t+h) - x}{h}) = 0\}$$

denotes the "contingent derivative" of P at $(t,x) \in$ Graph(P)
(see Aubin-Cellina [1984], Section 4.3). Indeed, Haddad's via-
bility theorem (see Haddad [1981], Aubin-Cellina [1984], Section
4.4) implies that under reasonable conditions, condition (0.6)
implies that

$$(0.8) \qquad \begin{cases} \forall x_o \in K, \text{ there exists a solution } x(\cdot) \text{ to (0.1) such} \\ \text{that } \forall t \geq 0, \ x(t) \in P(t) \end{cases}$$

According to the choice of the pipe P, we are able to infer pro-
perties of the behavior of some solutions to (0.1). The pipe
method has drawbacks analogous to Liapunov's method: when the
dynamics of a system are described by a map F, how can we find
its pipes?

However, we shall give several examples of pipes in the
following pages. We begin with pipes derived from potential func-
tions V, of the form

(0.9) $P(t) := \{x \in K \mid V(x(t)) \le w(t)\}$

where $w(t)$ is a function we shall construct in terms of F and V. Such pipes yield information analogous to the ones provided by Liapunov method.

We shall also characterize pipes of the form

(0.10) $P(t) := \{x \in K \mid w_-(t) \le V(x-c(t)) \le w_+(t)\}$

and more generally, of the form

(0.11) $P(t) := \{x \in K \mid \forall i=1,\ldots,p, V_i(\Phi(t,x)) \le w_i(t)\}$

We shall study pipes defined by constraints

(0.12) $P(t) := \{x \in K \mid A(t,x) \in M\}$

where A is a continuous map from $\mathbb{R}_+ \times \mathbb{R}^n$ to a vector space $\mathbb{R}^p$. Finally, we consider pipes of the form

(0.13) $P(t) := \Phi(t,C,D)$

where C and D are closed subsets of $\mathbb{R}^n$ and Φ is a differentiable map from $\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n$. In particular, when C is reduced to a point c, we investigate pipes of the form

(0.14) $P(t) := c + \phi(t)D$

Such pipes allow us to infer that for any $x_o \in c + \phi(0)D$, there exists a trajectory $x(\cdot)$ of (0.1) converging to c at time T when $\phi(T) = 0$.

We shall rely on results of non-smooth analysis as they are presented in Chapter 7 of Aubin-Cellina [1984] to give sufficient conditions for set-valued maps of the form (0.11), (0.12) or (0.13) to be pipes of a given set-valued map F.

1.   THE PIPE METHOD: A GENERAL DESCRIPTION AND BACKGROUND NOTES

Let us consider a dynamical system described by a set-valued map $(t,x) \in [0,T] \times \mathbb{R}^n \to F(t,x) \in \mathbb{R}^n$.

We consider the initial-value problem for the differential inclusion

$$(1.1) \qquad x'(t) \in F(t,x(t)) \; ; \; x(0) = x_o$$

where $(0,x_o) \in \text{Dom}(F)$ is given.

We recall the definitions of the <u>contingent cone</u> $T_K(x)$ to a subset K at $x \in K$:

$$(1.2) \qquad \begin{cases} T_K(x) := \{v \in \mathbb{R}^n \mid \underset{h \to 0+}{\lim \inf} \ \dfrac{d(x+hv,K)}{h} = 0\} \\[2mm] = \underset{\varepsilon,\alpha>0}{\cap} \ \underset{h \in]0,\alpha[}{\cup} (\dfrac{1}{h}(K-x) + \varepsilon B) \end{cases}$$

(see Aubin-Cellina [1984], pp.176-177). When G is a set-valued map from X to Y and when (x,y) belongs to the graph of G, the <u>contingent derivative</u> $DG(x,y)$ of G at (x,y) is the closed process (set-valued map whose graph is a closed cone) from X to Y defined by

$$(1.3) \qquad DG(x,y)(u) := \{v \in Y \mid \underset{\substack{h \to 0+ \\ u' \to u}}{\lim \inf} \ d(v, \frac{G(x+hu') - y}{h}) = 0\}$$

We observe that

$$(1.4) \qquad \text{GraphDG}(x,y) = T_{\text{Graph}(G)}(x,y)$$

When P is a set-valued map from $\mathbb{R}$ to X, it is enough to know the values of the contingent derivative $DP(t,x)$ of P at $(t,x) \in \text{Graph}(P)$ at 1, 0 and -1. We observe that

$$(1.5) \qquad DP(t,x)(1) = \{v \in X \mid \underset{h \to 0+}{\lim \inf} \ d(v, \frac{P(t+h) - x}{h}) = 0\}$$

and we shall often set

(1.6) $DP(t,x): = DP(t,x)(1)$

We also observe that

(1.7) $DP(t,x)(-1) = \{v \in X \mid \lim\limits_{h \to 0+} \inf d(v, \frac{P(t-h) - x}{h}) = 0\}$

and that

(1.8) $T_{P(t)}(x) \subset DP(t,x)(0)$

(Equality holds when P is Lipschitz around x.)

