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IIASA Working Paper

WP-83-065

July 1983

Working Paper

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FOREWORD

Low fertility levels in IIASA countries are creating aging populations whose demands for health care and income maintenance (social security) will increase to unprecedented levels, thereby calling forth policies that will seek to promote increased family care and worklife flexibility. The new Population Program will examine current patterns of population aging and changing life-styles in IIASA countries, project the needs for health and income support that such patterns are likely to generate during the next several decades, and consider alternative family and employment policies that might reduce the social costs of meeting these needs.

Multiregional and multistate demographic methods are being increasingly adopted in applied population research. This has stimulated a reexamination of a number of aspects of the basic methodology. One of these is the proper survivorship of the last two age groups in a population and an appropriate calculation of the corresponding survivorship proportions. This problem is particularly relevant in studies of the future age composition of the elderly population. This paper suggests several alternatives to improve the projection of the population in the last two age groups. The so-called preferred approach recommended by the authors may be readily integrated into the standard framework of multistate projections.

Andrei Rogers
Leader, Population Program

ABSTRACT

The aging of populations is a phenomenon which has become an important research topic. Demographers, however, have given inadequate attention to the projection of the number of old people and their future age composition. This paper shows that the conventional method for estimating the survivorship proportions of the very old tends to produce misleading results with respect to the size and composition of the aged. Several alternatives are suggested here to overcome these problems. An empirical example is used to point out the problems of the conventional approach and to evaluate the suggested improvements.

CONTENTS

AN IMPROVED METHOD OF COMPUTING MULTISTATE SURVIVORSHIP PROPORTIONS FOR THE TERMINAL AGE GROUPS

INTRODUCTION

In multistate demography, an arbitrary convention has been used to obtain the submatrices of survivorship (and out-migration) proportions for *the two oldest age groups* from the submatrices representing the age compositions of the multiregional life table (stationary) population. The convention mismatches two subpopulations and causes problems that have become particularly serious when one tries to construct a single-year projection model for a population with a high expectation of life. Since the convention was inherited from the common practice in single-state mathematical demography, the problem has been hidden or ignored for a long time. As population aging has become an important research topic in recent years, we find it timely to focus in this paper on pointing out the problems of this convention and proposing alternative solutions. To illustrate our arguments we shall draw on a multiregional example using Swedish data.

In section 1, we identify the problems of the conventional approach. Section 2 presents an alternative approach that reduces the severity of mismatching subpopulations by further splitting up the last age group into smaller ones. Section 3 shows our preferred alternative approach that yields sensible and interpretable results. Section 4 uses the 1974 Swedish data shown in Appendix A to evaluate the quality of the projection results that are generated by the different approaches. The last section summarizes the main points.

1. THE PROBLEM OF THE CONVENTIONAL APPROACH

Consider a population with N ($\geq$ 1) regions and w+1 age groups: (0,h),(h,2h),...,(x,x+h),...,(wh,∞). Let ${}_h\underset{\sim}{L}_x$ be an $N \times N$ matrix representing the place-of-residence-by-place-of-birth population distribution of the x age group (i.e., between exact ages x and x+h) in the multiregional life table population. Specifically, the element in the ith row and the jth column is the number of residents at exact ages between x and x+h in region j who were born in region i. Also let ${}_h\underset{\sim}{S}_x$ be the $N \times N$ submatrix of survivorship (and outmigration) proportions, where the element in row j and column i represents the proportion of the individuals in the ith region and the x+h age group who will reside in the jth region h years later. The diagonal elements of ${}_h\underset{\sim}{S}_x$ are *surviving stayer proportions*, whereas the off-diagonal elements of ${}_h\underset{\sim}{S}_x$ are *surviving outmigration proportions*. Without foreign migration, the constraint

$$0 \leq {}_hS_x^{ij} \leq 1 \tag{1}$$

must be satisfied for each element ${}_hS_x^{ij}$ of ${}_h\underset{\sim}{S}_x$

By definition, we have

$${}_h\underset{\sim}{L}_{x+h} = {}_h\underset{\sim}{S}_x \, {}_h\underset{\sim}{L}_x \qquad \text{for } x = 0,h,2h,\ldots,(w-1)h \tag{2}$$

and

$${}_\infty\underset{\sim}{L}_{wh} = {}_h\underset{\sim}{S}_{(w-1)h} \, {}_h\underset{\sim}{L}_{(w-1)h} + {}_h\underset{\sim}{S}_{wh} \, {}_\infty\underset{\sim}{L}_{wh} \tag{3}$$

The conventional approach is (i) to solve equation (2) for ${}_h\underset{\sim}{S}_x$ for the first w-1 age groups, (ii) to solve for ${}_h\underset{\sim}{S}_{(w-1)h}$ from the inappropriate equation

$${}_\infty\underset{\sim}{L}_{wh} = {}_h\underset{\sim}{S}_{(w-1)h} \, {}_h\underset{\sim}{L}_{(w-1)h} \tag{4}$$

and (iii) to set ${}_h\underset{\sim}{S}_{wh}$ to zero. The convention is found in most books that deal with uniregional as well as multiregional population projections (e.g., Keyfitz (1968 and 1977) and Rogers (1975)).

A problem of the conventional approach is that the last two ${}_h\underset{\sim}{S}_{wh}$ do not contain any demographically meaningful quantities. For example, in a single-region life table based on the 1974 Swedish female mortality data for 86 age groups (0,1,2,...,85+), we found that ${}_1L_{84} = 0.36424$ and ${}_\infty L_{85} = 1.79784$. According to the conventional approach, we get ${}_1S_{84} = {}_\infty L_{85}\ {}_1L_{84}^{-1} = 4.93592$ and ${}_1S_{85} = 0$. These are, of course, meaningless survivorship proportions. By disaggregating the female population into two regions (Stockholm and the rest of Sweden), and incorporating interregional migration information (see Appendix A), the conventional approach yields

$${}_1\underset{\sim}{S}_{84} = \begin{bmatrix} 5.193331 & 0.016234 \\ 0.057824 & 4.856811 \end{bmatrix} \tag{5}$$

and

$${}_1\underset{\sim}{S}_{85} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \tag{6}$$

Though they happen to be between zero and unity, the off-diagonal elements of ${}_1\underset{\sim}{S}_{84}$ are much too large to represent the surviving outmigration proportions of the relevant subpopulations. Furthermore, the diagonal elements of ${}_1\underset{\sim}{S}_{84}$ fall far above the upper bound of the constraint that must hold for the definition of surviving stayer proportions. In general, these elements tend to be farther above the upper bound, when the expectation of life at age wh is high, or when both w and h are small. Of course, one must not be serious about interpreting equation (6). With mortality and migration information available for all age groups, it is regrettable that the conventional approach ends up with meaningless submatrices of survivorship and outmigration proportions for the last two age groups. If the last age group of the raw data was 65+, then the ${}_h\underset{\sim}{S}_x$ submatrices would contain no useful information about the post-retirement subpopulations.

One may argue that the submatrices ${}_h\underset{\sim}{S}_x$ are constructed solely for the purpose of projection. Therefore, as long as the resulting projections are reasonably good, one need not care about interpretations. Unfortunately, the projections can turn out to be truly bad, particularly when a single-year projection model is used. Consider again the aforementioned biregional Swedish population. Applying the submatrix ${}_1\underset{\sim}{S}_{84}$ in equation (5) to the 1974 subpopulation in the 84th age group gives

$$ {}_\infty\underset{\sim}{K}_{85}(1975) = {}_1\underset{\sim}{S}_{84}\ {}_1\underset{\sim}{K}_{84}(1974) = \begin{bmatrix} 5.193331 & 0.016234 \\ 0.057824 & 4.856811 \end{bmatrix} \begin{bmatrix} 2132 \\ 10169 \end{bmatrix} = \begin{bmatrix} 11237 \\ 49512 \end{bmatrix} \qquad (7) $$

Comparing this vector with the observed initial subpopulation in the 85+ age group

$$ {}_\infty\underset{\sim}{K}_{85}(1974) = \begin{bmatrix} 8789 \\ 40714 \end{bmatrix} $$

we see that the conventional approach implies an annual growth rate for the last age group of 28% in the Stockholm region and 22% in the rest of Sweden. Lumping the two regions together gives an annual growth rate for this age group of 23%. These phenomenally high growth rates mean that the projection is totally misleading as far as the last age group is concerned. We will show later that the growth rate of the whole population generated by the conventional approach is also distorted quite badly.

