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THE GENERALIZED NUCLEOLUS AS A SOLUTION OF A COST ALLOCATION PROBLEM

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INTRODUCTION

It has recently become clear that ideas and methods from the theory of cooperative games can be used quite successfully to solve cost allocation problems. Among the extensive literature (see Loughlin, 1977) dealing with this subject we shall concentrate on an article based on work carried out at IIASA (Young et al., 1980). The authors of this article used game theory principles of rationality to solve the problem of sharing the cost of a joint municipal water supply system among the group of Swedish municipalities participating in the project.

In Section 1 we introduce the notion of the generalized nucleolus. The nucleolus and all its known modifications are special cases of our definition. Section 2 describes a method for calculating the nucleolus that can be readily implemented on a computer, and Section 3 puts forward an analytical criterion for testing the results. In Section 4 we describe some applications of the method to linear-fractional excess functions and convex games, and a number of formulas for three-person games are also given. Section 5 contains numerical results for seven modifications of the nucleolus for the six-person game discussed in Young et al. (1980).

1. DISCUSSION OF FUNDAMENTAL NOTIONS

A classical cooperative game concerns a pair $\langle J,v\rangle$ which consists of a set $J = \{1,2,\ldots,n\}$ of n players and a characteristic function $v(S)$ which maps every subset S of J, called a coalition, onto a nonnegative number (for example, the coalition's largest guaranteed payoff). The outcome of a cooperative game will be a coalition of n players and a payoff vector, i.e., all players will share a common payoff. Usually the payoff vector (imputation) is assumed to be individually rational, which means that no player will accept a value less than his own guaranteed payoff. The imputations are therefore elements of the following set:

$$X = \left\{x \in R^n \mid \sum_{i \in J} x_i = v(J)\ , x_i \geq v(i) \text{ for all } i \in J\right\} \ .$$

The main problem of cooperative game theory is to formalize the way in which the possibilities of each coalition (described by $v(S)$) may be converted to the possibilities of individual players (given in terms of imputations from the set X). The aim of the investigation is to calculate an imputation or a set of imputations that is in some sense optimal. The most popular principles of rationality used in cooperative game theory are the core, the von Neumann-Morgenstern solution, the Shapley value, and the nucleolus and its various modifications.

In the particular application of cooperative game theory considered here, the players are the users of water (regions, municipalities and so on) and the characteristic function is interpreted as the least cost $c(S)$ of serving coalition S of users, that is, the cost of constructing and operating a joint facility to supply coalition S. It is assumed that the amount $c(S)$ is sufficient to meet the water demand of each member of the coalition. The water demand of users is calculated on the basis of factors such as population, industrial requirements, and so on.

By analogy to the set X, let the set of cost allocations be defined as follows:

$$Y = \left\{ y \in R^n \mid \sum_{i \in J} y_i = c(J) \; , y_i \leq c(i) \text{ for all } i \in J \right\} \; .$$

Classical cooperative game theory as a rule operates with the game in 0-normalized form. This means that $v(i) = 0$, $i \in J$; $0 \leq v(S) \leq v(J)$, $S \subset J$. If, in addition, $v(J) = 1$, then the game is said to be in (0,1)-normalized form.

The following mapping is used to reduce the characteristic function $c(S)$ to 0-normalized form:

$$v(S) = \sum_{i \in S} c(i) - c(S) \quad . \tag{1}$$

A cost allocation y corresponding to $c(S)$ is associated with an imputation x by the following formula:

$$y_i = c(i) - x_i \quad , \qquad i \in J \quad . \tag{2}$$

One can easily see that the cost of serving any group of users may be less than the sum of the costs of serving them individually, especially when the users are neighbors. This case is represented as follows:

$$c(S) < \sum_{i \in S} c(i) \quad .$$

The function $c(S)$ is assumed to be superadditive, which in game theory means that

$$c(S) + c(T) \geq c(S \cup T) \tag{3}$$

where

$$S, T \subset J \quad , \qquad S \cap T = \emptyset \quad ,$$

because the possible ways of serving S together with T include the possibility of serving S alone and T alone.

It is shown by Young et al. (1980) that the nucleolus and its different modifications (see Schmeidler, 1969, and Shapley and Shubik, 1973) are the methods from game theory most applicable to cost allocation problems. In the present paper a definition of the generalized nucleolus is given (all known modifications of the nucleolus are special cases of this form) and a method for calculating it is suggested.