<u>Definition 1.1</u>: We shall say that a set-valued map P from $[0,T]$ to $\mathbb{R}^n$ is a <u>pipe of a set-valued map</u> $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ on $[0,T]$ if

(1.9) $\begin{cases} \text{(i) Graph (P) is closed and contained in Dom(F)} \\ \text{(ii) } \forall t \in [0,T[\,, \; \forall x \in P(t), \; F(t,x) \cap DP(t,x) \neq \emptyset \\ \text{(iii) } \forall x \in P(T), \; F(T,x) \cap DP(T,x)(0) \neq \emptyset \end{cases}$ ▲

Condition (1.9)iii) naturally disappears when we take $T: = +\infty$. Hence Haddad's viability theorem (see Haddad [1981], Aubin-Cellina [1984], Theorem 4.1.1, p.180) can be reformulated in the following way:

<u>Theorem 1.2</u>: Let us assume that

(1.10)   F is bounded, its graph is closed and its values are convex.

Let P be a pipe of F on $[0,T]$. Then, for any $x_o \in P(0)$, there exists a trajectory of (1.1) satisfying

(1.11) $\begin{cases} \text{(i) } \forall t \in [0,T], \; x(t) \in P(t) \\ \text{(ii) for almost all } t \in [0,T[\,, \; x'(t) \in DP(t,x(t)). \end{cases}$

and, when $T < +\infty$,

(1.12) $\forall t \in [T,\infty], \; x'(t) \in F(T,x(t))$ and $x(t) \in P(T).$ ▲

Proof: We introduce the following set-valued map G from Graph(P) to $\mathbb{R}_+ \times \mathbb{R}^n$ defined by

(1.13)
$$G(\tau,x) := \begin{cases} \{1\} \times F(\tau,x) & \text{if } \tau \in [0,T[\\ [0,1] \times F(\tau,x) & \text{if } \tau = T \\ \{0\} \times F(T,x) & \text{if } \tau > T \end{cases}$$

When $\tau < T$, condition (1.9)ii) implies the existence of v in $F(\tau,x) \cap DP(\tau,x)$. We then deduce that $(1,v)$ belongs to the contingent cone to Graph(P) at (τ,x). When $\tau \geq T$, condition (1.9)iii) states that $(0,v)$ belongs to the contingent cone to Graph (F) at (T,x). Hence, the set-valued map G satisfies

$$\forall(\tau,x) \in \text{Graph}(P), \quad G(\tau,x) \cap T_{\text{Graph}(p)}(\tau,x) \neq \emptyset.$$

Since G is bounded, its graph is closed and its values are convex, Haddad's theorem states that the differential inclusion

$$(\tau'(t),x'(t)) \in G(\tau(t),x(t)) \quad ; \quad (\tau(0),x(0)) = (0,x_o)$$

has a viable trajectory $t \to (\tau(t),x(t))$ in Graph(P). Since $\tau(t) = t$ when $t \in [0,T]$, $\tau(t) = T$ when $t \geq T$, we infer that $x(\cdot)$ is a solution to (1.1) satisfying (1.11). ∎

In order to check that a set-valued map P is a pipe of a given set-valued map F, we need to use some calculus on contingent cones and contingent derivatives of set-valued maps. Sooner or later, we shall need regularity assumptions of the type: $x \to T_K(x)$ is lower semicontinuous at some point x_o. This motivates the introduction of the Kuratowski lim inf of the contingent cones

(1.14)
$$\liminf_{y \to x} T_K(y) := \bigcap_{\varepsilon>0} \bigcup_{\alpha>0} \bigcap_{y \in B_K(x,\alpha)} (T_K(y) + \varepsilon B)$$

A theorem due to Cornet [1981], Penot [1981] (see Aubin-Ekeland [1984], Theorem 7.1.7, p.409) implies that this lim inf of the contingent cones is the tangent cone $C_K(x)$ to K at x introduced by Clarke [1975] (see also Clarke [1983], Aubin-Ekeland [1984], Definition 7.1.3, p.506) and defined by

$$(1.15) \quad \begin{cases} C_K(x) := \{v \in \mathbb{R}^n \mid \lim_{\substack{h \to 0+ \\ y \to x}} \dfrac{d(y+hv,K)}{h} = 0\} \\[2em] = \displaystyle\bigcap_{\varepsilon > 0} \; \bigcup_{\substack{\alpha,\beta > 0}} \; \bigcap_{\substack{y \in B_K(x,\alpha) \\ 0 < h \le \beta}} (\tfrac{1}{h}(K-y) + \varepsilon B) \end{cases}$$

It is a <u>closed convex subcone</u> of the contingent cone $T_K(x)$, which coincides with it when K is a C^1- manifold, when K is convex or more generally, when $y \to T_K(y)$ is lower semicontinuous at x.

When G is a set-valued map from X to Y and when (x,y) belongs to the graph of G, we define the <u>derivative</u> $CG(x,y)$ of G at (x,y) by

$$(1.16) \quad v \in CG(x,y)(u) \Leftrightarrow (u,v) \in C_{Graph(G)}(x,y)$$

When G is Lipschitz around x, we observe that

$$(1.17) \quad CG(x,y)(u) = \{v \in Y \mid \lim_{\substack{h \to 0+ \\ (x',y') \to (x,y) \\ Graph(G)}} d(v, \dfrac{G(x'+hu) - y'}{h}) = 0\}$$