In short, the conventionally constructed ${}_h\underset{\sim}{S}_x$ submatrices for the last two age groups are suitable neither for interpretation nor for projection. Therefore, we shall consider a few alternative approaches that may produce more satisfactory results.

2. THE DISAGGREGATION APPROACH

One can partly overcome the aforementioned problems by disaggregating the last open-ended age group into an arbitrary number (J) of age groups: ${}_h\hat{K}_{wh}$, ${}_h\hat{K}_{(w+1)h}$, ..., ${}_\infty\hat{K}_{(w+J)h}$. Any reasonable interpolation procedure such as a third-degree spline can be applied, as long as the constraint

$$ {}_\infty K_{wh} = \sum_{j=0}^{J-1} \hat{K}_{(w+j)h} + {}_\infty\hat{K}_{(w+J)h} \qquad (8) $$

is satisfied. Note that the age-composition of these new groups will not have a lasting effect on the projected population because the people in these age groups at the initial time will be totally replaced by the incoming cohorts after Jh years.

Under the assumption that the observed mortality and mobility rates of the last age group apply to all additional age groups, the submatrices of survivorship proportions can be computed as follows. The submatrix ${}_hS_{(w-1)h}$ in the conventional approach, expressed in terms of occurrence/exposure rates, is given by (Ledent 1978)

$$ {}_hS_{(w-1)h} = \frac{1}{h} M_{wh}^{-1} \left[I - \frac{h}{2} M_{(w-1)h}\right] \qquad (9) $$

where M_{wh} denotes the matrix of observed rates as set out in Willekens and Rogers (1978). This matrix is now replaced by a matrix ${}_h\hat{S}_{(w-1)h}$ calculated by

$$ {}_h\hat{S}_{(w-1)h} = \left[I + \frac{h}{2} M_{wh}\right]^{-1} \left[I - \frac{h}{2} M_{(w-1)h}\right] \qquad (10) $$

Assuming constant rates for all ages w+j (j = 0,...,J-2), the corresponding submatrices of survivorship proportions are given by

$$ {}_hS_{(w+j)h} = \left[I + \frac{h}{2} M_{wh}\right]^{-1} \left[I - \frac{h}{2} M_{wh}\right] \qquad j = 0 \text{ to } J-2 $$

$$ = {}_hP_{wh} \qquad (11) $$

The troublesome submatrix of survivorship proportions ${}_{h}\underset{\sim}{S}_{(w+J+1)h}$ for the open-ended age group can now be calculated in two ways. Either by applying the same procedure as for the other additional age groups and setting

$$ {}_{h}\underset{\sim}{S}_{(w+J+1)h} = {}_{h}\underset{\sim}{P}_{wh} \tag{12} $$

or using the conventional formula for the last survivorship matrix but with identical rates for the last and next to the last age groups. From equation (9) we then get

$$ {}_{h}\underset{\sim}{S}_{(w+J-1)h} = \frac{1}{h}\underset{\sim}{M}_{wh}^{-1}\left[\underset{\sim}{I} - \frac{h}{2}\underset{\sim}{M}_{wh}\right] $$

$$ = \frac{1}{h}\underset{\sim}{M}_{wh}^{-1} - \frac{1}{2}\underset{\sim}{I} \tag{13} $$

The computation based on equation (12) will further on be referred to as disaggregation approach (I), and the alternative involving equation (13) as disaggregation approach (II). For both alternatives, ${}_{h}\underset{\sim}{S}_{(w+J)h}$ is set to zero. The population can now be projected forward by the expanded growth matrix. The structure of this matrix is the same as that for the conventional approach: the $N \times N$ submatrices in the diagonal are all zero.

For the Swedish data in Appendix A, we have disaggregated the 85+ age group into 11 age groups (85,86,...,94,95+). According to equation (10), we now have

$$ {}_{1}\underset{\sim}{S}_{84} = \begin{bmatrix} 0.862548 & 0.000403 \\ 0.002809 & 0.858338 \end{bmatrix} \tag{14} $$

Although it looks much more sensible than the corresponding submatrix generated by the conventional approach (see equation (5)), this submatrix tends to understate the level of survivorship proportions, because in equation (10) the average mortality level of the 85+ age group is assumed to be applicable to the single-year age 85.

The submatrices computed according to equation (11) are

$$ {}_1\underset{\sim}{S}_{85} = {}_1\underset{\sim}{S}_{86} = \cdots = {}_1\underset{\sim}{S}_{93} = \begin{bmatrix} 0.833907 & 0.000472 \\ 0.001427 & 0.823267 \end{bmatrix} \qquad (15) $$

These submatrices also look sensible, except that the survivorship proportions tend to be understated for ${}_1\underset{\sim}{S}_{85}$ and overstated for ${}_1\underset{\sim}{S}_{93}$. For the next to the last age group, the disaggregation approach (I) yields

$$ {}_1\underset{\sim}{S}_{94} = \begin{bmatrix} 0.833907 & 0.000472 \\ 0.001427 & 0.823267 \end{bmatrix} \qquad (16) $$

whereas the disaggregation approach (II) gives

$$ {}_1\underset{\sim}{S}_{94} = \begin{bmatrix} 5.020865 & 0.016090 \\ 0.048612 & 4.658370 \end{bmatrix} \qquad (17) $$

While equation (16) does not give any information in addition to that of equation (15), equation (17) does not permit any meaningful interpretation. Therefore, the justification for the disaggregation approach lies in the hope of producing less distorted projections only.

3. THE PREFERRED APPROACH

Our preferred approach uses equation (2) to find the first w submatrices of survivorship and outmigration proportions and then uses the basic equation (3) to find ${}_h\underset{\sim}{S}_{wh}$. The solutions are

$$ {}_h\underset{\sim}{S}_x = {}_h\underset{\sim}{L}_{x+h}\ {}_h\underset{\sim}{L}_x^{-1} \qquad \text{for } x = 0,h,2h,\ldots,(w-1)h \tag{18} $$

and

$$ {}_h\underset{\sim}{S}_{wh} = ({}_\infty\underset{\sim}{L}_{wh} - {}_h\underset{\sim}{S}_{(w-1)h}\ {}_h\underset{\sim}{L}_{(w-1)h})\ {}_\infty\underset{\sim}{L}_{wh}^{-1} \tag{19} $$

To compute the submatrix ${}_h\underset{\sim}{S}_{(w-1)h}$ according to equation (18), we need to know the value of ${}_h\underset{\sim}{L}_{wh}$, which is different from the submatrix ${}_\infty\underset{\sim}{L}_{wh}$ that is computed in the conventional multiregional life table. To make the distinction perfectly clear, we write

$$ {}_h\underset{\sim}{L}_x = \int_0^h \underset{\sim}{\ell}_{x+\lambda}\ dx \tag{20} $$

and

$$ {}_\infty\underset{\sim}{L}_x = \int_0^\infty \underset{\sim}{\ell}_{x+\lambda}\ dx \tag{21} $$

where $\underset{\sim}{\ell}_{x+\lambda}$ is an $N \times N$ matrix showing the place-of-residence-by-place-of-birth population distribution at exact age $x+\lambda$ among the survivals of the arbitrarily specified birth cohort in $\underset{\sim}{\ell}_0$.* Failure to see the difference between ${}_h\underset{\sim}{L}_{wh}$ and ${}_\infty\underset{\sim}{L}_{wh}$ would lead us back to the conventional approach.

To compute ${}_h\underset{\sim}{L}_x$ and ${}_\infty\underset{\sim}{L}_x$ from the matrices of the observed mobility and mortality rates ($\underset{\sim}{M}_x$), an assumption about the underlying mathematical model must be made so that the integrals in equations (20) and (21) can be conveniently evaluated. Two well-known alternative assumptions are (i) that $\underset{\sim}{\ell}_x$ is piece-wise linear within the interval h, and (ii) that $\underset{\sim}{M}_x$ is a good approximation of the matrix of instantaneous mobility and mortality

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*For convenience, we usually let $\underset{\sim}{\ell}$ be such that most $\underset{\sim}{\ell}_x$ are invertible.

rates ($\underset{\sim}{\mu}_x$) which is in turn assumed to be piece-wise constant within the individual age groups. The approaches based on these two assumptions are called the linear and the exponential approaches, respectively. The former approach occasionally generates some nonsensical quantities like a negative outmigration proportion, whereas the latter does not have such a problem. Since it is widely used and does generate satisfactory results when single-year age groups are used, the linear approach shall be discussed first.

