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IIASA Working Paper

WP-84-086

November 1984

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November 1984
WP-84-86

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PREFACE

This paper reviews several duality results in the theory of linear vector optimization using an extended reformulation with general cone ordering. This generalization gives some insight into the relations between cone orderings.

This research was carried out while the author was visiting the Interactive Decision Analysis Project in the System and Decision Sciences Program.

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DUALITY IN LINEAR VECTOR OPTIMIZATION

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1. INTRODUCTION

For a given set X, let Re be a binary relation on X (i.e. $Re \subset X \times X$), with the indifference relation

$$I = Re \cap (-Re)$$

and the strict preference relation

$$P = Re \setminus I \quad .$$

Throughout this paper, we shall use the following notation:

$$x \succsim y \quad \leftrightarrow \quad (x,y) \in Re$$

$$x \sim y \quad \leftrightarrow \quad (x,y) \in I$$

$$x \succ y \quad \leftrightarrow \quad (x,y) \in P \quad .$$

A binary relation which is both transitive* and reflexive is called a *quasi-ordering relation*; if it is also antisymmetric it is called an *ordering relation*. A quasi-ordering relation with the property of connectedness is called a *weak ordering relation*.

Now let X be a mixture space, that is, a space with convex structure. In order to explain human behavior under risk in terms of a set of axioms, von Neumann and Morgenstern imposed a number of conditions which are essentially equivalent to the following:

1. $\succsim$ is a weak ordering relation.
2. $x \succsim y \leftrightarrow \alpha x + (1-\alpha)z \succsim \alpha y + (1-\alpha)z,\ 0 < \forall\alpha < 1,\ \forall z \in X$.
3. $\alpha x + (1-\alpha)y \succ z,\ \forall\alpha > 0 \rightarrow \text{not } z \succ y$.

Condition (2) is familiar as the independence condition, and (3) as the continuity condition. Under these conditions, it is possible to construct a utility function $u : X \rightarrow R$ such that

(a) $u(\alpha x + (1-\alpha)y) = \alpha u(x) + (1-\alpha)u(y)$

(b) $x \succ y \leftrightarrow u(x) > u(y)$

(c) $x \sim y \leftrightarrow u(x) = u(y)$.

Note that the expected utility hypothesis is derived from property (a). If we can extend the mixture space X into the n-dimensional Euclidean space R^n, the following theorem holds:

<u>Theorem 1.1</u> *(Aumann [1]). Condition (2) is equivalent to*

(i) $x \succsim y \rightarrow x + z \succsim y + z,\ \forall z$

(ii) $x \succsim y \rightarrow \alpha x \succsim \alpha y,\ \forall\alpha > 0$.

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*Transitivity: $x \succsim y,\ y \succsim z \rightarrow x \succsim z,\ \forall x,y,z \in X$
Reflexivity: $x \succsim x,\ \forall x \in X$
Antisymmetricity: $x \succsim y,\ y \succsim x \rightarrow x = y,\ \forall x,y \in X$
Connectedness: either $x \succsim y$ or $y \succsim x,\ \forall x,y \in X$.

Then condition (3) is equivalent to

(iii) $x \succ kz, \forall k > 0 \rightarrow \text{not } z \succ 0$.

Corollary 1.1. *The set* $D = \{x | x \succsim 0, x \in E^n\}$ *is a convex cone, and in addition* $x \succsim y \leftrightarrow x - y \in D$. *Moreover, letting* $T = \{x | x \succ 0, x \in E^n\}$, *the continuity condition (iii) of Theorem 1.1 can be rewritten as* $\bar{T} \cap (-T) = \emptyset$.

Remark 1.1. The set D defined above is called the *domination cone* of the decision maker's preferences.

We may consider the decision maker's preferences to be ranked according to a quasi-ordering relation, as suggested by Aumann. However, since this becomes an ordering relation on $X/\sim$, we shall suppose throughout the paper that the decision maker's preferences are ranked according to an ordering relation. Assuming that $\succsim$ in Corollary 1.1 is antisymmetric, we have $D \cap (-D) = \{0\}$, which implies that D is a *pointed* cone. It is then easily seen that it is sufficient for D to be closed to ensure the continuity of $\succsim$.