We shall now introduce some definitions from Young et al. (1980). It is natural to denote

$$y(S) = \sum_{i \in S} y_i \quad .$$

Assume

$$\overline{Y} = \{y \in R^n \mid y(J) = c(J)\} \quad .$$

Following Shapley and Shubik (1973), we define the least core as a set of vectors y which are optimal solutions of the following linear programming (l.p.) problem:

$$\min \varepsilon$$

$$y(S) \leq c(S) + \varepsilon \quad , \qquad S \subset J \tag{4}$$

$$y(J) = c(J) \quad .$$

The core (C) is the set of allocations y which satisfy all the constraints of problem (4) with $\varepsilon = 0$.

If problem (4) has multiple solutions the following tie-breaking device may be used. For any allocation y and coalition S, define the excess of S relative to y to be

$$e(S,y) = c(S) - y(S) \quad .$$

Let $\varepsilon_1(y)$ be the largest excess of any coalition relative to y, $\varepsilon_2(y)$ the next largest excess and so on. The nucleolus is a cost allocation $\overline{y}$ for which

$$\varepsilon_1(\overline{y}) \leqq \varepsilon_1(y) \qquad \text{for all } y \in \overline{Y} \tag{5}$$

$$\varepsilon_2(\overline{y}) \leqq \varepsilon_2(y) \qquad \text{for all } y \text{ satisfying (5)} \tag{6}$$

$$\varepsilon_3(\bar{y}) \leqq \varepsilon_3(y) \quad \text{for all } y \text{ satisfying (5) and (6)} \tag{7}$$

and so on.

We should perhaps comment on some aspects of this definition of the nucleolus.

1. The definition of the nucleolus given above may produce a vector which is not, in general, a cost allocation from Y but only from $\bar{Y}$ because the condition of individual rationality may not be observed. If the characteristic function is such that the core is not empty or, in other words, if the optimal ε in problem (4) is not positive, then every allocation $y \in \bar{Y}$ which is optimal in (4) is a member of Y. This is not necessarily the case when the optimal ε is positive.

This fact can be illustrated by the following simple example.

Let

$$J = \{1,2,3,4,5\} \ , \quad c(S) = \sum_{i \in S} c(i) \quad ,$$

where

$$S \neq \{2,3\},\ \{4,5\} \text{ and } c(J) - c(1) - c(2,3) - c(4,5) \overset{\text{def}}{=\!=} q > 0.$$

Then, for an allocation y to be optimal in problem (4), the following condition must be true:

$$y_1 = c(1) + \frac{1}{3} q \ > c(1) \quad .$$

To fulfill the condition of individual rationality we include the inequalities

$$y_i \leq c(i) \quad , \qquad i \in J$$

as constraints in problem (4).

2. We reproduce below a table from Menshikova (1974), which gives the nucleolus of an arbitrary three-person game in (0,1)-normalized form. Here $v(1,3) = a$, $v(1,2) = h$, $v(2,3) = d$ and without loss of generality it is assumed that

$$a \leq h \leq d \quad . \tag{8}$$

TABLE 1 The nucleolus of an arbitrary three-person game in (0,1)-normalized form.

Conditions		Nucleolus
$d \leq 1/3$		$(1/3,1/3,1/3)$
$d > 1/3$, $2a+2h-d > 1$		$\left(\frac{a+h-2d+1}{3}, \frac{d+h-2a+1}{3}, \frac{a+d-2h+1}{3}\right)$
$d > 1/3$, $2a+2h-d \leq 1$	for $2a+d \geq 1$	$\left(\frac{1-d}{2}, \frac{2h+d+1-2a}{4}, \frac{2a+d+1-2h}{4}\right)$
	for $2a+d < 1$, $2h+d \geq 1$	$\left(\frac{1-d}{2}, \frac{d+h}{2}, \frac{1-h}{2}\right)$
	for $2h+d < 1$	$\left(\frac{1-d}{2}, \frac{1+d}{4}, \frac{1+d}{4}\right)$

It can be seen from Table 1 that the least core consists of more than one point when

$$d > 1/3 \quad , \qquad 2a+2h-d \leq 1 \quad . \tag{9}$$

More precisely, the least core is the following set:

$$\left\{ x \in R^3 \,\middle|\, x_1 = \frac{1-d}{2}, x_2 + x_3 = \frac{1+d}{2}, \right.$$
$$\left. x_i \geq \max\left[\frac{1-d}{2}, v(1,i)\right], \ i = 2,3 \right\} \quad .$$

The set of parameters defined by condition (9) has full dimension and no small variation of the characteristic function can reduce the least core to a single point.