3.1 Linear Approach

Assuming a piece-wise linear $\underset{\sim}{\ell}_x$ over an integral h, equation (18) can be expressed in terms of observed occurrence/exposure rates as shown by equation (10). In particular, the survivorship proportions from the second last to the last age group is given by

$$_h\underset{\sim}{S}_{(w-1)h} = [\underset{\sim}{I} + \frac{h}{2} \underset{\sim}{M}_{wh}]^{-1} \; [\underset{\sim}{I} - \frac{h}{2} \underset{\sim}{M}_{(w-1)h}] \qquad (22)$$

In contrast, the submatrix $_h\underset{\sim}{S}_{(w-1)h}$ in the conventional approach was calculated according to equation (9).

Finally, the proportion surviving within the last age group has to be derived from formula (19). Replacing $_h\underset{\sim}{S}_{(w-1)h}$ in (19) by

$$_h\underset{\sim}{S}_{(w-1)h} = {_h\underset{\sim}{L}_{wh}} \; {_h\underset{\sim}{L}^{-1}_{(w-1)h}} \qquad (23)$$

leads to

$$_h\underset{\sim}{S}_{wh} = \underset{\sim}{I} - {_h\underset{\sim}{L}_{wh}} \; {_\infty\underset{\sim}{L}^{-1}_{wh}} \qquad (24)$$

Following Willekens and Rogers (1978), we can substitute $_\infty\underset{\sim}{L}_{wh}$ in (24) by

$$_\infty\underset{\sim}{L}_{wh} = \underset{\sim}{M}^{-1}_{wh} \; \underset{\sim}{\ell}_{wh} \qquad (25)$$

and finally get

$$ {}_h\underset{\sim}{S}_{wh} = \underset{\sim}{I} - {}_h\underset{\sim}{L}_{wh}\, \underset{\sim}{\ell}_{wh}^{-1}\, \underset{\sim}{M}_{wh} $$

Some further manipulations* show that the additional submatrix of survivorship proportions ${}_h\underset{\sim}{S}_{wh}$ is given by

$$ {}_h\underset{\sim}{S}_{wh} = \underset{\sim}{I} - \frac{h}{2}[{}_h\underset{\sim}{P}_{wh} + \underset{\sim}{I}]\underset{\sim}{M}_{wh} = {}_h\underset{\sim}{P}_{wh} \qquad (26) $$

with ${}_h\underset{\sim}{P}_{wh}$ to be calculated by

$$ {}_h\underset{\sim}{P}_{wh} = [\underset{\sim}{I} + \frac{h}{2}\underset{\sim}{M}_{wh}]^{-1}\, [\underset{\sim}{I} - \frac{h}{2}\underset{\sim}{M}_{wh}] \qquad (27) $$

Note that in the conventional approach, the zero matrix ${}_\infty\underset{\sim}{P}_{wh}$ is given but ${}_h\underset{\sim}{P}_{wh}$ is not computed. It can be shown that $\frac{h}{2}[{}_h\underset{\sim}{P}_{wh} + \underset{\sim}{I}] = \frac{h}{2}(\underset{\sim}{\ell}_{(w+1)h} + \underset{\sim}{\ell}_{wh})\underset{\sim}{\ell}_{wh}^{-1}$, which represents the average numbers of person-years lived in various regions during the h years beyond age wh by region of residence at exact age wh. This interpretation makes equation (26) intuitively clear. The survivorship proportions computed by formulas (22) and (26) have now to be placed into the last row of the growth matrix. Since an additional element ${}_h\underset{\sim}{S}_{wh}$ is used, the growth matrix is rewritten as

---

*First replacing ${}_h\underset{\sim}{L}_{wh}$ by

$$ {}_h\underset{\sim}{L}_{wh} = \frac{h}{2}[{}_h\underset{\sim}{P}_{wh} + \underset{\sim}{I}]\underset{\sim}{\ell}_{wh} $$

and then using the property (Ledent 1978)

$$ {}_h\underset{\sim}{P}_{wh} = [\underset{\sim}{I} - \frac{h}{2}\underset{\sim}{M}_{wh}][\underset{\sim}{I} + \frac{h}{2}\underset{\sim}{M}_{wh}]^{-1} $$

$$ = [\underset{\sim}{I} + \frac{h}{2}\underset{\sim}{M}_{wh}]^{-1}\, [\underset{\sim}{I} - \frac{h}{2}\underset{\sim}{M}_{wh}] $$

$$
\underset{\sim}{G} = \begin{bmatrix}
\underset{\sim}{0} & \underset{\sim}{0} & \underset{\sim}{B}_{\alpha h} \cdots \underset{\sim}{B}_{\beta h} & \cdots & \underset{\sim}{0} & \cdots & \underset{\sim}{0} \\
{}_h\underset{\sim}{S}_1 & \underset{\sim}{0} & & & \vdots & & \vdots \\
\vdots & {}_h\underset{\sim}{S}_2 & & & \underset{\sim}{0} & & \vdots \\
\vdots & \vdots & \ddots & & \vdots & & \vdots \\
\underset{\sim}{0} & \underset{\sim}{0} & & & {}_h\underset{\sim}{S}_{(w-1)h} & & {}_h\underset{\sim}{S}_{wh}
\end{bmatrix} \qquad (28)
$$

Applying the linear approach to the Swedish data in Appendix A yields

$$
{}_1\underset{\sim}{S}_{84} = \begin{bmatrix} 0.862548 & 0.000403 \\ 0.002809 & 0.858338 \end{bmatrix} \qquad (29)
$$

and

$$
{}_1\underset{\sim}{S}_{85} = \begin{bmatrix} 0.833907 & 0.000472 \\ 0.001427 & 0.823267 \end{bmatrix} \qquad (30)
$$

Both these matrices appear much more sensible than those of the conventional approach. Note that the ${}_1\underset{\sim}{S}_{84}$ submatrix does not differ between the preferred linear approach and the disaggregation approaches. The value of the submatrix ${}_1\underset{\sim}{S}_{85}$ of the preferred linear approach is identical to those of ${}_1\underset{\sim}{S}_{85}$, ${}_1\underset{\sim}{S}_{86}$, ..., ${}_1\underset{\sim}{S}_{93}$ of both disaggregation approaches.

3.2 Exponential Approach

The assumptions that the observed matrix of occurrence/exposure rates ($\underset{\sim}{M}_x$) is a good approximation of the matrix of the corresponding instantaneous rates ($\underset{\sim}{\mu}_x$) and that μ_x is piecewise constant within individual age groups imply that the model of the multiregional life table can be written as the differential equation

$$\dot{\ell}_{x+\lambda} = -M_x \ell_{x+\lambda} \quad \text{for } x = 0,h,2h,\ldots,wh \quad (31)$$

where $\dot{\ell}_{x+\lambda}$ is the derivative of $\ell_{x+\lambda}$ with respect to age, and λ is constrained by $0 \leq \lambda < h$ for the first w age groups and by $0 \leq \lambda$ for the last age group. The solution of the differential equation is

$$\ell_{x+\lambda} = e^{-\lambda M_x} \ell_x \quad \text{for } x = 0,h,2h,\ldots,wh \quad (32)$$

In other words, the matrix of probabilities of surviving between exact ages x and x+h is simply

$$_{\lambda}P_x = e^{-\lambda M_x} \quad \text{for } x = 0,h,2h,\ldots,wh \quad (33)$$

Substituting equation (32) and (33) into equations (20) and (21), expanding $e^{-\lambda M_x}$ into a Taylor series, and then integrating, we get

$$_{h}L_x = M_x^{-1} (I - {}_{h}P_x) \ell_x \quad (34)$$

$$= (I - {}_{h}P_x) M_x^{-1} \ell_x$$

and

$$_{\infty}L_{wh} = M_{wh}^{-1} \ell_{wh} \quad (35)$$

Substituting equations (34) and (35) into equations (18) and (19), we get

$$_{h}S_x = [I - {}_{h}P_{x+h}] M_{x+h}^{-1} \; {}_{h}P_x \; M_x [I - {}_{h}P_x]^{-1} \quad (36)$$

$$\text{for } x = 0,h,2h,\ldots,(w-1)h$$

and

$$_{h}S_{wh} = {}_{h}P_{wh} = e^{-hM_{wh}} \quad (37)$$

The exponential approach starts with computing all $_{h}P_x$ from M_x and then using equations (36) and (37) to compute all $_{h}S_x$. To obtain a highly accurate result for P_x, we recommend the following computation formula

$$
{}_h\underset{\sim}{P}_x = \left[\underset{\sim}{I} + \frac{1}{2}\,h\underset{\sim}{M}_x + \frac{5}{44}(h\underset{\sim}{M}_x)^2 + \frac{1}{66}(h\underset{\sim}{M}_x)^3 + \frac{1}{792}(h\underset{\sim}{M}_x)^4 + \frac{1}{15840}(h\underset{\sim}{M}_x)^5 + \frac{1}{665280}(h\underset{\sim}{M}_x)^6\right]^{-1}
$$

$$
\left[\underset{\sim}{I} - \frac{1}{2}\,h\underset{\sim}{M}_x + \frac{5}{44}(h\underset{\sim}{M}_x)^2 - \frac{1}{66}(h\underset{\sim}{M}_x)^3 + \frac{1}{792}(h\underset{\sim}{M}_x)^4 - \frac{1}{15840}(h\underset{\sim}{M}_x)^5 + \frac{1}{665280}(h\underset{\sim}{M}_x)^6\right] \quad (38)
$$

This formula was derived by the matrix continued fraction method of electrical engineers (Shieh et al. 1978). For an explanation of the logic underlying the method, see Liaw and Ledent (1980).