Example 1.1. For $X \subset R^2$, let $D' = \{(x_1, x_2) | (x_1 > 0) \text{ or } (x_1 = 0, x_2 \geq 0)\}$. Then D' is pointed, but not closed. Therefore, the preference relation defined by D' is not necessarily continuous. In fact, the preference relation defined by this D' is a lexicographic ordering relation.

In view of the above considerations, we shall assume that the decision maker's preference relation is defined by a pointed closed convex cone D. We shall consider the following vector optimization problem:

(P) D-maximize $f(x)$ over $x \in X \subset R^n$,

where $f = (f_1, \ldots, f_p)$ and $\hat{x}$ is said to be the *D-maximal solution* if there is no $x \in X$ such that $f(x) - f(\hat{x}) \in D \backslash \{0\}$. In this paper, we shall give an overview of some duality results for D-maximal solutions of linear vector optimization problems, that is, in cases where f is a linear vector-valued function and X is a polyhedral set. However, we shall use a cone-ordering relation which is more general than that of the non-negative orthant: For a cone S,

$$y \geqq_S \hat{y} \leftrightarrow y - \hat{y} \in S$$

$$y \geq_S \hat{y} \leftrightarrow y - \hat{y} \in S \setminus l(S)$$

$$y >_S \hat{y} \leftrightarrow y - \hat{y} \in \text{int } S ,$$

where $l(S)$ and int S denote the lineality space of S, $S \cap (-S)$, and the interior of S, respectively. The positive polar of $S \subset R^n$ is defined by

$$S^0 = \{x \in R^n \mid \langle x, s \rangle \geqq 0, \forall s \in S\} .$$

For a closed cone S, it is well known that int $S^0 \neq \emptyset$ if and only if S is pointed [11]. We shall also make frequent use of the following lemma:

Lemma 1.1. *Let* S_1 *and* S_2 *be cones in* R^n. *Then*

(i) $\quad (S_1 + S_2)^0 = S_1^0 \cap S_2^0 = (S_1 \cup S_2)^0$

(ii) $\quad (S_1 \cap S_2)^0 \subset S_1^0 + S_2^0$.

In particular, for convex polyhedral cones S_1' *and* S_2' ,

(ii') $\quad (S_1' \cap S_2')^0 = S_1'^0 + S_2'^0$.

2. DUALITY IN LINEAR VECTOR OPTIMIZATION

Possibly the first work on duality theory in multiobjective optimization was carried out by Gale, Kuhn and Tucker for linear cases [3]. They considered the following matrix optimization problem:

Let D, Q, M and N be pointed convex polyhedral cones in R^p, R^m, R^n and R^r, respectively. This means, in particular, that int $D^0 \neq \emptyset$. In what follows we shall suppose that int $N \neq \emptyset$. Further, we shall identify the set of all $m \times n$ matrices with $R^{m \times n}$. This relation also holds for matrices of other dimensions. Define

$$\bar{K}_+ := \{K \in R^{p \times r} : KN \subset D\}$$

$$K_+ := \{K \in R^{p \times r} : K(\text{int } N) \subset D \setminus \{0\}\} .$$

Then, the ordering relation $\geqq$ for $p \times r$ matrices is introduced as follows:

$$K^1 \geqq K^2 \text{ if and only if } K^1 - K^2 \in \bar{\mathcal{K}}_+ \ ,$$

$$K^1 \geq K^2 \text{ if and only if } K^1 - K^2 \in \mathcal{K}_+ \ .$$

Problem (P_{GKT})

$\mathcal{K}_+$-maximize K

subject to

$$Cx \geqq_D Ky$$

$$Ax \leqq_Q By$$

$$x \geqq_M 0$$

$$y >_N 0 \ .$$

Here $A \in R^{m \times n}$, $B \in R^{m \times r}$, $C \in R^{p \times n}$, $K \in R^{p \times r}$, $x \in R^n$ and $y \in R^r$.