The c-proportional least core is the set of allocations y which are optimal solutions of the l.p. problem:

$$\begin{aligned} &\min r \\ &y(S) \leq (1+r)c(S) \quad , \quad S \subset J \\ &y(J) = c(J) \quad . \end{aligned} \tag{10}$$

Finally, the weak least core is the set of optimal allocations for the following l.p. problem:

$$\begin{aligned} &\min \delta \\ &y(S) \leq c(S) + \delta|S| \quad , \quad S \subset J \\ &y(J) = c(J) \quad . \end{aligned} \tag{11}$$

Generalizing the definitions from Young et al. (1980) and taking into account point 2 above, we shall define the generalized least core as the following set:

$$L[d(S)] = \{y|y \text{ is optimal in } (12)\} \quad ,$$

where

$$\begin{aligned} &\min u \\ &y(S) \leq ud(S) + c(S) \quad , \quad S \subset J \\ &y(J) = c(J) \\ &y_i \leq c(i) \quad , \quad i \in J \quad . \end{aligned} \tag{12}$$

We shall now give an equivalent definition of the set $L[d(S)]$:

$$L[d(S)] = \{y(x)\,|\,x \text{ is optimal in } (13)\}$$

$$\begin{aligned} &\min u \\ &ud(S) + x(S) \geq v(S) \quad , \quad S \subset J \\ &x(J) = v(J) \\ &x \geqq 0 \quad , \end{aligned} \tag{13}$$

where $y(x)$ is the mapping (2) and $v(S)$ is calculated from (1). It is easy to see that the sets $L[1]$, $L[c(S)]$, $L[v(S)]$, and $L[|S|]$ are the least core, the c-proportional least core, the proportional least core from Young et al. (1980), and the weak least core, respectively.

It is natural to define the proportional, c-proportional, and weak nucleolus by analogy with the nucleolus. These modifications all come under the broad heading of the generalized nucleolus (Menshikova, 1976).

Definition of the generalized nucleolus

Fix some set $M \subset 2^J$ and a function $d\colon M \to R^1_+$, where M is the set of all permissible coalitions and $d(S) > 0$ is the normalizing multiplier for a coalition $S \in M$. For a fixed game $\langle J,v \rangle$ and vector x let $\theta(x)$ be a vector with components

$$e^o(S,x) = \frac{v(S) - x(S)}{d(S)} \quad ,$$

arranged according to their magnitude, where $S \in M$, i.e., $i < j$ implies $\theta_i(x) \geq \theta_j(x)$. We say that a vector $\theta(x)$ does not lexicographically exceed $\theta(z)$ ($\theta(x) \preccurlyeq \theta(z)$) if the first nonzero component of the vector $\theta(z) - \theta(x)$ is positive. Let the generalized nucleolus be the set $N[M,d]$ of points corresponding to the lexicographical minimum of the function $\theta(x)$ over the set X. More precisely,

$$N[M,d] = \{x \in X \mid \theta(x) \preccurlyeq \theta(z) \text{ for all } z \in X\} \quad .$$

It is obvious that when $M = 2^J$ and $d(S) \equiv 1$ the set $N[M,d]$ consists of one point - the nucleolus.