Applying the exponential approach to the Swedish data in Appendix A yields

$$
{}_1\underset{\sim}{S}_{84} = \begin{bmatrix} 0.861465 & 0.000408 \\ 0.002758 & 0.856931 \end{bmatrix} \quad (39)
$$

and

$$
{}_1\underset{\sim}{S}_{85} = \begin{bmatrix} 0.834322 & 0.000468 \\ 0.001415 & 0.823769 \end{bmatrix} \quad (40)
$$

Comparing equations (39) and (40) with equations (29) and (30) suggests that for single-year models, the linear and exponential approaches tend to yield similar results. By comparing these four equations with equations (5) and (6), our preferred approach is clearly better than the conventional approach, as far as interpretability is concerned. However, one must remember that the survivorship proportions for the second to last age group tend to understate their true values, since the average mortality of the open-ended age group tends to be higher than the true mortality of the first corresponding closed interval (wh, wh+h).

4. COMPARISON OF THE PROJECTION RESULTS

The survivorship submatrices generated by the conventional approach, the two versions of the disaggregation approach, and the two versions of our preferred approach are combined with reproduction submatrices to form alternative growth matrices for the 1974 Swedish data shown in Appendix A. Here we want to evaluate the projections generated by these growth matrices in terms of their effects on the changing population size of the 85+ age group and on the entire system's annual growth rate.

Right from the first projected year (1975), the conventional approach separates itself from the other approaches by forcing the 85+ age group to grow by an unrealistically high growth rate of 20.47%.* In contrast, the two versions of the preferred approach both imply a growth rate of 3.87%, whereas the first version of the disaggregation approach results in a growth rate of 3.44% and the second version a growth rate of 8.66%. Figure 1 shows that among all the approaches, the conventional one tends to produce most erratic projection for the last open-ended age group, because it amplifies the irregularities in the changing sizes of the incoming cohorts. The first version of the disaggregation approach shows a strong tendency to underproject the population size in the 85+ age group, whereas the second version of the disaggregation approach is moderately sensitive to the irregularities in the initial age composition but does not show a strong tendency to under- or over-project the size of the oldest age group. The two versions of our preferred approach produce practically identical results that neither amplify the irregularities of the incoming cohorts nor exhibit any tendency of over- or under-projection, assuming the age-specific rates are time-invariant.

The conventional approach not only yields unrealistic projections for the 85+ age group but also distorts significantly the growth rate of the entire population, although the proportion

*All the growth rates mentioned in this section are instantaneous rates per year.

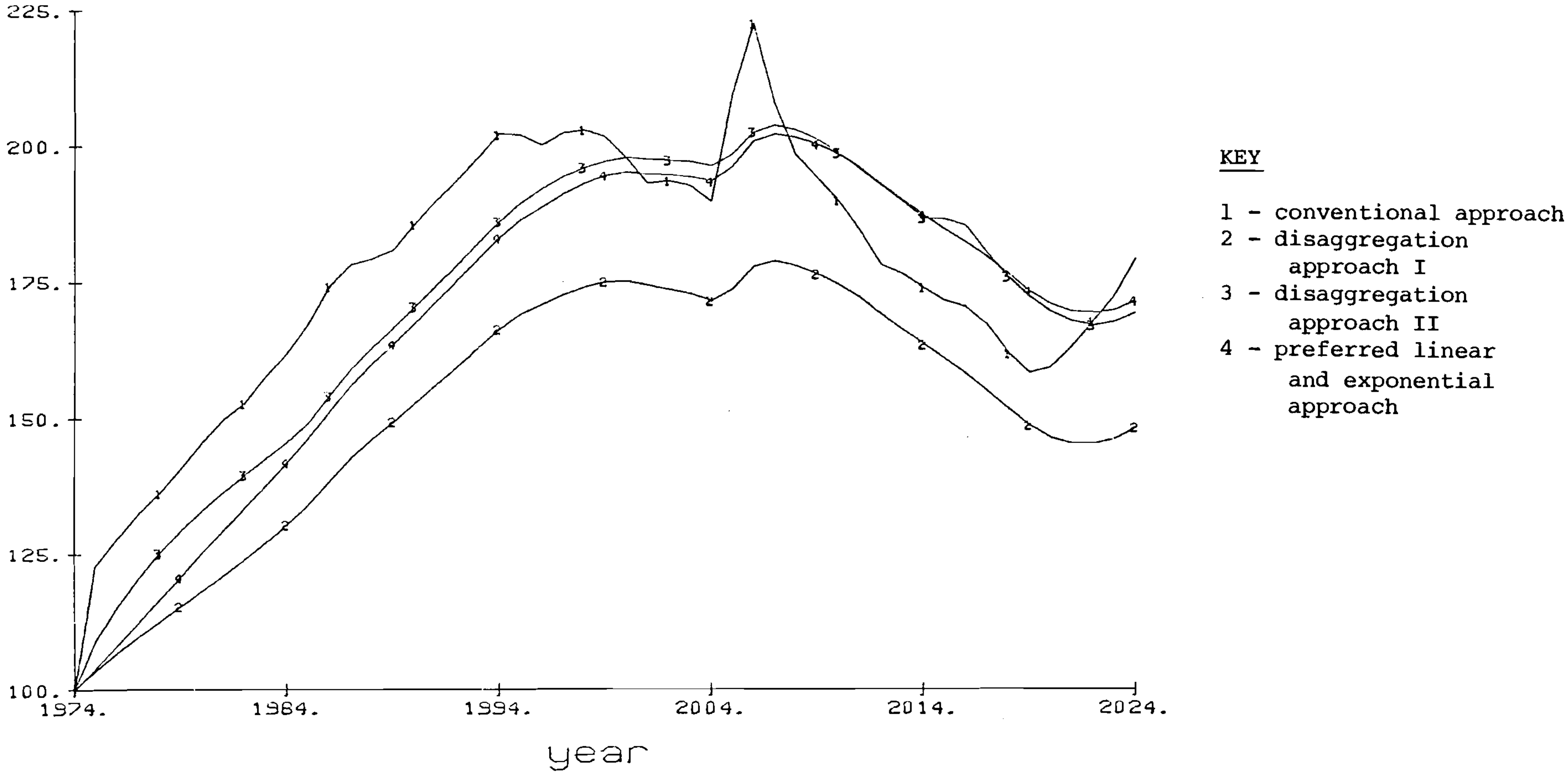
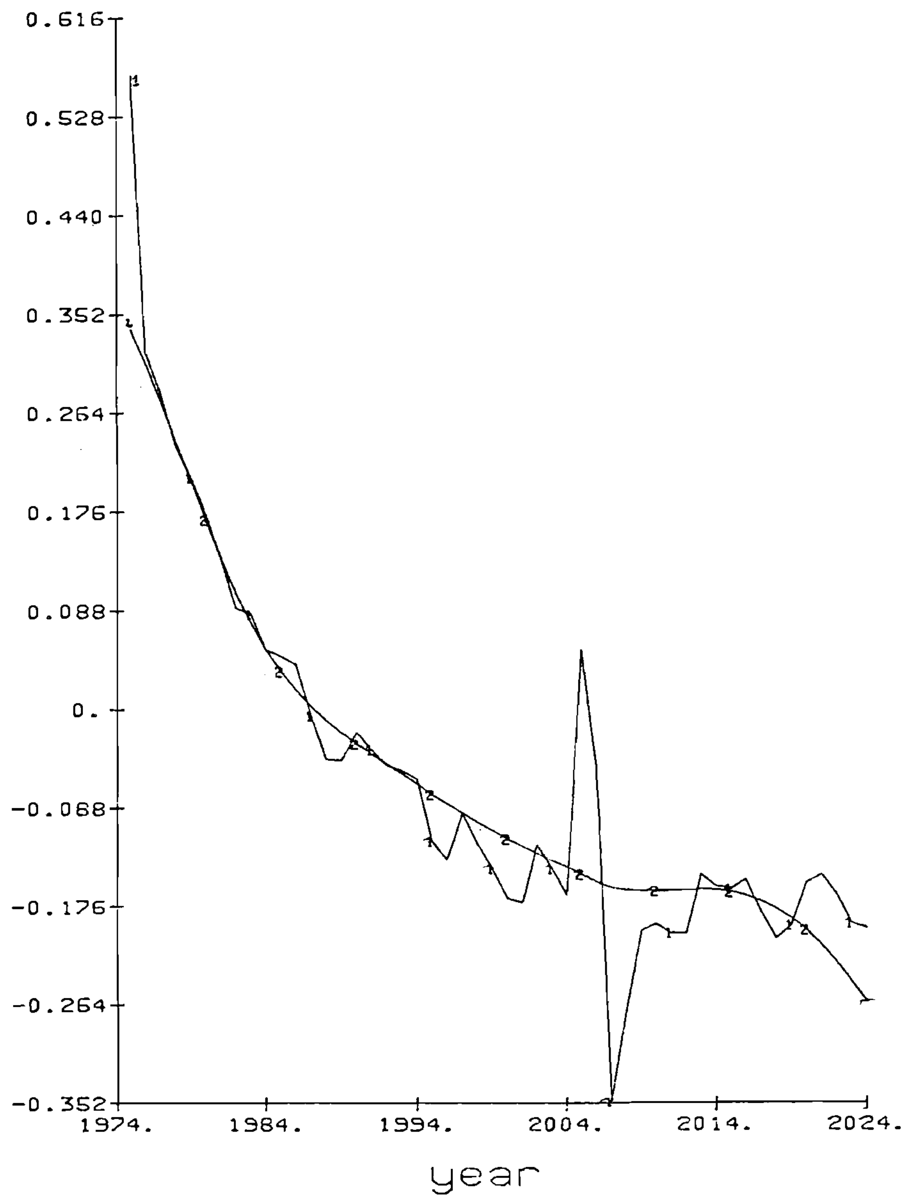