The dual problem associated with problem (P_{GKT}) is then

Problem (D_{GKT})

$\mathcal{K}_+$-minimize K

subject to

$$B^T \lambda \leqq_{N^0} K^T \mu$$

$$A^T \lambda \geqq_{M^0} C^T \mu$$

$$\lambda \in Q^0$$

$$\mu \in \operatorname{int} D^0 \ .$$

<u>Remark 2.1.</u> Problems (P_{GKT}) and (D_{GKT}) represent a class of matrix optimization problems of which vector optimization is a special case. In fact, in the case where B and K are vectors and y is a positive scalar, problem (P_{GKT}) reduces to the more usual formulation of vector optimization problems, that is,

<u>Problem (P'_{GKT})</u>

$$\text{Maximize } k$$

subject to

$$Cx \geq_D k$$

$$Ax \leq_Q b$$

$$x \geq_M 0 .$$

The dual problem associated with problem (P'_{GKT}) then becomes

<u>Problem (D'_{GKT})</u>

$$\text{Minimize } k$$

subject to

$$b^T\lambda \leq k^T\mu$$

$$A^T\lambda \geq_{M^0} C^T\mu$$

$$\lambda \in Q^0$$

$$\mu \in \text{int } D^0 .$$

Before proceeding to duality relations for the problem under consideration, we shall extend the well-known Minkowski-Farkas lemma.

<u>Lemma 2.1.</u> *For a matrix* $A \in R^{m\times n}$ *and a convex cone* $S \subset R^n$, *set*

$$AS := \{Ax\colon x \in S\} .$$

Then

$$(AS)^0 = \{\lambda \in R^m : A^T\lambda \in S^0\} .$$

Proof. Easy.

Lemma 2.2. *In order that* $\langle b,\lambda\rangle \geqq 0$ *for any* $\lambda \in Q^0$ *such that* $A^T\lambda \in M^0$, *it is necessary and sufficient that* $Ax \leqq_Q b$ *for some* $x \geqq_M 0$.

Proof. The given condition on λ is equivalent to

$$b \in ((AM)^0 \cap Q^0)^0 .$$

Further, since M and Q are convex polyhedral cones, we have

$$((AM)^0 \cap Q^0)^0 = AM + Q .$$

Finally, the given condition on λ is equivalent to

$$(b-Q) \cap AM \neq \emptyset ,$$

which is also equivalent to the given condition on x.

Remark 2.2. Extensions to cases with non-polyhedral Q and M are given by Fan [2] and Sposit and David [10].

Lemma 2.3. *For any two pointed convex cones* S *and* T, *with origin* 0, *the cone* S + T *is pointed if and only if* $S \cap (-T) = \{0\}$.

Proof. If a non-zero vector c is an element of $S \cap (-T)$, -c is also an element of T. Hence S + T contains $\alpha c + \beta(-c)$ for any $\alpha > 0$ and $\beta > 0$. This implies that S + T contains a non-trivial subspace, which means that S + T is not pointed.

Conversely, if S + T is not pointed, S + T contains non-zero elements c and -c. Let these be given by

$$c = s + t \quad , \quad s \in S,\ t \in T$$

$$-c = s' + t', \quad s' \in S,\ t' \in T .$$

Adding these equations, we obtain

$$0 = (s+s') + (t+t') . \qquad (2.1)$$

On the other hand, since S and T are pointed convex cones, $s+s' \neq 0$, $t+t' \neq 0$, $s+s' \in S$ and $t+t' \in T$. The relation (2.1) therefore implies $S \cap (-T) \neq \{0\}$. This completes the proof.

The following two lemmas are extensions of those given by Gale, Kuhn and Tucker [3].

Lemma 2.4. *In order that* $B^T\lambda \notin -N^0 \setminus \{0\}$ *for any* $\lambda \in Q^0$ *such that* $A^T\lambda \in M^0$, *it is necessary and sufficient that* $By \geqq_Q Ax$ *for some* $x \geqq_M 0$ *and* $y >_N 0$.