2. METHOD OF COMPUTATION

The computation of the nucleolus is a problem of linear lexicographical optimization and has been discussed in a large number of studies. For example, Kopelowitz (1967) suggested solving the sequence of l.p. problems generated by the definition of the nucleolus. In addition, Kohlberg (1971) and Owen (1974) have shown how to construct a single l.p. problem with the property that its unique solution is the nucleolus. This problem has not less than $4^n + 1$ constraints and $2^{n+1} + n$ variables, and the authors themselves underline the fact that a computer realization of this method is extremely complicated even for small values of n. Here we formulate a sequence of l.p. problems such that the set of solutions of the last problem coincides with the generalized nucleolus. We also consider the corresponding sequence of dual l.p. problems. Below we use $\chi(S)$ to denote the characteristic vector of a coalition S, i.e., $\chi_i(S) = \begin{cases} 1, & i \in S, \\ 0, & i \notin S \end{cases} \; i \in J.$

<u>Problem 1</u>

$$\min u$$

$$ud(S) + x(S) \geq v(S) \quad , \quad S \in M$$

$$-x(J) \geq -v(J)$$

$$x \geqq 0 \quad .$$

The dual formulation of this problem is

Problem 1'

$$
\begin{aligned}
&\max F_1(\lambda) \\
&\lambda: \sum_{S \in M} \lambda_S \chi(S) - \lambda_J \chi(J) \leqq 0 \\
&\qquad \sum_{S \in M} \lambda_S d(S) = 1 \quad , \qquad \lambda \geqq 0 \\
&F_1(\lambda) = \sum_{S \in M} \lambda_S v(S) - \lambda_J v(J) \quad .
\end{aligned}
\tag{14}
$$

Problem 1' has a solution, and from the first duality theorem (Udin and Golshtein, 1964) the optimal values of the objective functions in problems 1 and 1' must be equal (this value is denoted by T_1).

The S-th condition of problem 1' is described as free if there exists an optimal vector λ such that $\lambda_S > 0$, where $S \in M$. Let M_1 be the set of all free conditions of problem 1'. Then the second duality theorem (Udin and Golshtein, 1964) tells us that for any vector x optimal in problem 1 it is necessary that

$$\frac{v(S) - x(S)}{d(S)} = T_1 \quad , \qquad S \in M_1 \quad .$$

The set of all imputations which are optimal in problem 1 coincides with the generalized least core and can be described as the set of solutions to the following linear system:

$$
\begin{aligned}
&\frac{v(S) - x(S)}{d(S)} = T_1 \quad , \qquad S \in M_1 \\
&\frac{v(S) - x(S)}{d(S)} \leq T_1 \quad , \qquad S \in M \setminus M_1 \\
&x(J) = v(J) \quad , \qquad x \geqq 0 \quad .
\end{aligned}
$$

It is interesting that the extreme points of the polyhedron described by the constraints of problem 1' are vectors $\lambda \geqq 0$ that turn vector inequality (14) into an equality. In game theory these vectors are known as minimal balanced (m.b.) collections

(Bondareva, 1963), and their properties for games with a small number of players are well known. This allows us to obtain an analytical expression for the generalized nucleolus. Table 1, for example, was constructed in this way.

We use the following definition of a minimal balanced collection. A matrix A with ℓ rows, n columns and elements equal to 0 or 1 is called a balanced collection (of degree $\ell \times n$) if there is a positive vector λ and a positive number α such that

$$\lambda A = \alpha(1,1,\ldots,1) \quad .$$

A balanced collection can be described as minimal if the rows of the matrix are linearly independent.

If A is a minimal balanced collection and α is a fixed number, then vector λ is uniquely determined. This vector is also sometimes described as a minimal balanced collection.

For example, the set of all minimal balanced collections of degree $\ell \times 3$, where $\ell \leq 3$, is listed as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad .$$

If the optimal extreme point in problem 1' is a minimal balanced collection of degree $n \times n$ then the generalized least core consists of a single point which is the generalized nucleolus. The generalized least core and nucleolus also coincide when $M_1 = M$. In other cases it is necessary to consider the second l.p. problem in order to calculate the generalized nucleolus and so on.

Let us formulate the k-th l.p. problem of this sequence, assuming that the generalized nucleolus coincides with its solution.