Figure 1. Size index of population aged 85 and over (1974 = 100).

of the population in the last age group is usually relatively small. Table 1 shows that between 1974 and 1975 the growth rate of the entire population is projected to be 0.56% by the conventional approach and 0.34% by our preferred approaches. This difference is completely due to the different treatments of the open-ended age group. We also see from Figure 2 that the growth rate of the entire population generated by the conventional approach tends to fluctuate significantly in response to the changing sizes of the cohorts entering the last age group, whereas the corresponding growth rates generated by our preferred approaches exhibit a more regular pattern. The more irregular the initial age composition, the worse the conventional approach will perform. For countries that have more irregular age compositions than Sweden (e.g., the Soviet Union and the Federal Republic of Germany), the differences between the approaches would be more dramatic than what is revealed in our example. The patterns of population growth rates generated by the two disaggregation approaches do not differ much from those of our preferred approaches, as shown in Appendix B.

Table 1. Annual growth rate in percent, 1974-1975.

Approach	Population 85+	Total population
Conventional	20.47	0.56
Disaggregation I	3.44	0.33
Disaggregation II	8.66	0.40
Preferred linear	3.87	0.34
Preferred exponential	3.88	0.34



KEY

1 - conventional approach
2 - preferred linear and exponential approach

Figure 2. Annual growth rate (in percent): 1974-2024.

CONCLUSION

We have pointed out the problems of the conventional approach to population projection, arising from an inappropriate specification of the survivorship proportions for the last two age groups. After examining several alternative approaches that may remedy these problems, we are convinced that our preferred approaches described in section 3 are superior to the other alternatives in terms of (1) the interpretability of the survivorship proportions, (2) the reliability of the projection results, and (3) the computational effect involved.

APPENDIX A: DATA FOR SWEDEN 1974, FEMALES, STOCKHOLM AND REST OF THE COUNTRY

SWEDEN'74 - FEMALES - 2 REGIONS

STOCKHLM

age	population number	population - % -	births number	births - % -	deaths number	deaths - % -	arrivals number	arrivals - % -	departures number	departures - % -	observed rates (x 1000) birth	death	inmig	outmig	net mig
0	10166.	1.33	0.	0.	66.	0.99	242.	1.88	396.	2.69	0.	6.492	23.805	38.953	-15.149
1	10171.	1.33	0.	0.	15.	0.23	222.	1.72	399.	2.71	0.	1.475	21.827	39.229	-17.402
2	10423.	1.36	0.	0.	2.	0.03	225.	1.75	404.	2.74	0.	0.192	21.587	38.760	-17.174
3	10234.	1.34	0.	0.	6.	0.09	202.	1.57	318.	2.16	0.	0.586	19.738	31.073	-11.335
4	9620.	1.25	0.	0.	2.	0.03	175.	1.36	281.	1.91	0.	0.208	18.191	29.210	-11.019
5	9482.	1.24	0.	0.	2.	0.03	179.	1.39	268.	1.82	0.	0.211	18.878	28.264	-9.386
6	10027.	1.31	0.	0.	2.	0.03	178.	1.38	282.	1.91	0.	0.199	17.752	28.124	-10.372
7	10393.	1.36	0.	0.	4.	0.06	131.	1.02	242.	1.64	0.	0.385	12.605	23.285	-10.680
8	10466.	1.37	0.	0.	4.	0.06	154.	1.20	243.	1.65	0.	0.382	14.714	23.218	-8.504
9	10648.	1.39	0.	0.	0.	0.	129.	1.00	214.	1.45	0.	0.	12.115	20.098	-7.983
10	10248.	1.34	0.	0.	1.	0.02	118.	0.92	175.	1.19	0.	0.098	11.514	17.077	-5.562
11	9507.	1.24	0.	0.	2.	0.03	83.	0.64	145.	0.98	0.	0.210	8.730	15.252	-6.522
12	9074.	1.18	0.	0.	3.	0.05	111.	0.86	148.	1.00	0.	0.331	12.233	16.310	-4.078
13	8715.	1.14	0.	0.	2.	0.03	69.	0.54	148.	1.00	0.	0.229	7.917	16.982	-9.065
14	8511.	1.11	2.	0.02	0.	0.	80.	0.62	98.	0.67	0.235	0.	9.400	11.515	-2.115
15	8576.	1.12	4.	0.04	4.	0.06	106.	0.82	137.	0.93	0.466	0.466	12.360	15.975	-3.615
16	8805.	1.15	14.	0.14	4.	0.06	179.	1.39	148.	1.00	1.590	0.454	20.329	16.809	3.521
17	8955.	1.17	45.	0.45	4.	0.06	396.	3.07	208.	1.41	5.025	0.447	44.221	23.227	20.994
18	9132.	1.19	128.	1.28	2.	0.03	646.	5.01	277.	1.88	14.017	0.219	70.740	30.333	40.407
19	9342.	1.22	243.	2.43	4.	0.06	710.	5.51	313.	2.13	26.012	0.428	76.001	33.505	42.496
20	9902.	1.29	323.	3.23	2.	0.03	739.	5.74	422.	2.87	32.620	0.202	74.631	42.618	32.014
21	10659.	1.39	451.	4.51	4.	0.06	722.	5.60	438.	2.97	42.312	0.375	67.736	41.092	26.644
22	11171.	1.46	528.	5.28	3.	0.05	668.	5.18	489.	3.32	47.265	0.269	59.798	43.774	16.024
23	11720.	1.53	617.	6.18	4.	0.06	692.	5.37	542.	3.68	52.645	0.341	59.044	46.246	12.799
24	12614.	1.65	667.	6.68	10.	0.15	564.	4.38	551.	3.74	52.878	0.793	44.712	43.682	1.031
25	13615.	1.78	763.	7.64	8.	0.12	546.	4.24	622.	4.22	56.041	0.588	40.103	45.685	-5.582
26	14278.	1.86	871.	8.72	4.	0.06	458.	3.55	587.	3.99	61.003	0.280	32.077	41.112	-9.035
27	14596.	1.90	899.	9.00	6.	0.09	420.	3.26	533.	3.62	61.592	0.411	28.775	36.517	-7.742
28	14693.	1.92	871.	8.72	5.	0.08	340.	2.64	523.	3.55	59.280	0.340	23.140	35.595	-12.455
29	14712.	1.92	781.	7.82	10.	0.15	300.	2.33	412.	2.80	53.086	0.680	20.392	28.004	-7.613
30	14109.	1.84	673.	6.74	10.	0.15	263.	2.04	348.	2.36	47.700	0.709	18.641	24.665	-6.025
31	12844.	1.68	555.	5.55	10.	0.15	231.	1.79	348.	2.36	43.211	0.779	17.985	27.094	-9.109
32	11473.	1.50	400.	4.00	8.	0.12	151.	1.17	270.	1.83	34.864	0.697	13.161	23.534	-10.372
33	10147.	1.32	308.	3.08	8.	0.12	158.	1.23	212.	1.44	30.354	0.788	15.571	20.893	-5.322
34	9639.	1.26	212.	2.12	14.	0.21	137.	1.06	215.	1.46	21.994	1.452	14.213	22.305	-8.092
35	9522.	1.24	172.	1.72	9.	0.14	131.	1.02	195.	1.32	18.063	0.945	13.758	20.479	-6.721
36	9058.	1.18	145.	1.45	8.	0.12	87.	0.68	149.	1.01	16.008	0.883	9.605	16.450	-6.845
37	8659.	1.13	96.	0.96	12.	0.18	98.	0.76	143.	0.97	11.087	1.386	11.318	16.515	-5.197
38	8224.	1.07	77.	0.77	14.	0.21	80.	0.62	143.	0.97	9.363	1.702	9.728	17.388	-7.661
39	7866.	1.03	42.	0.42	10.	0.15	76.	0.59	107.	0.73	5.339	1.271	9.662	13.603	-3.941
40	7769.	1.01	35.	0.35	13.	0.20	68.	0.53	95.	0.64	4.505	1.673	8.753	12.228	-3.475
41	7936.	1.04	30.	0.30	12.	0.18	63.	0.49	94.	0.64	3.780	1.512	7.939	11.845	-3.906
42	8149.	1.06	15.	0.15	19.	0.29	67.	0.52	98.	0.67	1.841	2.332	8.222	12.026	-3.804
43	8206.	1.07	9.	0.09	17.	0.26	77.	0.60	62.	0.42	1.097	2.072	9.383	7.555	1.828
44	8248.	1.08	8.	0.08	15.	0.23	60.	0.47	82.	0.56	0.970	1.819	7.274	9.942	-2.667
45	8439.	1.10	6.	0.06	19.	0.29	58.	0.45	97.	0.66	0.711	2.251	6.873	11.494	-4.621
46	8599.	1.12	1.	0.01	19.	0.29	65.	0.50	87.	0.59	0.116	2.210	7.559	10.117	-2.558
47	8730.	1.14	0.	0.	22.	0.33	49.	0.38	78.	0.53	0.	2.520	5.613	8.935	-3.322
48	9125.	1.19	0.	0.	28.	0.42	74.	0.57	77.	0.52	0.	3.068	8.110	8.438	-0.329