When G is a C^1- single-valued map, $CG(x,G(x))$ coincides with the usual Jacobian $G'(x)$. When V is a function from $\mathbb{R}^n$ to $\mathbb{R} \cup \{+\infty\}$ and when we want to take into account the order relation (for defining pipes of the form

$$P(t) := \{x \in K \mid V(x) \le w(t)\}$$

for instance), we are led to introduce the epigraph of V defined by

$$(1.18) \quad \mathcal{E}p(V) := \{(x,w) \in Dom\ V \times \mathbb{R} \mid V(x) \le w\}$$

We then observe the following facts:

$$T_{\mathcal{E}p(V)}(x,V(x)) = \mathcal{E}p D_+V(x)$$

is the epigraph of the <u>epi-contingent derivative</u> $D_+V(x)$ defined by

(1.19) $\qquad D_+V(x)(u) = \lim_{\substack{h \to 0+ \\ u' \to u}} \inf \dfrac{V(x+hu) - Vx}{h}$

and that

$$C_{Ep(V)}(x,V(x)) = E_pC_+V(x)$$

is the epigraph of the epi-derivative $C_+V(x)$ of V at x, which is a lower semicontinuous, positively homogeneous convex function. We always have

(1.20) $\qquad D_+V(x) \leq C_+V(x)$

and equality holds when V is C^1 around x, or convex. When V is Lipschitz around x, we obtain simpler formulas

(1.21) $\qquad \begin{cases} \text{(i)} \quad D_+V(x)(u) = \lim_{h \to 0+} \inf \dfrac{V(x+hu) - V(x)}{h} \quad \text{(a Dini derivative)} \\[2em] \text{(ii)} \quad C_+V(x)(u) = \lim_{\substack{h \to 0+ \\ y \to x}} \sup \dfrac{V(y+hu) - V(y)}{h} \quad \text{(Clarke's} \\[1em] \qquad \text{directional derivative)} \end{cases}$

We observe also that when V is upper semicontinuous at x and $w > V(x)$, then $T_{Ep(V)}(x,w) = C_{Ep(V)}(x,w) = \mathbb{R}^n \times \mathbb{R}$.

We shall also use the notation:

(1.22) $\qquad C_-V(x)(u): = \lim_{\substack{h \to 0+ \\ y \to x}} \inf \dfrac{V(y+hu) - V(y)}{h}$

whose hypograph is the tangent cone to the hypograph of V at $(x,V(x))$ (when V is locally Lipschitz). Finally, we define the _generalized gradient_ $\partial V(x)$ of V at x as the (possibly empty) closed convex subset

(1.23) $\qquad \partial V(x): = \{p \in \mathbb{R}^n \mid \forall v \in \mathbb{R}^n, <p,v> \leq C_+V(x)(v)\}$

In particular, when $V: = \psi_K$ is the indicator of a subset $K (\psi_K(x) = 0$ when $x \in K$ and $\psi_K(x) = +\infty$ when $x \notin K\}$, then

(1.24) $\partial \psi_K(x) = (C_K(x))^- = : N_K(x)$

where $N_K(x)$ is the <u>normal cone</u> to K at x. We say that a solution $\bar{x}$ to

(1.25) $\bar{x} \in K$ and $0 \in \partial V(\bar{x}) + N_K(\bar{x})$

is a <u>critical</u> and <u>stationary</u> point of V on K. Elements $\bar{x} \in K$ minimizing V on K are critical points; the converse is true when V is convex.

2.   PIPES DERIVED FROM POTENTIAL FUNCTIONS

Let $K \subset \mathbb{R}^n$ be the viability domain and let us consider a "potential function" V from $\mathbb{R}^n$ to $\mathbb{R}_+ \cup \{+\infty\}$. We shall study in this section pipes of the form

(2.1) $P(t): = \{x \in K | V(x) \leq w(t)\}$

where w is a non-negative function defined on $[0,T]$. We shall begin by providing sufficient conditions on K,V,w and F implying that set-valued maps P of the form (2.1) are pipes of F. We obtain in this case the following result:

<u>Proposition 2.1</u>: Let us assume that K is closed and that V is locally Lipschitz around K. Let w be a C^1- function defined on a neighborhood of V such that

(2.2) $\begin{cases} \forall t \in [0,T[\text{ the elements of } P(t) \text{ are not critical} \\ \text{points of V on K} \end{cases}$

We posit the following condition

(2.3) $\begin{cases} \forall t \in [0,T[, \forall x \in K \text{ such that } V(x) = w(t), \\ \exists u \in F(t,x) \cap C_K(x) \text{ such that } C_+V(x)(u) \leq w'(t) \end{cases}$

and

$$(2.4) \quad \begin{cases} \text{If } x \in K \text{ satisfying } V(x) = w(T) \text{ is a critical} \\ \text{point of } V \text{ on } K, \text{ then } 0 \in F(T,x). \end{cases}$$

Then the set-valued map P defined by (2.1) is a pipe of F on $[0,T]$. ▲

Therefore, if F satisfies assumption (1.10), then for all $x_o \in K$ satisfying $V(x) \leq w(0)$, there exists a trajectory of the differential inclusion (1.1) satisfying

$$(2.5) \quad \begin{cases} \text{(i) } \forall t \in [0,T], \; x(t) \in K \text{ and } V(x(t)) \leq w(t), \\ \text{(ii) for almost all } t \geq 0, \\ x'(t) \in T_K(x(t)) \text{ and } D_+V(x(t))(x'(t)) \leq w'(t) \end{cases}$$ ▲