Proof. *Sufficiency.* Suppose, arguing by contradiction, that there exists a $\bar{\lambda} \in Q^0$ such that $A^T\bar{\lambda} \in M^0$ and $B^T\bar{\lambda} \in -N^0 \setminus \{0\}$. Then for any $x \in M$ and $y \in \text{int } N$, we have

$$\langle A^T\bar{\lambda}, x \rangle \geqq 0 > \langle B^T\bar{\lambda}, y \rangle .$$

On the other hand, the given condition for x and y implies that

$$\langle Ax - By, \bar{\lambda} \rangle \leqq 0$$

for some $x \in M$ and $y \in \text{int } N$, which contradicts the previous relation.

Necessity. Since Q, M and N are all pointed convex polyhedral cones, we have

the given condition for λ $\Leftrightarrow$ $-(BN)^0 \cap ((AM)^0 \cap Q^0) = \{0\}$ (from Lemma 2.1)
$\Leftrightarrow$ $(BN)^0 + ((AM)^0 \cap Q^0)$ is pointed (from Lemma 2.3)
$\Leftrightarrow$ $\text{int } (BN)^0 + ((AM)^0 \cap Q^0)^0 \neq \emptyset$ (from Lemma 1.1)
$\Leftrightarrow$ $\text{int } (BN) \cap (AM+Q) \neq \emptyset$
$\Leftrightarrow$ $B\,(\text{int } N) \cap (AM+Q) \neq \emptyset$
$\Leftrightarrow$ the given condition for x and y

Corollary 2.1. *One of the following statements*

(i) $B^T\lambda \in -N^0 \setminus \{0\}$ *for some* $\lambda \in Q^0$

(ii) $By \in Q$ *for some* $y >_N 0$

holds at all times but they cannot hold simultaneously.

Proof. The result is an extension of Gale's theorem and follows directly from Lemma 2.4.

Lemma 2.5. $\hat{K}$ *is a* K_+*-maximal solution of problem* (P_{GKT}) *if and only if*

(i) $C\bar{x} \geqq_D \hat{K}\bar{y}$ *holds for some* $\bar{x} \in M$ *and* $\bar{y} \in \text{int } N$ *such that* $A\bar{x} \geqq_Q B\bar{y}$, *and*

(ii) $Cx \ngeq_D \hat{K}y$ *holds for any* $x \in M$ *and* $y \in N$ *such that* $Ax \leqq_Q By$.
Similarly, $\hat{K}$ *is a* K_+*-minimal solution of problem* (D_{GKT}) *if and only if*

(ii') $B^T\bar{\lambda} \leqq_{N^0} \hat{K}^T\bar{\mu}$ *holds for some* $\bar{\lambda} \in Q^0$ *and* $\bar{\mu} \in \text{int } D^0$ *such that* $A^T\bar{\lambda} \geqq_{M^0} C^T\bar{\mu}$, *and*

(i') $B^T\lambda \nleq_{N^0} \hat{K}^T\mu$ *holds for any* $\lambda \in Q^0$ *and* $\mu \in D^0$ *such that* $A^T\lambda \geqq_{M^0} C^T\mu$.

Proof. We shall prove only the first part of the lemma; the proof of the second part may be obtained in a similar way.

If. Suppose that the $\hat{K}$ which satisfies (i) is not a solution of problem (P_{GKT}). Then there exists a matrix K' such that

$$K' \geq \hat{K}$$

and

$$C\bar{x} \geqq_D K'\bar{y}$$

for some $\tilde{x} \geqq_M 0$ and $\tilde{y} >_N 0$ such that $A\tilde{x} \leqq_Q B\tilde{y}$. Hence we have $C\tilde{x} \geqq_D K'\tilde{y} \geq_D \hat{K}\tilde{y}$, which contradicts condition (ii).