49	9513.	1.24	0.	0.	29.	0.44	45.	0.35	86.	0.58	0.	3.048	4.730	9.040	-4.310
50	9771.	1.27	0.	0.	20.	0.30	60.	0.47	78.	0.53	0.	2.047	6.141	7.983	-1.842
51	9973.	1.30	0.	0.	32.	0.48	49.	0.38	63.	0.43	0.	3.209	4.913	6.317	-1.404
52	10487.	1.37	0.	0.	30.	0.45	47.	0.36	88.	0.60	0.	2.861	4.482	8.391	-3.910
53	11211.	1.46	0.	0.	41.	0.62	45.	0.35	73.	0.50	0.	3.657	4.014	6.511	-2.498
54	10376.	1.35	0.	0.	60.	0.90	41.	0.32	78.	0.53	0.	5.783	3.951	7.517	-3.566
55	9342.	1.22	0.	0.	57.	0.86	44.	0.34	87.	0.59	0.	6.101	4.710	9.313	-4.603
56	9356.	1.22	0.	0.	57.	0.86	39.	0.30	81.	0.55	0.	6.092	4.168	8.658	-4.489
57	9088.	1.19	0.	0.	60.	0.90	37.	0.29	63.	0.43	0.	6.602	4.071	6.932	-2.861
58	8958.	1.17	0.	0.	63.	0.95	29.	0.23	73.	0.50	0.	7.033	3.237	8.149	-4.912
59	9067.	1.18	0.	0.	51.	0.77	38.	0.29	69.	0.47	0.	5.625	4.191	7.610	-3.419
60	9172.	1.20	0.	0.	62.	0.93	42.	0.33	73.	0.50	0.	6.760	4.579	7.959	-3.380
61	9375.	1.22	0.	0.	66.	0.99	39.	0.30	79.	0.54	0.	7.040	4.160	8.427	-4.267
62	9228.	1.20	0.	0.	76.	1.14	33.	0.26	89.	0.60	0.	8.236	3.576	9.645	-6.068
63	8907.	1.16	0.	0.	89.	1.34	40.	0.31	86.	0.58	0.	9.992	4.491	9.655	-5.164
64	8870.	1.16	0.	0.	91.	1.37	46.	0.36	99.	0.67	0.	10.259	5.186	11.161	-5.975
65	8708.	1.14	0.	0.	110.	1.66	59.	0.46	74.	0.50	0.	12.632	6.775	8.498	-1.723
66	8390.	1.09	0.	0.	117.	1.76	44.	0.34	66.	0.45	0.	13.945	5.244	7.867	-2.622
67	8023.	1.05	0.	0.	117.	1.76	33.	0.26	55.	0.37	0.	14.583	4.113	6.855	-2.742
68	7617.	0.99	0.	0.	109.	1.64	26.	0.20	57.	0.39	0.	14.310	3.413	7.483	-4.070
69	7131.	0.93	0.	0.	128.	1.93	39.	0.30	37.	0.25	0.	17.950	5.469	5.189	0.280
70	6732.	0.88	0.	0.	143.	2.15	21.	0.16	41.	0.28	0.	21.242	3.119	6.090	-2.971
71	6499.	0.85	0.	0.	151.	2.27	21.	0.16	32.	0.22	0.	23.234	3.231	4.924	-1.693
72	6305.	0.82	0.	0.	162.	2.44	24.	0.19	24.	0.16	0.	25.694	3.807	3.807	0.
73	6034.	0.79	0.	0.	183.	2.76	21.	0.16	28.	0.19	0.	30.328	3.480	4.640	-1.160
74	5570.	0.73	0.	0.	187.	2.82	16.	0.12	23.	0.16	0.	33.573	2.873	4.129	-1.257
75	5177.	0.68	0.	0.	184.	2.77	13.	0.10	30.	0.20	0.	35.542	2.511	5.795	-3.284
76	4840.	0.63	0.	0.	203.	3.06	16.	0.12	16.	0.11	0.	41.942	3.306	3.306	0.
77	4499.	0.59	0.	0.	207.	3.12	10.	0.08	27.	0.18	0.	46.010	2.223	6.001	-3.779
78	4201.	0.55	0.	0.	240.	3.62	14.	0.11	15.	0.10	0.	57.129	3.333	3.571	-0.238
79	3793.	0.49	0.	0.	237.	3.57	20.	0.16	10.	0.07	0.	62.484	5.273	2.636	2.636
80	3499.	0.46	0.	0.	254.	3.83	9.	0.07	13.	0.09	0.	72.592	2.572	3.715	-1.143
81	3258.	0.43	0.	0.	226.	3.40	9.	0.07	6.	0.04	0.	69.368	2.762	1.842	0.921
82	2945.	0.38	0.	0.	263.	3.96	7.	0.05	12.	0.08	0.	89.304	2.377	4.075	-1.698
83	2527.	0.33	0.	0.	262.	3.95	4.	0.03	10.	0.07	0.	103.680	1.583	3.957	-2.374
84	2132.	0.28	0.	0.	243.	3.66	4.	0.03	10.	0.07	0.	113.977	1.876	4.690	-2.814
85	8789.	1.15	0.	0.	1577.	23.75	23.	0.18	15.	0.10	0.	179.429	2.617	1.707	0.910
tot	766560.	100.00	9991.	100.00	6639.	100.00	12884.	100.00	14729.	100.00					
gross											0.817	1.221	1.251	1.431	
crude(x1000)											0.	8.661	16.808	19.214	-2.407
m.age		38.17		27.53		73.06		25.32		26.61	27.59	76.26	26.59	28.35	
e(0)												78.24			