We recall that when V is C^1,

$$(2.6) \quad D_+V(x)(u) = C_+V(x)(u) = \langle V'(x), u \rangle$$

and that when V is convex and continuous,

$$(2.7) \quad D_+V(x)(u) = C_+V(x)(u) = \inf_{h > 0} \frac{V(x+hu) - V(x)}{h}.$$ ■

We shall also study pipes of the form

$$P(t) := \{x \in K \,|\, w_-(t) \leq V(x-c(t)) \leq w_+(t)\}$$

where c is a function from $[0,T]$ to K and w_- and w_+ are non-negative functions, which define some kind of neighborhood around a function $t \to c(t)$, such as periodic trajectories of the dynamical system (1.1). They are special cases of pipes associated to p potential functions V_i by the formula

$$(2.8) \quad P(t) := \{x \in K \,|\, V_i(\Phi(t,x)) \leq w_i(t), \; i=1,\ldots,p)\}$$

where Φ is a smooth map from $[0,T] \times K$ to Dom $\vec{V}$. We shall then provide sufficient conditions on $\Phi, \vec{w}$ and the functions V_i for a set-valued P of this type to be a pipe for a given set-valued map F.

<u>Theorem 2.2</u>: Let us assume that K is closed, that Φ is C^1 around $[0,T] \times K$, that $\vec{w}$ is C^1 around $[0,T]$ and that the p potential functions V_i are locally Lipschitz on a neighborhood of $\Phi([0,T] \times K)$. Let us set

$$(2.9) \qquad I(t,x) := \{i=1,\ldots,p \,|\, V_i(\Phi(t,x)) = w_i(t)\}$$

We assume that

$$(2.10) \qquad \begin{cases} \forall t \in [0,T], \; \forall x \in P(t), \\[2mm] 0 \notin co[\; \underset{i \in I(t,x)}{\cup} \Phi_x'(t,x) * \partial V_i(\Phi(t,x))] + N_K(x) \end{cases}$$

and that

$$(2.11) \qquad \begin{cases} 0 \in F(t,x) \text{ for all } x \in P(T) \text{ such that there exists} \\[2mm] i \in I(T,x) \text{ such that } 0 \in \Phi_x'(T,x) * \partial V_i(\Phi(T,x)) + N_K(x) \end{cases}$$

We posit the following assumption

$$(2.12) \qquad \begin{cases} \text{(i)} \; \forall t \in [0,T[, \; \forall x \in P(t), \; \exists u \in F(t,x) \cap C_K(x) \text{ such that} \\[2mm] \qquad \forall i \in I(t,x), C_+V_i(\Phi(t,x))(\Phi_t'(t,x) + \Phi_x'(t,x)u) \leq w_i'(t) \\[2mm] \text{(ii)} \; \forall x \in P(T), \exists u \in F(T,x) \cap C_K(x) \text{ such that} \\[2mm] \qquad \forall i \in I(T,x), C_+V_i(\Phi(T,x))(\Phi_x'(T,x)u) \leq 0 \end{cases}$$

Then the set-valued map P defined by (2.8) is a pipe of F. ▲

If F satisfies assumption (1.10), then, for all $x_0 \in K$ satisfying $V_i(\Phi(0,x)) \leq w_i(0)$ $(i=1,\ldots,p)$, there exists a trajectory of the differential inclusion (1.1) satisfying

$$(2.13) \qquad \begin{cases} \text{(i)} \; \forall t \in [0,T[, \; \forall i=1,\ldots,p, \; V_i(\Phi(t,x(t))) \leq w_i(t) \\[2mm] \text{(ii)} \; \text{for almost all } t \in [0,T], \text{ for all } i \in I(t,x(t)), \\[2mm] \qquad x'(t) \in T_K(x(t)) \text{ and} \\[2mm] \qquad D_+V_i(\Phi(t,x(t)))(\Phi_t'(t,x(t)) + \Phi_x'(t,x(t))x_i'(t)) \leq w_i'(t) \end{cases}$$

▲

Remark: Observe that the elements $x \in K$ satisfying

$$0 \in \Phi_x'(t,x)^* \partial V_i(\Phi(t,x)) + N_K(x)$$

are the critical points of $x \to V_i(\Phi(t,x))$ on K. Assumption (2.11) states that critical points of some function $V_i(\Phi(T,\cdot))$ on K are equilibria of $F(T,\cdot)$. We can say that a solution to

$$(2.14) \qquad 0 \in co(\ \bigcup_{i=1}^{n}\ \Phi_x'(t,x)^* \partial V_i(\Phi(t,x)) + N_K(x)$$

is a <u>Pareto critical point</u> of the functions $V_i(\Phi(t,\cdot))$, (Pareto minima do satisfy this inclusion).

Remark: Observe also that if

$$(2.15) \qquad \forall t \in [0,T[\ ,\ \forall i=1,\ldots,n,\ w_i'(t) < 0,$$

then property (2.10) follows from (2.12). This is usually the interesting case, since we would like the pipes $P(t)$ to decrease when t increases.