Problem k

$$\min u$$

$$ud(S) + x(S) \geq v(S) \quad , \qquad S \in M \setminus \bigcup_{j=1}^{k-1} M_j$$

$$x(S) \geq v(S) - T_1 d(S) \quad , \qquad S \in M_1$$

....

$$x(S) \geq v(S) - T_{k-1} d(S) \quad , \; S \in M_{k-1}$$

$$-x(J) \geq -v(J)$$

$$x \geqq 0 \quad .$$

The dual formulation of this problem is

Problem k'

$$\max F_k(\lambda)$$

$$\lambda: \sum_{S \in M} \lambda_S \chi(S) - \lambda_J \chi(J) \leqq 0$$

$$\sum_{S \in M \setminus \bigcup_{j=1}^{k-1} M_j} \lambda_S d(S) = 1 \quad , \qquad \lambda \geqq 0$$

where

$$F_k(\lambda) = F_{k-1}(\lambda) - T_{k-1} \sum_{S \in M_{k-1}} \lambda_S d(S) \quad .$$

Let M_k be the set of all free conditions of problem k'. The generalized nucleolus is then the set of solutions of the following linear system:

$$\frac{v(S) - x(S)}{d(S)} = T_j \; , \quad S \in M_j \quad , \quad 1 \leq j \leq k$$

$$x(J) = v(J) \quad , \tag{15}$$

$$x \geqq 0 \quad ,$$

where either $M = \bigcup_{j=1}^{k} M_j$ or system (15) has a unique solution.

The above considerations therefore suggest that the generalized nucleolus can be calculated in two ways: by solving problems 1-k or problems 1'-k'.

We shall denote by $k[M,d(S)]$ the minimum number of l.p. iterations required to calculate the nucleolus.

The Stop Rule. $k[M,d(S)]$ is equal to the smallest ℓ for which the optimal extreme solutions of problems ℓ and $(\ell+1)$ coincide.

In our opinion, the second method of calculation (solution of problems 1'-k') is preferable to the first. We have already mentioned that it has an advantage in the construction of tables for the generalized nucleolus when n is small. The second advantage has to do with the fact that the sets of basic solutions in problems 1',2',...,k' change only slightly: these solutions are minimal balanced collections in all problems and only the multipliers α are changed from step to step. In addition, the set of feasible solutions to problems 1'-k' does not depend on $v(S)$ and hence is the same for all problems concerned with the same set M of coalitions. Finally, the second method of calculation is more suitable for computer realization.

3. ANALYTICAL CRITERION

The connection of the problems 1',2',...,k' with the notion of minimal balanced collections makes it possible to formulate a sequence of conditions necessary for a vector x to be from the generalized nucleolus. If we take these necessary conditions together we can obtain a sufficient condition for the generalized nucleolus to contain x.

Consider an imputation x for which

$$\theta(x) = (T_1(x),\ldots,T_1(x),\ldots,T_p(x),\ldots,T_p(x)) \quad .$$

Let $M_j(x)$ be the matrix with rows which are vectors $\chi(S)$ for S such that

$$\frac{v(S) - x(S)}{d(S)} = T_j(x) \quad .$$

The first necessary condition for $x \in N[M,d]$

Matrix $M_1(x)$ is a balanced collection and associated vector λ is a solution of the equation:

$$F_1(\lambda) = T_1(x) \quad .$$

Proposition 1. Let x be the unique solution of the system

$$\frac{v(S) - \xi(S)}{d(S)} = T_1(x) \quad , \qquad S \in M_1(x) \quad .$$

Then

$$N[M,d] = \{x\} \quad .$$

If the system

$$\frac{v(S) - \xi(S)}{d(S)} = T_1(x) \quad , \qquad S \in M_1(x)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (16)$$

$$\frac{v(S) - \xi(S)}{d(S)} = T_{k-1}(x) \quad , \; S \in M_{k-1}(x)$$

has multiple solutions and if

$$\bigcup_{j=1}^{k-1} M_j \neq M$$

the following necessary condition may be useful.

The k-th necessary condition for $x \in N[M,d]$

By adding certain rows of the matrices $M_j(x)$, $1 \leq j \leq k-1$ to matrix $M_k(x)$ it is possible to obtain a balanced collection λ for which $F_k(\lambda) = T_k(x)$.

Proposition 2. Let x be the unique solution of system (16). Then $N[M,d] = \{x\}$.

Thus we have a constructive method for testing whether a given imputation x is in the nucleolus. No computer is necessary when this method is used for problems with small n because in this case complicated calculations are not needed.