SWEDEN'74 - FEMALES - 2 REGIONS

REST

age	population number	population - % -	births number	births - % -	deaths number	deaths - % -	arrivals number	arrivals - % -	departures number	departures - % -	observed rates (x 1000) birth	death	inmig	outmig	net mig
0	44419.	1.33	0.	0.	324.	1.01	396.	2.69	242.	1.88	0.	7.294	8.915	5.448	3.467
1	43551.	1.31	0.	0.	54.	0.17	399.	2.71	222.	1.72	0.	1.240	9.162	5.097	4.064
2	44235.	1.33	0.	0.	20.	0.05	404.	2.74	225.	1.75	0.	0.452	9.133	5.086	4.047
3	43884.	1.32	0.	0.	13.	0.04	318.	2.16	202.	1.57	0.	0.296	7.246	4.603	2.643
4	43085.	1.29	0.	0.	10.	0.03	281.	1.91	175.	1.36	0.	0.232	6.522	4.062	2.460
5	44284.	1.33	0.	0.	16.	0.05	268.	1.82	179.	1.39	0.	0.361	6.052	4.042	2.010
6	47104.	1.41	0.	0.	22.	0.07	282.	1.91	178.	1.38	0.	0.467	5.987	3.779	2.208
7	48829.	1.47	0.	0.	19.	0.06	242.	1.64	131.	1.02	0.	0.389	4.956	2.683	2.273
8	49200.	1.48	0.	0.	12.	0.04	243.	1.65	154.	1.20	0.	0.244	4.939	3.130	1.809
9	49265.	1.48	0.	0.	6.	0.02	214.	1.45	129.	1.00	0.	0.122	4.344	2.618	1.725
10	47302.	1.42	0.	0.	6.	0.02	175.	1.19	118.	0.92	0.	0.127	3.700	2.495	1.205
11	44311.	1.33	0.	0.	4.	0.01	145.	0.98	83.	0.64	0.	0.090	3.272	1.873	1.399
12	42832.	1.29	0.	0.	8.	0.02	148.	1.00	111.	0.86	0.	0.187	3.455	2.592	0.864
13	42085.	1.26	0.	0.	11.	0.03	148.	1.00	69.	0.54	0.	0.261	3.517	1.640	1.877
14	42226.	1.27	1.	0.00	7.	0.02	98.	0.67	80.	0.62	0.024	0.166	2.321	1.895	0.426
15	42666.	1.28	13.	0.03	15.	0.05	137.	0.93	106.	0.82	0.305	0.352	3.211	2.484	0.727
16	43237.	1.30	92.	0.21	12.	0.04	148.	1.00	179.	1.39	2.128	0.278	3.423	4.140	-0.717
17	43706.	1.31	403.	0.93	18.	0.06	208.	1.41	396.	3.07	9.221	0.412	4.759	9.061	-4.301
18	43739.	1.31	776.	1.80	22.	0.07	277.	1.88	646.	5.01	17.742	0.503	6.333	14.769	-8.436
19	43488.	1.31	1333.	3.09	16.	0.05	313.	2.13	710.	5.51	30.652	0.368	7.197	16.326	-9.129
20	43813.	1.31	1830.	4.24	30.	0.09	422.	2.87	739.	5.74	41.768	0.685	9.632	16.867	-7.235
21	44573.	1.34	2276.	5.27	14.	0.04	438.	2.97	722.	5.60	51.062	0.314	9.827	16.198	-6.372
22	44653.	1.34	2767.	6.40	15.	0.05	489.	3.32	668.	5.18	61.967	0.336	10.951	14.960	-4.009
23	45690.	1.37	3009.	6.96	19.	0.06	542.	3.68	692.	5.37	65.857	0.416	11.863	15.146	-3.283
24	47906.	1.44	3320.	7.68	17.	0.05	551.	3.74	564.	4.38	69.302	0.355	11.502	11.773	-0.271
25	50074.	1.50	3580.	8.29	23.	0.07	622.	4.22	546.	4.24	71.494	0.459	12.422	10.904	1.518
26	51141.	1.53	3667.	8.49	21.	0.07	587.	3.99	458.	3.55	71.704	0.411	11.478	8.956	2.522
27	51745.	1.55	3460.	8.01	24.	0.07	533.	3.62	420.	3.26	66.866	0.464	10.301	8.117	2.184
28	52008.	1.56	3169.	7.33	26.	0.08	523.	3.55	340.	2.64	60.933	0.500	10.056	6.537	3.519
29	51299.	1.54	2801.	6.48	24.	0.07	412.	2.80	300.	2.33	54.601	0.468	8.031	5.848	2.183
30	49627.	1.49	2354.	5.45	28.	0.09	348.	2.36	263.	2.04	47.434	0.564	7.012	5.300	1.713
31	46364.	1.39	1992.	4.61	30.	0.09	348.	2.36	231.	1.79	42.964	0.647	7.506	4.982	2.524
32	42040.	1.26	1509.	3.49	15.	0.05	270.	1.83	151.	1.17	35.894	0.357	6.422	3.592	2.831
33	39102.	1.17	1153.	2.67	30.	0.09	212.	1.44	158.	1.23	29.487	0.767	5.422	4.041	1.381
34	38959.	1.17	870.	2.01	23.	0.07	215.	1.46	137.	1.06	22.331	0.590	5.519	3.517	2.002
35	38805.	1.16	702.	1.62	22.	0.07	195.	1.32	131.	1.02	18.090	0.567	5.025	3.376	1.649
36	37384.	1.12	602.	1.39	32.	0.10	149.	1.01	87.	0.68	16.103	0.856	3.986	2.327	1.658
37	36471.	1.09	426.	0.99	37.	0.12	143.	0.97	98.	0.76	11.681	1.015	3.921	2.687	1.234
38	35624.	1.07	362.	0.84	29.	0.09	143.	0.97	80.	0.62	10.162	0.814	4.014	2.246	1.768
39	34928.	1.05	235.	0.54	44.	0.14	107.	0.73	76.	0.59	6.728	1.260	3.063	2.176	0.888
40	34687.	1.04	204.	0.47	35.	0.11	95.	0.64	68.	0.53	5.881	1.009	2.739	1.960	0.778
41	35428.	1.06	117.	0.27	39.	0.12	94.	0.64	63.	0.49	3.302	1.101	2.653	1.778	0.875
42	36549.	1.10	78.	0.18	54.	0.17	98.	0.67	67.	0.52	2.134	1.477	2.681	1.833	0.848
43	37303.	1.12	51.	0.12	58.	0.18	62.	0.42	77.	0.60	1.367	1.555	1.662	2.064	-0.402
44	37498.	1.13	29.	0.07	62.	0.19	82.	0.56	60.	0.47	0.773	1.653	2.187	1.600	0.587
45	37856.	1.14	16.	0.04	77.	0.24	97.	0.66	58.	0.45	0.423	2.034	2.562	1.532	1.030
46	38254.	1.15	8.	0.02	74.	0.23	87.	0.59	65.	0.50	0.209	1.934	2.274	1.699	0.575
47	38464.	1.15	4.	0.01	91.	0.28	78.	0.53	49.	0.38	0.104	2.366	2.028	1.274	0.754
48	39602.	1.19	0.	0.	95.	0.30	77.	0.52	74.	0.57	0.	2.399	1.944	1.869	0.076