Corollary 2.3: Let K be a closed subset, V be a C^1 function from a neighborhood of $[0,T]$ to K, w_- and w_+ be C^1 non-negative functions satisfying

$$(2.16) \qquad \begin{cases} \forall t \in [0,T[\ ,\ 0 \leq w_-(t) < w_-(T) = w_+(T) < w_+(t) \\ \text{and } w_-'(t) > 0,\ w_+'(t) < 0 \end{cases}$$

We posit the following assumption:

$$(2.17) \qquad \begin{cases} \text{(i)} & \forall t \in [0,T[\ ,\ \forall x \text{ such that } V(x-c(t)) = w_+(t), \\ & \text{there exists } u \in F(t,x)\ \cap\ C_K(x) \text{ such that} \\ & C_+V(x-c(t))(u-c'(t)) \leq w_+'(t) \\ \\ \text{(ii)} & \forall t \in [0,T[\ ,\ \forall x \text{ such that } V(x-c(t)) = w_-(t), \\ & \text{there exists } u \in F(t,x)\ \cap\ C_K(x) \text{ such that} \\ & C_-V(x-c(t))(u-c'(t)) \geq w_-'(t) \\ \\ \text{(iii)} & \forall x \text{ such that } V(x-c(T)) = w_+(T) = w_-(T),\ 0 \in F(T,x) \end{cases}$$

Then the set-valued map P defined by

(2.18) $P(t) := \{x \in K \; ; \; w_-(t) \leq V(x-c(t)) \leq w_+(t)\}$
            is a pipe of F on $[0,T]$.

Proof of Theorem 2.2: We set $\text{Dom } \vec{V} = \overset{P}{\underset{i=1}{\cap}} \text{Dom } V_i$, $\vec{V}(x) := (V_1(x),$
$\dots, V_p(x))$ and $\text{Ep}(\vec{V}) := \{(x,w) \in \text{Dom } \vec{V} \times \mathbb{R}^p \,|\, V_i(x) \leq w_i \text{ for }$
$i=1,\dots,p)\}$. Let A be the C^1 map from a neighborhood of $[0,T] \times K$
to $\mathbb{R}^n \times \mathbb{R}^p$ defined by

(2.20) $A(t,x) := (\Phi(t,x), w(t))$

Then we can write

(2.21) $\text{Graph}(P) := \{(t,x) \in [0,T] \times K \,|\, A(t,x) \in \text{Ep}(\vec{V})\}$

We then use Proposition 7.6.3, p.440, of Aubin-Ekeland [1984].
It states that

(2.22)
$$\begin{cases} \{\tau \in T_{[0,T]}(t), u \in T_K(x) \,|\, A'(t,x)(\tau,u) \in T_{\text{Ep}(\vec{V})}(A(t,x))\} \\[2ex] \supset T_{\text{Graph}(P)}(t,x) \end{cases}$$

and that if the transversality condition

(2.23) $A'(t,x)(T_{[0,T]}(t) \times C_K(x)) - C_{\text{Ep}(\vec{V})}(A(t,x)) = \mathbb{R}^n \times \mathbb{R}^p$

then

(2.24)
$$\begin{cases} \{\tau \in T_{[0,T]}(t) \; , \; u \in C_K(x) \,| \\[2ex] A'(t,x)(\tau,u) \in C_{\text{Ep}(\vec{V})}(A(t,x))\} \subset C_{\text{Graph}(P)}(t,x) \end{cases}$$

Inclusion (2.22) implies that for all $t \in [0,T]$,

$$(2.25) \quad \begin{cases} DP(t,x) \\ \subset \{u \in T_K(x) \mid \forall i \in I(t,x), D_+V_i(\Phi(t,x))(\Phi'_t(t,x) + \Phi'_x(t,x)u) \\ \leq w'_i(t)\} \end{cases}$$

since

$$(2.26) \quad A'(t,x)(\tau,u) = (\Phi'_t(t,x)\tau + \Phi'_x(t,x)(u), w'(t)\tau),$$

and since

$$(2.27) \quad \begin{cases} T_{Ep(\vec{V})}(A(t,x)) = T_{Ep(\vec{V})}(\Phi(t,x), w(t)) \\ = \{(u,\lambda) \in \mathbb{R}^n \times \mathbb{R}^p \mid \forall i \in I(t,x), \lambda_i \geq D_+V_i(\Phi(t,x))(u)\} \end{cases}$$

In the same way, inclusion (2.24) can be rewritten in the following form

$$(2.28) \quad \begin{cases} \{u \in C_K(x) \mid \forall i \in I(t,x), C_+V_i(\Phi(t,x))(\Phi'_t(t,x)\tau + \Phi'_t(t,x)u) \\ \leq w'_i(t)\tau\} \subset CP(t,x)(\tau) \subset DP(t,x)(\tau). \end{cases}$$

This inclusion and assumption (2.12) imply that P is a pipe of F. It remains to check the transversality condition (2.23), which can be written in the following way:

$$\forall u_d \in \mathbb{R}^n, \forall \lambda_d \in \mathbb{R}^p, \exists u \in C_K(x), \exists \tau \in T_{[0,T]}(t)$$

such that

$$(2.29) \quad \begin{cases} \forall i \in I(t,x), w'_i(t)\tau \geq \\ C_+V_i(\Phi(t,x))(\Phi'(t,x)\tau + \Phi'_x(t,x)u - u_d) + \lambda_d \end{cases}$$

By assumption (2.10) and the separation theorem, there exists $\hat{u} \in C_K(x)$ such that

$$(2.30) \quad \forall i \in I(t,x), C_+V_i(\Phi(t,x))(\Phi'_x(t,x)\hat{u}) < 0$$

There exists η such that $C_+V_i(\Phi(t,x))(\Phi'_x(t,x)\hat{u} + v) \le 0$ when $v \in \eta B$. Let $\beta = 0$ if $\lambda_d \le 0$ and

$$\beta > \lambda_d / |C_+V_i(\Phi(t,x))(\Phi'_x(t,x)\hat{u})| \text{ if } \lambda_d > 0.$$

We take $\alpha = \beta + \eta|\|u_d\|$. Hence, $\tau: = 0$ and $u: = \alpha\hat{u}$ provide

a solution to (2.29).