4. APPLICATIONS OF THE METHOD

The above method for calculating the generalized nucleolus is not limited to excess of the type $e^o(S,x)$ as defined earlier. For example, Littlechild and Vaida (1966) propose to use the excess function

$$\tilde{e}(S,x) = \frac{x(J\setminus S) - v(J\setminus S)}{x(S) - v(S)} .$$

Proposition 3. For an arbitrary superadditive game $\langle J,v\rangle$ the set $N[M,\tilde{e}]$ coincides with $N[M,\gamma]$, where

$$\gamma(S,x) = \frac{v(S) - x(S)}{r(S)} , \quad r(S) = v(J) - v(S) - v(J\setminus S) .$$

Proposition 4. For any three-person game, the set $N[2^J,\gamma]$ is equal to the vector

$$\frac{v(J)}{\sum\limits_{i=1}^{3} r(i)} (r(1),r(2),r(3)) .$$

It is easy to generalize Proposition 4 for an n-person game with the following set of permissible coalitions:

$$\hat{M} = \{\{j\}\ , J\setminus\{j\} | j \in J\} \quad .$$

Proposition 5. For an arbitrary superadditive game $\langle J,v\rangle$ the set $N[\hat{M},\gamma]$ is equal to the vector

$$\frac{v(J)}{\sum_{i=1}^{3} r(i)} (r(1),r(2),\dots,r(n)) \quad .$$

The function $\gamma(S,x)$ has an interesting interpretation for convex games, i.e., games for which the inequality

$$v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$$

holds for any pair of coalitions S, T (Shapley, 1971).

The core is not empty in a convex game. The difference between maximum and minimum values of $x(S)$ when imputation x varies in the core is then called the range ($R(S)$) of a coalition S.

Thus

$$R(S) = \max_{x \in C} x(S) - \min_{x \in C} x(S) \quad .$$

It is easy to check that the functions $U(S) = \max_{x \in C} x(S)$ and $u(S) = \min_{x \in C} x(S)$ are related by the formula:

$$U(S) = v(J) - u(J\setminus S) \quad .$$

The range $R(S)$ of a coalition S is equal to the maximum surplus payoff above the guaranteed level $u(S) \geq v(S)$, when only imputations from the core are considered.

Now let us introduce a new excess function $\tilde{\gamma}(S,x)$ defined by

$$-\tilde{\gamma}(S,x) = \frac{x(S) - u(S)}{R(S)} = \frac{x(S) - u(S)}{v(J) - u(S) - u(J\setminus S)} \quad .$$

When x belongs to the core the value $-\tilde{\gamma}(S,x)$ is not greater than 1. This value is therefore a measure of how successfully a coalition S operates within the framework of the core.

For these reasons we suggest $N[M,\tilde{\gamma}]$ as a rationality principle for games with a non-empty core.

If a game is convex then $u(S) = v(S)$ (Shapley, 1971), so that $\tilde{\gamma}(S,x) = \gamma(S,x)$. Convex games also have another interesting feature:

Proposition 6. The optimal solution of problem 1 may be found in the class of minimal balanced collections A composed of partitions or their complements, i.e., for every pair $S,T \in A$ either $S \cap T = \emptyset$ or $(J \setminus S) \cap (J \setminus T) = \emptyset$.

This proposition has an important consequence. Of all the minimal balanced collections mentioned in Proposition 6, there are only two of degree $n \times n$:

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 & 1 & \dots & 1 \\ 1 & 0 & 1 & \dots & 1 \\ 1 & 1 & 0 & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 1 & 1 & \dots & 0 \end{pmatrix}.$$

Thus, as a rule, it is necessary to solve more than one l.p. problem to calculate the generalized nucleolus of a convex game.

Proposition 6 may be proved with the help of Proposition 7:

Proposition 7. Let A be a balanced collection. If for every pair $S,T \in A$ one of the following three conditions is true

1. $S \subset T$ or $T \supset S$
2. $S \cap T = \emptyset$
3. $S \cup T = J$

then $A \in \mathrm{Conv}\, \mathcal{B}$, where $\mathcal{B} = \{A \mid A$ defined in Proposition 6$\}$.*

---

*Conv P denotes the convex hull of set P.

Using minimal balanced collections we may prove Proposition 8:

Proposition 8. If for any coalition $S \subset J$ the following condition

$$\frac{\sum_{i \in S} d(i) - d(S)}{\sum_{i \in J} d(i)} \geq v(S)$$

is true, then the generalized nucleolus is given by the formula

$$N[M,d] = \frac{v(J)}{\sum_{i=1}^{n} d(i)} (d(1),d(2),\dots,d(n)) \quad ,$$

where M is a subset of 2^J which contains all one-person coalitions.