49	40522.	1.22	0.	0.	128.	0.40	86.	0.58	45.	0.35	0.	3.159	2.122	1.111	1.012
50	41198.	1.24	0.	0.	119.	0.37	78.	0.53	60.	0.47	0.	2.888	1.893	1.456	0.437
51	41925.	1.26	0.	0.	127.	0.40	63.	0.43	49.	0.38	0.	3.029	1.503	1.169	0.334
52	43739.	1.31	0.	0.	141.	0.44	88.	0.60	47.	0.36	0.	3.224	2.012	1.075	0.937
53	46601.	1.40	0.	0.	189.	0.59	73.	0.50	45.	0.35	0.	4.056	1.566	0.966	0.601
54	43789.	1.31	0.	0.	203.	0.63	78.	0.53	41.	0.32	0.	4.636	1.781	0.936	0.845
55	39550.	1.19	0.	0.	181.	0.56	87.	0.59	44.	0.34	0.	4.576	2.200	1.113	1.087
56	40034.	1.20	0.	0.	190.	0.59	81.	0.55	39.	0.30	0.	4.746	2.023	0.974	1.049
57	40281.	1.21	0.	0.	235.	0.73	63.	0.43	37.	0.29	0.	5.834	1.564	0.919	0.645
58	39980.	1.20	0.	0.	255.	0.79	73.	0.50	29.	0.23	0.	6.378	1.826	0.725	1.101
59	40712.	1.22	0.	0.	251.	0.78	69.	0.47	38.	0.29	0.	6.165	1.695	0.933	0.761
60	41265.	1.24	0.	0.	282.	0.88	73.	0.50	42.	0.33	0.	6.834	1.769	1.018	0.751
61	40990.	1.23	0.	0.	335.	1.04	79.	0.54	39.	0.30	0.	8.173	1.927	0.951	0.976
62	40621.	1.22	0.	0.	312.	0.97	89.	0.60	33.	0.26	0.	7.681	2.191	0.812	1.379
63	39956.	1.20	0.	0.	341.	1.06	86.	0.58	40.	0.31	0.	8.532	2.152	1.001	1.151
64	39934.	1.20	0.	0.	408.	1.27	99.	0.67	46.	0.36	0.	10.204	2.476	1.150	1.326
65	39657.	1.19	0.	0.	495.	1.54	74.	0.50	59.	0.46	0.	12.482	1.866	1.488	0.378
66	38323.	1.15	0.	0.	520.	1.62	66.	0.45	44.	0.34	0.	13.569	1.722	1.148	0.574
67	37081.	1.11	0.	0.	538.	1.68	55.	0.37	33.	0.26	0.	14.509	1.483	0.890	0.593
68	35944.	1.08	0.	0.	598.	1.86	57.	0.39	26.	0.20	0.	16.637	1.586	0.723	0.862
69	34727.	1.04	0.	0.	686.	2.14	37.	0.25	39.	0.30	0.	19.754	1.065	1.123	-0.058
70	33256.	1.00	0.	0.	700.	2.18	41.	0.28	21.	0.16	0.	21.049	1.233	0.631	0.601
71	32277.	0.97	0.	0.	748.	2.33	32.	0.22	21.	0.16	0.	23.174	0.991	0.651	0.341
72	31282.	0.94	0.	0.	848.	2.64	24.	0.16	24.	0.19	0.	27.108	0.767	0.767	0.
73	29672.	0.89	0.	0.	886.	2.76	28.	0.19	21.	0.16	0.	29.860	0.944	0.708	0.236
74	27644.	0.83	0.	0.	1004.	3.13	23.	0.16	16.	0.12	0.	36.319	0.832	0.579	0.253
75	25764.	0.77	0.	0.	1008.	3.14	30.	0.20	13.	0.10	0.	39.124	1.164	0.505	0.660
76	24061.	0.72	0.	0.	1117.	3.48	16.	0.11	16.	0.12	0.	46.424	0.665	0.665	0.
77	22174.	0.67	0.	0.	1157.	3.60	27.	0.18	10.	0.08	0.	52.178	1.218	0.451	0.767
78	20593.	0.62	0.	0.	1160.	3.61	15.	0.10	14.	0.11	0.	56.330	0.728	0.680	0.049
79	18879.	0.57	0.	0.	1228.	3.82	10.	0.07	20.	0.16	0.	65.046	0.530	1.059	-0.530
80	16993.	0.51	0.	0.	1197.	3.73	13.	0.09	9.	0.07	0.	70.441	0.765	0.530	0.235
81	15113.	0.45	0.	0.	1278.	3.98	6.	0.04	9.	0.07	0.	84.563	0.397	0.596	-0.199
82	13427.	0.40	0.	0.	1243.	3.87	12.	0.08	7.	0.05	0.	92.575	0.894	0.521	0.372
83	11768.	0.35	0.	0.	1247.	3.88	10.	0.07	4.	0.03	0.	105.965	0.850	0.340	0.510
84	10169.	0.31	0.	0.	1185.	3.69	10.	0.07	4.	0.03	0.	116.531	0.983	0.393	0.590
85	40714.	1.22	0.	0.	7870.	24.51	15.	0.10	23.	0.18	0.	193.300	0.368	0.565	-0.196
tot	3331970.	100.00	43209.	100.00	32113.	100.00	14729.	100.00	12884.	100.00					
gross											0.931	1.268	0.339	0.294	
crude(x1000)											0.	9.638	4.421	3.867	0.554
m.age		38.66		26.99		74.05		26.61		25.32	27.16	76.65	28.28	26.60	
e(0)												78.16			

APPENDIX B: PROJECTED GROWTH RATES OF SWEDISH FEMALES, 1975-2024

Year	1	2	3	4	5
1975	0.564200	0.333100	0.399800	0.338200	0.338500
1976	0.318300	0.298000	0.330900	0.308300	0.308600
1977	0.282100	0.261000	0.289200	0.275100	0.275400
1978	0.235100	0.223300	0.245800	0.239900	0.240100
1979	0.206700	0.186300	0.203100	0.204000	0.204100
1980	0.173100	0.151000	0.162300	0.168500	0.168600
1981	0.134300	0.118200	0.124600	0.134600	0.134700
1982	0.089500	0.088900	0.091000	0.103600	0.103600
1983	0.084500	0.063700	0.062400	0.076500	0.076400
1984	0.052600	0.043000	0.038700	0.053200	0.053100
1985	0.046400	0.025700	0.021800	0.033500	0.033400
1986	0.039000	0.010700	0.018000	0.016500	0.016400
1987	-0.006400	-0.004300	0.002900	0.002000	0.001800
1988	-0.046100	-0.017000	-0.011500	-0.010600	-0.010800
1989	-0.046200	-0.027800	-0.021400	-0.021500	-0.021700
1990	-0.021300	-0.037900	-0.031100	-0.031500	-0.031700
1991	-0.036900	-0.047500	-0.041100	-0.040900	-0.041200
1992	-0.051000	-0.056500	-0.052200	-0.050000	-0.050200
1993	-0.056200	-0.065000	-0.057700	-0.058900	-0.059000
1994	-0.063700	-0.074200	-0.068000	-0.067900	-0.068000
1995	-0.118300	-0.083200	-0.075000	-0.076900	-0.077000
1996	-0.135300	-0.092100	-0.082000	-0.085300	-0.085400
1997	-0.093700	-0.100900	-0.094800	-0.093500	-0.093600
1998	-0.120000	-0.108800	-0.106900	-0.101600	-0.101700
1999	-0.142400	-0.115600	-0.113000	-0.109300	-0.109400
2000	-0.169600	-0.122200	-0.115000	-0.116500	-0.116600
2001	-0.173500	-0.129100	-0.122400	-0.123100	-0.123200
2002	-0.121300	-0.135700	-0.129700	-0.129600	-0.129600
2003	-0.142600	-0.141900	-0.135400	-0.135800	-0.135900
2004	-0.166800	-0.147800	-0.141000	-0.141500	-0.141600
2005	0.052800	-0.154300	-0.154100	-0.147900	-0.148000
2006	-0.051700	-0.160600	-0.162800	-0.155200	-0.155300
2007	-0.350300	-0.164400	-0.160400	-0.160300	-0.160400
2008	-0.270200	-0.166400	-0.165100	-0.162200	-0.162400
2009	-0.197800	-0.166400	-0.168100	-0.162700	-0.162900
2010	-0.191900	-0.165200	-0.170500	-0.162400	-0.162600
2011	-0.200400	-0.163300	-0.169600	-0.161700	-0.162000
2012	-0.200600	-0.161200	-0.160000	-0.161000	-0.161200
2013	-0.147600	-0.161300	-0.162000	-0.160900	-0.161100
2014	-0.158700	-0.162000	-0.165800	-0.161800	-0.162000
2015	-0.160700	-0.163300	-0.133700	-0.163800	-0.163900
2016	-0.152000	-0.172000	-0.151700	-0.167200	-0.167400
2017	-0.181000	-0.179700	-0.201000	-0.172200	-0.172300
2018	-0.205700	-0.181300	-0.195200	-0.178800	-0.178800
2019	-0.193000	-0.186900	-0.192500	-0.187200	-0.187300
2020	-0.155200	-0.196600	-0.202400	-0.197900	-0.197900
2021	-0.148000	-0.209000	-0.216900	-0.211100	-0.211000
2022	-0.165300	-0.223400	-0.232400	-0.226500	-0.226400
2023	-0.191600	-0.239300	-0.241300	-0.243400	-0.243300
2024	-0.196500	-0.257900	-0.261000	-0.261700	-0.261500

<u>KEY</u>

1 - conventional approach
2 - disaggregation approach I
3 - disaggregation approach II
4 - preferred approach linear
5 - preferred approach exponential

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