Only if. Suppose, contrary to the assertion of the lemma, that $\hat{K}$ does not satisfy (ii). Then there exist some $x' \in M$ and $y' \in N$ such that

$$Cx' \geq_D \hat{K}y' \text{ and } Ax' \leqq_Q By' .$$

Taking (i) into account, it follows that

$$C(\bar{x}+x') \geq_D \hat{K}(\bar{y}+y')$$

for $\bar{x}+x' \in M$ and $\bar{y}+y' \in \text{int } N$ such that $A(\bar{x}+x') \leqq_Q B(\bar{y}+y')$. Choose a vector $d' \in D \setminus \{0\}$ such that $d' \leq_D C(\bar{x}+x') - \hat{K}(\bar{y}+y')$ and a matrix $\Delta K \in K_+$ such that $\Delta K(\bar{y}+y') = d'$. For a vector e in N^0 with $\langle e, \bar{y}+y' \rangle = 1$, a possible ΔK is given by $\Delta K := d_1'e, d_2'e, \ldots, d_r'e)^T$. Then

$$(\hat{K}+\Delta K)(\bar{y}+y') \leq_D C(\bar{x}+x')$$

and

$$K' := \hat{K} + \Delta K \geq \hat{K} \ ,$$

which implies that $\hat{K}$ cannot be a solution to problem (P_{GKT}).

Gale, Kuhn and Tucker have formulated a duality relation between problem (P_{GKT}) and problem (D_{GKT}) which can be stated in an extended form as follows:

<u>Theorem 2.1.</u>

(i) *A matrix $\hat{K}$ is a $\overline{K}_+$-maximal solution of problem (P_{GKT}) if and only if it is a $\overline{K}_+$-minimal solution of problem (D_{GKT}).*

(ii) *If $\hat{K}$ is a $\overline{K}_+$-maximal solution of problem (P_{GKT}) for some $\hat{x} \geqq_M 0$, and $\hat{y} >_N 0$, then we have $\hat{K}\hat{y} = C\hat{x}$.*

(iii) *If $\hat{K}$ is a feasible solution of problem (P_{GKT}) for some $\hat{x} \geqq_M 0$ and $\hat{y} >_N 0$, and is also a feasible solution of problem (D_{GKT}) for some $\hat{\lambda} \in Q^0$ and $\hat{\mu} \in \text{int } D^0$, and if*

$$\hat{K}\hat{y} = C\hat{x}$$

$$\hat{K}^T\hat{\mu} = B^T\hat{\lambda},$$

then $\hat{K}$ is an efficient solution of both problems (P_{GKT}) and (D_{GKT}).

<u>Proof.</u>

(i) It is easily shown, for any convex cones S and T, that

$$(S \oplus T)^0 = S^0 \oplus T^0,$$

where

$$S \oplus T := \{(s,t): s \in S,\ t \in T\} .$$

From Lemma 2.4 we have

(i) of Lemma 2.5 $\leftrightarrow$ $\begin{pmatrix} A \\ -C \end{pmatrix} \bar{x} \leq_{Q \oplus D} \begin{pmatrix} B \\ -\hat{K} \end{pmatrix} \bar{y}$ for some $\bar{x} \in M$ and $\bar{y} \in \operatorname{int} N$

$\leftrightarrow$ $(B^T, -\hat{K}^T) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \not\leq_{N^0} 0$ for any $(\lambda, \mu) \in Q^0 \oplus D^0$

such that

$(A^T, -C^T) \begin{pmatrix} \lambda \\ \mu \end{pmatrix} \geq_{M^0} 0$

$\leftrightarrow$ (i') of Lemma 2.5.

Similarly,
(ii') of Lemma 2.5 $\leftrightarrow$ $\begin{pmatrix} -A^T \\ B^T \end{pmatrix} \bar{\lambda} \leq_{M^0 \oplus N^0} \begin{pmatrix} -C^T \\ \hat{K}^T \end{pmatrix} \bar{\mu}$ for some $\bar{\lambda} \in Q^0$ and $\bar{\mu} \in \operatorname{int} D^0$

$\leftrightarrow$ $(-C, \hat{K}) \begin{pmatrix} x \\ y \end{pmatrix} \not\leq_D 0$ for any $(x,y) \in M \oplus N$

such that

$(-A, B) \begin{pmatrix} x \\ y \end{pmatrix} \geq_Q 0$

$\leftrightarrow$ (ii) of Lemma 2.5.

Proof of (ii) and (iii) follows directly from Lemma 2.5. This completes the proof.

Another vector optimization formulation with more reciprocity was suggested by Kornbluth [6]:

Problem (P_K)

D-maximize Cx

subject to

$Az \leqq_Q By$

$x \geqq_M 0$

$y >_N 0$.