Then this transversality condition holds true for all $t \in [0,T[$ and all $x \in P(t)$. When it fails to be true for some $x \in P(T)$, we then assume that such an x is an equilibrium of $F(T,\cdot)$. ∎

3. PIPES DEFINED BY TIME-DEPENDENT CONSTRAINTS

Let us consider a continuous single valued map A from a neighborhood of $[0,T] \times K$ to a vector space $\mathbb{R}^p$ and a subset M of $\mathbb{R}^p$. We shall provide sufficient conditions for a set-valued map P of the form

$$(3.1) \qquad P(t): = \{x \in K | A(t,x) \in M\}$$

to be a pipe of a set-valued map F. We begin with the case when A is continuously differentiable.

Proposition 3.1: Let K and M be closed subsets and A be continuously differentiable. We assume that for all $t \in [0,T]$, $\forall x \in P(t)$,

$$(3.2) \qquad A'_x(t,x)C_K(x) - C_M(A(t,x)) = \mathbb{R}^p$$

If for any $t \in [0,T]$ and any $x \in P(t)$, there exists $v \in F(t,x) \cap C_K(x)$ satisfying

$$(3.3) \qquad \begin{cases} \text{(i)} \quad A'_x(t,x)v \in C_M(A(t,x)) - A'_t(t,x) \text{ when } t < T \\[2mm] \text{(ii)} \quad A'_x(T,x)v \in C_M(A(T,x)) \qquad\qquad \text{when } t = T \end{cases}$$

then the set-valued map P defined by (3.1) is a pipe of F on $[0,T]$ and

(3.4) $DP(t,x) \subset \{u \in T_K(x) | A'_x(t,x)u \in T_M(A(t,x)) - A'_t(t,x)\}$ ▲

We can relax the assumption that A is continuously differentiable and replace the Jacobian of A by the derivative $CA(t,x) := CA(t,x,A(t,x))$ whose graph is the tangent cone to the graph of A at (t,x).

Then Proposition 3.1 follows from

<u>Proposition 3.2</u>:   Let K and M be closed subsets and A be a continuous map. We assume that for all $t \in [0,T], \forall x \in P(t)$,

(3.5)
$$\begin{cases} \text{(i) Dom } CA(t,x) = \mathbb{R} \times \mathbb{R}^n \\ \text{(ii) } CA(t,x)(0,C_K(x)) - C_M(A(t,x)) = \mathbb{R}^p \end{cases}$$

If for any $t \in [0,T]$ and any $x \in P(t)$ there exists $v \in F(t,x) \cap C_K(x)$ satisfying

(3.6)
$$\begin{cases} \text{(i) } CA(t,x)(1,v) \in C_M(A(t,x)) \text{ when } t < T \\ \text{(ii) } CA(T,x)(0,v) \in C_M(A(T,x)) \text{ when } t = T, \end{cases}$$

then the set-valued map P defined by (3.1) is a pipe of F on $[0,T]$ and

(3.7) $DP(t,x) \subset \{u \in T_K(x) | DA(t,x)(1,u) \cap T_M(A(t,u)) \neq \emptyset\}$ ▲

<u>Proof</u>:   The graph of P is the projection onto $\mathbb{R} \times \mathbb{R}^n$ of the subset

$$L := ([0,T] \times K \times M) \cap \text{Graph } A \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$$

By Proposition 7.6.3, p.440 of Aubin-Ekeland [1984], we know that

(3.8)
$$\begin{cases} (T_{[0,T]}(t) \times T_K(x) \times T_M(Ax)) \cap \text{Graph } DA(t,x)) \\ \supset T_L(t,x,A(t,x)) \end{cases}$$

This implies inclusion (3.7).

We also know that the transversality condition

$$(3.9) \qquad T_{[0,T]}(t) \times C_K(x) \times C_M(x) - \text{graph } CA(t,x) = \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p,$$

implies that

$$(3.10) \qquad C_L(t,x,A(t,x)) \supset T_{[0,T]}(t) \times C_K(x) \times C_M(Ax) \cap \text{Graph } CA(t,x)$$

We then observe that

$$(3.11) \qquad \{u \in C_K(x) \mid CA(t,x)(\tau,u) \cap C_M(A(t,x))) \neq \emptyset\} \subset CP(t,x)(\tau)$$

Indeed, let w belong to $CA(t,x)(\tau,u) \cap C_M(A(t,x))$ and let $t_n \to t$, $x_n \to x$ and $h_n \to 0+$. Since (τ,u,w) belongs to $C_L(t,x,A(t,x))$, there exist sequences $\tau_n \to \tau$, $u_n \to u$ and $w_n \to w$ such that $(t_n + h_n\tau_n, \ x_n + h_n u_n, \ A(t_n,x_n) + h_n w_n) \in L$, i.e. such that $x_n + h_n u_n \in P(t_n + h_n\tau_n)$ for all n. This implies that u belongs to $C(t,x)(\tau)$. It remains to check the transversality condition (3.9).