Corollary. If v(S) satisfies the condition

$$v(S) \leq \frac{|S| - 1}{n}$$

then the nucleolus is equal to the vector

$$\frac{v(J)}{n} (1,1,\dots,1) \quad .$$

Table 2 gives the weak nucleolus of an arbitrary three-person game (a,h, and d have the same meaning as in Table 1).

TABLE 2 The weak nucleolus of an arbitrary three-person game.

Conditions	Weak nucleolus
$a+h \geq d$	$\left(\frac{a+h-2d+1}{3}, \frac{d+h-2a+1}{3}, \frac{a+d-2h+1}{3}\right)$
$a+h < d$	$\left(\frac{1-d}{3}, \frac{2+d+3h-3a}{6}, \frac{2+d+3a-3h}{6}\right)$

We shall now give a table for the c-proportional nucleolus assuming that c(S) is superadditive and that some technical condition

$$c_1 + c_{23} \leq c_2 + c_{13} \leq c_3 + c_{12} \quad ,$$

is always true for some ordering of players.

TABLE 3 The c-proportional nucleolus of an arbitrary three-person game.

Conditions	c-Proportional nucleolus
$\frac{c_{12}+c_{13}+c_{23}}{2} \leq c_1+c_{23}$	$\frac{1}{c_{12}+c_{13}+c_{23}}\left(c_{13}+c_{12}-c_{23}\, , c_{12}+c_{23}-c_{13}\, , c_{13}+c_{23}-c_{12}\right)$
$\frac{c_{12}+c_{13}+c_{23}}{2} > c_1+c_{23}$	$\frac{c_1}{c_1+c_{23}}\, , \frac{c_1(c_{12}-c_{13})+c_{23}c_{12}}{(c_1+c_{23})(c_{12}+c_{13})}\, , \frac{c_1(c_{13}-c_{12})+c_{23}c_{13}}{(c_1+c_{23})(c_{12}+c_{13})}$

5. NUMERICAL EXAMPLE

We shall now consider in detail the game from Young et al. (1980), which arose from the problem of sharing water costs among a group of Swedish municipalities. A careful study of local conditions led to the grouping of the 18 municipalities into 6 independent units A, H, K, L, M, T, consisting respectively of 5, 4, 2, 1, 3, 3 municipalities. The characteristic function c(S) for this 6-person game is given in Young et al. and the corresponding values reproduced in Table 4, together with the values of v(S) calculated from equation (1).

Our definition of the generalized nucleolus permits us to limit our attention to some subset M of the set of all coalitions 2^J rather than considering the whole set. We shall use this fact and the characteristic function v(S) in the calculation that follows.

Assume

$$M = \{S \subset J \mid \text{ either } |S| = 1 \text{ or } v(S) > 0\} \quad .$$

TABLE 4 Values of c(S) and v(S) for various coalitions S.

S	c(S)	v(S)	S	c(S)	v(S)
A	21.95	0	HKM	42.55	6.25
H	17.08	0	HKT	44.94	5.03
K	10.91	0	HLM	45.81	7.96
L	15.88	0	HLT	46.98	7.96
M	20.81	0	HMT	56.49	3.38
T	21.98	0	KLM	42.01	5.59
			KLT	48.77	0
AH	34.69	4.34	KMT	50.32	3.38
AK	32.86	0	LMT	51.46	7.21
AL	37.83	0			
AM	42.76	0	AHKL	48.95	16.87
AT	43.93	0	AHKM	60.25	10.5
HK	22.96	5.03	AHKT	62.72	9.2
HL	25.00	7.96	AHLM	64.03	11.69
HM	37.89	0	AHLT	65.20	11.69
HT	39.06	0	AHMT	74.10	7.69
KL	26.79	0	AKLM	63.96	5.59
KM	31.45	0.27	AKLT	70.72	0
KT	32.89	0	AKMT	72.27	3.38
LM	31.10	5.59	ALMT	73.41	7.21
LT	37.86	0	HKLM	48.07	16.61
MT	39.41	3.38	HKLT	49.24	16.61
			HKMT	59.35	11.43
			HLMT	64.41	11.34
			KLMT	56.61	12.97
AHK	40.74	9.2			
AHL	43.22	11.69	AHKLM	69.76	16.87
AHM	55.50	4.34	AHKLT	70.93	16.87
AHT	56.67	4.34	AHKMT	77.42	15.31
AKL	48.74	0	AHLMT	83.00	14.70
AKM	53.40	5.59	AKLMT	73.97	17.56
AKT	54.84	0	HKLMT	66.46	20.20
ALM	53.05	5.59			
ALT	59.81	0			
AMT	61.36	3.38			
HKL	27.26	16.61	AHKLMT	83.82	24.79