Problem (D_K)

D-minimize $B^T\lambda$

subject to

$A^T\lambda \geqq_{M^0} C^T\mu$

$\lambda \in Q^0$

$\mu \in \text{int } D^0$.

Theorem 2.2. *There exists a D-maximal solution* $\hat{x}$ *to problem* (P_K) *for some* $y=\hat{y}$ *if and only if there exists a D-minimal solution* $\hat{\lambda}$ *to problem* (D_K) *for some* $\mu=\hat{\mu}$.

Proof. See [6].

The following relationship between problems (P_{GKT}) and (P_K) $((D_{GKT})$ and $(D_K))$ is a simple extension of that revealed by Rödder [9].

Theorem 2.3.

(i) *If* $(\hat{K},\hat{x},\hat{y})$ *solves problem* (P_{GKT}), *then* $\hat{x}$ *is a D-maximal solution of problem* (P_K) *for* $y=\hat{y}$.

(ii) *If* $\hat{x}$ *is a D-maximal solution of problem* (P_K) *for* $y=\hat{y}$, *then there exists a matrix* $\hat{K}$ *such that* $(\hat{K},\hat{x},\hat{y})$ *solves problem* (P_{GKT}).

(iii) *Statements analogous to* (i) *and* (ii) *hold for problem* (D_{GKT}) *and problem* (D_K).

<u>Proof</u>. (i) is obvious. Since (iii) is dual to (ii), we need only proof (ii) here. Suppose that $\hat{x}$ is an efficient solution of problem (P_K). It may readily be shown that there exists a $\hat{\mu} \in \text{int } D^0$ such that

$$\hat{\mu}^T C\hat{x} \geqq \hat{\mu}^T Cx$$

for all $x \geqq 0$ such that

$$Ax \leqq_Q B\hat{y} \quad .$$

Considering the dual problem associated with this scalarized linear programming problem, it follows that there exists a $\hat{\lambda} \in Q^0$ such that

$$\hat{y}^T B^T \hat{\lambda} \leqq \hat{y}^T B^T \lambda$$

for any $\lambda \in Q^0$ such that

$$A^T \lambda \geqq_{M^0} C^T \hat{\mu} \quad .$$

From the well-known duality theorem of linear programming, we have

$$\hat{y}^T B^T \hat{\lambda} = \hat{\mu}^T C\hat{x} \quad .$$

This condition implies that the two equations

$$D\hat{y} = C\hat{x}$$

and

$$\hat{\mu}^T D = \hat{\lambda}^T B$$

have a common solution $\hat{D}$ (see, for example, Penrose [8]). Hence, it follows immediately from (iii) of Theorem 2.1 that $\hat{D}$ is an efficient solution of problem (P_{GKT}). This completes the proof.

Isermann [4,5] has given a more attractive formulation which does not include auxiliary parameters such as y and μ. We shall now consider it in an extended form. Let $\mathcal{U}_0$ be a class of $p \times m$ matrices U such that there exists a $\mu \in \text{int } D^0$ with $U^T\mu \in Q^0$. The primal and dual problems are then defined as follows:

Problem (P_I)

$$\text{D-maximize } \{Cx: \quad x \in X\}$$

where

$$X := \{x \in M: \quad Ax \leqq_Q b\} \ .$$

Problem (D_I)

$$\text{D-minimize } \{Ub: \quad U \in \mathcal{U}_0\}$$

where

$$\mathcal{U}_0 := \{U: \quad \text{there exists a } \mu \in \text{int } D^0 \text{ such that}$$
$$A^TU^T\mu \geqq_{M^0} C^T\mu \text{ and } U^T\mu \in Q^0\} \ .$$

The following duality properties hold for these problems:

Theorem 2.4.

(i) $Ub \not\leq_D Cx$ *for all* $(U,x) \in \mathcal{U}_0 \times X$.

(ii) *Suppose that* $\bar{U} \in \mathcal{U}_0$ *and* $\bar{x} \in X$ *satisfy* $\bar{U}b = C\bar{x}$. *Then* $\bar{U}$ *is a D-minimal solution of the dual problem and* $\bar{x}$ *is a D-maximal solution of the primal problem* (P_I).