Let τ,u,w be given in $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^p$. Since $\text{Dom } CA(t,x) = \mathbb{R} \times \mathbb{R}^n$, there exists $v \in CA(t,x)(-\tau,-u)$. By assumption (3.5)ii), there exist $u_1 \in C_K(x)$ and $w_1 \in C_M(A(t,x))$ such that $-w-v \in CA(0,u_1) - w_1$. Hence $-w$ belongs to $CA(-\tau,u_1-u) - w$, i.e., $(0-\tau,u_1-u,w_1-w)$ belongs to Graph (A) and $(0,u_1,w_1)$ to $C_{[0,T] \times K \times M}(t,x,A(t,x))$.

4. HOMOTOPIC TRANSFERS

Let us consider two closed subsets C and D of $\mathbb{R}^n$ and a differentiable map Φ from a neighborhood of $[0,T] \times C \times D$ to $\mathbb{R}^n$. We consider pipes of the form

$$(4.1) \qquad P(t) := \Phi(t,C,D)$$

Proposition 4.1: Let us assume that

$$\forall t \leq T \ \forall x \in P(t), \ \exists(y,z) \in C \times D \text{ such that } \Phi(t,y,z) = x,$$
$$\exists(u,v) \in T_{C \times D}(y,z) \text{ such that}$$

$$(4.2) \begin{cases} \text{(i) if } t < T, \Phi'_y(t,y,z)u + \Phi'_z(t,y,z)v \in F(t,x) - \Phi'_t(t,y,z) \\ \\ \text{(ii) if } t = T, \Phi'_y(T,y,z) + \Phi'_z(T,y,z)v \in F(T,x) \end{cases}$$

Then the set-valued map P defined by (4.1) is a pipe of F on $[0,T]$. ▲

Proof: We observe that Graph(P) is the image of $[0,T]$ x C x D under the map ψ defined by $\psi(t,y,z) = (t,\Phi(t,y,z))$.

By Proposition 7.6.2, p.430 of Aubin-Ekeland [1984], $\psi'(t,y,z)T_{[0,T] \text{ x C x D}}(t,y,z) \subset T_{\text{Graph}(P)}(\psi(t,y,z))$. We deduce that condition (4.1)i) implies property (1.9)ii). We proceed in the same way to show that (4.1)ii) implies (1.9)iii) since $P(T) = \Phi(T,C,D)$. ∎

When C and D are closed and convex, we can characterize pipes of the form (4.1) through dual conditions. If K is a subset of $\mathbb{R}^n$, we denote by

$$(4.3) \qquad \sigma(K,p) := \sup_{x \in K} \langle p,x \rangle$$

its support function.

Proposition 4.2: Let us assume that the values of F are compact and convex and that the subsets C and D are closed and convex. If for any $t \in [0,T], \forall x \in P(t)$, there exists $(x,y) \in C \text{ x } D$ satisfying $\Phi(t,y,z) = x$ and for all

$$p \in \Phi'_y(t,y,z)^{*-1}N_C(y) \cap \Phi'_z(t,y,z)^{*-1}N_D(z),$$

we have

$$(4.4) \begin{cases} \text{(i) } \forall t < T, \langle p,\Phi'_t(t,y,z) \rangle + \sigma(F(t,\Phi(t,y,z)),-p) \geq 0 \\ \\ \text{(ii) for } t = T, \sigma(F(T,\Phi(T,y,z)),-p) \geq 0 \end{cases}$$

then the set-valued map P defined by (4.1) is a pipe of F on $[0,T]$. ▲

<u>Proof</u>: When C and D are convex, $T_{C \times D}(y,z) = T_C(y) \times T_D(z)$ so that conditions (4.2)i) and ii) can be written

$$(4.5) \begin{cases} \text{(i)} \quad (F(t,\Phi) - \Phi'_t(t,y,z)) \cap (\Phi'_y(t,y,z)T_C(y) + \Phi'_z(t,y,z)T_D(z)) \neq \emptyset \\[2ex] \text{(ii)} \quad F(T,\Phi) \cap (\Phi'_y(T,y,z)T_C(y) + \Phi'_z(T,y,z)T_D(z)) \neq \emptyset \end{cases}$$

The separation theorem shows that they are equivalent to conditions (4.4). ∎

<u>Corollary 4.3</u>: Let us assume that C and D are closed convex subsets and that the values of F are convex and compact. Let $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a differentiable function satisfying either one of the following equivalent conditions:

For any $t \geq 0$, $x \in P(t)$, there exists $y \in C, z \in D$ such that $x = y + \phi(t)z$ and either

$$(4.6) \begin{cases} \text{(i)} \quad (F(t,y+\phi(t)z) - \phi'(t)z) \cap (T_C(y)+T_D(z)) \neq \emptyset \text{ if } t < T \\[2ex] \text{(ii)} \quad F(T,y+\phi(T)z)) \cap (T_C(y)+T_D(z)) \neq \emptyset \text{ if } t = T \end{cases}$$

or

$$(4.7) \begin{cases} \forall p \in N_C(y) \cap N_D(z), \\[2ex] \text{(i)} \quad \phi'(t) \, \sigma_D(p) + \sigma(F(t,y+\phi(t)z),-p) \geq 0 \text{ if } t < T \\[2ex] \text{(ii)} \quad \sigma(F(T,y+\phi(T)z),-p) \geq 0 \quad \text{ if } t = T \end{cases}$$

Then the set-valued map P defined by

$$(4.8) \qquad P(t) := C + \phi(t)D$$

is a pipe of F on $[0,T]$. ▲

Let us consider the instance when $C = \{c\}$ and when 0 belongs to the interior of the closed convex subset D.