It is evident that

$$N[M,1] = N[2^J,1] \quad ,$$

so we can reduce the number of coalitions under consideration from 62 to 48.

An algorithm based on the above method was implemented in the LP-BESM-6 system and used to compute the generalized nucleolus. The numerical results obtained are given in Table 5, together with the number k of l.p. problems solved in each case.

TABLE 5 The values of y and x for the nucleolus, the weak nucleolus, the c-proportional nucleolus, and the proportional nucleolus.

Method	A	H	K	L	M	T	k
Nucleolus							4
y	20.35,	12.06,	5.00,	8.61,	18.32,	19.49	
x	1.6,	5.02,	5.91,	7.27,	2.49,	2.49	
Weak Nucleolus							6
y	20.03,	12.52,	3.94,	9.07,	18.54,	19.71	
x	1.92,	4.56,	6.97,	6.81,	2.27,	2.27	
c-Proportional Nucleolus							7
y	19.81,	12.57,	4.35,	9.25,	18.37,	19.47	
x	2.14,	4.51,	6.56,	6.63,	2.44,	2.51	
Proportional Nucleolus							5
y	20.36	12.46	3.47	8.67	18.845	20.015	
x	1.59	4.62	7.44	7.21	1.965	1.965	

It is reasonable to consider two more variants of the generalized nucleolus. The following variants of the function d(S) are based on additional information about the problem. Let P_i represent the population and D_i the water demand of the i-th group of municipalities. Then

$$P(S) = \sum_{i \in S} P_i \quad \text{and} \quad D(S) = \sum_{i \in S} D_i \quad .$$

The function d(s) is additive in both of the above cases as well as when $d(S) = |S|$, and so the corresponding nucleoli are monotonic (in the sense meant by Young et al.). We therefore suggest that the first approach should be called the proportional-to-population nucleolus rather than the proportional-to-population allocation method (as in Young et al.) and, similarly, the second approach should be known as the proportional-to-demand nucleolus instead of the proportional-to-demand allocation method. In general, the function d(S) makes it possible to use more statistics without losing attractive game theoretical features.

To conclude, we shall give a table of all the modifications of the nucleolus calculated for two values of v(J).

TABLE 6 Results obtained for all modifications of the nucleolus, calculated for v(J) = 24.79 and v(J) = 20.79.

d(S)	N[2^J,d(S)]						k[d(S)]	v(J)
1	1.6	5.02	5.91	7.27	2.49	2.49	4	24.79
	1.19	3.83	6.4	6.08	1.65	1.65	7	20.79
|S|	1.92	4.56	6.97	6.81	2.27	2.27	6	24.79
	1.25	3.89	6.3	6.14	1.6	1.6	6	20.79
c(S)	2.14	4.51	6.56	6.63	2.44	2.51	7	24.79
	1.2	3.91	6.35	6.19	1.56	1.58	7	20.79
v(S)	1.59	4.62	7.44	7.21	1.965	1.965	5	24.79
	1.34	3.88	6.24	6.05	1.64	1.64	5	20.79
v(J)-v(S)-v(J-S)	2.26	5.32	5.73	6.83	2.45	2.20	3	24.79
	0.68	3.74	6.46	6.35	1.79	1.76	5	20.79
P(S)	1.77	4.83	6.52	6.59	3.45	1.62	6	24.79
	1.29	3.83	6.37	6.2	1.82	1.29	6	20.79
D(S)	1.90	4.65	6.68	6.54	3.14	1.87	7	24.79
	1.26	3.87	6.33	6.21	1.76	1.36	7	20.79

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