(iii) $\text{Max}_D\ (P_I) = \text{Min}_D\ (D_I)$.

Proof.

(i) Suppose, contrary to the assertion of the theorem, that there exist some $\bar{x} \in X$ and $\bar{U} \in \mathcal{U}_0$ such that

$$\bar{U}b \lneq_D C\bar{x} \quad . \tag{2.2}$$

Note here that by definition there exists a $\bar{\mu} \in \text{int } D^0$ such that

$$A^T\bar{U}^T\bar{\mu} \geqq_{M^0} C^T\bar{\mu}$$

$$\bar{U}^T\bar{\mu} \in Q^0 \quad .$$

Therefore, since $\bar{x} \in M$, we have

$$\langle A^T\bar{U}^T\bar{\mu}, \bar{x} \rangle \geqq \langle C^T\bar{\mu}, \bar{x} \rangle \, . \tag{2.3}$$

Furthermore, from (2.2)

$$\langle \bar{\mu}^T, \bar{U}b \rangle < \langle \bar{\mu}^T, C\bar{x} \rangle \, . \tag{2.4}$$

It then follows from (2.3) and (2.4) that

$$\langle \bar{\mu}^T, \bar{U}A\bar{x} \rangle > \langle \bar{\mu}, \bar{U}b \rangle \, . \tag{2.5}$$

However, since $\bar{U}^T\bar{\mu} \in Q^0$ and $A\bar{x} \leqq_a b$, we have $\langle \bar{\mu}, \bar{U}A\bar{x} \rangle \leqq \langle \bar{\mu}, \bar{U}b \rangle$, which contradicts (2.5).

(ii) Suppose, contrary to the assertion of the theorem, that $\bar{U}b \notin \text{Min}_D(D_I)$. Then there exists a $\hat{U} \in \mathcal{U}_0$ such that $\hat{U}b \leq_D \bar{U}b = C\bar{x}$, which contradicts result (i). Therefore, $\bar{U}$ is a D-minimal solution of the dual problem. In a similar fashion, we can conclude that $\bar{x}$ is a D-maximal solution of the primal problem.

(iii) We shall first prove $\text{Max}_D(P_I) \subset \text{Min}_D(D_I)$. Let $\hat{x}$ be a D-maximal solution of problem (P_I). Then it is well-known that there exists some $\hat{\mu} \in \text{int } D^0$ such that $\langle \hat{\mu}, C\hat{x} \rangle \geqq \langle \hat{\mu}, Cx \rangle$ for all $x \in X$. It is sufficient to prove the statement in the case where $\hat{x}$ is a basic solution. Transform the original inequality constraints $Ax \leqq_Q b$ into

$$Ax + y = b$$

$$y \geqq_Q 0 \, .$$

Let B denote the submatrix of [A,-I] which consists of m columns corresponding to the basic variables. Then from the initial simplex tableau

$$\left[\begin{array}{c|c|c} A & I & b \\ \hline \hat{\mu}^T C & 0 & 0 \end{array}\right] ,$$

we obtain the final tableau

$$\left[\begin{array}{c|c|c} B^{-1}A & B^{-1} & B\text{-}b \\ \hline \hat{\mu}^T(C-C_B B^{-1}A) & -\hat{\mu}^T C_B B^{-1} & \hat{\mu}^T C_B B^{-1} b \end{array}\right]$$

using the simplex method.

From the well-known properties of linear programming problems, we have

$$\hat{\mu}^T(C-C_B B^{-1}A) \leq_{M^0} 0$$

$$\hat{\mu}^T C_B B^{-1} \geq_{Q^0} 0 .$$

Setting $\hat{U} = C_B B^{-1}$, these relations can be rewritten as

$$\hat{\mu}^T C \leq_{M^0} \hat{\mu}^T \hat{U} A$$

$$\hat{\mu}^T \hat{U} \in Q^0 ,$$

which shows that $\hat{U} \in \mathcal{U}_0$.