We introduce the function a_0 defined by

(4.9)
$$\begin{cases} a_0(t,w) := \\[2ex] \sup_{\substack{z \in D \\ \sigma_D(p) = 1}} \sup_{p \in N_D(z)} \inf_{v \in F(t,c+wz)} \langle p,v \rangle \\[2ex] = \sup_{z \in D} \inf_{v \in F(t,c+wz)} \sup_{\substack{p \in N_D(z) \\ \sigma_D(p) = 1}} \langle p,v \rangle \end{cases}$$

(The last equation follows from the minimax theorem.)

Let us assume that there exists a continuous function $a : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$, satisfying $a(t,0) = 0$ for all $t \geq 0$, such that

(4.10) $\quad \forall (t,w) \in \mathbb{R}_+ \times \mathbb{R}_+, \ a(t,w) \geq a_0(t,w)$

Let ϕ be a solution to the differential equation

(4.11) $\quad \phi'(t) = a(t,\phi(t)), \ \phi(0) = \phi_0$ given

satisfying

(4.12) $\quad a(T,\phi(T)) = 0$

Since $\sigma_D(p) > 0$ for all $p \neq 0$, we deduce that for all $z \in D$ and all $p \in N_D(z)$,

$$\phi'(t)\sigma_D(p) \geq a(t,\phi(t)\sigma_D(p) \geq a_0(t,\phi(t))\sigma_D(p)$$

$$\geq -\sigma_D(p) \sup_{v \in F(t,c+\phi(t)z)} \langle -\frac{p}{\sigma_D(p)}, v \rangle$$

$$= -\sigma(F(t,c+\phi(t)z),-p)$$

Hence, condition (4.7)i) is satisfied. Also

$$0 = a(T,\phi(T)) \geq a_0(T,\phi(T)) \geq \frac{-1}{\sigma_{D(p)}} \sigma(F(T,c+\phi(T)z),-p)$$

Then

$$(4.13) \qquad P(t): = c + \phi(t)D$$

defines a pipe of F.

For instance, if D: = B is the unit ball, then $\sigma_B(p) =$

$\|p\|$ and $N_B(z) = \lambda z$ for all $z \in S: = \{x \mid \|x\| = 1\}$. Hence, in this case we have

$$(4.15) \qquad a_0(t,w): = \sup_{\|z\|=1} \inf_{v \in F(t,c+wz)} <v,z>$$

In other words, the function a_0 defined by (4.9) conceals all the information needed to check whether a given subset D can generate a pipe P.

<u>Remark</u>: When a is non-positive and satisfies a(t,0) = 0 for all $t \geq 0$, then there exists a non-negative non-increasing solution $\phi(\cdot)$ of the differential equation (4.11)

When T = ∞, we infer that $\int_0^\infty a(\tau,\phi(\tau))d\tau$ is finite. Let us assume that for 0 all $w_* \in \mathbb{R}_+$,

$$(4.16) \qquad \lim_{\substack{t \to \infty \\ w \to w_*}} a(t,w) = a_*(w_*)$$

Then the limit ϕ_* of $\phi(t)$ when $t \to \infty$ satisfies the equation

$$a_*(\phi_*) = 0$$

Otherwise, there would exist $\varepsilon > 0$ and T such that $a_*(\phi_*) + \varepsilon < 0$ and for all $t > T$, $a(t,\phi(t)) \leq a_*(\phi_*) + \varepsilon$ by definition of a_*. We deduce the contradiction

$$\phi(t) - \phi(T) = \int_T^t a(\tau,\phi(\tau))d\tau \leq (t-T)(a_*(\phi_*) + \varepsilon)$$

when t is large enough.

Example: Let us consider the case when F does not depend upon t. We set

$$(4.17) \qquad \rho_o := \sup_{\lambda \in \mathbb{R}} \inf_{w > 0} (\lambda w - a_o(w))$$

Assume also that $\lambda_o \in \mathbb{R}$ achieves the supremum. We can take $\psi(w) := \lambda_o w - \rho_o$.

If $\rho_o > 0$, the function

$$(4.18) \qquad \phi_T(t) := \begin{cases} \dfrac{\rho_o}{\lambda_o}(1-\exp(\lambda_o(t-T))) & \text{if } \lambda_o \neq 0 \\[2mm] -\rho_o(t-T) & \text{if } \lambda_o = 0 \end{cases}$$

is such that $P(t) := \{c+\phi_T(t)D\}$ is a pipe of F such that $P(T) = \{c\}$. If $\rho_o \leq 0$ and $\lambda_o < 0$, then the functions

$$(4.19) \qquad \phi(t) := \frac{1}{\lambda_o}(\rho_o - e^{\lambda_o t})$$

are such that $P(t) := c+\phi_C(t)D$ defines a pipe of F on $[0, \infty[$ such that $P(t)$ decreases to the set $P_\infty := c+\dfrac{\rho_o}{\lambda_o}D$.

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