On the other hand,

$$\hat{U}b = C_B B^{-1} b = C_B x_B = C(x_B, 0) = C\hat{x} .$$

In view of result (ii), the last relation implies that $\hat{U}$ is a D-minimal solution of problem (D_I). Hence we have

$$\mathrm{Max}_D(P_I) \subset \mathrm{Min}_D(D_I) .$$

We shall now prove $\mathrm{Max}_D(P_I) \supset \mathrm{Min}_D(D_I)$. Suppose that $\hat{U}$ is a D-minimal solution of problem (D_I). Then it is clear that for every $\mu \in \mathrm{int}\, D^0$ there cannot exist any $U \in \mathcal{U}_0$ with $U^T\mu \in Q^0$ such that

$$\mu^T UA \geqq_{M^0} \mu^T C$$

$$\mu^T Ub < \mu^T \hat{U} b \; .$$

Setting $\lambda = U^T\mu$, it follows that

$$\left.\begin{aligned} &\lambda^T A \geqq_{M^0} \mu^T C \\ &\lambda^T b < \mu^T \hat{U} b \end{aligned}\right\} \qquad (2.6)$$

cannot hold for any $\lambda \in Q^0$ and any $\mu \in \mathrm{int}\, D^0$. More strongly, we can see that there is no $\lambda \in Q^0$ and no $\mu \in D^0$ for which (2.6) is satisfied. In fact, arguing by contradiction, suppose that some $\lambda' \in Q^0$ and $\mu' \in D^0$ exist such that

$$\lambda'^T A \geqq_{M^0} \mu'^T C$$

$$\lambda'^T b < \mu'^T \hat{U} b \; .$$

But since $\hat{U}$ is a solution of problem (D_I), there exist $\hat{\mu} \in \mathrm{int}\, D^0$ and $\hat{\lambda} \equiv \hat{U}^T \hat{\mu} \in Q$ such that

$$\hat{\lambda}^T A \geqq_{M^0} \hat{\mu}^T C$$

$$\hat{\lambda}^T b = \hat{\mu}^T \hat{U}^T b \; .$$

Therefore, we have

$$(\hat{\lambda} + \lambda')^T A \geqq_{M^0} (\hat{\mu} + \mu')^T C$$

$$(\hat{\lambda} + \lambda')^T b < (\hat{\mu} + \mu')^T \hat{U} b \; .$$

This implies the existence of solutions $\hat{\mu} + \mu' \in \text{int } D^0$, $\hat{\lambda} + \lambda' \in Q^0$ to (2.6), which contradicts our earlier assumption.

Rewriting (2.6), we may say that there is no $\lambda \in Q^0$ and no $\mu \in D^0$ for which

$$(\lambda^T, \mu^T) \begin{pmatrix} -A & b \\ C & -\hat{U}b \end{pmatrix} \in -(M^0 \oplus R_+^{1\,0} \setminus \{0\})$$

is satisfied. Thus, from Corollary 2.1, there exists a solution $(\hat{x}, \alpha) \in \text{int } (M^0 \oplus R_+^{1\,0})^0 = \text{int } (M \oplus R_+^1)$ satisfying

$$\begin{pmatrix} -A & b \\ C & -\hat{U}b \end{pmatrix} \begin{pmatrix} \hat{x} \\ \alpha \end{pmatrix} \in Q \oplus D \, .$$

Since $\alpha > 0$, we finally have

$$A\hat{x} \leqq_Q b \qquad (2.7)$$

$$C\hat{x} \geqq_D \hat{U}b \, .$$

Using result (i), the last relation reduces to

$$C\hat{x} = \hat{U}b \, . \qquad (2.8)$$

From result (ii), relations (2.7) and (2.8) imply that $\hat{x}$ is a D-maximal solution of problem (P_I). This completes the proof.

3. CONCLUDING REMARKS

This paper reviews several duality results in linear vector optimization using an extended reformulation with general cone ordering. This generalization gives some insight into the relations between cone orderings. In a previous paper [7], the author discusses duality in nonlinear vector optimization in a geometrically unified way. All of the results given in this paper can be adapted to nonlinear situations by treating this as a special case and using a vector-valued Lagrangian. This will be treated in a subsequent paper.

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